Exercise 1. Let $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^* I$ be the group of biholomorphisms of $\mathrm{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, where the class of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ acts by the homography

$$z \mapsto \frac{az+b}{cz+d} \qquad (*)$$

a) Prove that $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\}$. In other words, one can always choose $ad - bc = 1$ in $(*)$.

b) Show that if $G \subset \mathrm{PGL}_2(\mathbb{C})$ is a subgroup (with the discrete topology) whose action on $\mathrm{P}^1(\mathbb{C})$ is proper, G is finite.

c) Let α, β be the elements of $\mathrm{PGL}_2(\mathbb{C})$ defined by $\alpha(z) = 1/z$, $\beta(z) = 1-z$. Prove that $(\alpha\beta)^3 = \mathrm{id}$. Deduce that the subgroup G they generate is finite. What is its order ?

d) Let $f : \mathrm{P}^1(\mathbb{C}) \to \mathrm{P}^1(\mathbb{C})$ be the holomorphic map defined by the rational fraction

$$f(z) = (z^2 - z + 1)^3/(z(1-z))^2.$$

What is the degree of f ? Show that $f = f \circ \alpha = f \circ \beta$.

e) Deduce that f factorizes as the quotient map $p : \mathrm{P}^1(\mathbb{C}) \to \mathrm{P}^1(\mathbb{C})/G$ followed by a biholomorphism $\mathrm{P}^1(\mathbb{C})/G \to \mathrm{P}^1(\mathbb{C})$.

f) Let $G \subset \mathrm{PSL}_2(\mathbb{C})$ be a finite subgroup. Prove that G is conjugate to a subgroup of $\mathrm{PSU}_2 = \mathrm{SU}_2/\{\pm I\}$ (hint : show that its preimage $\widetilde{G}$ in SU_2 is finite, and average the standard hemitian metric of $\mathbb{C}^2$ under the action of $\widetilde{G}$).

g) Show that $\mathrm{SU}_2 \subset \mathrm{SL}_2(\mathbb{C})$ is the set of matrices $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$. It may be identified with the 3-sphere $\mathbb{S}^3 \subset \mathbb{C}^2$.

h) Let V be the real vector space of hermitian matrices of size 2 and trace 0, i.e. of the form

$$h = \begin{pmatrix} x & \overline{y} \\ y & -x \end{pmatrix}$$

with $(x, y) \in \mathbb{R} \times \mathbb{C}$. Note that $-\det(h) = x^2 + |y|^2$ is a euclidean norm on V.

Show that SU_2 acts linearly on V by $(g, h) \mapsto ghg^{-1}$, and deduce from this a group homomorphism

$$\sigma : \mathrm{SU}_2/\{\pm I\} \to \mathrm{SO}_3 = \mathrm{SO}(V, -\det).$$

Prove that σ is an isomorphism (hint : show that $\mathrm{diag}(e^{-it/2}, e^{it/2}) \in \mathrm{SU}_2$ acts on $V \simeq \mathbb{R} \times \mathbb{C}$ by $(x, y) \mapsto (x, e^{it}y)$, a rotation with axis $L = \mathbb{R}(1, 0)$; conjugate this by a suitable $g \in \mathrm{SU}_2$ to obtain the rotations with an arbitrary fixed axis in V).

Thus the finite subgroups of $\mathrm{Aut}(\mathrm{P}^1(\mathbb{C}))$ are the same as the finite subgroups of the three dimensional rotation group SO_3. We will now match the spaces $\mathrm{P}^1(\mathbb{C})$ and $\mathbb{S}^2$ on which they act.

i) Let $Z = {}^t(z_1, z_2) \in \mathbb{S}^3 \subset \mathbb{C}^2$ (a column vector of norm 1), and define $\varphi(Z)$ as the matrix $2ZZ^* - I$. Check that $\varphi(Z)$ belongs to V, and $-\det(\varphi(Z)) = 1$, so that $\varphi(Z) \in \mathbb{S}^2 \subset V$. Prove that $\varphi(Z)$ depends only on the line $\mathbb{C}Z \subset \mathbb{C}^2$, and conversely that it determines this line (consider the $+1$ eigenspace).

j) Conclude by defining a bijection $\psi : \mathbb{S}^3/\mathbb{S}^1 \simeq \mathrm{P}^1(\mathbb{C}) \to \mathbb{S}^2$ such that for all $g \in \mathrm{PSU}_2$

$$\psi \circ g = \sigma(g) \circ \psi.$$

Exercise 2.

a) Let X be a compact connected Riemann surface, G a finite subgroup of $\mathrm{Aut}(X)$, and $\pi : X \to Y = X/G$ the quotient map of Riemann surfaces.

Show that for each $y \in Y$, there is $m_y \geq 1$ dividing the order $|G|$ of G, such that all points $x \in \pi^{-1}(y)$ have $e_\pi(x) = m_y$, and that $\operatorname{card} \pi^{-1}(y) = |G|/m_y$.

b) Assume now that $X = \mathrm{P}^1(\mathbb{C})$, so that Y is isomorphic to $\mathrm{P}^1(\mathbb{C})$. Let $B \subset Y$ be the finite set of points y with $m_y \geq 2$ (the branching set of π).

Using the Riemann-Hurwitz formula, show that B has either 0, 2 or 3 elements, with the first two cases corresponding respectively to G trivial or cyclic.

Assuming $|B| = 3$, let m_1, m_2, m_3 be the values of m_y for $y \in B$, in increasing order. Show that (m_1, m_2, m_3) and $|G|$ are necessarily one of the following

$$(2, 2, n) \qquad |G| = 2n \quad (n \geq 2)$$

$$(2, 3, 3) \qquad |G| = 12$$
$$(2, 3, 4) \qquad |G| = 24$$
$$(2, 3, 5) \qquad |G| = 60.$$

To construct such G in $\mathrm{Aut}(\mathrm{P}^1(\mathbb{C}))$, one can also by exercise 1 look for them in SO_3 acting on $\mathbb{S}^2$. Let us show how to achieve this for two examples, leaving the two others as challenges for the interested.

c) Show that the subgroup G of SO_3 generated by two half turns R, S whose axes are at angle π/n is of order $2n$. Show that on $\mathbb{S}^2$, G has two orbits with stabilizers of order 2 and one with stabilizers of order n.

d) Show that the group $G \subset \mathrm{SO}_3$ preserving the cube $[-1, 1]^3$ is of order 24 (hint : consider its action on oriented edges). Show that G has on $\mathbb{S}^2$ one orbit with stabilizers of order 2, one with stabilizers of order 3, and one with stabilizers of order 4.