**Exercise 1**. Let $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half-plane in $\mathbb{C}$.
Recall that the biholomorphisms of $H$ are the maps

$$z \mapsto \gamma(z) = \frac{az+b}{cz+d},$$

with the real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ determined up to a scalar factor in $\mathbb{R}^*$ and $ad - bc > 0$. Denote by $\mathrm{Aut}(H)$ the corresponding group, isomorphic to $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ (the above transformations with $ad - bc = 1$).

**a)** If $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup, show that its image $\overline{\Gamma}$ in $\mathrm{Aut}(H)$ acts freely on $H$ (i.e. with trivial stabilizers) if and only if it has no torsion (hint : use that the action is proper).

**b)** Show that if $A \in \mathrm{SL}_2(\mathbb{Z})$ is of finite order $n$, then $n$ divides 4 or 6, and that $n = 2$ only for $A = -I$.

**c)** The discrete subgroup $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{R})$ acts properly on $H$, whith quotient Riemann surface denoted by $Y(1) = H/\overline{\Gamma}(1)$, and projection $p : H \to Y(1)$.
Determine the branch points of the map $p$ in $Y(1)$ (hint : find the orbits of points of $H$ with non-trivial stabilizer in $\overline{\Gamma}(1)$).

**d)** For $N \geq 1$, let $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$ be kernel of the reduction modulo $N$ morphism to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Show that its image $\overline{\Gamma}(N)$ in $\mathrm{Aut}(H)$ has no torsion for $N \geq 2$.

**Exercise 2**. Let $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half-plane in $\mathbb{C}$.

**a)** Show that the group $\mathrm{Aut}(H)$ is generated by the following transformations
— the translations $z \mapsto z + t$, $t \in \mathbb{R}$
— the scalings $z \mapsto \lambda z$, $\lambda \in \mathbb{R}_+^*$
— the symmetry $z \mapsto -1/z$.

**b)** Show that the subgroup $B \subset \mathrm{Aut}(H)$ of affine transformations $z \mapsto az + b$, $a \in \mathbb{R}_+^*$, $b \in \mathbb{R}$ acts transitively on $H$ with trivial stabilizers (it is "simply transitive" on $H$).

**c)** Determine the stabilizer of the point $i \in H$ in $\mathrm{Aut}(H)$, and its action on the tangent space $T_i H$.

**d)** Show that for all points $z_1, z_2 \in H$, there exists $g \in \mathrm{Aut}(H)$ such that $g(z_1) = i$ and $g(z_2) = iy$ with $y \in [1, +\infty[$.

**e)** Check that the Riemannnian metric

$$\eta = \frac{|dz|^2}{\mathrm{Im}(z)^2} = \frac{dx^2 + dy^2}{y^2}$$

is invariant by $\mathrm{Aut}(H)$. One calls $(H, \eta)$ the *Poincaré upper half-plane*.

**f)** Show that the positive vertical axis is a minimizing geodesic in $(H, \eta)$, i.e. that intervals in it realize the minimal $\eta$-length of $C^1$ paths between their endpoints. Show that moreover they are the *unique* (non-parametrized) geodesic segments between their endpoints. Compute the Riemannian distance $d_\eta(iy_1, iy_2)$ for $y_1, y_2 \in \mathbb{R}_+^*$.

**g)** What is the image by $z \mapsto -1/z$ of the vertical half-line $1 + i\mathbb{R}_+^*$ ?

**h)** Deduce that the vertical half-lines and the half-circles centered on the real axis are minimizing geodesics in $(H, \eta)$, and that they represent all geodesics in $(H, \eta)$ (you can use, or admit, that a geodesic segment is determined by one of its endpoint and its tangent direction there).

**i)** Consider the "ideal boundary" $\partial_\infty H = \mathbb{R} \cup \{\infty\} = P^1(\mathbb{R})$ of $H$, with the topology induced by its embedding in $P^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, and note that $\mathrm{Aut}(H)$ is a subgroup of $\mathrm{Aut}(P^1(\mathbb{C}))$.

Show that $\mathrm{Aut}(H)$ acts transitively on $\partial_\infty H$, with stabilizer of the point $\infty$ equal to the affine subgroup $B$ (question **??**).

**j)** Show that $\mathrm{Aut}(H)$ acts transitively on the couples of distinct point of $\partial_\infty H$, with stabilizer of $(0, \infty)$ equal to the group of scalings (question **??**).

**k)** Show that the oriented bi-infinite geodesics in $H$ are uniquely determined by the ordered pair of their two "ideal" endpoints in $\partial_\infty H$. By the previous question, $\mathrm{Aut}(H)$ acts transitively on them.

**l)** Show that $\mathrm{Aut}(H)$ acts sharply 3-transitively on $\partial_\infty H$, i.e. transitively on 3-tuples of distinct points, with trivial stabilizers.

**m)** Associated to $\eta$ is its (oriented) Riemannian area measure $dA_\eta$. Show that it equals $\frac{dx \wedge dy}{y^2}$. Also express it as a suitable multiple of $dz \wedge d\bar{z}$ for $z = x + iy$.

**n)** Compute the area of the "ideal" (geodesic) triangle $T = \{z \in H \mid -1 < \mathrm{Re}(z) < 1, |z| > 1\}$ ?

**o)** Deduce that all "ideal triangles" in $H$ have the same area $\pi$.

**p)** Compute the area of the triangle $T' = \{z \in H \mid -1/2 < \mathrm{Re}(z) < 1/2, |z| > 1\}$.