Correction of exercise 1

Exercise 1. Let $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half-plane in $\mathbb{C}$.
Recall that the biholomorphisms of H are the maps

$$z \mapsto \gamma(z) = \frac{az+b}{cz+d},$$

with the real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ determined up to a scalar factor in $\mathbb{R}^*$ and $ad - bc > 0$. Denote by $\mathrm{Aut}(H)$ the corresponding group, isomorphic to $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ (the above transformations with $ad - bc = 1$).

a) *If $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup, show that its image $\overline{\Gamma}$ in $\mathrm{Aut}(H)$ acts freely on H (i.e. with trivial stabilizers) if and only if it has no torsion (hint : use that the action is proper).*

By properness of the action, stabilizers are finite, so if Γ has no torsion, they are trivial. For the converse, it is enough to show that a torsion element $\gamma \neq \mathrm{id}$ of $\mathrm{Aut}(H)$ has a fixed point in H. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix in $\mathrm{SL}_2(\mathbb{R})$ defining γ, and $n > 1$ the order of γ, A^n is scalar, hence equals $\pm I$. So $A^{2n} = I$, and A is not scalar since $\gamma \neq \mathrm{id}$. In particular, A is diagonalizable over $\mathbb{C}$ (its minimal polynomial has no multiple root), and its two eigenvalues λ, λ^{-1} are different (otherwise A would be scalar). They are roots of unity, necessarily of order > 2, so are not real, and $\lambda^{-1} = \overline{\lambda}$.
In particular A has no real eigenvector, and if $v = (v_1, v_2) \in \mathbb{C}^2 \setminus \{0\}$ is an eigenvector, this implies that $v_1, v_2 \neq 0$, otherwise $(1, 0)$ or $(0, 1)$ would be a real eigenvector, and $z = v_1/v_2 \in \mathbb{C}$ is not real, otherwise $(v_1/v_2, 1)$ would be a real eigenvector.
But the equation $Av = \lambda v$ implies $\gamma(z) = z$ and since A is real, the other eigendirection is generated by $\overline{v} = (\overline{v}_1, \overline{v}_2)$, giving $\gamma(\overline{z}) = \overline{z}$. One of the fixed points $z, \overline{z}$ of γ is then in H.

b) *Show that if $A \in \mathrm{SL}_2(\mathbb{Z})$ is of finite order n, then n divides 4 or 6, and that $n = 2$ only for $A = -I$.*

Being of finite order, A is diagonalizable over $\mathbb{C}$, and let λ, λ^{-1} be the eigenvalues of A. They are roots of unity of order equal to n, and $\lambda + \lambda^{-1} = \mathrm{tr}(A)$ is an integer. Hence either $\lambda = \lambda^{-1} = \pm 1$ and $A = \pm I$, $n = 1$ or 2, or λ, λ^{-1} are non-real, with real part in $\frac{1}{2}\mathbb{Z}$, so necessarily in $\{0, \pm 1/2\}$, and this implies $\lambda \in \{\pm i, \pm j^{\pm 1}\}$, so $n \in \{3, 4, 6\}$.

c) *The discrete subgroup $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{R})$ acts properly on H, whith quotient Riemann surface denoted by $Y(1) = H/\overline{\Gamma}(1)$, and projection $p : H \to Y(1)$.*
Determine the branch points of the map p in $Y(1)$ (hint : find the orbits of points of H with non-trivial stabilizer in $\overline{\Gamma}(1)$).

The branch points of the quotient map $p : H \to H/\overline{\Gamma}(1)$ are the images of the ramification points, which are the $z \in H$ with $e_p(z) > 1$, and $e_p(z)$ is equal to the order of the stabilizer of z. Hence the branch points of p are the images of points with non-trivial stabilizers, so they correspond bijectively to the $\overline{\Gamma}(1)$-orbits of such points. More precisely, $b \in H/\overline{\Gamma}(1)$ is a branch point if and only if $p^{-1}(b)$ is an orbit with non-trivial stabilizers.
By the previous questions, they are the eigendirections in $z \in H \subset \mathrm{P}^1(\mathbb{C})$ of the $A \in \mathrm{SL}_2(\mathbb{Z})$ which are of order 3, 4 or 6 (recall that $\mathrm{P}^1(\mathbb{C})$ is the set of lines in $\mathbb{C}^2$, with z representing the line generated by $(z, 1) \in \mathbb{C}^2$).
If $z \in H$ is such a point, its orbit under $\Gamma(1)$ intersects the fundamental domain

$$D = \{z \in H \mid -1/2 \leq \mathrm{Re}(z) \leq 1, |z| \geq 1\}$$

(proof given in lecture).

So let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ having eigenvector $(z, 1)$, with $z \in D$, and eigenvalue λ of order 3, 4 or 6, so that $\mathrm{Re}(\lambda) \in \{0, \pm 1/2\}$. Replacing if necessary A by $-A$, we may assume $c \geq 0$.

From $cz + d = \lambda$, we deduce $c\,\mathrm{Im}(z) = \mathrm{Im}(\lambda) \leq 1$, and since $\mathrm{Im}(z) \geq \sqrt{3}/2$, one deduces $c = 1$ and $\mathrm{Im}(\lambda) > 0$. But then $z = \lambda - d$, and this is in D only if $d = 0$ and $z = \lambda$, or $d = \pm 1$ and $z = \pm 1/2 + i\sqrt{3}/2$. In both cases, z is in the $\overline{\Gamma}(1)$-orbit of one of the points i, j, which have non-trivial stabilizers of orders 2 and 3.

Hence the branch points of p are $p(i)$ and $p(j)$.

d) *For $N \geq 1$, let $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$ be kernel of the reduction modulo N morphism to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Show that its image $\overline{\Gamma}(N)$ in $\mathrm{Aut}(H)$ has no torsion for $N \geq 2$.*

Let $A = I + NB \in \Gamma(N)$, with $B \in M_2(\mathbb{Z})$. If its image in $\mathrm{PSL}_2(\mathbb{Z})$ is non-trivial and of finite order, we know that A must be of order $n \in \{3, 4, 6\}$, and $\mathrm{tr}(A) \in \{0, \pm 1\}$. But $\mathrm{tr}(A) = 2 + N\,\mathrm{tr}(B)$, so if $N \geq 4$ this is impossible.

To also treat the cases $N \in \{2, 3\}$, note that $\det(I + NB) = 1 + N\,\mathrm{tr}(B) + N^2 \det(B)$ (value at 1 of the characteristic polynomial of $-NB$). This must be 1, so N divides $\mathrm{tr}(B)$. But then $\mathrm{tr}(A) = 2 + N\,\mathrm{tr}(B) \equiv 2 \pmod{N^2}$, and this is impossible if $N \geq 2$.