Exercise 1. For an integer $N \geq 1$ let

$$\Gamma(N) = \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv I \pmod N\}$$

and denote by $\overline{\Gamma}(N)$ the image of the group $\Gamma(N)$ in $\mathrm{PSL}_2(\mathbb{Z}) \subset \mathrm{Aut}(H)$, where H denotes Poincaré's upper half-plane. Let also

$$p_N : H \to H/\overline{\Gamma}(N) = Y(N)$$

denote the quotient map and quotient Riemann surface.

a) Show that $\Gamma(N)$ is a finite index normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and that it contains $-I$ iff $N \leq 2$. Deduce that $\Gamma(N)$ is isomorphic to $\overline{\Gamma}(N)$ for $N > 2$.

b) Show that there is a unique map $\varphi_N : Y(N) \to Y(1)$ such that $\varphi_N \circ p_N = p_1$.

c) Show that the (finite) quotient group

$$G_N = \overline{\Gamma}(1)/\overline{\Gamma}(N) \simeq \Gamma(1)/(\{\pm I\}.\Gamma(N))$$

acts on $Y(N)$, and that there is an isomorphism $i_N : Y(N)/G_N \simeq Y(1)$ such that $i_N \circ q_N = \varphi_N$, where $q_N : Y(N) \to Y(N)/G_N$ denotes the quotient map.

d) Show that $\Gamma(2)$ is of index 6 in $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ (hint : show that the "reduction mod 2" morphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ is surjective [1]).

e) Show that $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to the symmetric group S_3.

f) Deduce that $\varphi_2 : Y(2) \to Y(1)$ is a proper holomorphic map of degree 6.

g) Let

$$D = \{z \in H \mid |\mathrm{Re}(z)| \leq 1, |z \pm 1/2| \geq 1/2\}.$$

Show that for all $z \in H$, there exists $\gamma \in \overline{\Gamma}(2) \subset \mathrm{Aut}(H)$ such that $\gamma(z) \in D$. In words, D meets every orbit of $\overline{\Gamma}(2)$ on H (hint : maximize $\mathrm{Im}(\gamma(z))$ for $\gamma \in \overline{\Gamma}(2)$, then use translations and the two elements $z \mapsto z/(\pm 2z+1)$ of $\overline{\Gamma}(2)$).

h) Show that D also meets all the orbits in H of the subgroup $\langle \alpha, \beta \rangle$ of $\overline{\Gamma}(2)$ generated by

$$\alpha = T^2 : z \mapsto z+2, \quad \beta = ST^{-2}S : z \mapsto z/(2z+1),$$

where $S : z \mapsto -1/z$ and $T : z \mapsto z+1$ are in $\overline{\Gamma}(1)$ (hint : re-use the proof of previous question).

i) Show that the elements of

$$E = \{1, S, T, TS, TST, (TS)^2\} \subset \overline{\Gamma}(1)$$

have distinct classes in $\overline{\Gamma}(1)/\overline{\Gamma}(2) \simeq \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$, and represent all elements of this group.

j) Let

$$D_1 = \{z \in H \mid |\mathrm{Re}(z)| \leq 1/2, |z| \geq 1\}$$

denote the "standard" fundamental domain of $\overline{\Gamma}(1)$ in H. Show that

$$D_2 = \bigcup_{g \in E} gD_1$$

is a weak fundamental domain of $\Gamma(2)$ in H. Here by weak fundamental domain we mean that all z in a dense open subset of D_2 are the unique point of their $\overline{\Gamma}(2)$-orbit in D_2

1. More generally, $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective, but here it is very easy to verify directly

k) Let
$$D_1' = \{z \in H \mid 0 \le \mathrm{Re}(z) \le 1/2, |z| \ge 1\}.$$
It the right half of the "standard" fundamental domain of $\overline{\Gamma}(1)$. Introduce the (antiholomorphic) reflections in the sides of D_1'

$$s_0 : z \mapsto -\overline{z}, \quad s_1 : z \mapsto 1/\overline{z}, \quad s_2 : z \mapsto 1 - \overline{z},$$

which fix respectively the points with $\mathrm{Re}(z) = 0$, $|z| = 1$, $\mathrm{Re}(z) = 1/2$. Then $D_1' \cup s_0 D_1'$ is the above standard fundamental domain D_1 of $\overline{\Gamma}(1)$, and $s_0 s_1 = S$, $s_2 s_0 = T$, $s_2 s_1 = TS$.

Show that D and D_2 are both unions of images of D_1' by twelve compositions of the maps s_0, s_1, s_2.

l) Deduce that D is also a (weak) fundamental domain for $\overline{\Gamma}(2)$ (study the decompositions of D and D_2 into images of D_1' obtained in the previous question, and move some pieces by elements of $\overline{\Gamma}(2)$ to go from D_2 to D).

m) Deduce that α, β generate the group $\overline{\Gamma}(2)$ (hint : choose $z_0 \in D$ which the unique point of d in its orbit under $\overline{\Gamma}(2)$, and for any $\gamma \in \overline{\Gamma}(2)$, find $g \in \langle \alpha, \beta \rangle$ such that $g(\gamma(z_0))) = z_0$; then use that $\overline{\Gamma}(2)$ acts freely on H — it has no torsion, see previous exercise sheet).

n) Consider the action of $\overline{\Gamma}(2)$ on $P^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. Show that it has three orbits, represented by $0, 1$ and ∞.