Riemann surfaces
Final examination (duration : 3 hours)

NB. In the hidden part of your sheets, please write your name and some student number. In the non-hidden part, please write only the student number.

The only authorized documents are the notes of the course. You can compose in English or in French. The arguments must be sufficiently detailed. Accuracy and clarity will also be taken into consideration.

Exercise 1

We first consider the Riemann surface $X = \mathbf{P}^1(\mathbf{C})$.

1. Determine the dimension and a basis of the Riemann–Roch space $\mathcal{L}([0]+[1])$.
2. Let D and D' be two divisors on X. Define a natural $\mathbf{C}$-linear map
$$\mu : \mathcal{L}(D) \otimes_{\mathbf{C}} \mathcal{L}(D') \to \mathcal{L}(D+D').$$
3. Show that if D and D' are effective divisors $(D, D' \geq 0)$ then μ is surjective.

We now look at the Riemann surface $X = \mathbf{C}/\Lambda$, where Λ is a lattice in $\mathbf{C}$. We consider the divisors $D = n[0]$ and $D' = n'[0]$, where $n, n' \geq 1$ are two positive integers.

4. Let $\mu : \mathcal{L}(D) \otimes_{\mathbf{C}} \mathcal{L}(D') \to \mathcal{L}(D+D')$ be the map defined as in Question 2. Under which conditions on n, n' is the map μ surjective ? (You may give examples or counterexamples.)

Exercise 2

Let σ be the holomorphic involution of $\mathbf{P}^1(\mathbf{C})$ defined by $\sigma(z) = 1/z$. The group $G = \{1, \sigma\}$ acts on $\mathbf{P}^1(\mathbf{C})$ and leaves stable $\mathbf{C}^\times$.

1. Determine the quotient $\mathbf{C}^\times/G$ as a Riemann surface.
2. Find a nonzero meromorphic differential ω on $\mathbf{P}^1(\mathbf{C})$ which is invariant under σ, in other words satisfies $\sigma^*\omega = \omega$.
3. Compute $\mathrm{ord}_a(\omega)$ for every $a \in \mathbf{P}^1(\mathbf{C})$.

Problem

The aim of the problem is to show Hurwitz's theorem, asserting that a compact connected Riemann surface of genus $g \geq 2$ can have at most $84(g-1)$ automorphisms. We then investigate some properties of the Riemann surfaces attaining this bound.

Let X be a compact connected Riemann surface of genus $g \geq 2$. We admit that the group $G = \mathrm{Aut}(X)$ of automorphisms of X is finite, and we denote by N its order. Let X/G be the quotient Riemann surface, and let $\pi : X \to X/G$ be the quotient map.

1. Let $p \in X$. Show that the ramification index $e_\pi(p)$ depends only on $q = \pi(p) \in X/G$. We denote this integer by e_q.

2. Let $q_1, \ldots, q_r$ be the branch points of π, and let $e_i = e_{q_i}$. Using the Riemann–Hurwitz formula, prove that
$$2g - 2 = N \cdot (2g' - 2 + S)$$
where g' is the genus of X/G, and $S = \displaystyle\sum_{i=1}^{r} \big(1 - \frac{1}{e_i}\big)$.
3. Show that if $g' > 1$, then $N \leq g - 1$.
4. Show that if $g' = 1$, then $N \leq 4(g-1)$.

We assume from now on that $g' = 0$.

5. Prove that $r \geq 3$.
6. In the case $r \geq 5$, show that $N \leq 4(g-1)$.
7. In the case $r = 4$, show that $\sum_{i=1}^{4} \frac{1}{e_i} \leq \frac{11}{6}$ and deduce $N \leq 12(g-1)$.
8. In the case $r = 3$, show that $\sum_{i=1}^{3} \frac{1}{e_i} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7}$ and deduce Hurwitz's theorem.

The compact Riemann surface of genus $g \geq 2$ with the maximal number $84(g-1)$ of automorphisms are known as the *Hurwitz surfaces*. We study here some of their properties.

Let X be a Hurwitz surface of genus $g \geq 2$, and let $G = \mathrm{Aut}(X)$.

9. Using the previous questions, prove that there exists a ramified covering $f : X \to \mathbf{P}^1(\mathbf{C})$ with Galois group G and branch points $B = B(f) = \{0, 1, \infty\}$.

Let q be a fixed base point of $\mathbf{P}^1(\mathbf{C}) \setminus B$, and let $\Gamma = \pi_1(\mathbf{P}^1(\mathbf{C}) \setminus B, q)$ be the fundamental group. We choose a labelling $f^{-1}(q) = \{p_1, \ldots, p_N\}$ and denote by $\rho_f : \Gamma \to \mathcal{S}_N$ the monodromy representation of f.

10. Let γ be a loop in $\mathbf{P}^1(\mathbf{C}) \setminus B$ which is based at q, and let $\tilde{\gamma}$ be the unique lift of γ starting at p_1. Demonstrate that there exists a unique $g \in G$ such that $\tilde{\gamma}(1) = g(p_1)$.
11. Show that $\gamma \mapsto g^{-1}$ defines a surjective group morphism $\psi : \Gamma \to G$.
12. Prove that there are loops $\gamma_0, \gamma_1, \gamma_\infty$ in $\mathbf{P}^1(\mathbf{C}) \setminus \{0, 1, \infty\}$ around $0, 1, \infty$ respectively, such that $\gamma_0\gamma_1\gamma_\infty = 1$ in Γ.
13. Show that G is generated by an element σ of order 2 and an element τ of order 3 such that their product $\sigma\tau$ has order 7.

We admit that $H = \mathrm{PSL}_2(\mathbf{Z}/7\mathbf{Z})$ is the unique group of order 168 satisfying the condition of Question 13.

14. Establish that up to an automorphism of H, there exists a unique pair (σ, τ) of elements of H satisfying the conditions in Question 13.
15. Deduce that every Hurwitz surface of genus 3 is isomorphic to the Klein quartic.