Riemann surfaces

Partial examination (duration : 2 hours)

The only authorized documents are the notes of the course. You can write the answers in French or English. The accuracy and clarity of the arguments will be taken into consideration. Marks (only indicative) : 6 points (Ex. 1), 4 points (Ex. 2), 20 points (Problem).

Exercise 1

(1) Determine a rational function $f(z) \in \mathbf{C}(z)$ such that its divisor is given by $\mathrm{div}(f) = [0] + [\infty] - [1] - [-1]$ (here we view f as a meromorphic function on $\mathbf{P}^1(\mathbf{C})$).

We normalise f so that $f(z) \sim z$ when $z \to 0$. From now on, we view f as a holomorphic map from $\mathbf{P}^1(\mathbf{C})$ to $\mathbf{P}^1(\mathbf{C})$.

(2) Express f in homogeneous coordinates : find homogeneous polynomials P, Q in $\mathbf{C}[x, y]$ such that $f((x : y)) = (P(x, y) : Q(x, y))$ for every $(x : y) \in \mathbf{P}^1(\mathbf{C})$.

(3) What is the degree of f ?

(4) Compute the ramification points of f, the associated ramification indices, and the branch points of f.

(5) Using the formula for the ramification index of a composite map, determine the ramification points and ramification indices of $f \circ f$.

Exercise 2 (NB. The two questions are independent.)

Let Λ be a lattice in $\mathbf{C}$, and let $X = \mathbf{C}/\Lambda$ be the associated complex torus.

(1) Let $z_0 \in \mathbf{C}$ such that $z_0 \notin \frac{1}{2}\Lambda$. Compute the divisor of the elliptic function $f(z) = \wp(z + z_0) - \wp(z - z_0)$, where $\wp(z) = \wp_\Lambda(z)$ is the Weierstrass function.

(2) Let S be a finite subset of X, and let $p \in X \setminus S$. Prove that there exists a meromorphic function f on X with a simple zero at p and no zeroes or poles at any point of S.

Problem

In this problem, we study the number of fixed points of automorphisms of Riemann surfaces.

Let X be a compact connected Riemann surface. We denote by $\mathrm{Aut}(X)$ the group of (holomorphic) automorphisms of X.

We first study the case $X = \mathbf{P}^1(\mathbf{C})$. We recall that any automorphism h of $\mathbf{P}^1(\mathbf{C})$ is of the form $h(z) = (az + b)/(cz + d)$ for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{C})$.

(1) Let $M \in \mathrm{GL}_2(\mathbf{C})$, and let h be the corresponding automorphism of X. Show that $h = \mathrm{id}_X$ if and only if M is scalar.

(2) Assume $h \neq \mathrm{id}_X$. Show that the fixed points of h in X are in bijection with the eigenspaces of M.

(3) Show that h has either 1 or 2 fixed points in X.

(4) Prove that if h has exactly one fixed point in X, then h is conjugate in $\mathrm{Aut}(X)$ to an automorphism of the form $z \mapsto z + b$ with $b \in \mathbf{C}$, $b \neq 0$.

We now consider the case $X = \mathbf{C}/\Lambda$, where Λ is a lattice in $\mathbf{C}$.

(5) Let $\alpha \in \mathbf{C}^\times$ be such that $\alpha\Lambda = \Lambda$. Show that the map $\mathbf{C} \to \mathbf{C}$ given by $z \mapsto \alpha z$ induces an automorphism h_α of X.

(6) Determine the fixed points of h_{-1}.

(7) For any $\beta \in \mathbf{C}/\Lambda$, we denote by t_β the automorphism of X given by $x \mapsto x + \beta$. Show that if $\alpha \neq 1$, then $t_\beta \circ h_\alpha$ and h_α have the same number of fixed points.
Hint : Show that $t_\beta \circ h_\alpha$ is conjugate to h_α in $\mathrm{Aut}(X)$.

(8) Show that $G = \{\alpha \in \mathbf{C}^\times : \alpha\Lambda = \Lambda\}$ is a finite subgroup of $\mathbf{C}^\times$.

(9) In the case $\Lambda = \mathbf{Z} + i\mathbf{Z}$, determine G and the fixed points of h_α for each $\alpha \in G$, $\alpha \neq 1$.

In the final part of this problem, we investigate the general case where X is a compact connected Riemann surface of genus g. The aim is to show that any non-trivial automorphism of X has at most $2g + 2$ fixed points.

We recall (a consequence of) the Riemann–Roch theorem, which you can freely use : for any divisor D on X, we have the inequality $\dim_{\mathbf{C}} \mathcal{L}(D) \geq \deg(D) - g + 1$.

Let $h \in \mathrm{Aut}(X)$, $h \neq \mathrm{id}_X$. We fix a point $p \in X$ such that $h(p) \neq p$.

(10) Prove that there exists a non-constant meromorphic function f on X which has a pole of order at most $g + 1$ at p, and is holomorphic on $X \setminus \{p\}$.

(11) Determine the poles of the meromorphic function $F = f - f \circ h$. What can you say about their orders ?

(12) Deduce that h has at most $2g + 2$ fixed points in X.

Remark (not part of the exam). Let $\hat{X}$ be the compactification of the Riemann surface $X : y^2 = P(x)$ where $P \in \mathbf{C}[x]$ is a polynomial of degree $d \geq 1$ with simple roots. The genus of $\hat{X}$ is $g = \lfloor \frac{d-1}{2} \rfloor$. One can show that the involution $\sigma : (x, y) \mapsto (x, -y)$ extends to $\hat{X}$ and has exactly $2g + 2$ fixed points in $\hat{X}$. In fact, any compact connected Riemann surface of genus g having an involution with $2g + 2$ fixed points must be of this form (we will prove this later).