Étale cohomology

Notes of the Master 2 course
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François Brunault
francois.brunault@ens-lyon.fr

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The course assumes familiarity with sheaves on topological spaces and with schemes, as seen in the M1 course on Algebraic geometry [Fu].

Contents

1 Categories

The reference for this material is Grothendieck's article in Tohoku [Gro57] for abelian categories, and [Tam94] for Grothendieck topologies, presheaves and sheaves.

1.1 Definition

A category is a collection of objects with various maps between them. More formally:

Definition 1.1. A *category* $\mathcal{C}$ is the data of:

- a set $\mathrm{Obj}(\mathcal{C})$: the objects of $\mathcal{C}$;
- for each $X, Y \in \mathrm{Obj}(\mathcal{C})$, a set $\mathrm{Hom}(X, Y)$ whose elements are called the morphisms from X to Y (in $\mathcal{C}$);
- for each $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, a map $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ called composition of morphisms,

such that the composition of morphisms is associative and admits identity elements $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ for each $X \in \mathrm{Obj}(\mathcal{C})$ (the identity elements are then unique).

We define the isomorphisms in $\mathcal{C}$ as those morphisms which admit a two-sided inverse.

We will often write $X \in \mathcal{C}$ instead of $X \in \mathrm{Obj}(\mathcal{C})$, and $f : X \to Y$ instead of $f \in \mathrm{Hom}(X, Y)$. Of course, this does not necessarily mean that f is a map in the usual sense – in general X and Y are not even sets to begin with.

Remark 1.2. The collection of objects of a category should really be a class instead of a set (as is well-known, considering the "set of all sets" in naïve set theory leads to paradoxes). In this course we will not care about such fundational issues.

Here are some examples of categories: Set (sets), Grp (groups), Ab (abelian groups), Rng (commutative rings), R-Mod (modules over a ring R), Top (topological spaces), Sch (schemes), Sch$/S$ (schemes over a given scheme S). In all these cases the morphisms are the usual ones.

The following construction will be useful. Let $(I, \leq)$ be a partially ordered set (= ensemble ordonné). Then I can be made into a category: the objects are the elements of I, and there is a unique morphism $i \to j$ whenever $i \leq j$.

In this course, we will define and use extensively categories of sheaves.

1.2 Products and coproducts

Definition 1.3. Let $(X_i)_{i \in I}$ be a family of objects of $\mathcal{C}$. A *product* of the family $(X_i)_{i \in I}$ is an object P of $\mathcal{C}$ endowed with morphisms $P \to X_i$ for each $i \in I$, such that P enjoys the following universal property:

For every $Y \in \mathcal{C}$ and every family of morphisms $f_i : Y \to X_i$ $(i \in I)$, there exists a unique morphism $f : Y \to P$ making the obvious diagrams commute.

In a given category, products may or may not exist. If a product exists, then the universal property makes it unique up to isomorphism (where an isomorphism between two products is defined in the natural way). The product of $(X_i)_{i \in I}$ is then denoted by $\prod_{i \in I} X_i$, and the morphisms $\prod_{i \in I} X_i \to X_i$ are called the *canonical projections*.

Example 1.4. Products always exist in the categories Set, Grp, Ab, Rng, R-Mod, Top and the notation just introduced coincides with the usual one. In the category of schemes (resp. schemes over a base S), finite products exist and are given by $X \times Y = X \times_{\mathrm{Spec}\,\mathbf{Z}} Y$ (resp. $X \times_S Y$), but infinite products don't exist in general.

Dually, a *coproduct* of a family $(X_i)_{i\in I}$ of objects of $\mathcal{C}$ is an object S of $\mathcal{C}$ endowed with morphisms $X_i \to S$ for each $i \in I$, such that for every $Y \in \mathcal{C}$, the natural map

$$\operatorname{Hom}(S, Y) \to \prod_{i\in I} \operatorname{Hom}(X_i, Y)$$

is bijective. If a coproduct exists, then it is unique up to isomorphism and is denoted by $\coprod_{i\in I} X_i$. The morphisms $X_i \to \coprod_{i\in I} X_i$ are called the *canonical injections*.

Remark 1.5. In additive or abelian categories (to be defined later), we will rather use the notation $\bigoplus_{i\in I} X_i$, and refer to coproducts as *direct sums*.

Exercise 1. Determine what is the coproduct $X \coprod Y$ in the following categories: Set, Grp, Ab, CRng (commutative rings), R-Mod, Top, Sch, Sch$/S$.

Finally, we will also need the notion of *fibre product* in an arbitrary category $\mathcal{C}$. Given two morphisms $f : X \to T$ and $g : Y \to T$ in $\mathcal{C}$, a fibre product of X and Y over T is an object P of $\mathcal{C}$ endowed with two projections $p : P \to X$ and $q : P \to Y$, such that:

(a) The following diagram commutes

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & T \end{tikzcd} \tag{1}$$

(b) For every pair of morphisms $u : Z \to X$ and $v : Z \to Y$ with $fu = gv$, there exists a unique $w : Z \to P$ such that $u = pw$ and $v = qw$.

If the properties (a) and (b) hold, we say that (1) is a *Cartesian square*, and we signify this by inscribing a small square in it.

A more conceptual way to define $X \times_T Y$ is to consider the category $\mathcal{C}/T$ of *objects over* T, consisting of pairs (X, f) with $X \in \mathcal{C}$ and $f : X \to T$, and then to form the product in this category.

Example 1.6. In the category Set, the fibre product of $f : X \to T$ and $g : Y \to T$ is given by $X \times_T Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$. In the case $Y = \{t\}$ for some $t \in T$ and $g : \{t\} \to T$ is the inclusion, the fibre product $X \times_T Y$ is simply the fibre $f^{-1}(t)$, whence the terminology.

Example 1.7. In the category Sch of schemes, finite fibre products exist, as seen in [Fu].

1.3 Subobjects and quotients

Let $\mathcal{C}$ be an arbitrary category.

Definition 1.8. A morphism $f : X \to Y$ in $\mathcal{C}$ is called a *monomorphism* (resp. *epimorphism*) if for every $Z \in \mathcal{C}$, the map $\operatorname{Hom}(Z, X) \to \operatorname{Hom}(Z, Y)$ (resp. $\operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$), given by composing with f, is injective.

Example 1.9. In the categories Set, Ab, R-Mod (where all objects have underlying sets), the monomorphisms are exactly the injective maps, and the epimorphisms are exactly the surjective maps.

Isomorphisms in $\mathcal{C}$ are always mono and epi, but the converse is not true in general.

Exercise 2. Find a morphism of commutative rings $f : R \to R'$ which is an epimorphism but is not surjective as a map.

Let us fix an object $X \in \mathcal{C}$.

Definition 1.10. Two monomorphisms $i : A \to X$ and $i' : A' \to X$ are said to be *equivalent* is there exists an isomorphism $\phi : A \xrightarrow{\cong} A'$ such that $i = i'\phi$.

This clearly defines an equivalence relation on the set of monomorphisms with values in X. Let us choose a representative in each equivalence class. Then these representatives are called the *subobjects* of X.

Of course, two subobjects of X can be abstractly isomorphic without being equivalent. For example, in the abelian group $\mathbf{Z}$ the subgroups $2\mathbf{Z}$ and $3\mathbf{Z}$ are isomorphic but not equivalent.

Exercise 3. Let $i : A \to X$ and $i' : A' \to X$ be two monomorphisms in $\mathcal{C}$. We say that i is contained in i' (written $i \leq i'$) if there exists a morphism $\phi : A \to A'$ such that $i = i'\phi$.

(a) Show that such a ϕ is uniquely determined, and is mono.

(b) Show that i and i' are equivalent if and only if $i \leq i'$ and $i' \leq i$.

(c) Show that $\leq$ induces a partial order (ensemble ordonné) on the set of subobjects of X.

Note that Exercise 3(b) gives a way to check whether two monomorphisms are equivalent.

Similarly, we define the *quotients* of X as representatives of the equivalence classes of epimorphisms $X \to B$.

Exercise 4. In the category Set, describe explicitly the quotients of a given set X.

1.4 Additive and abelian categories

Roughly speaking, an abelian category is a category which behaves like the category of (left) R-modules, where R is a ring. In particular, one can do diagram chasing and homological algebra in such a category.

Definition 1.11. A category $\mathcal{C}$ is *additive* if the following axioms hold:

(1) For each $A, B \in \mathcal{C}$, the set $\mathrm{Hom}(A, B)$ is endowed with the structure of an abelian group, and the composition of morphisms is bilinear.

(2) The product and coproduct of any two objects exist in $\mathcal{C}$.

(3) The category $\mathcal{C}$ has a zero object, i.e. an object which is both initial and final in $\mathcal{C}$.

Remark 1.12. Axiom (2) implies that finite products and coproducts exist in $\mathcal{C}$. Any two zero objects are canonically isomorphic. Therefore, there is no harm in denoting 0 any zero object.

Exercise 5. Show that in an additive category $\mathcal{C}$, products and coproducts coincide: for every $A, B \in \mathcal{C}$, the canonical map $A \coprod B \to A \prod B$ is an isomorphism.

We usually denote by $\oplus$ (direct sum) the product or coproduct in an additive category.

We now define kernels and cokernels in additive categories.

Definition 1.13. Let $\mathcal{C}$ be an additive category. Let $u : A \to B$ a morphism in $\mathcal{C}$. A *kernel* of u is a subobject $i : K \to A$ such that:

(1) $u \circ i = 0$;

(2) for every morphism $v : C \to A$, we have $u \circ v = 0$ if and only if $v = i \circ w$ for some $w : C \to K$.

$$\begin{tikzcd} K \arrow[r, "i"] & A \arrow[r, "u"] & B \\ & C \arrow[ul, dashed, "w"] \arrow[u, "v"] \arrow[ur, "0"'] & \end{tikzcd}$$

If the kernel of u exists, then it is unique, and we denote it by $\ker(u)$.

Remark 1.14. By taking $w = \mathrm{id}_K$, we see that condition (2) implies condition (1).

Exercise 6. Define similarly cokernels. Show that if the cokernel of $u : A \to B$ exists, then it is unique.

We denote by $\mathrm{coker}(u)$ the cokernel of u, if it exists.
We now define images and coimages.

Definition 1.15. Let $\mathcal{C}$ be an additive category having kernels and cokernels. Let $u : A \to B$ be a morphism in $\mathcal{C}$. The *image* of u is defined by $\mathrm{im}(u) = \ker(\mathrm{coker}(u))$. Similarly, the *coimage* of u is defined by $\mathrm{coim}(u) = \mathrm{coker}(\ker(u))$.

Example 1.16. Consider the additive category $\mathcal{C} = R$-Mod where R is a ring. Then kernels and cokernels always exist and coincide with the usual ones. Therefore, the image and coimage are well-defined. Let us compute them for an R-linear map $u : M \to N$. The cokernel of u is the epimorphism $N \to N/u(M)$. Thus, the image of u in the categorical sense is the kernel of this morphism, in other words the monomorphism $u(M) \to N$. Thus it coincides with the usual image. Similarly, the kernel of u is the monomorphism $\ker(u) \to M$, hence the coimage of u is the epimorphism $M \to M/\ker(u)$. Note that by the isomorphism theorem, there is a canonical isomorphism $M/\ker(u) \xrightarrow{\cong} u(M)$. In other words, the image and coimage of u are canonically isomorphic.

The situation in the previous example generalizes: the following lemma says that in an arbitrary additive category, there is a canonical morphism from the coimage to the image (when these two objects exist). But it is not always an isomorphism.

Lemma 1.17. *Let $\mathcal{C}$ be an additive category having kernels and cokernels, and let $u : A \to B$ be a morphism in $\mathcal{C}$. Then there exists a unique morphism $\overline{u} : \mathrm{coim}(u) \to \mathrm{im}(u)$ such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & B \\ \mathrm{coim}(u) \arrow[r, "\overline{u}"] & \mathrm{im}(u) \arrow[u] \end{tikzcd} \tag{2}$$

The diagram (2) is called the *canonical factorisation* of u.
We finally define abelian categories.

Definition 1.18. An *abelian cateogry* is an additive category $\mathcal{C}$ satisfying:

(Ab1) Kernels and cokernels exist in $\mathcal{C}$.

(Ab2) For every morphism u in $\mathcal{C}$, the canonical morphism $\overline{u} : \mathrm{coim}(u) \to \mathrm{im}(u)$ is an isomorphism.

In an abelian category, the morphisms which are both mono and epi are exactly the isomorphisms.

Given an object A in an abelian category $\mathcal{C}$, there is a natural bijection between the subobjects of A and the quotients of A:

$$\{\text{subobjects of } A\} \xrightarrow{\cong} \{\text{quotients of } A\}$$
$$B \mapsto A/B.$$

The reciprocal bijection sends an epimorphism $q : A \to Q$ to its kernel.

Exercise 7. Let A be an object of an abelian category $\mathcal{C}$. Given two subobjects B, C of A, define their sum $B + C$ and their intersection $B \cap C$.

The interest of introducing abelian categories is that we can do homological algebra.

Definition 1.19. A sequence $A \xrightarrow{u} B \xrightarrow{v} C$ in an abelian category $\mathcal{C}$ is called *exact* if $\mathrm{im}(u) = \ker(v)$.

Exercise 8.

1. Consider a sequence $A \xrightarrow{u} B \xrightarrow{v} C$. Show that $v \circ u = 0 \Leftrightarrow \mathrm{im}(u) \subset \ker(v)$.
2. Show that $0 \to A \xrightarrow{u} B$ (resp. $A \xrightarrow{u} B \to 0$) is exact if and only if u is mono (resp. epi).
3. Show that a sequence $0 \to A \to B \to C$ is exact in $\mathcal{C}$ if and only if
$$0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C)$$
is an exact sequence of abelian groups for every $X \in \mathcal{C}$.

In abelian categories, we can do diagram chasing as with R-modules. In particular, we can use various results as the snake lemma, the five lemma, the nine lemma...Most of the time, we will chase diagrams as if we were working with R-modules, and won't do the necessary verifications using only the axioms. In fact, these verifications can be avoided, thanks to *Mitchell's embedding theorem*: every abelian category $\mathcal{C}$ is equivalent to a full subcategory of R-Mod for some ring R (not necessarily commutative), and the kernels and cokernels in $\mathcal{C}$ coincide with those in R-Mod.

1.5 Functors

Definition 1.20. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A *covariant functor* $F : \mathcal{C} \to \mathcal{C}'$ is the data of:

- for each object $X \in \mathcal{C}$, an object $F(X) \in \mathcal{C}'$;
- for each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$,

such that the map $f \mapsto F(f)$ is compatible with composition and preserves the identity morphisms.

A contravariant functor is defined the same way except that the arrows are reversed: for each $f : X \to Y$ we have $F(f) : F(Y) \to F(X)$. A contravariant functor from $\mathcal{C}$ to $\mathcal{C}'$ is the same thing as a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}'$, where $\mathcal{C}^{\mathrm{op}}$ denotes the *opposite category* of $\mathcal{C}$: this is the category with the same objects but arrows reversed, so that $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$. When we say "functor" without precision, we mean a covariant functor.

As an example, let $\mathcal{C}$ be a category and $X_0 \in \mathcal{C}$ be an arbitrary object. For any $X \in \mathcal{C}$, consider the set $F(X) = \mathrm{Hom}_{\mathcal{C}}(X_0, X)$. For any morphism $f : X \to Y$ in $\mathcal{C}$, define $F(f) : \mathrm{Hom}(X_0, X) \to \mathrm{Hom}(X_0, Y)$ to be the map sending g to $f \circ g$. Then $F : \mathcal{C} \to \mathrm{Set}$ is a covariant functor. Similarly, we may define a functor $G : \mathcal{C} \to \mathrm{Set}$ by putting $G(X) = \mathrm{Hom}(X, X_0)$, but this time G is contravariant.

Functors arise naturally in mathematics, and we give here only a few examples.

Example 1.21. In topology, the fundamental group is a functor from pointed topological spaces to groups. Indeed, for any continuous map $f : X \to Y$ between topological spaces and any $x \in X$, there is associated a group morphism $\pi_1(f) : \pi_1(X, x) \to \pi_1(Y, f(x))$. Similarly, the singular homology (resp. cohomology) groups of topological spaces define covariant (resp. contravariant) functors.

Example 1.22. Let R be a ring and S be an R-algebra. We have a *base change* functor R-Mod $\to$ S-Mod defined by $M \mapsto M \otimes_R S$. We also have the *restriction of scalars* S-Mod $\to$ R-Mod. If M is any R-module, we have functors R-Mod $\to$ R-Mod given by $\cdot \otimes M$, $\mathrm{Hom}_R(M, \cdot)$ and $\mathrm{Hom}_R(\cdot, M)$. The first two are covariant, while the third is contravariant. The tensor powers, symmetric powers and exterior powers also define functors.

Example 1.23. Let G be a group. The theory of group cohomology gives covariant functors $H^i : \mathbf{Z}G\text{-Mod} \to \mathrm{Ab}$ for any $i \geq 0$. We may also consider $H^i(G, \mathbf{Z})$ as a function of G. In this way, one gets a contravariant functor $\mathrm{Grp} \to \mathrm{Ab}$.

Example 1.24. Let $\mathcal{C}$ be a category, and let $(I, \leq)$ be a partially ordered set. A *commutative diagram* of shape I in $\mathcal{C}$ is a covariant functor $F : I \to \mathcal{C}$.

For example, if I is the set of vertices of a square, and is properly ordered, then a diagram of shape I in $\mathcal{C}$ is simply a commuting square in $\mathcal{C}$. The commutativity of the square follows from the uniqueness of the morphism $i \to j$ when $i \leq j$.

More generally, if $\mathcal{I}$ is an arbitrary category, a *diagram* of shape $\mathcal{I}$ in $\mathcal{C}$ is a covariant functor $\mathcal{I} \to \mathcal{C}$.

Definition 1.25. Let $F, G : \mathcal{C} \to \mathcal{C}'$ be two functors. A *morphism of functors* (or natural transformation) $f : F \to G$ is the data, for each object $X \in \mathcal{C}$, of a morphism $f(X) : F(X) \to G(X)$ in $\mathcal{C}'$ such that, for every morphism $u : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{f(X)} & G(X) \\ \big\downarrow{\scriptstyle F(u)} & & \big\downarrow{\scriptstyle G(u)} \\ F(Y) & \xrightarrow{\overline{u}} & G(Y). \end{array} \tag{3}$$

We say that f is an *isomorphism of functors* if $f(X)$ is an isomorphism for each $X \in \mathcal{C}$.

Definition 1.26. A functor $F : \mathcal{C} \to \mathrm{Set}$ (resp. $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$) is called *representable* if F is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(X_0, \cdot)$ (resp. $\mathrm{Hom}_{\mathcal{C}}(\cdot, X_0)$) for some object $X_0 \in \mathcal{C}$.

Given a representable functor F, the object X_0 representing F is uniquely determined up to isomorphism (this is the consequence of the *Yoneda lemma*).

Definition 1.27. Let $R : \mathcal{C} \to \mathcal{C}'$ be a functor between arbitrary categories. A functor $L : \mathcal{C}' \to \mathcal{C}$ is called *left adjoint* to R if for each pair of objects $X' \in \mathcal{C}'$ and $Y \in \mathcal{C}$, there exists an isomorphism

$$\mathrm{Hom}(L(X'), Y) \cong \mathrm{Hom}(X', R(Y)) \tag{4}$$

which is functorial in X' and Y. If the left adjoint of R exists, then it is unique up to a unique isomorphism, and we denote it by ${}^{\mathrm{ad}}R$.

Note that the isomorphisms (4) are part of the data of an adjunction.

Remark 1.28. In order to show that a given functor $R : \mathcal{C} \to \mathcal{C}'$ has a left adjoint, it suffices to check that for every object $X' \in \mathcal{C}'$, the functor

$$\begin{aligned} \mathcal{C} &\to \mathrm{Set} \\ Y &\mapsto \mathrm{Hom}(X', R(Y)) \end{aligned}$$

is representable. Indeed, defining $L(X') = Z$, one checks that L is naturally a functor and that the isomorphisms (4) are automatically also functorial in X'.

Exercise 9. Show that "base change" and "restriction of scalars" of modules (introduced above) are adjoint functors.

Exercise 10. Consider the functors $\mathrm{Ab} \xrightarrow{F} \mathrm{Grp} \xrightarrow{G} \mathrm{Set}$ given by "forgetting the structure".

1. Determine the left adjoints of F, G and $G \circ F$. Check that ${}^{\mathrm{ad}}(G \circ F) = {}^{\mathrm{ad}}F \circ {}^{\mathrm{ad}}G$.

2. Do F, G and $G \circ F$ have right adjoints?

Definition 1.29. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between arbitrary categories. We say that F is an *equivalence of categories* if there exists a functor $G : \mathcal{C}' \to \mathcal{C}$ such that the composed functors $GF : \mathcal{C} \to \mathcal{C}$ and $FG : \mathcal{C}' \to \mathcal{C}'$ are isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{C}'}$ respectively.

One can show that a functor is an equivalence of categories without exhibiting an inverse:

Exercise 11. Show that $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if and only if the following properties are satisfied:

- F is *fully faithful*: for any $X, Y \in \mathcal{C}$, the map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(FX, FY)$ is bijective;

- F is *essentially surjective*: every object $X' \in \mathcal{C}'$ is isomorphic to $F(X)$ for some $X \in \mathcal{C}$.

Example 1.30. (An example of equivalence of categories.) A conceptual way to look at covering spaces is as follows. Let X be a topological space and $x_0 \in X$ a base point. Let $G = \pi_1(X, x_0)$ be the fundamental group. Then we have a functor $F : \mathrm{Cov}_X \to G\text{-Set}$ from the category of covering spaces of X to the category of right G-sets. The functor F is defined as follows: to any covering space $p : Y \to X$, we associate the fibre $p^{-1}(x_0)$. By the path lifting property, this fibre is endowed with a right action of G. If X is path-connected and locally simply connected, then F is an equivalence of categories. Moreover, F induces an equivalence between the subcategory of those covering spaces which are path-connected, and the subcategory of transitive G-sets (i.e. the action of G is transitive). The latter category is equivalent to the category of subgroups of G, by associating to any subgroup H of G the transitive G-set $H\backslash G$.

The theory of the *étale fundamental group* generalizes these results to schemes. This uses the *étale toplogy*, which we will define later.

Definition 1.31. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. The category $\mathcal{H}om(\mathcal{C}, \mathcal{C}')$ is defined as follows: the objects are the (covariant) functors $\mathcal{C} \to \mathcal{C}'$, and the morphisms are the natural transformations between them. This category is also denoted by $\mathcal{C}'^{\mathcal{C}}$.

Similarly, the contravariant functors $\mathcal{C} \to \mathcal{C}'$ form a category denoted by $\mathcal{H}om(\mathcal{C}^{\mathrm{op}}, \mathcal{C}')$.

In the case the target $\mathcal{C}'$ is abelian, the resulting category $\mathcal{H}om(\mathcal{C}, \mathcal{C}')$ is abelian:

Proposition 1.32. *Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, with $\mathcal{C}'$ abelian. Then $\mathcal{H}om(\mathcal{C}, \mathcal{C}')$ can be made into an abelian category. Moreover, a sequence $F \to G \to H$ is exact in $\mathcal{H}om(\mathcal{C}, \mathcal{C}')$ if and only if for every $X \in \mathcal{C}$, the sequence $F(X) \to G(X) \to H(X)$ is exact in $\mathcal{C}'$.*

Proof. See [Tam94, 1.3.1]. □

We now study functors between additive or abelian categories.

Definition 1.33. A functor $F : \mathcal{C} \to \mathcal{C}'$ between additive categories is called *additive* if the induced maps on the Hom-sets are linear: for all morphisms $u, v : A \to B$ in $\mathcal{C}$, we have $F(u + v) = F(u) + F(v)$.

Exercise 12. Show that additive functors preserve zero objects and direct sums.

Definition 1.34. Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories, and let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant additive functor. We say that F is *left exact* (resp. *right exact*) if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, the sequence

$$0 \to F(A) \to F(B) \to F(C) \qquad (\text{resp. } F(A) \to F(B) \to F(C) \to 0) \tag{5}$$

is exact in $\mathcal{C}'$. We say that F is *exact* if it is both left and right exact, in other words F sends short exact sequences in $\mathcal{C}$ to short exact sequences in $\mathcal{C}'$.

As an example, Exercise 8 shows that for any object X of an abelian category $\mathcal{C}$, the functor $\mathrm{Hom}(X, -) : \mathcal{C} \to \mathrm{Ab}$ is left exact.

Exercise 13. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact covariant additive functor. Show that the following stronger property holds: for every exact sequence $0 \to A \to B \to C$ in $\mathcal{C}$, the sequence $0 \to FA \to FB \to FC$ is exact.

Remark 1.35. Similarly, a contravariant functor $G : \mathcal{C} \to \mathcal{C}'$ is called left exact (resp. right exact) if under the same assumptions, the resulting sequence

$$0 \to G(C) \to G(B) \to G(A) \qquad (\text{resp. } G(A) \to G(B) \to G(A) \to 0) \tag{6}$$

is exact in $\mathcal{C}'$.

Exercise 14. Let Y be an object of an abelian category $\mathcal{C}$. Show that the contravariant functor $\mathrm{Hom}(-, Y) : \mathcal{C} \to \mathrm{Ab}$ is left exact.

Many natural functors are either left or right exact, but they are usually not exact, and this is precisely what makes things interesting: one tries to extend (5) and (6) to long exact sequences. This gives rise to the so-called *derived functors*, which we will discuss in Section 4.

1.6 Inductive limits

Inductive limits are a generalisation of the notion of infinite increasing union. For example, if $(A_n)_{n\geq 0}$ is an increasing family of subgroups inside an abelian group A, then $\varinjlim_{n\geq 0} A_n = \bigcup_{n\geq 0} A_n$. But we can define inductive limits in a much more general setting.

Before doing this, let us discuss briefly infinite direct sums. By definition, finite direct sums $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ always exist in abelian categories. But infinite direct sums don't exist in general. For example, consider the category of *finitely generated* R-modules, where R is a noetherian ring. By the Noetherian assumption, this category has kernels and cokernels, and it satisfies (Ab2), so it is an abelian category. Yet the infinite direct sum $\bigoplus_{\mathbf{N}} R$ does not exist in this category. We say that the abelian category $\mathcal{C}$ has *arbitrary direct sums* if all coproducts as defined in Section 1.2 exist.

Definition 1.36. Let $\mathcal{C}$ be an abelian category having arbitrary direct sums. For any object $A \in \mathcal{C}$ and any family $(A_i)_{i\in I}$ of subobjects of A, we define the *sum* of the A_i by

$$\sum_{i\in I} A_i = \mathrm{im}\Big(\bigoplus_{i\in I} A_i \to A\Big).$$

We now define inductive systems.

Definition 1.37. Let $\mathcal{C}$ be an abelian category, and let $\mathcal{I}$ be a category. An *inductive system in $\mathcal{C}$ indexed by $\mathcal{I}$* is a covariant functor $\mathcal{I} \to \mathcal{C}$.

As seen in 1.5, the category $\mathcal{C}^{\mathcal{I}}$ of inductive systems indexed by $\mathcal{I}$ is an abelian category.

For example, if I is a partially ordered set, then an inductive system indexed by I is a family $(A_i)_{i\in I}$ of objects of $\mathcal{C}$ together with compatible maps $A_i \to A_j$ for each pair (i, j) with $i \leq j$.

Definition 1.38. Let $\mathcal{I}$ be a category and let $A \in \mathcal{C}$. The *constant functor with value* A is the functor $c_A : \mathcal{I} \to \mathcal{C}$ assigning to each $i \in \mathcal{I}$ the object A, with the identity map id_A as transition morphisms.

Definition 1.39. Let $F = (A_i)_i \in \mathcal{C}^{\mathcal{I}}$ be an inductive system in $\mathcal{C}$ indexed by a category $\mathcal{I}$. An *inductive limit* of F is an object $A \in \mathcal{C}$ together with a morphism of functors $F \to c_A$ which is universal: every morphism of functors $F \to c_B$ with $B \in \mathcal{C}$ factors uniquely through $F \to c_A$. If the inductive limit exists, then we denote it by $\varinjlim_{\mathcal{I}} A_i$.

Exercise 15. State the universal property of the inductive limit in more concrete terms in the case $\mathcal{I} = I$ is a partially ordered set.

Theorem 1.40. *Let $\mathcal{C}$ be an abelian category having arbitrary direct sums. Then $\mathcal{C}$ has arbitrary inductive limits. Moreover $\varinjlim_{\mathcal{I}} : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ is a right exact additive functor.*

Proof. Let $F = (A_i)_{i \in \mathcal{I}}$ be an inductive system. Let $S = \bigoplus_{i \in \mathcal{I}} A_i$ be the direct sum of the A_i. We will define the inductive limit of F as a quotient of S.

For each morphism $u : i \to j$, let $f_u : A_i \to A_i \oplus A_j$ be the morphism defined by $f_u = (\mathrm{id}_{A_i}, -F(u))$, and let R_u denote the image of f_u in S. Define $R = \sum_u R_u$ where the sum is over all possible u. Then the quotient S/R together with the obvious maps $A_i \to S/R$ satisfies the universal property. The second part of the theorem is not hard, and left as an exercise. $\square$

Example 1.41. If I is an arbitrary set, then the direct sum $\bigoplus_I$ indexed by I can be seen as the inductive limit over the category with objects the elements of I, and no arrow except the identity morphisms. We also have $\bigoplus_I = \varinjlim_{J \subset I} \bigoplus_J$ where J runs over the finite subsets of I (with the obvious transition maps). So infinite direct sums are a particular case of inductive limits. This shows that the first implication in Theorem 1.40 is in fact an equivalence.

Mnemonic. It is always easy to go *to* an inductive limit, but it is non-trivial to go *from* an inductive limit (more precisely, it uses the universal property).

Here is a special case where the inductive limit has a more explicit description.

Definition 1.42. A partially ordered set $(I, \leq)$ is said to be *directed* (or *filtered*) if for every $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$. We also say that $(I, \leq)$ is a directed set.

Example 1.43.

- Let $\mathbf{N}^* = \{1, 2, 3, \ldots\}$. Then $(\mathbf{N}^*, \leq)$ and $(\mathbf{N}^*, |)$ are directed sets.
- Let X be a topological space and $x \in X$. Then the set of open neighborhoods of x, ordered with $U \leq V \Leftrightarrow U \supset V$, is directed.

As explained before, every poset defines a category.

Exercise 16. Let $(I, \leq)$ be a directed set, and let $(A_i)_{i \in I}$ be an inductive system of abelian groups. Show that $\varinjlim_{i \in I} A_i$ is in bijection with the quotient of $\bigsqcup_{i \in I} A_i$ (set-theoretic disjoint union) by the equivalence relation $\sim$ defined as follows: $a_i \sim a'_j$ if and only if there exists $k \in I$ such that $i, j \leq k$ and the images of a_i and a'_j in A_k coincide.

In particular, every element of the inductive limit $\varinjlim_{i \in I} A_i$ comes from some A_i (depending on the element), and an element $a_i \in A_i$ is zero in the inductive limit if and only if there exists $j \geq i$ such that the image of a_i in A_j is zero.

We now give a condition under which the inductive limit is an exact functor.

Theorem 1.44. *Let $\mathcal{C}$ be an abelian category having arbitrary direct sums, and let I be a* directed *partially ordered set. Assume that $\mathcal{C}$ satisfies the following axiom:*

(Ab5) *For every $A \in \mathcal{C}$, every increasing family of subobjects $(A_i)_{i\in I}$ of A (that is, we have $A_i \subset A_j$ whenever $i \leq j$) and every compatible system of morphisms $u_i : A_i \to B$ into a fixed object $B \in \mathcal{C}$ (this means that u_i is induced by u_j whenever $i \leq j$), there exists a (unique) morphism $u : \sum_i A_i \to B$ inducing the u_i.*

Then the functor $\varinjlim_I : \mathcal{C}^I \to \mathcal{C}$ is exact.

Example 1.45. The category Ab satisfies (Ab5). Therefore directed inductive limits of abelian groups are exact functors.

Exercise 17.

1. Show that (Ab5) holds in Ab and R-Mod.

2. Show that (Ab5) does not hold in the opposite category of Ab.

Exercise 18. Let I be the partially ordered set $\{\bullet \leftarrow \bullet \to \bullet\}$ (which is not directed).

(a) Describe the functor $\varinjlim_I : \mathrm{Ab}^I \to \mathrm{Ab}$.

(b) Show that the condition (Ab5) does not hold and that $\varinjlim_I$ is not exact.

Remark 1.46. Although we will not need it, we may define *projective limits* in a completely similar way, reversing the arrows and replacing subobjects by quotients. If an abelian category $\mathcal{C}$ has arbitrary products then it has arbitrary projective limits, and the corresponding functors are left exact. Moreover, if the condition dual to (Ab5) is satisfied (that is, (Ab5) holds in the abelian category $\mathcal{C}^{\mathrm{op}}$, see [Gro57]), then directed projective limits are exact functors.

We will need a permanence property for the axiom (Ab5).

Lemma 1.47. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, with $\mathcal{D}$ abelian. Assume that $\mathcal{D}$ has arbitrary direct sums. Then $\mathcal{H}om(\mathcal{C}, \mathcal{D})$ also has arbitrary direct sums.*

Proof. Given a family $(F_i)_{i\in I}$ of functors $F_i : \mathcal{C} \to \mathcal{D}$, we define the direct sum $\bigoplus_{i\in I} F_i$ component-wise, by $(\bigoplus_{i\in I} F_i)(X) := \bigoplus_{i\in I} F_i(X)$ for every $X \in \mathcal{C}$. The reader may check that $\bigoplus_{i\in I} F_i$ is indeed a functor and satisfies the universal property of the coproduct. □

Proposition 1.48. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, with $\mathcal{D}$ abelian. Assume that $\mathcal{D}$ has arbitrary direct sums and satisfies (Ab5). Then the same holds true for $\mathcal{H}om(\mathcal{C}, \mathcal{D})$.*

Proof. See [Tam94, 1.4.3]. □

2 Presheaves and sheaves

2.1 The topological setting

Presheaves and sheaves were defined in [Fu] in the framework of topological spaces. It turns out that presheaves can also be interpreted as functors, leading to a natural generalization of this notion for arbitrary categories. Before doing that, we recall the definition of sheaves and presheaves, and their basic properties.

Definition 2.1. Let X be a topological space. A *presheaf of abelian groups* (or *abelian presheaf*) F on X is the data of:

- for each open subset U of X, an abelian group $F(U)$;
- for each pair of open subsets $V \subset U$ of X, a linear map $\rho_{U,V} : F(U) \to F(V)$,

such that:

- for every open subset U of X, we have $\rho_{U,U} = \mathrm{id}_{F(U)}$;
- for every open subsets $W \subset V \subset U$ of X, we have $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

Notation 2.2. The elements of $F(U)$ are called *sections* of F on U. Given $s \in F(U)$ and $V \subset U$, the section $\rho_{U,V}(s) \in F(V)$ is called the *restriction* of s to V and is denoted by $s|_V$.

Using category theory, the definition of presheaves is very short. Let $\mathcal{T}_X$ be the topology of X, that is the set of open subsets of X. It is partially ordered by inclusion, hence defines a category. Then an abelian presheaf on X is simply a contravariant functor $F : \mathcal{T}_X \to \mathrm{Ab}$. Moreover, a morphism of presheaves $f : F \to G$ is a natural transformation from F to G.

Notation 2.3. We denote by $\mathcal{P}_X$ the category of abelian presheaves on X.

By Proposition 1.32, $\mathcal{P}_X$ is an abelian category, and by Lemma 1.47, it has arbitrary direct sums. Note that in $\mathcal{P}_X$, the kernel, cokernel, image, coimage, direct sum, sum ... are all defined "component-wise", in other words they are the obvious thing on each open subset of X. In particular, a sequence of abelian presheaves $0 \to F \to G \to H \to 0$ is exact if and only $0 \to F(U) \to G(U) \to H(U) \to 0$ is exact for every open subset U of X.

Definition 2.4. Let X be a topological space and let F be an abelian presheaf on X. Then F is a *sheaf* if and only if for every open set $U \subset X$ and every open covering $(U_i)_{i\in I}$ of U, the following "gluing" condition holds:

For every family of sections $s_i \in F(U_i)$ $(i \in I)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for every i, j, there exists a unique section $s \in F(U)$ such that $s|_{U_i} = s_i$ for every i.

A morphism of sheaves $f : F \to G$ is a morphism of presheaves.

Remark 2.5. Using the gluing condition with the empty family $(I = \emptyset)$, we see that any sheaf F on X satisfies $F(\emptyset) = 0$. This is not necessarily true for presheaves.

Notation 2.6. We denote by $\mathcal{S}_X$ the category of abelian sheaves on X.

By definition, $\mathcal{S}_X$ is a full subcategory of $\mathcal{P}_X$ (i.e. the morphisms are the same in $\mathcal{S}_X$ and $\mathcal{P}_X$). We will see later that $\mathcal{S}_X$ is an abelian category.

Example 2.7. The presheaf F on X defined by $F(U) = \mathbf{Z}^U = \{\text{functions } U \to \mathbf{Z}\}$ (with the obvious restriction maps) is a sheaf.

Exercise 19. (Two examples of presheaves which are not sheaves.)

(a) Define a presheaf F on X by $F(U) = \mathbf{Z} = \{\text{constant functions } U \to \mathbf{Z}\}$, with the obvious restriction maps. Show that the gluing does not always exist, so that F is not a sheaf in general.

(b) Define a presheaf F on X by $F(U) = 0$ if $U \neq X$, and $F(X) = \mathbf{Z}$. Show that the gluing exists but is not always unique, so that F is not a sheaf in general.

Definition 2.8. The *stalk* of a presheaf F at $x \in X$ is defined by $F_x := \varinjlim_U F(U)$, where the inductive limit is taken over all open neighborhoods of x in X.

Given $s \in F(U)$ and $x \in U$, the *germ* s_x of s at x is the canonical image of s in F_x.

Exercise 20. Let $0 \to F \to G \to H \to 0$ be a sequence of abelian presheaves on X. Show that if this sequence is exact, then for every $x \in X$, the sequence $0 \to F_x \to G_x \to H_x \to 0$ is also exact. Show that the converse is false by taking X to be a discrete topological space with two points.

This show that for presheaves, the stalks do not contain "enough information". The situation is better for sheaves, as the following exercise shows.

Exercise 21. (a) (Sections of sheaves are determined by their germs) Let F be a sheaf on X. Show that for any $U \subset X$, the canonical map $F(U) \to \prod_{x \in U} F_x$ is injective.

(b) (Isomorphisms are determined by stalks) Let $f : F \to G$ be a morphism of sheaves on X. Show that f is an isomorphism if and only if $f_x : F_x \to G_x$ is an isomorphism for all $x \in X$. Note: this does not say that if two sheaves have isomorphic stalks, then they are isomorphic.

So the exactness of a sequence of presheaves *cannot* be tested on the stalks. We will see, however, that exactness of sequences of sheaves (a notion yet to be defined) *can* be tested on the stalks. (Beware that the notion of exactness differs in the categories $\mathcal{P}_X$ and $\mathcal{S}_X$!)

Since presheaves and sheaves play a prominent role in this course, make sure you understand very well the following definition and universal property of sheafification.

Definition 2.9. Let F be a presheaf on X. For every open set $U \subset X$, we define $F^\sharp(U)$ to be the set of maps $s : U \to \bigsqcup_{x \in U} F_x$ (set-theoretic disjoint union) which are locally given by sections of F: there exists an open covering $U = \bigcup_{i \in I} U_i$ (depending a priori on s) and sections $s_i \in F(U_i)$ such that $s(x) = (s_i)_x$ for every $x \in U_i$. Then $F^\sharp$ is a sheaf on X called the *sheafification* of F.

Exercise 22. Determine the sheafifications of the presheaves considered in Exercise 19.

Proposition 2.10. *Let F be a presheaf on X, and let $i : F \to F^\sharp$ be its sheafification. Then for every sheaf G on X and every morphism of presheaves $f : F \to G$, there exists a unique morphism $f^\sharp : F^\sharp \to G$ such that $f = f^\sharp \circ i$. In other words, the canonical map*

$$\mathrm{Hom}(F^\sharp, G) \to \mathrm{Hom}(F, G)$$

given by composing with i, is a bijection.

Proof. We only explain how to construct the inverse map $\mathrm{Hom}(F, G) \to \mathrm{Hom}(F^\sharp, G)$, leaving the remaining details to the reader.

Let $f : F \to G$ be a morphism of presheaves. Let U be an open subset of X, and let $s \in F^\sharp(U)$. By definition s is a map $U \to \bigsqcup_{x \in U} F_x$ which is locally given by sections of F. In other words U is covered by open sets $(U_i)_{i \in I}$ (depending on s) such that for every $i \in I$, there exists $s_i \in F(U_i)$ such that $s(x) = (s_i)_x$ for every $x \in U_i$.

Let us define $t_i = f(s_i) \in G(U_i)$. We want to show that the sections t_i glue to a section over U. Let $i, j \in I$ and $x \in U_i \cap U_j$. Then $(t_i)_x = f_x(s(x)) = (t_j)_x$, so that t_i and t_j coincide on an open neighborhood of x. Since G is a sheaf, it follows that t_i and t_j coincide on $U_i \cap U_j$ (unicity of the gluing). Therefore there exists $t \in F(U)$ such that $t|_{U_i} = t_i$ for every $i \in I$ (existence of the gluing). We finally define $f^\sharp(s) := t$ and check that $f^\sharp$ is a morphism of sheaves. □

The sheafification process is functorial: the association $F \mapsto F^\sharp$ defines a functor $\mathcal{P}_X \to \mathcal{S}_X$. Proposition 2.10 can be reformulated very simply by saying that the sheafification $\sharp : \mathcal{P}_X \to \mathcal{S}_X$ and the inclusion $\mathcal{S}_X \to \mathcal{P}_X$ are adjoint functors.

The fact that $\mathcal{S}_X$ is an abelian category (which we haven't proved yet) is not obvious. Indeed, given a morphism of sheaves $f : F \to G$, the image of f in the category $\mathcal{P}_X$ is a presheaf but need not be a sheaf a general. This can already be seen in the classical topological setting. We recall the following examples:

Example 2.11. Let $X = \mathbf{R}/\mathbf{Z}$ be the 1-dimensional torus. Let C^∞ be the sheaf of real-valued C^∞ functions on X. Consider the morphism of sheaves $d : C^\infty \to C^\infty$ given by $d(f) = f'$ for any C^∞ function on an open subset of X. First, what is the kernel of d? In the category of presheaves, we have

$$\begin{aligned}\ker(d)(U) &= \{f : U \to \mathbf{R} : f' = 0\} \\ &= \{f : U \to \mathbf{R} \text{ locally constant}\}.\end{aligned}$$

Note that $\ker(d)$ is a sheaf because "locally constant" is a local property. In particular, the kernels of d in the categories $\mathcal{P}_X$ and $\mathcal{S}_X$ coincide.

What is the image of d? In the category of presheaves, we have

$$\mathrm{im}(d)(U) = \mathrm{im}(C^\infty(U) \xrightarrow{d} C^\infty(U)).$$

Now, consider the constant function $1 \in C^\infty(X)$. It is easy to see that there is no C^∞ function f on X such that $f' = 1$. So $1 \notin \mathrm{im}(d)(X)$. But locally, the function 1 certainly admits primitives. This shows that the image of d in the category $\mathcal{P}_X$ is not a sheaf. As we will define later, the image of d *in the category* $\mathcal{S}_X$ consists of those C^∞ functions which are *locally* the derivative of a C^∞ function. This implies that the image of d in $\mathcal{S}_X$ is the whole sheaf C^∞. We can summarize by saying that we have an exact sequence of *sheaves*

$$0 \to \underline{\mathbf{R}} \to C^\infty \xrightarrow{d} C^\infty \to 0. \tag{7}$$

Here $\underline{\mathbf{R}}$ denotes the sheaf of locally constant $\mathbf{R}$-valued functions. (Note that the sequence is not exact in $\mathcal{P}_X$.)

Exercise 23. Let $X = \mathbf{C}^\times$ endowed with the classical topology. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X (endowed with addition), and let $\mathcal{O}_X^\times$ be the sheaf of non-vanishing holomorphic functions on X (endowed with multiplication). Consider the morphism of abelian sheaves

$$\exp : \mathcal{O}_X \to \mathcal{O}_X^\times$$

sending an holomorphic function f to $\exp(f)$.

1. Show that the function $g \in \mathcal{O}(X)^\times$ defined by $g(z) = z$ is not a section of the presheaf $\mathrm{im}(\exp)$.

2. Deduce that the presheaf $\mathrm{im}(\exp)$ is not a sheaf.

3. Show that the sheafification of $\mathrm{im}(\exp)$ is the whole sheaf $\mathcal{O}_X^\times$.

So, we have an exact sequence of sheaves (valid actually for any Riemann surface X)

$$0 \to 2\pi i \underline{\mathbf{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0.$$

Again, this is not an exact sequence in $\mathcal{P}_X$.

In these examples, we see that the image of a morphism of sheaves does not coincide with the "naive" image (that is, the image in the category of presheaves). In fact, returning to our Example 2.11 with $X = \mathbf{R}/\mathbf{Z}$, taking the global sections of (7) gives a sequence

$$0 \to \mathbf{R} \to C^\infty(X) \xrightarrow{d} C^\infty(X) \tag{8}$$

As explained, this sequence is not exact on the right. The idea of cohomology is that we can extend this to a long exact sequence, as follows:

$$0 \to \mathbf{R} \to C^\infty(X) \xrightarrow{d} C^\infty(X) \to H^1(X, \underline{\mathbf{R}})$$

The cohomology group $H^1(X, \mathbf{R})$ measures the defect of the exactness of (8). It turns out that the sheaf cohomology $H^1(X, \underline{\mathbf{R}})$ coincides with the de Rham cohomology

$$H^1(X, \mathbf{R}) = \frac{\{\text{closed 1-forms on } X\}}{\{\text{exact 1-forms on } X\}}.$$

The vector space $H^1(X, \mathbf{R})$ is generated by the class of the differential form dx on $\mathbf{R}/\mathbf{Z}$ (it is well-defined because dx is a 1-form on $\mathbf{R}$ which is invariant under translation). The form dx is closed but not exact, for the exact same reason as above: it does not admit a primitive.

More generally, any exact sequence of sheaves $0 \to F \to G \to H \to 0$ on a topological space X will give rise to a long exact sequence

$$0 \to F(X) \to G(X) \to H(X) \to H^1(X, F) \to H^1(X, G) \to H^1(X, H) \to H^2(X, F) \to \cdots$$

2.2 The categorical setting

Recall that the abelian presheaves on a topological space X are the contravariant functors $\mathcal{T}_X \to \mathrm{Ab}$. This leads naturally to the following definition.

Definition 2.12. Let $\mathcal{C}$ be an arbitrary category. A *presheaf of sets* (resp. *abelian groups*) on $\mathcal{C}$ is a contravariant functor $F : \mathcal{C} \to \mathrm{Set}$ (resp. $F : \mathcal{C} \to \mathrm{Ab}$). A *morphism of presheaves* $f : F \to G$ on $\mathcal{C}$ is a morphism of functors.

Presheaves of abelian groups are also called *abelian presheaves*.

Let F be a presheaf on $\mathcal{C}$. By definition, for every morphism $\varphi : V \to U$ in $\mathcal{C}$, we have a map $F(\varphi) : F(U) \to F(V)$. When the morphism φ is clear from the context, we will denote the image of $s \in F(U)$ in $F(V)$ simply by $s|_V$, in analogy with the topological situation.

Notation 2.13. We denote by $\mathcal{P}_{\mathcal{C}}$ the category of abelian presheaves on $\mathcal{C}$.

Example 2.14. If $\mathcal{C}$ is the category with a single object and a single arrow, then $\mathcal{P}_{\mathcal{C}} \cong \mathrm{Ab}$.

Note that $\mathcal{P}_{\mathcal{C}}$ is an abelian category by 1.5. Concretely, a sequence $F \to G \to H$ in $\mathcal{P}_{\mathcal{C}}$ is exact if and only if for every object $X \in \mathcal{C}$, the sequence $F(X) \to G(X) \to H(X)$ is an exact sequence of abelian groups.

By Lemma 1.47, the abelian category $\mathcal{P}_{\mathcal{C}}$ has arbitrary direct sums. Thus, by Theorem 1.40, $\mathcal{P}_{\mathcal{C}}$ has arbitrary inductive limits.

Exercise 24. Let (F_i) be an inductive system of abelian presheaves on $\mathcal{C}$, indexed by an arbitrary category $\mathcal{I}$. Show that its inductive limit $F = \varinjlim_{\mathcal{I}} F_i$ is given by $F(X) = \varinjlim_{\mathcal{I}} F_i(X)$ for every $X \in \mathcal{C}$.

By Proposition 1.48, the category $\mathcal{P}_{\mathcal{C}}$ satisfies (Ab5), so directed limits define exact functors: for every directed set I, the functor $\varinjlim_I : \mathcal{P}_{\mathcal{C}}^I \to \mathcal{P}_{\mathcal{C}}$ is exact.

It is not yet clear how to define the notion of sheaf on a category $\mathcal{C}$. We will see later how adding an extra structure on $\mathcal{C}$ (namely, a Grothendieck topology) enables one to define sheaves on $\mathcal{C}$.

2.3 Direct and inverse images

Recall the definition of direct and inverse images of presheaves in the classical topological setting. Let $f : X \to Y$ be a continuous maps between topological spaces. Given an abelian presheaf F on X, we define

$$(f_* F)(V) = F(f^{-1}(V))$$

for every open subset V of Y, together with the natural restriction maps. This defines a presheaf f_*F on Y. Conversely, given an abelian presheaf G on Y, we define

$$(f^*G)(U) = \varinjlim_{V \supset f(U)} G(V)$$

for any open subset U of X, where the inductive limit is taken with respect to the open subsets V of Y containing $f(U)$, ordered with (reverse) inclusion. Together with the natural restriction maps, this defines a presheaf f^*G on X.

Exercise 25. 1. Check that f_*F (resp. f^*G) is a presheaf on Y (resp. X).

2. Show that $f_* : \mathcal{P}_X \to \mathcal{P}_Y$ and $f^* : \mathcal{P}_Y \to \mathcal{P}_X$ are additive functors.
3. Show that (f^*, f_*) are adjoint functors.
4. Compute the stalks of the presheaf f^*G on X.

We define similarly direct and inverse images of general presheaves.

Definition 2.15. Let $\mathcal{C}, \mathcal{D}$ be arbitrary categories, and let $f : \mathcal{C} \to \mathcal{D}$ be a (covariant) functor. For any abelian presheaf G on $\mathcal{D}$, the inverse image f^*G is the abelian presheaf on $\mathcal{C}$ defined by $(f^*G)(X) = G(f(X))$ for every object $X \in \mathcal{C}$.

Note that $f^*G = G \circ f$ so that f^*G is indeed a contravariant functor, hence defines a presheaf on $\mathcal{C}$. Moreover, for each morphism $u : G \to G'$ of presheaves on $\mathcal{D}$, we define $f^*u : f^*G \to f^*G'$ by $(f^*u)(X) = u(f(X)) : G(f(X)) \to G'(f(X))$ for every $X \in \mathcal{C}$. Thus we get a functor $f^* : \mathcal{P}_\mathcal{D} \to \mathcal{P}_\mathcal{C}$, called the *inverse image functor*.

One should be careful that a continuous map $f : X \to Y$ as above induces a functor $\mathcal{T}_Y \to \mathcal{T}_X$, so what is called a direct image in the topological setting becomes an inverse image in the categorical setting.

Lemma 2.16. *The inverse image functor $f^* : \mathcal{P}_\mathcal{D} \to \mathcal{P}_\mathcal{C}$ is additive, exact and commute with inductive limits.*

Proof. If $u, v : G \to G'$ are two morphisms of abelian presehaves on $\mathcal{D}$, then $f^*(u+v) = f^*u + f^*v$ on every object of $\mathcal{C}$, so that f^* is additive.

If $G' \to G \to G''$ is an exact sequence in $\mathcal{P}_\mathcal{D}$, then the resulting sequence $f^*G' \to f^*G \to f^*G''$ is exact on every object of $\mathcal{C}$, so that f^* is exact.

Let (G_i) be an inductive system of abelian presheaves on $\mathcal{D}$. Using Exercise 24, we have the following formal computation, where everything is functorial:

$$f^*(\varinjlim G_i)(X) = (\varinjlim G_i)(f(X)) = \varinjlim G_i(f(X)) = \varinjlim (f^*G_i)(X) = (\varinjlim f^*G_i)(X).$$

□

We now define direct images of abelian presheaves.

Theorem 2.17. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between arbitrary categories. Then the inverse image functor $f^* : \mathcal{P}_\mathcal{D} \to \mathcal{P}_\mathcal{C}$ has a left adjoint $f_* : \mathcal{P}_\mathcal{C} \to \mathcal{P}_\mathcal{D}$. The functor f_* is right exact and commutes with inductive limits.*

Proof. To show that f^* has a left adjoint, we need to show the following. For every $F \in \mathcal{P}_\mathcal{C}$, there should exist an object $f_*(F) \in \mathcal{P}_\mathcal{D}$, and for each $G \in \mathcal{P}_\mathcal{D}$, an isomorphism of abelian groups

$$\mathrm{Hom}(f_*F, G) \cong \mathrm{Hom}(F, f^*G) \tag{9}$$

which is functorial in G. The requirement that the isomorphism is linear will ensure that f_* is additive. Let F be an abelian presheaf on $\mathcal{C}$. We first define the abelian group $f_*F(V)$ for all $V \in \mathcal{D}$. Consider all pairs (U, ϕ) where U is an object of $\mathcal{C}$, and $\phi : V \to f(U)$ is a morphism in $\mathcal{D}$. The collection of all these pairs forms a category $\mathcal{I}_V$ if we define a morphism $(U_1, \phi_1) \to (U_2, \phi_2)$ to be a morphism $\psi : U_1 \to U_2$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} & f(U_1) \arrow[dd, "f(\psi)"] \\ V \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & \\ & f(U_2) \end{tikzcd}$$

The assignment $(U, \phi) \mapsto F(U)$ gives a contravariant functor $F_V : \mathcal{I}_V \to \text{Ab}$, and thus an inductive system of abelian groups indexed by $\mathcal{I}_V^{\text{op}}$. We may thus form the inductive limit, and define

$$f_*F(V) = \varinjlim_{(U,\phi)} F(U) \stackrel{\text{def}}{=} \varinjlim_{\mathcal{I}_V^{\text{op}}} F_V.$$

Let $h : V' \to V$ be a morphism in $\mathcal{D}$. We have a functor $\mathcal{I}_V \to \mathcal{I}_{V'}$ defined by mapping $(U, \phi : V \to f(U))$ to $(U, \phi \circ h : V' \to f(U))$. We deduce a morphism at the level of the inductive limits

$$\varinjlim_{\mathcal{I}_V^{\text{op}}} F_V \to \varinjlim_{\mathcal{I}_{V'}^{\text{op}}} F'_V,$$

which in turns gives a linear map $f_*F(V) \to f_*F(V')$.

Therefore, we have constructed an abelian presheaf f_*F on $\mathcal{D}$. It remains to show the existence of isomorphisms (9) which are functorial in G. We will construct the required maps in both directions.

Let $u : f_*F \to G$ be a morphism is $\mathcal{P}_{\mathcal{D}}$. For each $U \in \mathcal{C}$, we get a morphism

$$u(f(U)) : f_*F(f(U)) \to G(f(U)) = f^*G(U).$$

Now the pair $(U, \text{id}_{f(U)})$ is an object of the category $\mathcal{I}_{f(U)}$, so there is a canonical morphism

$$F(U) = F_{f(U)}(U, \text{id}_{f(U)}) \to \varinjlim F_{f(U)} = f_*F(f(U)).$$

By composing, we get the desired map $F(U) \to f^*G(U)$, and it is clearly functorial in U. Thus we get a morphism of abelian presheaves $v : F \to f^*G$. Moreover, the corresponding map $\text{Hom}(f_*F, G) \to \text{Hom}(F, f^*G)$ is linear, and functorial in G.

Conversely, let $v : F \to f^*G$ be a morphism in $\mathcal{P}_{\mathcal{C}}$. Let $V \in \mathcal{D}$. We want to construct a map

$$f_*F(V) = \varinjlim_{(U,\phi)} F(U) \to G(V).$$

For this, we need to construct compactible maps $F(U) \to G(V)$ for each $(U, \phi : V \to f(U)) \in \mathcal{I}_V$. But we have maps

$$F(U) \xrightarrow{v(U)} f^*G(U) = G(f(U)) \xrightarrow{G(\phi)} G(V).$$

These maps are compatible, and the universal property gives $f_*F(V) \to G(V)$. This is functorial in V and produces $u : f_*F \to G$. We let the reader check that the maps $u \leftrightarrow v$ are inverse to each other.

Let us check that f_* is right exact. Let $0 \to F \to G \to H \to 0$ be a short exact sequence in $\mathcal{P}_\mathcal{C}$. We want to prove that for each $V \in \mathcal{D}$, the sequence $f_*F(V) \to f_*G(V) \to f_*H(V) \to 0$ is exact. For each pair $(U, \phi) \in \mathcal{I}_V$, we have by definition an exact sequence $0 \to F(U) \to G(U) \to H(U) \to 0$. Taking the inductive limit and using Theorem 1.40 with the category Ab, we get the required right-exactness.

Finally, let us prove that f_* commutes with inductive limits. It is a general fact in category theory that left (resp. right) adjoint functors commute with inductive (resp. projective) limits. The argument is purely formal. We need the following auxiliary isomorphism, valid in an arbitrary category provided the limits exist (proof left to the reader):

$$\mathrm{Hom}(\varinjlim X_i, X) = \varprojlim \mathrm{Hom}(X_i, X).$$

Now, let (F_i) be an inductive system of abelian presheaves on $\mathcal{C}$, and let $F = \varinjlim F_i \in \mathcal{P}_\mathcal{C}$. For any $G \in \mathcal{P}_\mathcal{D}$, we have canonical isomorphisms

$$\begin{aligned}
\mathrm{Hom}(f_* \varinjlim F_i, G) &\cong \mathrm{Hom}(\varinjlim F_i, f^*G) \\
&\cong \varprojlim \mathrm{Hom}(F_i, f^*G) \\
&\cong \varprojlim \mathrm{Hom}(f_*F_i, G) \\
&\cong \mathrm{Hom}(\varinjlim f_*F_i, G).
\end{aligned}$$

The last step is provided by the Yoneda Lemma, which ensures that $f_* \varinjlim F_i$ and $\varinjlim f_*F_i$ are canonically isomorphic (check this step by yourself). □

The function $f_* : \mathcal{P}_\mathcal{C} \to \mathcal{P}_\mathcal{D}$ is called the *direct image functor*.

Remark 2.18. In the topological setting, the category $\mathcal{I}_V$ introduced in the above proof is simply the set of open neighborhoods of $f(V)$ (where $f : X \to Y$ is our continuous map). This set is directed because the intersection of two open neighborhoods is still an open neighborhood. This implies that f_* is an exact functor (see Exercise 26). In general however, the category $\mathcal{I}_V$ is not a *directed category*, so f_* need not be an exact functor.

Exercise 26. In the topological setting, show that f_* recovers the inverse image functor. Show that f_* is exact in this case.

Exercise 27. Define direct and inverse images of presheaves of *sets*. Show that if a presheaf of sets F is represented by an object Z, then its direct image f_*F is represented by the object $f(Z)$. Is it true that the inverse image of a representable presheaf is representable?

Exercise 28. Let $\mathcal{C}_0$ be the category with a single object and a single arrow. We mentioned earlier that $\mathcal{P}_{\mathcal{C}_0} \cong$ Ab. Let $\mathcal{C}$ be a category, and let $X \in \mathcal{C}$. Let $i : \mathcal{C}_0 \to \mathcal{C}$ be the functor mapping to the object X. Compute the functors $i^* : \mathcal{P}_\mathcal{C} \to$ Ab and $i_* :$ Ab $\to \mathcal{P}_\mathcal{C}$.

The abelian presheaf $i_*(\mathbf{Z})$ in the last exercise will be useful later.

2.4 Grothendieck topologies

So far, we have defined presheaves on categories, but not yet sheaves. To do this we need additional data on the category.

In order to get some intuition, let us first consider the situation for a topological space X. Sheaves on X are defined as those presheaves on X which satisfy the gluing condition with respect to open coverings. How does this generalize to an arbitrary category $\mathcal{C}$? Grothendieck's idea is that instead of trying to define what it means for a family of morphisms in $\mathcal{C}$ to be a covering, one should think of the properties we want the coverings to satisfy, and *axiomatize* the notion of covering.

In the topological setting, open coverings satisfy the following formal properties:

(1) If U is covered by the family (U_i), and $V \subset U$, then V is covered by the family $(U_i \cap V)$.

(2) If U is covered by the family (U_i), and each U_i is covered by the family $(V_{i,j})$, then U is covered by the family of all $V_{i,j}$.

(3) U is covered by itself.

The definition of a Grothendieck topology is modelled on these properties.

Definition 2.19. Let $\mathcal{C}$ be an arbitrary category. A *Grothendieck topology* on $\mathcal{C}$ is the data, for each object $U \in \mathcal{C}$, of a set $\mathrm{cov}(U)$ of families $(\varphi_i : U_i \to U)_{i \in I}$ of morphisms in $\mathcal{C}$, called the *coverings* of U, satisfying the following axioms:

(T1) Given a covering $(U_i \to U)_{i \in I} \in \mathrm{cov}(U)$ and a morphism $V \to U$ in $\mathcal{C}$, all fibre products $U_i \times_U V$ exist in $\mathcal{C}$, and the family $(U_i \times_U V \to V)_{i \in I}$ belongs to $\mathrm{cov}(V)$.

(T2) Given a covering $(U_i \to U)_{i \in I} \in \mathrm{cov}(U)$ and coverings $(V_{i,j} \to U_i)_{j \in J_i} \in \mathrm{cov}(U_i)$ for each $i \in I$, the family $(V_{i,j} \to U)_{i \in I, j \in J_i}$ belongs to $\mathrm{cov}(U)$.

(T3) If $\varphi : U' \to U$ is an isomorphism in $\mathcal{C}$ then the family $\{\varphi\}$ belongs to $\mathrm{cov}(U)$.

A *site* $\mathcal{T}$ is the data of a category $\mathrm{cat}(\mathcal{T})$ together with a Grothendieck topology on $\mathrm{cat}(\mathcal{T})$. The set of all coverings in $\mathcal{T}$ is denoted by $\mathrm{cov}(\mathcal{T})$.

We insist on the fact that the coverings are part of the data: in particular, there may be several Grothendieck topologies on a given category, just like there may be several topologies on a given set.

Examples 2.20. (1) Let X be a topological space. The category of open subsets of X together with the usual coverings, i.e. families $(U_i \to U)_{i \in I}$ with $U = \bigcup_{i \in I} U_i$, defines a Grothendieck topology. Indeed, in this category the fibre product of two open subsets U and V is simply the intersection $U \cap V$ (where the fibre product is taken over any open subset containing both U and V, for example over X).

(2) Let X be again a topological space. Let Top/X be the category of *spaces over* X: the objects are pairs (Y, f) where Y is a topological space and $f : Y \to X$ is a continuous map. The morphisms in Top/X are the continuous maps $Y \to Z$ making the obvious diagram commute. Let Y be a space over X. We say that a family of continuous maps $(f_i : Y_i \to Y)_{i \in I}$ is a covering if $Y = \bigcup_{i \in I} f_i(Y_i)$.

Exercise 29. (a) Describe the fibre products in Top and Top/X.

(b) Show that the data above defines a Grothendieck topology on Top/X.

(c) Show that the same holds if we require moreover the $f_i : Y_i \to Y$ to be open immersions.

We refer to (1) as the *small site* associated to X, and to (2) as the *big site* associated to X.

We will later generalize these sites in the case X is a scheme. There are plenty of useful topologies on the category of schemes and we will study them later.

2.5 Sheaves

As mentioned before, the notion of Grothendieck topology gives us a good setting to define sheaves on arbitrary categories. We now explain this definition.

Let $\mathcal{T} = (\mathrm{cat}(\mathcal{T}), \mathrm{cov}(\mathcal{T}))$ be a site. We already know what is a presheaf of sets (resp. abelian groups) on $\mathcal{T}$: it is a contravariant functor $F : \mathrm{cat}(\mathcal{T}) \to \mathrm{Set}$ (resp. $F : \mathrm{cat}(\mathcal{T}) \to \mathrm{Ab}$). Note that this definition depends only on the underlying category and not on the Grothendieck topology.

In order to define sheaves on $\mathcal{T}$, we need the following auxiliary definition.

Definition 2.21. Let X, Y, Z be sets, and let $\alpha : X \to Y$ and $\beta, \gamma : Y \to Z$ be maps. We say that the diagram

$$X \xrightarrow{\alpha} Y \overset{\beta}{\underset{\gamma}{\rightrightarrows}} Z \tag{10}$$

is *exact* if α is injective and the image of α is equal to the equalizer of (β, γ), in other words $\operatorname{im}(\alpha) = \{y \in Y : \beta(y) = \gamma(y)\}$.

Note that if X, Y, Z are abelian groups and α, β, γ are linear, then the diagram (10) is exact if and only if the sequence

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta-\gamma} Z$$

is exact.

Definition 2.22 (Sheaves on a site)**.** Let $\mathcal{T} = (\operatorname{cat}(\mathcal{T}), \operatorname{cov}(\mathcal{T}))$ be a site, and let F be a presheaf of sets or abelian groups on $\mathcal{T}$. We say that F is a *sheaf* if for every covering $(U_i \to U)_{i \in I}$ in $\operatorname{cov}(\mathcal{T})$, the diagram

$$F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I^2} F(U_i \times_U U_j) \tag{11}$$

is exact, where the two arrows on the right are given by $(s_i)_i \mapsto (s_i|_{U_i \times_U U_j})_{i,j}$ and $(s_i)_i \mapsto (s_j|_{U_i \times_U U_j})_{i,j}$ respectively.

A *morphism of sheaves* on $\mathcal{T}$ is defined as a morphism of presheaves.

Notation 2.23. Let $\mathcal{T}$ be a site. We denote by $\mathcal{P}_{\mathcal{T}}$ the category of abelian presheaves on $\mathcal{T}$, and by $\mathcal{S}_{\mathcal{T}}$ the category of abelian sheaves on $\mathcal{T}$. When the context is clear, we simply write $\mathcal{P}$ and $\mathcal{S}$.

By definition $\mathcal{S}$ is a full subcategory of $\mathcal{P}$: every object of $\mathcal{S}$ is an object of $\mathcal{P}$, and the morphisms in $\mathcal{S}$ coincide with those in $\mathcal{P}$.

Lemma 2.24. *The category $\mathcal{S}$ is additive.*

Proof. For each $F, G \in \mathcal{S}$, we have $\operatorname{Hom}_{\mathcal{S}}(F, G) = \operatorname{Hom}_{\mathcal{P}}(F, G)$ by definition, so $\operatorname{Hom}_{\mathcal{S}}(F, G)$ is an abelian group, and the composition of morphisms is bilinear. Also, the zero object 0 in $\mathcal{P}$, defined by $0(U) = \{0\}$ for every $U \in T$, is a sheaf and is a zero object in $\mathcal{S}$. Finally, let $F, G \in \mathcal{S}$. Consider $F \oplus G \in \mathcal{P}$. It is easy to see that $F \oplus G$ is a sheaf. Since the morphisms are the same in $\mathcal{S}$ and $\mathcal{P}$, we deduce that $F \oplus G$ is the product and also the coproduct of F and G in $\mathcal{S}$. □

We will prove later that the category $\mathcal{S}$ is abelian.

2.6 Sheafification

In this section, we define a sheafification functor $\sharp : \mathcal{P} \to \mathcal{S}$ for arbitrary sites. This is an essential tool to show that the category of abelian sheaves on a site is abelian.

Let $\mathcal{T}$ be a site. We denote by $\mathcal{P}$ (resp. $\mathcal{S}$) the category of abelian presheaves (resp. sheaves) on $\mathcal{T}$.

Theorem 2.25. *The inclusion functor $i : \mathcal{S} \to \mathcal{P}$ has a left adjoint $\sharp : \mathcal{P} \to \mathcal{S}$, called the* sheafification.

Given an abelian presheaf F, we will construct its sheafification $F^\sharp$ by applying twice a certain functor $\dagger : \mathcal{P} \to \mathcal{P}$.

Definition 2.26. Let F be an abelian presheaf on $\mathcal{T}$. For every object $U \in T$ and every covering $\mathcal{U} = (U_i \to U)_{i \in I} \in \operatorname{cov}(U)$, we define

$$H^0(\mathcal{U}, F) = \Big\{ (s_i) \in \prod_{i \in I} F(U_i) : \forall i, j \in I, s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \Big\}.$$

Note that we have a canonical map $F(U) \to H^0(\mathcal{U}, F)$. We have not assumed that F is a sheaf, so that this map is neither injective nor surjective in general. We may view $H^0(\mathcal{U}, F)$ as a first approximation of the sheafification of F. However $H^0(\mathcal{U}, F)$ does not depend on U alone. So we need to vary the covering $\mathcal{U}$. To this end, we introduce the notion of refinement of a covering.

Definition 2.27. Let $\mathcal{U} = (U_i \to U)_{i \in I}$ be a covering of U in $\mathcal{T}$. A *refinement* of $\mathcal{U}$ is the data of a covering $\mathcal{U}' = (U'_j \to U)_{j \in J}$ together with a map $\alpha : J \to I$ and, for each $j \in J$, a morphism $f_j : U'_j \to U_{\alpha(j)}$ in $\operatorname{cat}(\mathcal{T})$. For brevity, we denote by $f : \mathcal{U}' \to \mathcal{U}$ this refinement.

Let $U \in T$. The collection of all coverings of U together with the refinement maps as morphisms forms a category. We still denote by $\operatorname{cov}(U)$ this category.

Note that for every refinement $f : \mathcal{U}' \to \mathcal{U}$ in $\operatorname{cov}(U)$, we get a canonical map

$$H^0(f, F) : H^0(\mathcal{U}, F) \to H^0(\mathcal{U}', F)$$

Thus $H^0(\cdot, F)$ defines a contravariant functor $\operatorname{cov}(U) \to \mathrm{Ab}$.

Definition 2.28. Let F be an abelian presheaf of $\mathcal{T}$. For every $U \in T$, we define

$$F^\dagger(U) = \varinjlim_{\mathcal{U} \in \operatorname{cov}(U)} H^0(\mathcal{U}, F) = \varinjlim_{\operatorname{cov}(U)^{\mathrm{op}}} H^0(\cdot, F).$$

In other words, we consider finer and finer coverings $\mathcal{U}$ of U, and take the inductive limit of the resulting groups $H^0(\mathcal{U}, F)$.

Remark 2.29. The group $F^\dagger(U)$ is also denoted by $\check{H}^0(U, F)$ and known as the zeroth Čech cohomology group of F over U. The Čech cohomology groups $\check{H}^q(U, F)$ are a generalization of this construction, by considering fibre products of the form $U_{i_0} \times_U \cdots \times_U U_{i_q}$ for arbitrary $q \geq 0$ and making a certain cochain complex out of them, see [Tam94, I.2.2].

Let $V \to U$ be a morphism in $\mathcal{T}$. If $\mathcal{U} = (U_i \to U)_{i \in I}$ is a covering of U, then $\mathcal{V} = (U_i \times_U V \to V)_{i \in I}$ is a covering of V, and we get a morphism $H^0(\mathcal{U}, F) \to H^0(\mathcal{V}, F)$. Passing to the inductive limit, we get a morphism $F^\dagger(U) \to F^\dagger(V)$. This gives to $F^\dagger$ the structure of an abelian presheaf on $\mathcal{T}$.

The presheaf $F^\dagger$ is not a sheaf in general, but at least we have the uniqueness part in the gluing property. In order to show this, we need the following preparatory lemma.

Lemma 2.30. *Let F be a presheaf on $\mathcal{T}$. Let $\mathcal{U}$ and $\mathcal{U}'$ be coverings of $U \in T$, and let $f, g : \mathcal{U}' \to \mathcal{U}$ be two refinement maps. Then the maps $H^0(\mathcal{U}, F) \to H^0(\mathcal{U}', F)$ induced by f and g coincide.*

Proof. Write $\mathcal{U} = (U_i \to U)_{i \in I}$ and $\mathcal{U}' = (U'_j \to U)_{j \in J}$. Let $f_j : U'_j \to U_{\alpha(j)}$ and $g_j : U'_j \to U_{\beta(j)}$ be the refinements, with $\alpha, \beta : J \to I$. Let $s = (s_i) \in H^0(\mathcal{U}, F)$. Then $H^0(f, F)(s)$ is the family $(s_{\alpha(j)}|_{U'_j})$, and similarly $H^0(g, F)(s) = (s_{\beta(j)}|_{U'_j})$. So we have to show the sections $s_{\alpha(j)}$ and $s_{\beta(j)}$ coincide on U'_j. But they already coincide on $U_{\alpha(j)} \times_U U_{\beta(j)}$ by assumption on s, and there is a canonical morphism $U'_j \to U_{\alpha(j)} \times_U U_{\beta(j)}$ through which f_j and g_j factor. □

This shows that the inductive limit defining $F^\dagger(U)$ can be taken over the *set* (and not the category) of coverings of U. More precisely, let us endow the set $\operatorname{cov}(U)$ with the following partial order: $\mathcal{U} \leq \mathcal{U}'$ if and only if there exists at least one refinement $\mathcal{U}' \to \mathcal{U}$. By Lemma 2.30, if $\mathcal{U} \leq \mathcal{U}'$, then the induced map $H^0(\mathcal{U}, F) \to H^0(\mathcal{U}', F)$ does not depend on the refinement $f : \mathcal{U}' \to \mathcal{U}$. So $F^\dagger(U)$ is actually an inductive limit over the *poset* $\operatorname{cov}(U)$.

Exercise 30. Using axioms (T1) and (T2), show that $\operatorname{cov}(U)$ is a directed set.

In particular, when working with $F^\dagger(U)$ we may use its explicit description given by Exercise 16.

Proposition 2.31. *Let F be an abelian presheaf on $\mathcal{T}$. Then for every $U \in T$ and every covering $\mathcal{U} = (U_i \to U)_{i\in I}$ in $\mathcal{T}$, the canonical map*

$$F^\dagger(U) \to \prod_{i\in I} F^\dagger(U_i)$$

is injective.

In other words, sections of $F^\dagger$ are determined by their restrictions to an open covering. We say that $F^\dagger$ is a *separated* presheaf.

Proof. Let $\overline{s} \in F^\dagger(U)$ such that for every $i \in I$, we have $\overline{s}|_{U_i} = 0$. Since $F^\dagger(U)$ is a directed inductive limit, there exists a covering $\mathcal{V}$ of U such that $\overline{s}$ is represented by $s \in H^0(\mathcal{V}, F)$. Write $\mathcal{V} = (V_j \to U)_{j\in J}$. Then $\overline{s}|_{U_i}$ is represented by the image s_i of s under the canonical map $H^0(\mathcal{V}, F) \to H^0(\mathcal{V}_i, F)$, where $\mathcal{V}_i$ is the covering of U_i defined by the family $(V_j \times_U U_i \to U_i)_{j\in J}$, which makes sense by axiom (T1).

Since $F^\dagger(U_i)$ is a directed inductive limit and since $\overline{s}|_{U_i} = 0$, there exists a refinement $f_i : \mathcal{W}_i \to \mathcal{V}_i$ such that $H^0(f_i, F)(s_i) = 0$. Using axiom (T2), the collection of coverings $\mathcal{W}_i \in \operatorname{cov}(U_i)$ defines a covering $\mathcal{W}$ of U. Moreover, the refinements f_i together provide a refinement $f : \mathcal{W} \to \mathcal{V}$. By construction, the image of s under the map $H^0(\mathcal{V}, F) \to H^0(\mathcal{W}, F)$ is zero. This shows that the image of s in the inductive limit is zero, so that $\overline{s} = 0$. $\square$

Note that for every $U \in T$, we have a canonical map $F(U) \to F^\dagger(U)$. This gives a morphism of abelian presheaves $F \to F^\dagger$.

Lemma 2.32. *If F is a sheaf on $\mathcal{T}$, then $F \to F^\dagger$ is an isomorphism.*

Proof. In fact, since F is a sheaf, we have $H^0(\mathcal{U}, F) \cong F(U)$ for every covering $\mathcal{U}$ of U. $\square$

Proposition 2.33. *If F is a separated presheaf on $\mathcal{T}$, then $F \to F^\dagger$ is a monomorphism in $\mathcal{P}_\mathcal{T}$, and $F^\dagger$ is a sheaf on $\mathcal{T}$.*

We need the following lemma.

Lemma 2.34. *Let F be a separated presheaf on $\mathcal{T}$, and let $U \in T$. Then for every covering $\mathcal{U} \in \operatorname{cov}(U)$ and every refinement $f : \mathcal{U}' \to \mathcal{U}$, the canonical map $H^0(\mathcal{U}, F) \to H^0(\mathcal{U}', F)$ is injective. As a consequence, the canonical map $H^0(\mathcal{U}, F) \to F^\dagger(U)$ is injective.*

Proof. Write $\mathcal{U} = (U_i \to U)_{i\in I}$ and $\mathcal{U}' = (U'_j \to U)_{j\in J}$. Consider the family $\mathcal{V} = (U_i \times_U U'_j \to U)_{i\in I, j\in J}$, which is a covering by axioms (T1) and (T2). This covering refines both $\mathcal{U}$ and $\mathcal{U}'$, so there are natural refinement maps $p : \mathcal{V} \to \mathcal{U}$ and $p' : \mathcal{V} \to \mathcal{U}'$. Lemma 2.30 ensures that

$$H^0(p, F) = H^0(f \circ p', F) = H^0(p', F) \circ H^0(f, F).$$

So it suffices to shows that $H^0(p, F)$ is injective. This map is induced by the map

$$\prod_{i\in I} F(U_i) \to \prod_{(i,j)\in I\times J} F(U_i \times_U U'_j)$$

given by restricting from U_i to $U_i \times_U U'_j$. But F is separated and the $U_i \times_U U'_j$ cover U_i, so this map is injective.

The second claim of the Lemma follows from the explicit description of a directed inductive limit. $\square$

Proof of Proposition 2.33. Let F be a separated presheaf on $\mathcal{T}$. Let $U \in T$. Using Lemma 2.34 with the trivial covering $(U \xrightarrow{\mathrm{id}_U} U)$, we see that $F(U) \to F^\dagger(U)$ is injective, so that $F \to F^\dagger$ is a monomorphism.

Now, let us show that $F^\dagger$ is a sheaf. We already know from Proposition 2.31 that $F^\dagger$ is separated, so we are left to prove the existence of gluings. Let $\mathcal{U} = (U_i \to U)_{i\in I}$ be a covering of $U \in T$. Let $(\overline{s_i})_i \in \prod_{i\in I} F^\dagger(U_i)$ be a family of sections satisfying the gluing condition. We need to construct a section of $F^\dagger$ over U restricting to the $\overline{s_i}$. Represent each $\overline{s_i}$ as an element $s_i \in H^0(\mathcal{V}_i, F)$, where $\mathcal{V}_i = (V_{i,\mu} \to U_i)_\mu$ is a certain covering of U_i. We have

$$s_i = (t_{i,\mu})_\mu \in \prod_\mu F(V_{i,\mu}).$$

Let us fix $i, j \in I$. Then $(V_{i,\mu} \times_U U_j)_\mu$ is a covering of $U_i \times_U U_j$. Let

$$\sigma_{i,j} \in \prod_\mu F(V_{i,\mu} \times_U U_j)$$

denote the canonical image of s_i. Similarly $(U_i \times_U V_{j,\nu})_\nu$ is a covering of $U_i \times_U U_j$, and we denote by

$$\tau_{i,j} \in \prod_\nu F(U_i \times_U V_{j,\nu})$$

the canonical image of s_j. By assumption, we know that $\sigma_{i,j}$ and $\tau_{i,j}$ coincide in $F^\dagger(U_i \times_U U_j)$. By Lemma 2.34, we deduce that $\sigma_{i,j}$ and $\tau_{i,j}$ coincide on every common refinement of the coverings $(V_{i,\mu} \times_U U_j)_\mu$ and $(U_i \times_U V_{j,\nu})_\nu$. In particular, they coincide on the covering

$$(V_{i,\mu} \times_U V_{j,\nu} \to U_i \times_U U_j)_{\mu,\nu}.$$

This means that the sections $t_{i,\mu}$ and $t_{j,\nu}$ coincide on $V_{i,\mu} \times_U V_{j,\nu}$. Now the $V_{i,\mu}$ cover U. Denoting by $\mathcal{W}$ this covering, we get that the family $(t_{i,\mu})_{i,\mu}$ defines an element of $H^0(\mathcal{W}, F)$, and thus an element of $F^\dagger(U)$. By construction, this section restricts to the sections $\overline{s_i}$. $\square$

Using Propositions 2.31 and 2.33, we see that given a presheaf F, the presheaf $(F^\dagger)^\dagger$ is always a sheaf. This is our sheafification, but we need to show it is functorial.

Given a covering $\mathcal{U} \in \mathrm{cov}(U)$, the group $H^0(\mathcal{U}, F)$ is functorial in F: for each morphism $F \to G$ in $\mathcal{P}$, we have a natural map $H^0(\mathcal{U}, F) \to H^0(\mathcal{U}, G)$, and passing to the inductive limit, a natural map $F^\dagger(U) \to G^\dagger(U)$. So we get a morphism $F^\dagger \to G^\dagger$ in $\mathcal{P}$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ F^\dagger & \longrightarrow & G^\dagger. \end{array} \tag{12}$$

We thus have defined an additive functor $\dagger\colon \mathcal{P} \to \mathcal{P}$.

Definition 2.35. The *sheafification* is the additive functor $\sharp : \mathcal{P} \to \mathcal{S}$ defined by $F^\sharp = (F^\dagger)^\dagger$.

Lemma 2.36. *Let F be a presheaf and G be a sheaf on $\mathcal{T}$. Every morphism of presheaves $u : F \to G$ factors uniquely as $F \to F^\dagger \to G$.*

Proof. The existence of the factorisation follows from the diagram (12). Let us show uniqueness. It suffices to do the case $u = 0$. Let $v : F^\dagger \to G$ such that all the induced maps $F(U) \to G(U)$ are zero. Let $\mathcal{U} = (U_i \to U)_i \in \operatorname{cov}(U)$. The commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{U}, F) & \xrightarrow{\subset} & \prod_i F(U_i) \\ \downarrow & & \downarrow \\ G(U) \cong H^0(\mathcal{U}, G) & \xrightarrow{\subset} & \prod_i G(U_i) \end{array}$$

shows that $H^0(\mathcal{U}, F) \to G(U)$ is the zero map. Passing to the inductive limit, the same is true for $v(U) : F^\dagger(U) \to G(U)$. $\square$

We can finally prove Theorem 2.25. We show that $\sharp : \mathcal{P} \to \mathcal{S}$ is a left adjoint of the inclusion functor $\mathcal{S} \to \mathcal{P}$. This amounts to say that for any abelian presheaf F and any abelian sheaf G, every morphism $u : F \to G$ factors uniquely as $F \to F^\sharp \to G$. This follows immediately from Lemma 2.36. $\square$

2.7 The category of abelian sheaves

In this section we prove that the category of abelian sheaves is an abelian category.

Theorem 2.37. *Let $\mathcal{T}$ be a site. Then the category $\mathcal{S}$ of abelian sheaves on $\mathcal{T}$ is an abelian category.*

Proof. We need to show (Ab1) (existence of kernels and cokernels) and (Ab2). Let $f : F \to G$ be a morphism of abelian sheaves on $\mathcal{T}$. Let K be the kernel of f in $\mathcal{P}$. We claim that K is a sheaf. Indeed, let $(U_i \to U)_i$ be a covering of U. We have a commutative diagram

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ K(U) & \longrightarrow & \prod_i K(U_i) & \rightrightarrows & \prod_{i,j} K(U_i \times_U U_j) \\ \downarrow & & \downarrow & & \downarrow \\ F(U) & \longrightarrow & \prod_i F(U_i) & \rightrightarrows & \prod_{i,j} F(U_i \times_U U_j) \\ \downarrow & & \downarrow & & \downarrow \\ G(U) & \longrightarrow & \prod_i G(U_i) & \rightrightarrows & \prod_{i,j} G(U_i \times_U U_j) \end{array}$$

where the columns are exact, and the second and third rows are exact. This implies that the first rows is also exact. So K is a sheaf. Let us show that K is a kernel of f in $\mathcal{S}$. For every $X \in \mathcal{P}$, we have an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{P}}(X, K) \to \operatorname{Hom}_{\mathcal{P}}(X, F) \to \operatorname{Hom}_{\mathcal{P}}(X, G)$$

In particular, this holds for $X \in \mathcal{S}$, and since $\mathcal{S}$ is a full subcategory of $\mathcal{P}$, we deduce that K is the kernel of f in $\mathcal{S}$.

Now let C be the cokernel of f in $\mathcal{P}$. Trying to proceed as before fails (both for the existence and the uniqueness of the gluing), which hints to the fact that C is not a sheaf in general. Let us show that $C^\sharp$ is a cokernel of f, more precisely that the composite map $G \to C \to C^\sharp$ is a cokernel of f in $\mathcal{S}$. For every $X \in \mathcal{S}$, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(C,X) \arrow[r] & \mathrm{Hom}(G,X) \arrow[r] & \mathrm{Hom}(F,X) \\ & \mathrm{Hom}(C^\sharp,X) \arrow[u, "\cong"] \arrow[ur] & & \end{tikzcd}$$

where the vertical map is an isomorphism by the universal property of sheafification. This shows that $C^\sharp$ is the cokernel of f in $\mathcal{S}$.

So kernels, cokernels, images and coimages in $\mathcal{S}$ are all defined. In order to go further, we need the following lemmas.

Lemma 2.38. *The additive functor* $\dagger\colon \mathcal{P} \to \mathcal{P}$ *is left exact.*

Proof. Let $0 \to F \to G \to H \to 0$ be an exact sequence in $\mathcal{P}$. We want to show that for every $U \in T$, the sequence $0 \to F^\dagger(U) \to G^\dagger(U) \to H^\dagger(U)$ is exact. First, given a covering $\mathcal{U}$ of U, it is not hard to show that

$$0 \to H^0(\mathcal{U}, F) \to H^0(\mathcal{U}, G) \to H^0(\mathcal{U}, H) \tag{13}$$

is an exact sequence of abelian groups (exercise!). This sequence is clearly functorial in $\mathcal{U}$, so it defines an exact sequence in the category of inductive systems of abelian groups indexed by $\mathrm{cov}(U)$ (for the notion of exactness in this category, see Proposition 1.32). Moreover $\mathrm{cov}(U)$ is a directed set, so by Theorem 1.44, the functor $\varinjlim_{\mathrm{cov}(U)} : \mathrm{Ab}^{\mathrm{cov}(U)} \to \mathrm{Ab}$ is exact. Applying this functor to (the inductive version of) the sequence (13), we get the desired exact sequence. □

Lemma 2.39. *Let $f : F \to G$ be a morphism of abelian sheaves. Then the image of f in $\mathcal{S}$ is canonically isomorphic to the sheafification of the image of f in $\mathcal{P}$.*

Proof. We denote by $\mathrm{im}_\mathcal{P}$ (resp. $\mathrm{im}_\mathcal{S}$) the image in the category $\mathcal{P}$ (resp. $\mathcal{S}$), and similarly for the other notions. Let $C = \mathrm{coker}_\mathcal{P}(f)$ and $I = \mathrm{im}_\mathcal{P}(f)$. We have an exact sequence in $\mathcal{P}$

$$0 \to I \to G \to C \to 0 \tag{14}$$

By Lemma 2.38 and Exercise 13, the functor $\dagger \circ \dagger\colon \mathcal{P} \to \mathcal{P}$ is left exact. Applying this functor to (14), we get

$$0 \to I^\sharp \to G^\sharp \to C^\sharp$$

still in $\mathcal{P}$. Since the kernel is the same in $\mathcal{P}$ and $\mathcal{S}$, this means that $I^\sharp$ is the kernel of $G = G^\sharp \to C^\sharp$ in $\mathcal{S}$, which gives $I^\sharp \cong \mathrm{im}_\mathcal{S}(f)$ as desired. □

We can finally prove (Ab2) in $\mathcal{S}$. Let $f : F \to G$ be a morphism in $\mathcal{S}$. Let

$$F \to \mathrm{coim}_\mathcal{P}(f) \xrightarrow{\overline{f}} \mathrm{im}_\mathcal{P}(f) \to G$$

be the canonical factorisation of f, so that $\overline{f}$ is an isomorphism in $\mathcal{P}$. Applying the sheafification functor to this factorisation, we get

$$F \to \mathrm{coim}_\mathcal{P}(f)^\sharp \xrightarrow{\overline{f}^\sharp} \mathrm{im}_\mathcal{P}(f)^\sharp \to G$$

This is still a factorisation of f, but this time in $\mathcal{S}$. We have $\mathrm{coim}_\mathcal{P}(f)^\sharp \cong \mathrm{coim}_\mathcal{S}(f)$ by definition, and $\mathrm{im}_\mathcal{P}(f)^\sharp \cong \mathrm{im}_\mathcal{S}(f)$ by Lemma 2.39. Moreover, since $\overline{f}$ was an isomorphism, the same is true for $\overline{f}^\sharp$. This proves (Ab2). □

Theorem 2.40. *The inclusion functor $i : \mathcal{S} \to \mathcal{P}$ is left exact, and the sheafification functor $\sharp : \mathcal{P} \to \mathcal{S}$ is exact.*

Proof. If $0 \to F \to G \xrightarrow{f} H$ is exact in $\mathcal{S}$, then F is the kernel of f in S, which is the same as the kernel in $\mathcal{P}$. So the sequence is still exact in $\mathcal{P}$. Hence i is left exact.

The functor $\sharp$ is left adjoint to i so by general principles, it is right exact. Moreover, we have seen that $i \circ \sharp = \dagger \circ \dagger$ is left exact. Since i is fully faithful, it follows that $\sharp$ is also left exact. □

Remark 2.41. In Section 2.1, we saw examples showing that in general $i : \mathcal{S} \to \mathcal{P}$ is not exact.

Theorem 2.42. *The category $\mathcal{S}$ has arbitrary direct sums, and satisfies (Ab5).*

Proof. Given a family $(F_i)_{i\in I}$ of abelian sheaves, the direct sum $\bigoplus_{i\in I} F_i$ in $\mathcal{P}$ is not a sheaf in general, but its sheafification is a direct sum of the family $(F_i)_{i\in I}$ in $\mathcal{S}$.

For the property (Ab5), see [Tam94, I.3.2.1]. □

2.8 Constant sheaves

Let $\mathcal{T} = (\mathrm{cat}(\mathcal{T}), \mathrm{cov}(\mathcal{T}))$ be a site.

Definition 2.43. Let A be an abelian group. The *constant presheaf* on $\mathcal{T}$ associated to A is the presheaf $X \mapsto A$ on $\mathrm{cat}(\mathcal{T})$. More precisely, it is the functor $F_A : \mathrm{cat}(\mathcal{T}) \to \mathrm{Ab}$ defined by $F_A(X) = A$ for every object $X \in \mathrm{cat}(\mathcal{T})$, with the restriction maps given by the identity map of A.

Definition 2.44. Let A be an abelian group. The *constant sheaf* on $\mathcal{T}$ associated to A is the sheafification of the constant presheaf $X \mapsto A$.

Notation 2.45. This constant sheaf is denoted by $\underline{A}$ or $A_{\mathcal{T}}$.

In order to gain some intuition, it is worth spelling this out for topological spaces.

Let X be a topological space, and $\mathcal{T}_X$ the site given by the topology on X. Let A be any abelian group. Let F_A be the constant presheaf on $\mathcal{T}_X$ associated to A. Let C_A be the presheaf of constant A-valued functions on X, namely

$$C_A(U) = \{f : U \to A \text{ constant}\}$$

for any open subset U of X. Finally, let L_A be the presheaf of *locally constant* A-valued functions on X, namely

$$L_A(U) = \{f : U \to A \text{ locally constant}\}$$

for any $U \subset X$. (This is the same as requiring that $f : U \to A$ is continuous with respect to the discrete topology on A.) It is not hard to show that L_A is actually a sheaf, and that C_A is not a sheaf in general.

We will show that $F_A^\sharp \cong L_A$. First note that the presheaves F_A and C_A are almost identical, except that $F_A(\emptyset) = A$ while $C_A(\emptyset) = \{0\}$.

Exercise 31. Show that $F_A^\dagger$ is canonically isomorphic to C_A.

So $F_A^\sharp = C_A^\dagger$, and it remains to show that $C_A^\dagger \cong L_A$, which is an important property in itself.

Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering of $U \subset X$. One can see that

$$H^0(\mathcal{U}, C_A) = \{f : U \to A : \forall i \in I, f|_{U_i} \text{ is constant}\}$$

so we have natural inclusion $H^0(\mathcal{U}, C_A) \subset L_A(U)$. But conversely, given a locally constant function $f : U \to A$, there certainly exists an open covering $U = \bigcup_{i\in I} U_i$ such that f is constant on each U_i. So $L_A(U)$ is the union of its subspaces $H^0(\mathcal{U}, A)$ when $\mathcal{U}$ runs through the coverings of U. Now the presheaf C_A is separated, so the inductive limit $C_A^\dagger(U) = \varinjlim_{\mathcal{U}} H^0(\mathcal{U}, C_A)$ can be described as a filtered union, and is isomorphic to $L_A(U)$. The isomorphisms $C_A^\dagger(U) \cong L_A(U)$ are compatible with the restriction maps, which shows that $C_A^\dagger \cong L_A$.

In conclusion, the constant sheaf $\underline{A} = A_X$ on X is equal to the sheaf of locally constant A-valued functions on X. Note that the sections of the *constant* sheaf are only *locally constant* functions.

3 Étale morphisms

Good references for the theory of schemes include Görtz–Wedhorn [GW10], Liu [Liu02], Vakil [Vak].

Étale morphisms of schemes provide a generalization of the notions of "local homeomorphism" and "covering space" in topology. We could try to mimick the topological definition by taking those morphisms of schemes which are local isomorphisms, but unfortunately, there are too few such morphisms, so that the theory of covering spaces with respect to Zariski topology would be rather dull. Similarly, the Zariski topology is too coarse: there are not enough open subsets, and the open subsets are too big. For example, if X is an irreducible scheme, then any two non-empty open subsets meet, so that every locally constant function on X is actually contant. This means that constant sheaves do not carry enough information and do not give rise to an interesting cohomology theory[1].

The idea of étale topology is that we should consider "more" open subsets. More precisely, we should *replace* the Zariski open subsets of a scheme by the étale morphisms to this scheme. This gives a Grothendieck topology and then we are in good position to apply tools like cohomology of sheaves.

We first recall basic definitions about algebraic varieties.

Definition 3.1. Let k be a field.

1. An *affine variety over* k is a closed subscheme of the affine space $\mathbf{A}^n_k$ for some $n \geq 0$ (in other words, it is the spectrum of a finitely generated k-algebra).

2. A *projective variety over* k is a closed subscheme of the projective space $\mathbf{P}^n_k$ for some $n \geq 0$.

3. An *algebraic variety over* k (or simply k-variety) is a scheme X which is locally of finite type over k.

In other words, an algebraic variety over k is a scheme X/k which admits an open cover by affine varieties over k. (Some authors add separateness, reducedness or finiteness conditions). In particular, affine and projective varieties are algebraic varieties. Any Zariski open subset of an algebraic variety is an algebraic variety. A Zariski open subset of a projective variety is called a *quasi-projective variety.*

If X is an algebraic variety over an algebraically closed field k, then the set $X(k)$ of k-valued points of X is in natural bijection with the set of closed points of X.

[1] On the other hand, the cohomology of coherent $\mathcal{O}_X$-modules with respect to the Zariski topology does give something interesting.

Let X be a scheme over a base scheme S, and let $f : X \to S$ be the structural morphism. For any $s \in S$, the *fibre of f above s* is the scheme $X_s := X \times_S \operatorname{Spec} k(s)$. We also denote it by $f^{-1}(s)$. The *geometric fibre* of f above s is the scheme $X_{\overline{s}} := X \times_S \operatorname{Spec} \overline{k(s)}$.

More generally, let $\overline{s}$ be a geometric point of S, by which we mean a morphism $\overline{s} : \operatorname{Spec} k \to S$ where k is some algebraically closed field (this is the same as giving a point $s \in S$ together with an extension of fields $k(s) \to k$). Then the geometric fibre of f at $\overline{s}$ is defined by $X_{\overline{s}} := X \times_S \operatorname{Spec} k$ where the fibre product is taken with respect to $\overline{s}$. We also denote by $f^{-1}(\overline{s})$ this geometric fibre.

3.1 Local properties

The following definition was already seen in [Fu].

Definition 3.2. Let (P) be a property of schemes (resp. affine schemes). We say that a scheme X is *locally* (P) if there exists an open cover (resp. affine open cover) $X = \bigcup_{i \in I} U_i$ such that each U_i has the property (P).

For example, a scheme X is locally Noetherian if and only if it is covered by open subsets which are spectra of Noetherian rings.

We now come to properties of morphisms of schemes.

Definition 3.3. Let (P) be a property of morphisms of schemes (resp. morphisms of affine schemes). We say that a morphism $f : X \to Y$ is *locally* (P) if there exist open covers (resp. affine open covers) $Y = \bigcup_{j \in J} V_j$ and $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that for every $j \in J$ and $i \in I_j$, the induced morphism $f : U_i \to V_j$ has the property (P).

Definition 3.4. Let (P) be a property of morphisms of schemes.

1. We say that (P) is *stable by composition* if the composition of any two morphisms satisfying (P) also satisfies (P).
2. We say that (P) is *stable by base change* if for every morphism $f : X \to Y$ satisfying (P) and every morphism $\varphi : Y' \to Y$, the induced morphism $f' : X \times_Y Y' \to Y'$ satisfies (P).

Exercise 32. The property of being an open (resp. closed) immersion is stable by composition and base change.

Definition 3.5. Let (P) be a property of morphisms of schemes.

1. We say that (P) is *local on the source* if for every morphism of schemes $f : X \to Y$ and every open cover $X = \bigcup_{i \in I} U_i$, we have
$$f \text{ satisfies } (P) \Leftrightarrow \forall i \in I, f|_{U_i} : U_i \to Y \text{ satisfies } (P).$$
2. We say that (P) is *local on the target* if for every morphism of schemes $f : X \to Y$ and every open cover $Y = \bigcup_{j \in J} V_j$, we have
$$f \text{ satisfies } (P) \Leftrightarrow \forall j \in J, f|_{f^{-1}(V_j)} : f^{-1}(V_j) \to V_j \text{ satisfies } (P).$$
3. We say that (P) is *local* if (P) is both local on the source and on the target.

Exercise 33. Let (P) be a property of morphisms of rings. Consider the property of morphisms of schemes defined by "locally (P)". Give conditions on (P) ensuring that "locally (P)" is local on the source (resp. on the target).

3.2 Non-singularity

In this section we first explain the notion of non-singularity for varieties over algebraically closed fields. Before we give the definition, let us try to give some geometric intuition.

Let $P \in \mathbf{C}[x_1, \ldots, x_n]$ be a non-constant polynomial, and let $V(P) = \{(x_1, \ldots, x_n) \in \mathbf{C}^n : P(x_1, \ldots, x_n) = 0\}$ denote the hypersurface defined by P in $\mathbf{C}^n$. This hypersurface defines a complex analytic subvariety of $\mathbf{C}^n$ as soon as it is *non-singular*, meaning that for every point $x \in V(P)$, the partial derivatives $(\frac{\partial P}{\partial x_1}, \ldots, \frac{\partial P}{\partial x_n})$ don't simultaneously vanish at x. If this is the case, then the implicit function theorem (in the holomorphic setting) tells us that $V(P)$ is a complex analytic variety of dimension $n-1$.

More generally, let k be an algebraically closed field, and let $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ be polynomials. Let $V(f_1, \ldots, f_r) = \{x \in k^n : f_1(x) = \cdots = f_r(x) = 0\}$ denote the zero locus of these polynomials in k^n. We want to find a differential condition on the map $f = (f_1, \ldots, f_r) : k^n \to k^r$ ensuring that the variety $V(f_1, \ldots, f_r)$ is non-singular (say in the complex analytic sense, when $k = \mathbf{C}$). From the point of view of schemes, we are studying the affine variety $X = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ over k, and the set of closed points of X can be identified with $X(k) = V(f_1, \ldots, f_r)$.

Definition 3.6. Let $x \in X(k)$ be a closed point. The *tangent space* $T_x X$ of X at x is the k-vector space given by the kernel of the Jacobian matrix

$$J_f(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial x_1}(x) & \cdots & \frac{\partial f_r}{\partial x_n}(x) \end{pmatrix} \in M_{r,n}(k) \tag{15}$$

Intuitively, the Jacobian matrix is the matrix of the "differential" of f at x. We may now define non-singularity.

Definition 3.7. Let X be an affine algebraic variety over an algebraically closed field k. Assume that X is irreducible of dimension d. We say that a point $x \in X(k)$ is *non-singular* (or that X is *non-singular at x*) if the k-vector space $T_x X$ has dimension d. We say that X is *non-singular* if every point of $X(k)$ is non-singular.

Example 3.8. Let us consider the curve $C : y^2 = x^3 + x^2 = x^2(x+1)$ over k. Assume $\operatorname{char}(k) \neq 2$. Intuitively, the point $p = (0,0) \in C(k)$ is singular because there are two tangents at p, namely the (distinct) lines $y = \pm x$ in k^2. This can be made precise: because of its definition, the tangent space $T_p C$ must contain these lines, and since it is a vector space, it must be equal to k^2. This means that $\dim_k T_p C = 2$ and thus p is a singular point. In this case we say p is an ordinary double point (or a node). You may check that p is also a singular point in the case $\operatorname{char}(k) = 2$.

Another example is given by $C' : y^2 = x^3$ over k and the point $p = (0,0)$. Although there seems to be only one tangent line at p, the tangent space $T_p C'$ is also equal to k^2, so that p is a singular point of C'. In this case we say p is a cusp.

Under the assumptions of Definition 3.7, one can show that the dimension of $T_x X$ is always at least d. In the particular case where X has dimension $n - r$ (in other words, the number of defining equations of X is equal to the codimension – this is not always true for arbitrary affine varieties), the variety X is non-singular at x if and only if $J_f(x)$ has rank r, which means that the "differential" of f at x is surjective (one may think of the map f being something like a submersion at x).

An unpleasant feature of this definition of non-singularity is that it depends a priori on the defining equations of X. The following important exercise shows that in fact, non-singularity is an intrinsic notion.

Exercise 34. Let $X = \operatorname{Spec} A$ be an affine variety over an algebraically closed field k, and let $x \in X(k)$. Let $\mathcal{O}_{X,x}$ be the local ring of X at x, with maximal ideal $\mathfrak{m}_x$. The *cotangent space* $T_x^* X$ of X at x is the k-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$.

(a) Let $\mathfrak{m}$ be the maximal ideal of A corresponding to x. Show that the k-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is isomorphic to $\mathfrak{m}/\mathfrak{m}^2$.

(b) Show that the cotangent space $T_x^* X$ is canonically isomorphic to the dual of the tangent space $T_x X$.

(c) In the case X is irreducible, deduce that non-singularity as defined in Definition 3.7 does not depend on the presentation of X.

Tangent vectors have also a scheme-theoretic interpretation using *dual numbers*. For any field k, the algebra of dual numbers over k is defined by $k[\varepsilon] = k[t]/(t^2)$, where ε denotes the class of t (note that $\varepsilon^2 = 0$).

Exercise 35. Let X be an affine variety over an algebraically closed field k, and let $x \in X(k)$. Show that the tangent space $T_x X$ is naturally in bijection with the set of morphisms of schemes $\operatorname{Spec} k[\varepsilon] \to X$ whose image is equal to $\{x\}$.

The scheme $\operatorname{Spec} k[\varepsilon]$ is called a *thick point* over k. As a topological space, it is just one point, so the thick point is a scheme of dimension 0, but its tangent space is a line.

There are two generalizations of non-singularity in the theory of schemes: regularity and smoothness. Regularity is a property of schemes (it is an absolute notion), while smoothness is a property of morphisms of schemes (it is a relative notion). For an affine variety over an algebraically closed field, non-singularity is equivalent to regularity, and also equivalent to the structural morphism being smooth. However, in general regularity and smoothness are two distinct notions.

3.3 Regularity

It is straightforward to generalize the notion of cotangent space for arbitrary schemes. This leads us naturally to the definition of regular schemes.

Definition 3.9. Let X be a scheme, and let $x \in X$. Let $\mathcal{O}_{X,x}$ be the local ring of X at x, with maximal ideal $\mathfrak{m}_x$ and residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. The *cotangent space* $T_x^* X$ of X at x is the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Definition 3.10. Let X be a locally Noetherian scheme, and let $x \in X$. We say that x is a *regular point* of X if $\dim_{k(x)} T_x^* X = \dim \mathcal{O}_{X,x}$. We say that X is *regular* if every point of X is regular.

The following proposition is almost a restatement of what was proved in Exercise 34.

Proposition 3.11. *Let X be an irreducible affine variety over an algebraically closed field k.*

1. *A point $x \in X(k)$ is regular if and only if it is non-singular.*

2. *The scheme X is regular if and only if the variety X is non-singular.*

Proof. The first assertion follows from Exercise 34 together with the fact [Fu] that the dimension of the local ring $\mathcal{O}_{X,x}$ is equal to the dimension of X.

The second assertion uses the following theorem of Auslander–Buchsbaum and Serre [Mat89, Theorem 19.3]: the localisation of a regular local ring at a prime ideal is again regular. □

With the above theory at hand, it is now possible to define non-singularity for general varieties, without assuming that the variety is irreducible or even affine.

Definition 3.12. Let X be an arbitrary algebraic variety over an algebraically closed field k. A point $x \in X(k)$ is said to be *non-singular* if X is regular at x. We say that X is non-singular if every point of $X(k)$ is non-singular.

Let X be an algebraic variety over k, and let $x \in X(k)$. By Exercise 34, the dimension of the cotangent space T_x^*X may be computed by taking any affine chart of X containing x and then determining the kernel of the Jacobian matrix (15). Furthermore, by [Fu], the dimension of the local ring $\mathcal{O}_{X,x}$ is equal to the maximum of the dimensions of the irreducible components of X passing through x. So the definition of non-singularity we have given is actually quite concrete.

Exercise 36. Let X be an algebraic variety over an algebraically closed field k. Show that the following conditions are equivalent:

(1) X is non-singular.

(2) X is regular.

(3) No two irreducible components of X meet, and these components are non-singular.

(4) The connected components of X are irreducible and non-singular.

We now have defined non-singular varieties, but also the much more general notion of regular schemes. Here are some algebraic properties of Noetherian local rings which are useful when dealing with regularity.

Proposition 3.13. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k.*

1. *(Nakayama's lemma) Let $x_1, \dots, x_r$ be elements of $\mathfrak{m}$. Then $x_1, \dots, x_r$ generate the ideal $\mathfrak{m}$ if and only if $\bar{x}_1, \dots, \bar{x}_r$ generate the k-vector space $\mathfrak{m}/\mathfrak{m}^2$. In particular, the minimal number of generators of $\mathfrak{m}$ is equal to the dimension of $\mathfrak{m}/\mathfrak{m}^2$.*

2. *We have $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$.*

Proof. See [AM69, Proposition 2.8] and [AM69, Corollary 11.15]. □

Corollary 3.14. *Let X be a locally Noetherian scheme, and let $x \in X$. Let d be the dimension of the local ring $\mathcal{O}_{X,x}$. The following conditions are equivalent:*

(1) X is regular at x.

(2) The ideal $\mathfrak{m}_x$ can be generated by d elements (in other words, $\mathcal{O}_{X,x}$ is a regular local ring).

(3) The $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ can be generated by d elements.

Definition 3.15. Let X be a scheme. Let x be a regular point of X and $d = \dim \mathcal{O}_{X,x}$. A generating family $(t_1, \dots, t_d)$ of $\mathfrak{m}_x$ is called a *regular sequence at x*. We also say that $t_1, \dots, t_d$ are local coordinates for X at x.

Example 3.16. Let C be a curve over an algebraically closed field k, and let $x \in C(k)$. Then x is a non-singular point of C if and only if $\mathcal{O}_{C,x}$ is a discrete valuation ring. In this case, a local coordinate at x is simply a uniformizer of $\mathcal{O}_{C,x}$.

Exercise 37. Show that a locally Noetherian scheme X is regular if and only if every closed point of X is regular.

Exercise 38. Let p be a prime number. Consider the closed subscheme X of $\mathbf{A}^1_{\mathbf{Z}}$ defined by $X = V(x^2 - p) = \operatorname{Spec} \mathbf{Z}[x]/(x^2 - p)$. We view X as a scheme over $\operatorname{Spec} \mathbf{Z}$. Let $X_p = X \times_{\mathbf{Z}} \mathbf{F}_p$ be the reduction of X modulo p.

(a) Let P_0 be the point $x = 0$ in X_p. Show that X_p is not regular at P_0.

(b) Show nevertheless that X is regular at P_0.

(c) Show that the scheme X is regular is and only if $p = 2$ or $p \equiv 3 \bmod 4$. In the case $p \equiv 1 \bmod 4$, determine the regular locus of X.

3.4 Smoothness

We want to generalize the notion of non-singularity to varieties over arbitrary fields, or even to schemes over general bases. Of course, we could still use regularity as a definition. But it turns out that regularity, in addition to being subtle, is not always well-behaved. For example, the following example shows that regularity is not preserved under base change.

Example 3.17. Let $k = \mathbf{F}_p(t)$ and let X be the affine variety $\{x^p = t\} \subset \mathbf{A}^1_k$, in other words $X = \operatorname{Spec} k[x]/(x^p - t)$. Using the Eisenstein criterion, the polynomial $x^p - t$ is irreducible in $k[x]$, so that $k[x]/(x^p - t)$ is a field and X is regular. On the other hand, consider $X_{\overline{k}} = X \times_k \overline{k}$. Let $t^{1/p}$ be a p-th root of t in $\overline{k}$. The polynomial $x^p - t$ factors as $(x - t^{1/p})^p$ in $\overline{k}[x]$, so that

$$X_{\overline{k}} = \operatorname{Spec} \overline{k}[x]/(x - t^{1/p})^p \cong \operatorname{Spec} \overline{k}[y]/(y^p).$$

The ring $\overline{k}[y]/(y^p)$ is a Noetherian local ring of dimension 0 with maximal ideal $\mathfrak{m} = (y)$. We see that $\mathfrak{m}/\mathfrak{m}^2$ is 1-dimensional, so that $X_{\overline{k}}$ is not regular.

An algebraic variety X over k is called *geometrically regular* if $X \times_k \overline{k}$ is regular. The following exercise gives examples (due to Zariski and Chevalley) of curves which are regular but not geometrically regular.

Exercise 39. Let k be a field of characteristic $p > 0$, and let $a \in k$ be an element which is not a p-th power in k.

(a) Consider the affine algebraic curve Z over k defined by $Z : x^p + y^p = a$. Show that Z is a regular scheme.

(b) However, let $k' = k(a^{1/p})$. Show that the base change $Z' = Z \times_k k'$ is not regular.

(c) Same questions with the curve $C : x^p + y^2 = a$.

(d) Show that $C \times_k \overline{k}$ is reduced and irreducible while $Z \times_k \overline{k}$ is irreducible but not reduced.

The right thing to do is to define a notion of "non-singularity" for *morphisms* of schemes, rather than schemes alone. The good notion is called *smoothness*. Intuitively, the smooth morphisms are those which look like submersions in differential geometry (i.e. the differential is everywhere surjective). In differential geometry, this condition ensures that the fibres are subvarieties, in other words do not have singular points.

We first define standard smoothness for ring morphisms.

Definition 3.18. Let R be a ring. An R-algebra S is *standard smooth* if it admits a presentation $S \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $c \leq n$ and the following condition is satisfied:

$$\det\Big(\frac{\partial f_i}{\partial x_j}\Big)_{1 \leq i,j \leq c} \in S^\times. \tag{16}$$

Note that (16) is really a property of the presentation, not of the algebra: a finitely presented R-algebra S may be standard smooth although its defining presentation does not satisfy (16) – it may just be the case that S admits a “better” presentation. In order to avoid confusion, we will also say that a given presentation is standard smooth (or not).

Exercise 40. (a) Show that if L/K is a finite separable field extension, then $\operatorname{Spec} L \to \operatorname{Spec} K$ is standard smooth.

(b) Let R be a ring, let $n \geq 1$ be an integer and let $S = R[T]/(T^n - 1)$. Show that this presentation is standard smooth if and only if $n \in R^{\times}$.

The following lemma shows that over algebraically closed fields, standard smoothness implies non-singularity.

Lemma 3.19. *Let k be an algebraically closed field. If A is a standard smooth k-algebra, then $X = \operatorname{Spec} A$ is a non-singular affine variety. Moreover, if $A \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a standard smooth presentation, then for every point $x = (a_1, \ldots, a_n) \in X(k)$, the family $(x_i - a_i)_{c+1 \leq i \leq n}$ is a regular sequence at x.*

Proof. Exercise. □

The converse of Lemma 3.19 is not true: if X is non-singular then A may not be standard smooth over k. However, it is a theorem that if X is non-singular then it may be covered by affine open subsets which are spectra of standard smooth k-algebras.

Remark 3.20. Let $S \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a standard smooth algebra, and let $f : \operatorname{Spec} S \to \operatorname{Spec} R$ be the associated morphism. We could be tempted to use condition (16) to show that locally f looks like the projection $\mathbf{A}_R^{n-c} \to \operatorname{Spec} R$, using the last $(n-c)$ coordinates in S. However, this is not true in general: there is no implicit function theorem in algebraic geometry, the reason being that the Zariski topology is too coarse. The étale topology remedies in some sense to this problem: we will see that *locally for the étale topology*, f indeed looks like $\mathbf{A}_R^{n-c} \to \operatorname{Spec} R$.

We now define smooth morphisms of schemes.

Definition 3.21. Let $f : X \to Y$ be a morphism of *affine* schemes. We say that f is *standard smooth* if the induced ring map $\mathcal{O}(Y) \to \mathcal{O}(X)$ is standard smooth.

Definition 3.22. Let $f : X \to Y$ be a morphism between arbitrary schemes.

1. We say that f is *smooth at $x \in X$* if there exist affine open subsets $U \subset X$ and $V \subset Y$ with $x \in U$ and $f(U) \subset V$, such that the induced map $f|_U : U \to V$ is standard smooth.

2. We say that f is *smooth* if it is smooth at every point of X.

In other words, a morphism $f : X \to Y$ is smooth if and only if it is locally standard smooth. From the definition, standard smooth morphisms are smooth (as indicated above, the converse is not true).

One advantage of Definitions 3.18 and 3.22 is that they are quite concrete. However with these definitions, how to prove that a morphism is not smooth, or that an algebra is not standard smooth? We need to check that no presentation ever satisfies the Jacobian condition, which seems an impossible task...We will give later a characterization of smoothness which is “coordinate-free” (see Theorem 3.30) and enables one to show that a given morphism is not smooth.

We now give standard properties of smooth morphisms.

Proposition 3.23. *1. Every smooth morphism is locally of finite presentation.*

2. The smooth locus of a morphism $f : X \to Y$ is an open subset of X.

3. Open immersions are smooth.

4. Smoothness is a local property.

5. Smoothness is stable by composition and base change.

Proof. 1. This is because standard smooth morphisms are of finite presentation.

2. This follows from the definition.

3. This follows from the fact that isomorphisms are standard smooth.

4. We will need the following lemma.

Lemma 3.24. *Let R be a ring.*

(a) For any $f \in R$, the R-algebra $R[1/f] = R[T]/(fT-1)$ is standard smooth.

(b) If S is a standard smooth R-algebra and T is a standard smooth S-algebra, then T is a standard smooth R-algebra.

(c) If S is a standard smooth R-algebra, then for any R-algebra R', the R'-algebra $S' = S \otimes_R R'$ is standard smooth.

Proof. (a) We have $\frac{\partial}{\partial T}(fT-1) = f \in R[\frac{1}{f}]^\times$.

(b) Choose standard smooth presentations of $R \to S$ and $S \to T$. Using a suitable ordering of the variables, the Jacobian matrix of $R \to T$ is block-triangular and the diagonal blocks have determinant in $T^\times$, so that $R \to T$ is standard smooth.

(c) Let $\varphi : R \to R'$ be the stuctural morphism. If $S \cong R[x_1, \dots, x_n]/(f_1, \dots, f_c)$ is a standard smooth presentation then $S' \cong R'[x_1, \dots, x_n]/(\varphi(f_1), \dots, \varphi(f_c))$ is also standard smooth.

□

Let $f : X \to Y$ be a morphism of schemes, and let $X = \bigcup_i U_i$ be an open covering. Let $f_i = f|_{U_i}$. If the f_i are smooth, then so is f. Conversely, assume that f is smooth. We want to show that f_i is smooth. Let $x \in U_i$. Let $U \subset X$ and $V \subset Y$ be affine open subsets such that $x \in U$, $f(U) \subset V$ and $U \to V$ is standard smooth. The principal open subsets of U form a basis of the topology of U, so we may write $U \cap U_i$ as a union of principal open subsets. Choose one, say $D(h)$, which contains x. Using Lemma 3.24(a)(b), we see that $D(h) \to V$ is standard smooth. So smoothness is local on the source.

Now let $Y = \bigcup_j V_j$ be an open covering, and let $f_j : f^{-1}(V_j) \to V_j$. If the f_j are smooth, then so is f. Conversely, assume that f is smooth. Let $x \in f^{-1}(V_j)$. Let $U \subset X$ and $V \subset Y$ be affine open subsets such that $x \in U$, $f(U) \subset V$ and $\tilde{f} : U \to V$ is standard smooth. Let $D(g)$ be a principal open subset of V contained in $V \cap V_j$ and containing $f(x)$. Then $D(h) = \tilde{f}^{-1}(D(g))$ is a principal open subset of U contained in $f^{-1}(V_j)$ and containing x. Moreover $D(h) \to D(g)$ is standard smooth by Lemma 3.24(c) applied with $R' = \mathcal{O}(V)[\frac{1}{g}]$. So smoothness is also local on the target.

5. Let us show that smoothness is stable by composition. Since smoothness is local, we are reduced to show that the composition of two standard smooth morphisms is smooth. But this composition is even standard smooth by Lemma 3.24(b).

   Let us show that smoothness is stable by base change. Let $f : X \to Y$ be a smooth morphism of schemes, and let $g : Y' \to Y$ be an arbitrary morphism. Let $X' = X \times_Y Y'$. We want to show that $f' : X' \to Y'$ is smooth. Since smoothness is local, we may assume that X, Y and Y' are affine and that f is standard smooth (check that!). But then f' is standard smooth by Lemma 3.24(c).

$\square$

We now give a characterisation of smoothness in terms of thickenings. This may serve as a more intrinsic definition.

For any scheme X, we denote by $|X|$ the topological space underlying X. For any morphism of schemes $f : X \to Y$, we denote by $|f| : |X| \to |Y|$ the underlying continuous map between topological spaces. Recall the following definition.

Definition 3.25. A morphism of schemes $i : X \to Y$ is a *closed immersion* if $|i|$ identifies $|X|$ with a closed subspace of $|Y|$, and the morphism of sheaves $i^\sharp : \mathcal{O}_Y \to i_*\mathcal{O}_X$ is surjective.

For example, if R is any ring and I is an ideal of R, then $i : \operatorname{Spec} R/I \to \operatorname{Spec} R$ is a closed immersion. The set-theoretic image of i is the closed subset $V(I)$ of $\operatorname{Spec} R$. Note that i is not determined by $|i|$, as in general the ideal I carries more information than the set $V(I)$.

Definition 3.26. We say that a morphism of schemes $i : X \to X'$ is a *thickening* (or that X' is a thickening of X) if i is a closed immersion and $|i|$ is a homeomorphism. We say that i is a *thickening of order 1* if moreover the sheaf of ideals

$$\mathcal{I} = \ker(\mathcal{O}_{X'} \to i_*\mathcal{O}_X)$$

defining the closed subscheme X in X', satisfies $\mathcal{I}^2 = 0$.

Using the notations above, if I is a nilpotent ideal then $\operatorname{Spec} R/I \to \operatorname{Spec} R$ is a thickening. If $I^2 = 0$, then it is a thickening of order 1. For example, if k is a field and $k[\varepsilon] = k[t]/(t^2)$ are the dual numbers over k, then $\operatorname{Spec} k \to \operatorname{Spec} k[\varepsilon]$ is a thickening of order 1, because $\varepsilon^2 = 0$ in $k[\varepsilon]$.

Theorem 3.27. *Let $f : X \to S$ be a morphism of schemes. Then f is smooth if and only if*

(1) f is locally of finite presentation;

(2) f is formally smooth: for any commutative diagram

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ T \arrow[urr, "g"] \arrow[r] & T' \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

where T' is a thickening of order 1 of T, there exists, locally for the Zariski topology on T', at least one morphism $g' : T' \to X$ making the diagram commute.

This property conveys the intuition that the differential of f is everywhere surjective. To see this, take a geometric point $\bar{x} : \operatorname{Spec} k \to X$, mapping under f to the geometric point $\bar{s} : \operatorname{Spec} k \to S$. Let t be a tangent vector on S at $\bar{s}$, which by Exercise 35 can be seen as a morphism $\operatorname{Spec} k[\varepsilon] \to S$ extending $\bar{s}$. Taking $T = \operatorname{Spec} k$, $T' = \operatorname{Spec} k[\varepsilon]$ and $g = \bar{x}$, the property (2) above says that there should exist at least one tangent vector on X at $\bar{x}$ mapping, under the differential of f, to our tangent vector t.

We now come to the most useful characterization of smoothness.

We need to introduce the notion of flatness for morphisms of schemes. Intuitively, flatness of a morphism $f : X \to Y$ guarantees some kind of "continuity" of the fibres X_y when y varies in Y.

Definition 3.28. Let R be a ring. We say that an R-module M is *flat* if the functor F_M : R-Mod $\to$ R-Mod defined by $N \mapsto M \otimes_R N$ is exact. We say that an R-algebra S is *flat* if it is flat as an R-module.

Note that the functor $N \mapsto M \otimes_R N$ is always right exact.

For example, any free R-module is flat. Flatness can be checked locally: an R-module M is flat if and only if for every prime ideal $\mathfrak{p}$ of R, the localisation $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module. Thus, any locally free R-module (e.g. any projective R-module) is flat. A typical example of non-flat module is the $\mathbf{Z}$-module $\mathbf{Z}/n\mathbf{Z}$ with $n > 1$.

Definition 3.29. Let $f : X \to Y$ be a morphism of schemes. Let $x \in X$ and $y = f(x)$. We say that f is *flat at x* if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-algebra. We say that f is *flat* if it is flat at every point of X.

Since flatness is defined by means of the stalks, it is clear that flatness is a local property. So in order to show that a morphism of schemes $f : X \to Y$ is flat, it is enough to show that it is locally given by ring maps $R \to S$ satisfying the property that S is a flat R-module.

The main theorem about smooth morphisms is the following.

Theorem 3.30. *A morphism of schemes $f : X \to Y$ is smooth if and only if f is locally of finite presentation, flat and for every $y \in Y$, the geometric fibre $X_{\overline{y}} = X \times_Y \operatorname{Spec} \overline{k(y)}$ is a non-singular variety.*

We will not give a proof but rather references. For the direct implication, we may assume that f is a standard smooth morphism between affine schemes. Then f is of finite presentation, and its geometric fibres are non-singular varieties because the condition with the Jacobian matrix is satisfied. Moreover f is flat, see [Sta19, Lemma 01VD]. The reverse implication is more difficult, see [Sta19, Lemma 01V7 and Lemma 01V8].

Theorem 3.30 means that if we restrict to those morphisms which are locally of finite presentation and flat, then *smoothness is a fibral property.* So informally, a smooth morphism is a (flat) family of non-singular algebraic varieties.

An immediate consequence of Theorem 3.30 is the following link between smoothness and non-singularity.

Theorem 3.31. *Let X be an algebraic variety over an algebraically closed field k, and let $f : X \to \operatorname{Spec} k$ be the structural morphism.*

1. *f is smooth at $x \in X(k)$ if and only if x is a non-singular point of X.*
2. *f is smooth if and only if X is non-singular.*

For algebraic varieties over arbitrary fields, Theorem 3.30 says the following. Take $Y = \operatorname{Spec} k$ where k is an arbitrary field. If X is an algebraic variety over k, then $X \to \operatorname{Spec} k$ is automatically locally of finite presentation and flat. So we get the following result.

Theorem 3.32. *Let X be an algebraic variety over an arbitrary field k. The structural morphism $X \to \operatorname{Spec} k$ is smooth if and only if $X \times_k \overline{k}$ is a non-singular variety.*

As the following proposition shows, smoothness also shows up naturally in algebraic number theory.

Proposition 3.33. *Let $K \subset L$ be number fields, and let $\mathcal{O}_K \subset \mathcal{O}_L$ be the associated rings of integers. Let $f : \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$ be the corresponding morphism.*

1. *Let $\mathfrak{q}$ be a prime ideal in L, lying over the prime ideal $\mathfrak{p}$ in K. The morphism f is smooth at $\mathfrak{q}$ if and only if $e(\mathfrak{q}/\mathfrak{p}) = 1$, where $e(\cdot)$ denotes the ramification index.*

2. *The morphism f is smooth if and only if L/K is everywhere unramified.*

Proof. Let $d = [L : K]$. The $\mathcal{O}_K$-algebra $\mathcal{O}_L$ is finitely generated and locally free, hence f is of finite presentation and flat. So it remains to check the condition of non-singularity for the geometric fibres. The geometric fibre of f at the generic point is given by $\mathcal{O}_L \otimes_{\mathcal{O}_K} \overline{K} = L \otimes_K \overline{K}$ which is isomorphic to the product of d copies of $\overline{K}$, so this fibre is always non-singular.

Let us now look at the fibres over the closed points of $\operatorname{Spec} \mathcal{O}_K$. Let $\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$ be the prime decomposition of $\mathfrak{p}$ in $\mathcal{O}_L$. The fibre of f at $\mathfrak{p}$ is given by $\mathcal{O}_L \otimes_{\mathcal{O}_K} k(\mathfrak{p}) = \mathcal{O}_L/\mathfrak{p}\mathcal{O}_L$ which by the Chinese remainder theorem is isomorphic to $\prod_{i=1}^{r} \mathcal{O}_L/\mathfrak{P}_i^{e_i}$. So the fibre of f at $\mathfrak{p}$ is the disjoint union of the $\operatorname{Spec} \mathcal{O}_L/\mathfrak{P}_i^{e_i}$. Let $A_i = \mathcal{O}_L/\mathfrak{P}_i^{e_i}$, which is a $k(\mathfrak{p})$-algebra. We want to know whether the variety $X_i = \operatorname{Spec}(A_i \otimes \overline{k(\mathfrak{p})})$ is non-singular. If $e_i = 1$, this is true because $\mathcal{O}_L/\mathfrak{P}_i$ is a finite field, thus a finite separable extension of $k(\mathfrak{p})$, so that X_i is a finite set of points of the form $\operatorname{Spec} \overline{k(\mathfrak{p})}$. Let us now assume that $e_i \geq 2$. Note that X_i has dimension 0, and every non-singular affine variety of dimension 0 over an algebraically closed field k is a finite disjoint union of points $\operatorname{Spec} k$ (by Nakayama's Lemma, any regular local ring of dimension 0 must be a field). So if X_i were non-singular, then $A_i \otimes \overline{k(\mathfrak{p})}$ would be a product of fields. Since A_i injects into $A_i \otimes \overline{k(\mathfrak{p})}$, this would imply that A_i is reduced (every nilpotent element is zero), which is a contradiction.

Note that for every prime ideal $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ which is unramified in L, the geometric fibre of f at $\mathfrak{p}$ consists of d points (each of the form $\operatorname{Spec} \overline{k(\mathfrak{p})}$), so we may think geometrically of f as a morphism of degree d, and the ramified prime ideals correspond exactly to those geometric fibres which have less than d points, exactly like what happens for curves. $\square$

Exercise 41 (Continuation of Exercise 40). (a) Let L/K be an extension of fields. Show that $\operatorname{Spec} L \to \operatorname{Spec} K$ is smooth if and only if L/K is finite and separable.

(b) Let R be a ring and $n \geq 1$ be an integer. Let $\mu_{n,R} = \operatorname{Spec} R[T]/(T^n - 1)$ denote the R-scheme of n-th roots of unity. Show that $\mu_{n,R} \to \operatorname{Spec} R$ is smooth if and only if $n \in R^\times$.

Exercise 42. Let S be a finitely presented R-algebra, $S \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Assume that the ideal of S generated by the $m \times m$ minors of the matrix $(\frac{\partial f_i}{\partial x_j})_{1 \leq i \leq m, 1 \leq j \leq n}$ is equal to S. Show that $\operatorname{Spec} S \to \operatorname{Spec} R$ is smooth.

Theorem 3.31 gives the link between smoothness and non-singularity. More generally, we may ask if there is a relation between smoothness and regularity. We have the following result.

Theorem 3.34. *Let $f : X \to Y$ be a morphism of schemes, where Y is a regular scheme. If f is smooth, then X is regular.*

The converse does not hold, as Example 3.17 and Exercise 39 already show. However, we have the following result.

Theorem 3.35. *Let k be a* perfect *field, and let X be a k-scheme. If X is regular, then the structural morphism $X \to \operatorname{Spec} k$ is smooth.*

Here are some more examples coming from geometry and number theory.

Example 3.36. Consider the map $f : \mathbf{A}^1_{\mathbf{C}} \to \mathbf{A}^1_{\mathbf{C}}$ given by $f(x) = x^2$. The fibre of f above 0 is a double point, more precisely $\operatorname{Spec} \mathbf{C}[x]/(x^2)$. This is a singular variety, hence f is not smooth. However $\mathbf{A}^1_{\mathbf{C}}$ is regular.

Example 3.37. Consider the affine variety $Q : z = xy$ in $\mathbf{A}^3_{\mathbf{C}}$ and the projection $\pi : Q \to \mathbf{A}^1_{\mathbf{C}}$ given by $\pi(x, y, z) = z$. The fibre of π over $z \neq 0$ is a hyperbola so is non-singular, but the fibre over $z = 0$ is the variety $\{xy = 0\} \subset \mathbf{A}^2_{\mathbf{C}}$, which is the union of the lines $\{x = 0\}$ and $\{y = 0\}$. The point $(x, y) = (0, 0)$ is singular, so that π is not smooth. However, Q is regular as it is isomorphic to $\mathbf{A}^2_{\mathbf{C}}$ using the map $(x, y) \mapsto (x, y, xy)$. Another way to present this example is to say that the map $f : \mathbf{A}^2_{\mathbf{C}} \to \mathbf{A}^1_{\mathbf{C}}$ given by $f(x, y) = xy$ is not smooth.

Example 3.38. (Arithmetic analogue of Example 3.37) Let p be a prime number. Consider the scheme $X = V(xy - p) \subset \mathbf{A}^2_{\mathbf{Z}}$ and the structural morphism $f : X \to \operatorname{Spec} \mathbf{Z}$. Let us investigate the fibres of f. First, the fibre of f over the generic point of $\operatorname{Spec} \mathbf{Z}$ is the variety $\{xy = p\} \subset \mathbf{A}^2_{\mathbf{Q}}$, which (after extending scalars to $\overline{\mathbf{Q}}$) is non-singular. The fibre of f over a prime $q \neq p$ is the variety $\{xy = \overline{p}\} \subset \mathbf{A}^2_{\mathbf{F}_q}$, where $\overline{p}$ is the reduction of p mod q, so these fibres are also non-singular. But the fibre of f over p is again $\{xy = 0\}$, for which $(0, 0)$ is a singular point, hence f is not smooth. However, let us show that X is regular. Indeed, the only point where f is not smooth is the singular point in the fibre over p. Algebraically, this is the maximal ideal $\mathfrak{m} = (x, y, p)$ in $R = \mathbf{Z}[x, y]/(xy - p)$. The residue field is $\mathbf{F}_p$. Since R has dimension 2, we need to show that the $\mathbf{F}_p$-vector space $\mathfrak{m}R_{\mathfrak{m}}/(\mathfrak{m}R_{\mathfrak{m}})^2 = \mathfrak{m}/\mathfrak{m}^2$ can be generated by 2 elements (here $R_{\mathfrak{m}}$ is the localisation of R at $\mathfrak{m}$). This is true because $p = xy \in \mathfrak{m}^2$. So X is regular, but f is not smooth.

Exercise 43.
(a) Find the monic polynomials $P \in \mathbf{Z}[x]$ such that the closed subscheme $\{P(x) = 0\}$ of $\mathbf{A}^1_{\mathbf{Z}}$ is smooth over $\mathbf{Z}$.
(b) Can you find polynomials $P \in \mathbf{Z}[x, y]$ such that the closed subscheme $\{P(x, y) = 0\}$ of $\mathbf{A}^2_{\mathbf{Z}}$ is smooth over $\mathbf{Z}$?

3.5 Étale morphisms

As explained before, étale morphisms will play the role of local homeomorphisms in topology. We first need to recall the notion of dimension (resp. relative dimension) of a scheme (resp. morphism of schemes), which was introduced in [Fu].

Recall that the *dimension* of a scheme X is the Krull dimension of the underlying topological space $|X|$. We say that X has *pure dimension* $d \geq 0$ if every irreducible component of X has dimension d; in this case, we say that X is *equidimensional*.

Example 3.39. If K is a number field, then $\operatorname{Spec} K$ has pure dimension 0 and $\operatorname{Spec} \mathcal{O}_K$ has pure dimension 1.

It was seen in [Fu, TD 6, 0.4.2.b] that the dimension of an algebraic variety does not depend on the base field, in the following sense: if X is an algebraic variety over k and K/k is an arbitrary field extension, then $\dim(X \otimes_k K) = \dim X$. In particular, dimension can be computed by passing to the algebraic closure $\overline{k}$.

Definition 3.40. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. We say that f has *relative dimension* $d \geq 0$ if every non-empty fibre X_y with $y \in Y$ has pure dimension d.

Remark 3.41. Since we assume that f is locally of finite type, the non-empty fibres X_y are algebraic varieties over $k(y)$. So f has relative dimension d if and only if for every $y \in Y$, all the irreducible components of X_y have dimension d. Again, this can be checked by passing to the geometric fibre $X_{\overline{y}}$.

Example 3.42. For every integer $n \geq 0$, the morphisms $\mathbf{A}^n_S \to S$ and $\mathbf{P}^n_S \to S$ have relative dimension n. For every ring R and any integer $n \geq 1$, the morphism $f : \mathbf{A}^1_R \to \mathbf{A}^1_R$ given by $x \mapsto x^n$ has relative dimension 0. More generally, for any finite R-algebra S, the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ has relative dimension 0.

Definition 3.43. Let $f : X \to Y$ be a morphism of schemes. We say that f is *étale* if f is smooth of relative dimension 0.

An R-algebra S is said to be étale if $\operatorname{Spec} S \to \operatorname{Spec} R$ is étale.

Proposition 3.44. *1. Open immersions are étale.*

2. *The property of being étale is local.*

3. *The property of being étale is stable by base change.*

4. *The property of being étale is stable by composition.*

5. *Let $f : X \to Y$ be a morphism of schemes. If f is a local isomorphism for the Zariski topology, then f is étale.*

Proof. 1. An open immersion is smooth and clearly of relative dimension 0.

2. Smoothness and "being of relative dimension 0" are both local properties.

3. Smoothness and relative dimension are both stable by base change.

4. Let $f : X \to Y$ and $g : Y \to Z$ be étale morphisms. Since f and g are smooth, so is $g \circ f$. Let us show that $g \circ f$ has relative dimension 0. Let $z \in Z$. We may write the fibre X_z as $X \times_Y Y_z$. Since f has relative dimension 0, the base change $X \times_Y Y_z \to Y_z$ also has relative dimension 0. Since Y_z has dimension 0, it follows that X_z has dimension 0.

5. This follows from 2. and the fact that isomorphisms are étale morphisms.

□

The converse of Proposition 3.44.5 is not true, as the following basic (but important) example shows. Let $f : \mathbf{A}^1_{\mathbf{C}} \setminus \{0\} \to \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ be the morphism defined by $f(x) = x^2$. This morphism is easily seen to be standard smooth, hence smooth, and every geometric fibre of f consists of two distinct points, thus f has relative dimension 0. It follows that f is étale. However, let us show that f is not a local isomorphism. Note that $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ is irreducible, so every non-empty open subset of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ contains the generic point ξ. Moreover $f(\xi) = \xi$ and the induced field extension at ξ is $\mathbf{C}(x)/\mathbf{C}(x^2)$. Since this extension is non-trivial, it follows that f cannot be a local isomorphism (otherwise, it would be an isomorphism on some neighbourhood of ξ).

The following proposition describes the schemes which are étale over a field.

Proposition 3.45. *Let X be a scheme over $\operatorname{Spec} k$. Then $X \to \operatorname{Spec} k$ is étale if and only if $X = \bigsqcup_{i \in I} \operatorname{Spec} k_i$ where each k_i is a finite separable extension of k.*

Proof. $\Leftarrow$ Since the property of being étale is local on the source, it suffices to show that if k' is a finite separable extension of k, then $\operatorname{Spec} k' \to \operatorname{Spec} k$ is étale. By the primitive element theorem, we have $k' \cong k[T]/(f)$ with $f \in k[T]$ monic and separable. Thus k' is standard smooth over k, and clearly $\operatorname{Spec} k' \to \operatorname{Spec} k$ has relative dimension 0.

$\Rightarrow$ Assume that $X \to \operatorname{Spec} k$ is étale. Let $U = \operatorname{Spec} A$ be any affine open subset of X. Since being étale is local, A is étale over k. Let $\overline{k}$ be an algebraic closure of k. Since being étale is stable under base change, $A_{\overline{k}} = A \otimes_k \overline{k}$ is étale over $\overline{k}$. In particular the affine variety $\operatorname{Spec} A_{\overline{k}}$

has dimension 0 and is non-singular. By Nakayama's Lemma (see the proof of Proposition 3.33), we have $A_{\overline{k}} \cong \overline{k}^n$ for some $n \geq 0$. In particular A is reduced and finite dimensional over k, which implies that $A \cong k_1 \times \cdots \times k_m$ where each k_i is a finite extension of k. But $k_i \otimes_k \overline{k}$ is reduced if only if k_i/k is separable, which gives what we want. $\square$

Here is an example from algebraic number theory. By Proposition 3.33, if $K \subset L$ are number fields, then $\operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$ is étale if and only if L/K is everywhere unramified. In general, if $\Delta = \Delta_{L/K}$ denotes the relative discriminant ideal in $\mathcal{O}_K$, then $\operatorname{Spec} \mathcal{O}_L\big[\frac{1}{\Delta}\big] \to \operatorname{Spec} \mathcal{O}_K\big[\frac{1}{\Delta}\big]$ is always étale. Here the localisation is defined by $\mathcal{O}_K\big[\frac{1}{\Delta}\big] = \big\{\frac{a}{b} : a, b \in \mathcal{O}_K, \operatorname{Supp}(b) \subset \operatorname{Supp}(\Delta)\big\}$.

Proposition 3.46. *If $f : X \to X'$ and $g : Y \to Y'$ are étale S-morphisms, then $f \times_S g : X \times_S Y \to X' \times_S Y'$ is étale.*

Proof. We may write $f \times_S g$ as the composition $X \times_S Y \to X' \times_S Y \to X' \times_S Y'$. Each of these maps is étale by Proposition 3.44.3, so the composition is étale by Proposition 3.44.4. $\square$

Theorem 3.47. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ and g are étale, then f is étale.*

Proof. See [Gro67, IV.17.3.4]. $\square$

Theorem 3.48. *Every étale morphism $f : X \to Y$ is universally open: for every morphism $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ is open. In particular, f is open.*

Proof. It suffices to show that an étale morphism is open. More generally, any morphism locally of finite presentation and flat is (universally) open, see [Sta19, Proposition 00I1]. $\square$

The following concrete characterisation of étale morphisms is useful in practice.

Proposition 3.49. *A morphism of schemes $f : X \to Y$ is étale if and only if f is locally of finite presentation, flat and for every $y \in Y$, the fibre $X_y = X \times_Y \operatorname{Spec} k(y)$ is isomorphic as a $k(y)$-variety to $\bigsqcup_{i \in I} \operatorname{Spec} k_i$ where each k_i is a finite separable extension of $k(y)$.*

In the case the schemes are affine, we would also like to characterise étale morphisms using presentations.

Definition 3.50. Let R be a ring. An R-algebra S is called *standard étale* if S is isomorphic to $(R[T]/(f))_g$ where $f, g \in R[T]$ with f monic and f' invertible in $(R[T]/(f))_g$.

Lemma 3.51. *If S is a standard étale R-algebra, then* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is étale.*

Proof. Write $S = (R[T]/(f))_g$. We have $S \cong R[T, U]/(f(T), g(T)U - 1)$. The Jacobian matrix associated to this presentation is

$$\begin{pmatrix} f'(T) & 0 \\ g'(T)U & g(T) \end{pmatrix}.$$

Its determinant is equal to $f'g$ which is invertible in S. Thus S is standard smooth over R and the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ is smooth. It remains to show that the morphism has relative dimension 0. But the fibres are of the form $(k[T]/(f))_g$ with k a field and $f \in k[T]$ monic. Since $\operatorname{Spec} k[T]/(f)$ is finite and discrete, the result follows. $\square$

More generally, if the R-algebra $S = R[T_1, \dots, T_n]/(f_1, \dots, f_n)$ satisfies the Jacobian condition $\det\big(\frac{\partial f_i}{\partial T_j}\big) \in \mathrm{GL}_n(S)$ (so that S is standard smooth over R), the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ is étale. In other words, every standard smooth algebra with as many equations as variables gives rise to an étale morphism.

The converse of Lemma 3.51 does not hold, but we have the following.

Theorem 3.52. *Let $f : \operatorname{Spec} S \to \operatorname{Spec} R$ be a morphism of affine schemes. The following conditions are equivalent:*

(1) f is étale.

(2) Locally on the source, f is of the form $\operatorname{Spec} A \to \operatorname{Spec} R$ where A is a standard étale R-algebra.

(3) There exists an integer $n \geq 0$ and polynomials $f_1, \ldots, f_n \in R[T_1, \ldots, T_n]$ such that $S \cong R[T_1, \ldots, T_n]/(f_1, \ldots, f_n)$ and $J = \left(\dfrac{\partial f_i}{\partial T_j}\right)_{1 \leq i,j \leq n} \in \mathrm{GL}_n(S)$.

Proof. The implication (2) $\Rightarrow$ (1) follows from Lemma 3.51. For (1) $\Rightarrow$ (2) (Chevalley's Theorem), see [Sta19, Lemma 02GT] or [Gro67, IV.18.4.6]. For (1) $\Leftrightarrow$ (3), see [Sta19, Lemma 00U9]. $\square$

A more intrinsic way to define étale morphisms is to use thickenings as in Theorem 3.27.

Theorem 3.53. *Let $f : X \to S$ be a morphism of schemes. Then f is étale if and only if*

(1) f is locally of finite presentation;

(2) f is formally étale: for any commutative diagram

$$\begin{tikzcd}
& & X \arrow[d, "f"] \\
T \arrow[urr, "g"] \arrow[r] & T' \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

where T' is a thickening of order 1 of T, there exists a unique morphism $g' : T' \to X$ making the diagram commute.

Informally, f being formally étale means that the differential of f is everywhere bijective: given $x \in X$ and a tangent vector t at $s = f(x)$, there exists a unique tangent vector at x whose image under the differential of f is equal to t.

We may now show that smooth morphisms "look like" affine spaces, making precise Remark 3.20.

Proposition 3.54 (Smooth schemes are étale-locally like affine spaces)**.** *Let $f : X \to Y$ be a morphism of schemes. Then f is smooth if and only if locally on the source and target, f can be written as follows:*

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathbf{A}^d_Y \arrow[d, "\pi"] \\
& Y
\end{tikzcd}$$

where $d \geq 0$ is an integer, φ is étale, and π is the canonical projection.

There is a close relation between étale morphisms and complex analytic geometry. Indeed, let X be an algebraic variety over $\mathbf{C}$. Then the set of complex points $X(\mathbf{C})$ has the structure of a (complex) *analytic space*: it is locally given by the common vanishing locus of holomorphic functions $f_1, \ldots, f_r : \mathbf{C}^n \to \mathbf{C}$ (indeed, we may take $f_1, \ldots, f_r$ to be polynomials). Furthermore, if $f : X \to Y$ is a morphism between algebraic varieties over $\mathbf{C}$, then the induced map $f_{\mathbf{C}} : X(\mathbf{C}) \to Y(\mathbf{C})$ is a morphism of analytic spaces, in other words it is given locally by holomorphic functions. We then have the following result.

Theorem 3.55. *Let $f : X \to Y$ be a morphism between algebraic varieties over $\mathbf{C}$. Then f is étale if and only if $f_{\mathbf{C}}$ is locally biholomorphic.*

Thus the étale morphisms correspond exactly to our intuition of local isomorphisms in the classical setting.

3.6 The étale topology

We may finally define the étale topology for schemes.

Definition 3.56. Let S be a scheme. An *étale S-scheme* is a S-scheme X such that the structural morphism $X \to S$ is étale. A morphism between two étale S-schemes X and Y is a morphism $X \to Y$ making the obvious triangle commute.

We denote by $\text{Ét}/S$ the category of étale S-schemes. Note the following facts:

- By Proposition 3.46, the category $\text{Ét}/S$ has finite fibre products.
- By Proposition 3.47, any morphism between étale S-schemes is itself étale.
- The scheme S is a final object in $\text{Ét}/S$.

Definition 3.57. Let X be an étale S-scheme. A family of morphisms $(\varphi_i : U_i \to X)_{i \in I}$ in $\text{Ét}/S$ is said to be a *covering* if $X = \bigcup_{i \in I} \varphi_i(U_i)$.

Lemma 3.58. *The coverings in $\text{Ét}/S$ satisfy axiom (T1), (T2) and (T3).*

Proof. (T2) and (T3) are clear. For (T1), it suffices to prove that if $\varphi : U \to X$ and $f : X' \to X$ are étale, then the image of $\varphi' : U \times_X X' \to X'$ is equal to $f^{-1}(\varphi(U))$. This is true without assumption of f (Exercise). □

Definition 3.59. Let S be a scheme. The *étale site* $S_{\text{ét}}$ is the category of étale S-schemes, endowed with the Grothendieck topology given by the above coverings.

Remark 3.60. The site $S_{\text{ét}}$ is the small étale site. We can also define the *big étale site*: it is the category Sch/S of *all* S-schemes, and the coverings are the families $(\varphi_i : U_i \to X)_{i \in I}$ where the φ_i are étale morphisms and $X = \bigcup_{i \in I} \varphi_i(U_i)$. Since every morphism between étale S-schemes is étale, there is a canonical morphism from $S_{\text{ét}}$ to the big étale site of S. It turns out that for cohomology of abelian sheaves, it doesn't matter whether one uses the small or the big site.

There are other useful topologies on the category of schemes. Here are some examples.

- The *big fppf site* on Sch/S: a family $(\varphi_i : U_i \to X)_{i \in I}$ is said to be a fppf covering[2] if each φ_i is flat and locally of finite presentation, and $X = \bigcup_{i \in I} \varphi_i(U_i)$.
- The *big fpqc site* on Sch/S: a family $(\varphi_i : U_i \to X)_{i \in I}$ is said to be a fpqc covering[3] if each φ_i is flat, and for every open affine $V \subset X$, there exists a finite subset $J \subset I$ and open affine subsets $V_j \subset U_j$ for each $j \in J$, such that $V = \bigcup_{j \in J} \varphi_j(V_j)$.

A morphisms of schemes $f : X \to Y$ is called *weakly étale* if f is flat and $\Delta_f : X \to X \times_Y X$ is flat.

- The *big pro-étale site* on Sch/S: a family $(\varphi_i : U_i \to X)_{i \in I}$ is said to be a pro-étale covering if each φ_i is weakly étale, and for every open affine $V \subset X$, there exists a finite subset $J \subset I$ and open affine subsets $V_j \subset U_j$ for each $j \in J$, such that $V = \bigcup_{j \in J} \varphi_j(V_j)$.

One may check that these are indeed Grothendieck topologies on Sch/S.

Let us describe the coverings of $\operatorname{Spec} k$ where k is a field, for the different choices of topology.

[2] fidèlement plat de présentation finie

[3] fidèlement plat quasi-compact

- The Zariski coverings of $\operatorname{Spec} k$ are the families $(U_i)_{i\in I}$ with $U_i = \emptyset$ or $\operatorname{Spec} k$, and at least one of the U_i is $\operatorname{Spec} k$. In particular, the trivial covering $\operatorname{Spec} k \to \operatorname{Spec} k$ refines every Zariski covering.

- The étale coverings of $\operatorname{Spec} k$ are the families $(U_i \to \operatorname{Spec} k)_{i\in I}$ where each U_i is the disjoint union of spectra of finite separable extensions of k, and at least one U_i is not empty. In particular, any étale covering of $\operatorname{Spec} k$ has a refinement of the form $\operatorname{Spec} K \to \operatorname{Spec} k$, where K is a finite separable extension of k.

- Let $\mathcal{U} = (U_i \to \operatorname{Spec} k)_{i\in I}$ be a fppf covering. Choose $i \in I$ such that $U_i \neq \emptyset$. Then $U_i \to \operatorname{Spec} k$ is locally of finite type, and for all closed points $x \in U_i$, the extension $k(x)/k$ is finite. So $\mathcal{U}$ has a refinement of the form $\operatorname{Spec} K \to \operatorname{Spec} k$ where K/k is a finite extension.

- Similarly, any fpqc covering of $\operatorname{Spec} k$ has a refinement of the form $\operatorname{Spec} K \to \operatorname{Spec} k$ where K/k is an arbitrary field extension.

- Finally, any pro-étale covering of $\operatorname{Spec} k$ has a refinement of the form $\operatorname{Spec} K \to \operatorname{Spec} k$ where K/k is a separable algebraic (but not necessarily finite) extension. In particular, we see that $\operatorname{Spec} k$ has a "universal" pro-étale covering, namely $\operatorname{Spec} k^s \to \operatorname{Spec} k$ where k^s is the separable closure of k (the "largest" algebraic separable extension of k).

3.7 The étale fundamental group

We give here a brief discussion of the theory of fundamental groups for schemes.

Definition 3.61. Let X be a scheme. We denote by $\mathrm{F\acute{E}t}/X$ the category of *finite* étale X-schemes, with the morphisms given by the X-morphisms.

We fix a point $\overline{x} : \operatorname{Spec} k \to X$ where k is a *separably closed* field. Consider the functor

$$\begin{aligned} F_{\overline{x}} : \mathrm{F\acute{E}t}/X &\to \mathrm{Set} \\ Y &\mapsto Y_{\overline{x}} = \operatorname{Hom}_X(\overline{x}, Y) \end{aligned}$$

which associates to any finite étale cover of X its fibre over $\overline{x}$.

The functor $F_{\overline{x}}$ is usually not representable. For example, consider $X = \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$. One may prove that the only finite étale covers of X are the maps $f_n : X \to X$ given by $t \mapsto t^n$ for some $n \geq 1$. Contrary to the topological case, there is no "universal" cover of X, in other words no étale cover which dominates all the f_n. Note that the topological universal cover is given by the exponential map $\exp : \mathbf{C} \to \mathbf{C}^\times$, which is not an algebraic map. On the other hand, every finite étale cover of X is dominated by some f_n. This holds in general, in the following sense.

Theorem 3.62. *The functor $F_{\overline{x}}$ is pro-representable: there exists a projective system $\tilde{X} = (X_i)_{i\in I}$ of finite étale morphisms $X_i \to X$, indexed by a directed set I, such that for every finite étale cover Y of X, we have*

$$F_{\overline{x}}(Y) = \operatorname{Hom}_X(\tilde{X}, Y) \overset{def}{=} \varinjlim_{i\in I} \operatorname{Hom}_X(X_i, Y).$$

One can always choose the X_i to be Galois coverings of X: this means that the cardinality of $\operatorname{Aut}_X(X_i)$ is equal to the degree of $X_i \to X$, where the degree of a finite étale morphism is the cardinality of any geometric fiber.

Definition 3.63. The *étale fundamental group of X at $\overline{x}$* is

$$\pi_1^{\text{ét}}(X, \overline{x}) = \operatorname{Aut}_X(\tilde{X}) \stackrel{\text{def}}{=} \varprojlim_{i \in I} \operatorname{Aut}_X(X_i).$$

Since each $\operatorname{Aut}_X(X_i)$ is a finite group, the étale fundamental group is a profinite group.

Remark 3.64. We could have worked with a geometric point $\overline{x} : \operatorname{Spec} k \to X$ where k is algebraically closed. This gives the same fundamental group, essentially because $\operatorname{Aut}(\overline{k}/k) \cong \operatorname{Gal}(k^s/k)$ for any field k.

Remark 3.65. Since every X_i is finite étale over X, the transition maps $X_i \to X_j$ are also finite étale by Theorem 3.47 and [Sta19, Lemma 035D]. In particular they are affine. It follows that the inverse limit $\hat{X} = \varprojlim X_i$ exists as a scheme, see [Sta19, Section 01YV]. But usually $\hat{X}$ is not locally of finite presentation over X, so cannot be étale over X. However $\hat{X} \to X$ is always a pro-étale covering, because a filtered inductive limit of étale algebras is weakly étale [BS15, Prop. 2.3.3].

Examples 3.66.

- Let k be a field and let $\iota : k \to k^s$ be a separable closure. Then we have $\pi_1^{\text{ét}}(\operatorname{Spec} k, \iota) = \operatorname{Gal}(k^s/k)$.
- Let $X = \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$, and let $f_n : X \to X$ be the finite étale cover given by $x \mapsto x^n$. Then $\operatorname{Aut}(f_n) \cong \mu_n(\mathbf{C})$, so that
$$\pi_1^{\text{ét}}(X) = \varprojlim_{n \geq 1} \mu_n(\mathbf{C}) \cong \hat{\mathbf{Z}}.$$
- Let X be a smooth quasi-projective variety over $\mathbf{C}$. By a theorem of Grauert and Remmert (generalization of Riemann's existence theorem), we have an equivalence of categories between $\mathrm{FÉt}/X$ and the category of finite topological coverings of $X(\mathbf{C})$. Hence $\pi_1^{\text{ét}}(X)$ and $\pi_1(X(\mathbf{C}))$ have the same finite quotients. This implies that the étale fundamental group $\pi_1^{\text{ét}}(X)$ is isomorphic to the profinite completion of $\pi_1(X(\mathbf{C}))$.

Finally, we determine the fundamental group of $\operatorname{Spec} \mathbf{Z}$.

Theorem 3.67. $\operatorname{Spec} \mathbf{Z}$ *is simply connected.*

Proof. This means that the only connected finite étale cover of $\operatorname{Spec} \mathbf{Z}$ is the identity map $\operatorname{Spec} \mathbf{Z} \to \operatorname{Spec} \mathbf{Z}$. We take as geometric point the morphism $\operatorname{Spec} \overline{\mathbf{Q}} \to \operatorname{Spec} \mathbf{Z}$ induced by the inclusion $\mathbf{Z} \subset \overline{\mathbf{Q}}$. Let X be a scheme finite étale over $\operatorname{Spec} \mathbf{Z}$. By definition $X = \operatorname{Spec} A$ is affine and A is a finitely generated $\mathbf{Z}$-module. In particular A is integral over $\mathbf{Z}$. Moreover A is flat over $\mathbf{Z}$, which means that A is torsion free. Since $A \otimes \mathbf{Q}$ is étale over $\mathbf{Q}$, Proposition 3.45 implies that $A \otimes \mathbf{Q} \cong K_1 \times \cdots \times K_r$ where the K_i are number fields. Since A injects into $A \otimes \mathbf{Q}$ and is integral over $\mathbf{Z}$, we deduce that A is a subring of $A' = \mathcal{O}_{K_1} \times \cdots \times \mathcal{O}_{K_r}$.

The idea is now to use discriminants. Recall that if R is a ring and S is an R-algebra which is finite free as an R-module, the discriminant $\delta(S/R)$ is the determinant of the matrix $(\operatorname{Tr}_{S/R}(s_i s_j))_{i,j}$, where (s_i) is an R-basis of S. It is well-defined up to multiplication by $(R^\times)^2$. The discriminant is compatible with base change maps $R \to R'$. For example, the discriminant of a number field K is $\delta(\mathcal{O}_K/\mathbf{Z})$, which is a well-defined integer.

In our case, consider $\delta = \delta(A/\mathbf{Z}) \in \mathbf{Z}$. If p is prime, $A \otimes \mathbf{F}_p$ is étale over $\mathbf{F}_p$, hence is a product of finite separable extensions of $\mathbf{F}_p$. It follows that $\delta(A \otimes \mathbf{F}_p/\mathbf{F}_p)$ is nonzero, and thus $\delta \not\equiv 0 \bmod p$. Since this is true for every p, we deduce that $\delta = \pm 1$. Moreover, we have

$$\delta(A/\mathbf{Z}) = (A' : A)^2 \delta(A'/\mathbf{Z}) = (A' : A)^2 \prod_{i=1}^{r} \Delta_{K_i}.$$

It follows that $A = A'$ and $\Delta_{K_i} = \pm 1$ for every i. By the Hermite–Minkowski theorem, we get $K_i = \mathbf{Q}$ for every i, and thus $A \cong \mathbf{Z}^r$. So X is a finite union of copies of $\operatorname{Spec} \mathbf{Z}$. $\square$

Exercise 44. For $X = \operatorname{Spec} \mathbf{Z}$ and $\overline{x}$ as above, make explicit the category $\mathrm{F\acute{E}t}/X$ and the functor $F_{\overline{x}}$. Show that $F_{\overline{x}}$ is represented by $\operatorname{Spec} \mathbf{Z}$ and deduce that $\pi_1^{\text{ét}}(\operatorname{Spec} \mathbf{Z}, \overline{x}) = \{1\}$.

Exercise 45. Let K be an arbitrary number field, and let $\iota : K \to \overline{K}$ be an algebraic closure. Show that $\pi_1^{\text{ét}}(\operatorname{Spec} \mathcal{O}_K, \iota)$ is canonically isomorphic to $\operatorname{Gal}(K^{\mathrm{ur}}/K)$ where K^{ur} is the maximal unramified extension of K inside $\overline{K}$.

Exercise 46. Let A be a complete discrete valuation ring with quotient field K and residue field k. Let $\overline{x}$ be the point of $X = \operatorname{Spec} A$ induced by the inclusion $A \subset K^s$. Show that there are isomorphisms $\pi_1^{\text{ét}}(X, \overline{x}) \cong \operatorname{Gal}(K^{\mathrm{ur}}/K) \cong \operatorname{Gal}(k^s/k)$ where K^{ur} is the maximal unramified extension of K inside K^s.

4 Cohomology

4.1 Injective objects

Let $\mathcal{C}$ be an abelian category. For every object $M \in \mathcal{C}$, the contravariant functor $A \mapsto \operatorname{Hom}(A, M)$ is left exact.

Definition 4.1. An object $M \in \mathcal{C}$ is called *injective* if the functor $A \mapsto \operatorname{Hom}(A, M)$ is exact.

In other words M is injective if every morphism $A' \to M$ where A' is a subobject of A, extends to a morphism $A \to M$.

Let us determine the injective objects in the category of abelian groups.

Proposition 4.2. *An abelian group M is injective if and only if M is divisible: for every integer $n \geq 1$, the multiplication-by-n map $M \to M$ is surjective.*

Proof. Let M be a divisible abelian group. Let A be an abelian group and A' be a subgroup of A. Let $u : A' \to M$ be a linear map. We wish to extend u' to a linear map $A \to M$. Let us consider the set E of all extensions $\tilde{u} : \tilde{A} \to M$ of u, where $\tilde{A}$ is an intermediate subgroup between A' and A. The set E is partially ordered, and every chain in E has an upper bound. By Zorn's Lemma, E has at least one maximal element $\tilde{u} : \tilde{A} \to M$. We claim that $\tilde{A} = A$. Assume the contrary, and let $a \in A \setminus \tilde{A}$. Consider $\bar{a} \in \tilde{A}/A$. If $\bar{a}$ has infinite order, then $\langle \tilde{A}, a\rangle \cong A \oplus \mathbf{Z}$ and $\tilde{u}$ can be extended to $\langle \tilde{A}, a\rangle$, which is a contradiction. So $\bar{a}$ must have finite order $n \geq 2$. Write $n\bar{a} = b$ with $b \in \tilde{A}$. Choose an element $\tilde{m} \in M$ such that $n\tilde{m} = \tilde{u}(b)$. Then $\tilde{u}$ extends to a linear map $\langle \tilde{A}, a\rangle \to M$ sending a to $\tilde{m}$, which is again a contradiction. So M is injective.

The converse is left as an exercise. □

For example, $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are divisible abelian groups and thus injective objects in Ab.

Definition 4.3. We say that $\mathcal{C}$ has *enough injectives* if for every object $A \in \mathcal{C}$, there exists a monomorphism $A \to M$ where M is an injective object of $\mathcal{C}$.

Theorem 4.4. *The abelian category* Ab *has enough injectives.*

Proof. We know that the abelian group $\mathbf{Q}/\mathbf{Z}$ is injective. We will need the following lemma.

Lemma 4.5. *Let A be an abelian group and $a \in A \setminus \{0\}$. Then there exists $f \in \operatorname{Hom}(A, \mathbf{Q}/\mathbf{Z})$ such that $f(a) \neq 0$.*

Proof. Let $A' = \mathbf{Z}a$. Distinguishing the cases where a has finite or infinite order, it is not hard to construct a morphism $f' : A' \to \mathbf{Q}/\mathbf{Z}$ such that $f'(a) \neq 0$. Since $\mathbf{Q}/\mathbf{Z}$ is injective, f' extends to a morphism $f : A \to \mathbf{Q}/\mathbf{Z}$, and we have $f(a) \neq 0$. □

Now, let A be an abelian group. We wish to embed A into an injective abelian group. Consider the following map:

$$\begin{aligned} \phi : A &\to (\mathbf{Q}/\mathbf{Z})^{\mathrm{Hom}(A,\mathbf{Q}/\mathbf{Z})} \\ a &\mapsto (f(a))_{f\in\mathrm{Hom}(A,\mathbf{Q}/\mathbf{Z})}. \end{aligned}$$

Thanks to Lemma 4.5, the map ϕ is injective. Now $\mathbf{Q}/\mathbf{Z}$ is injective and an arbitrary product of injectives is again injective, so we are done. $\square$

Exercise 47. Show more generally that for any ring R, the category R-Mod has enough injectives. *Hint:* Given an R-module M, construct an embedding $M \hookrightarrow K^{\mathrm{Hom}(M,\mathbf{Q}/\mathbf{Z})}$ where $K = \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})$.

Let us give a sufficient condition under which an abelian category possesses enough injectives.

Definition 4.6. Let $\mathcal{C}$ be an abelian category having arbitrary direct sums. We say that $Z \in \mathcal{C}$ is a *generator* of $\mathcal{C}$ if for every object $A \in \mathcal{C}$, there exists an epimorphism

$$\bigoplus Z \to A \to 0$$

where $\bigoplus Z$ denotes the direct sum of arbitrarily many copies of Z.

For example, the abelian group $\mathbf{Z}$ generates the category of abelian groups. Indeed, for any abelian group A, we have a canonical surjective map $\bigoplus_{a\in A} \mathbf{Z} \to A$.

Theorem 4.7. *Let $\mathcal{C}$ be an abelian category. If $\mathcal{C}$ has arbitrary direct sums, satisfies (Ab5) and has a generator, then $\mathcal{C}$ has enough injectives.*

Proposition 4.8. *Let $\mathcal{C}$ be a category and $\mathcal{D}$ be an abelian category. If $\mathcal{D}$ has arbitrary direct sums, satisfies (Ab5) and has a generator, then $\mathcal{H}om(\mathcal{C},\mathcal{D})$ also satisfies these properties. In particular, $\mathcal{H}om(\mathcal{C},\mathcal{D})$ has enough injectives.*

Corollary 4.9. *For any category $\mathcal{C}$, the category $\mathcal{P}_{\mathcal{C}}$ of abelian presheaves on $\mathcal{C}$ has enough injectives.*

If $\mathcal{C}$ is an abelian category with enough injectives, then every object $A \in \mathcal{C}$ has an injective resolution, that is a long exact sequence

$$0 \longrightarrow A \longrightarrow M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} \cdots$$

where each object M^i is injective. Furthermore, the injective resolution $A \to M^*$ is unique up to homotopy. For the precise meaning and a proof of this, see [CE99, Chap. 5] or [Sta19, Section 010V], which gives more details on the notion of homotopy. We will not insist on this as a similar result (with projective resolutions) was seen in [Gil]. It will also appear later in Section 4.3.

4.2 ∂-functors

As a motivation, consider the theory of group cohomology. Let G be a group and G-Mod be the category of G-modules. As seen in [Gil], there are cohomology functors $H^i : G\text{-Mod} \to \mathrm{Ab}$ for every $i \geq 0$, the functor H^0 being simply $M \mapsto M^G$. Moreover, every short exact sequence of G-modules $0 \to M' \to M \to M'' \to 0$ gives rise to a long exact sequence

$$0 \to H^0(M') \to H^0(M) \to H^0(M'') \xrightarrow{\partial} H^1(M') \to H^1(M) \to H^1(M'') \xrightarrow{\partial} \cdots$$

where the ∂ are called the connecting maps. We say that the collection of functors $(H^i)_{i\geq 0}$ is an exact ∂-functor. In general, cohomology can be defined using ∂-functors.

Definition 4.10. Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories. A *∂-functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a collection $(T^i)_{i\geq 0}$ of covariant additive functors $T^i : \mathcal{C} \to \mathcal{C}'$, together with connecting morphisms $\partial : T^i(A'') \to T^{i+1}(A')$, defined for every $i \geq 0$ and every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{C}$. This data should satisfy the following properties:

- Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{C}$, the sequence
$$T^0(A') \to T^0(A) \to T^0(A'') \xrightarrow{\partial} T^1(A') \to T^1(A) \to T^1(A'') \xrightarrow{\partial} \cdots \tag{17}$$
is a complex in $\mathcal{C}'$ (i.e. the composition of two successive maps is zero).
- Given a morphism of short exact sequences in $\mathcal{C}$
$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0, \end{tikzcd}$$
the diagram
$$\begin{tikzcd} T^i(A'') \arrow[r, "\partial"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ T^i(B'') \arrow[r, "\partial"] & T^{i+1}(B') \end{tikzcd}$$
is commutative for all $i \geq 0$. In other words, a morphism of short exact sequences in $\mathcal{C}$ gives rise to a morphism in the abelian category of (bounded below) complexes in $\mathcal{C}'$.

Definition 4.11. We say that a ∂-functor $(T^i)_{i\geq 0}$ from $\mathcal{C}$ to $\mathcal{C}'$ is *exact* if for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{C}$, the resulting long sequence (17) is exact in $\mathcal{C}'$.

Given two ∂-functors $S = (S^i)_{i\geq 0}$ and $T = (T^i)_{i\geq 0}$ from $\mathcal{C}$ to $\mathcal{C}'$, we define a morphism $f : S \to T$ as a collection of natural transformations $f^i : S^i \to T^i$ satisfying the obvious commutativity constraints for the connecting maps.

The following natural question is fundamental.

Question 4.12. Let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant additive functor. Does there exist a ∂-functor T from $\mathcal{C}$ to $\mathcal{C}'$ extending F in the sense that $T^0 = F$? Does there exist a universal such T?

In other words, given $F : \mathcal{C} \to \mathcal{C}'$, we ask whether there exists a ∂-functor S from to $\mathcal{C}$ to $\mathcal{C}'$ extending F such that for every ∂-functor T from $\mathcal{C}$ to $\mathcal{C}'$ extending F, there exists a unique morphism of ∂-functors $f : S \to T$. If such an S exists, then the universal property makes it unique up to a unique isomorphism of ∂-functors, and we say that S is the *universal ∂-functor extending F*. The S^i are also known as the *right satellite functors of F*.

Example 4.13. If the functor $F : \mathcal{C} \to \mathcal{C}'$ is exact, then the universal ∂-functor extending F exists. It is given by $S^0 = F$ and $S^i = 0$ for every $i \geq 1$.

The satellite functors exist in great generality. We give here a sufficient condition for their existence.

Theorem 4.14. *Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories. Assume that $\mathcal{C}$ has enough injectives. Then for every covariant additive functor $F : \mathcal{C} \to \mathcal{C}'$, there exists a universal ∂-functor $S = (S^iF)_{i\geq 0}$ extending F. Moreover, S is exact if and only if F is half exact*[4].

[4]This means that for every short exact sequence $0 \to A' \to A \to A'' \to 0$, the sequence $F(A') \to F(A) \to F(A'')$ is exact.

Proof. We only explain the construction of $S^1F : \mathcal{C} \to \mathcal{C}'$, as the other functors are defined inductively by $S^{i+1}F = S^1(S^iF)$.

Let $A \in \mathcal{C}$. Take a short exact sequence $0 \to A \to M \to Q \to 0$ where M is an injective object, and define

$$S^1F(A) = F(Q)/\mathrm{im}(F(M)).$$

Up to canonical isomorphism, the object $S^1F(A)$ of $\mathcal{C}'$ doesn't depend on the choice of the sequence (exercise). This construction defines an additive functor $S^1F : \mathcal{C} \to \mathcal{C}'$. Moreover, by the very definition of S^1F, we have a connecting map $\partial : F(Q) \to S^1F(A)$ for every short exact sequence as above.

We leave to the reader the fact that the ∂-functor $(S^i)_{i\geq 0}$ has the expected properties (see [CE99, Chap. 3] for the solution). We also refer to [Gro57, 2.2] for a version of this theorem with the weaker assumption that every object of $\mathcal{C}$ has an injective effacement. □

4.3 Derived functors

The reference for this section is [Wei94, Chap. 2]. The more modern (and more general) approach to derived functors uses the language of derived categories [Wei94, Chap. 10], but we won't cover this here.

Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories.

Definition 4.15. Assume that $\mathcal{C}$ has enough injectives. Let $F : \mathcal{C} \to \mathcal{C}'$ be a *left exact* covariant additive functor. The *right derived functor of* F is the universal ∂-functor from $\mathcal{C}$ to $\mathcal{C}'$ extending F. We denote it by $RF = (R^iF)_{i\geq 0}$.

By Theorem 4.14, the right derived functor RF exists, is unique up to a unique isomorphism, and is exact. Each functor R^iF is a covariant additive functor $\mathcal{C} \to \mathcal{C}'$, and for each short exact sequence $0 \to A' \to A \to A''$ in $\mathcal{C}$, there is a long exact sequence in $\mathcal{C}'$

$$0 \to F(A') \to F(A) \to F(A'') \xrightarrow{\partial} R^1F(A') \to R^1F(A) \to R^1F(A'') \xrightarrow{\partial} \cdots \tag{18}$$

(it is also exact at $F(A')$ by left exactness of F). For $i \geq 1$, the functor R^iF is half-exact, but in general is neither left nor right exact, as can be seen from the sequence (18).

The following proposition shows that RF can be computed using injective resolutions.

Proposition 4.16. *Assume that $\mathcal{C}$ has enough injectives, and that $F : \mathcal{C} \to \mathcal{C}'$ is left exact. Let A be an object of $\mathcal{C}$. Choose an injective resolution of A in $\mathcal{C}$*

$$0 \longrightarrow A \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \cdots$$

Then $R^iF(A)$ is the i-th cohomology group of the complex $F(M^*)$.

Proof. Let Z^i (resp. B^i) denote the cocycles (resp. coboundaries) in degree i of the complex $F(M^*)$. We first do the case $i = 1$, namely $R^1F(A) \cong Z^1/B^1$. We have a short exact sequence

$$0 \longrightarrow M^0/A \longrightarrow M^1 \longrightarrow M^2.$$

Since F is left exact, we get $Z^1 \cong F(M^0/A)$. Moreover, B^1 is the image of $F(M^0)$ in $F(M^1)$. Under the previous isomorphism, the image of B^1 in Z^1 coincides with the image of $F(M^0)$ in $F(M^0/A)$. By definition of the satellite functor, we get $R^1F(A) \cong Z^1/B^1$.

To prove the general case, we do an induction on i. Note that we have an injective resolution

$$0 \longrightarrow M^0/A \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \cdots$$

By the induction hypothesis applied to this resolution, we have $R^iF(M^0/A) \cong Z^{i+1}/B^{i+1}$ for $i \geq 1$. Now consider the short exact sequence

$$0 \longrightarrow A \longrightarrow M^0 \longrightarrow M^0/A \longrightarrow 0.$$

Applying the ∂-functor RF, we get a long exact sequence

$$\cdots \longrightarrow R^iF(M^0) \longrightarrow R^iF(M^0/A) \longrightarrow R^{i+1}F(A) \longrightarrow R^{i+1}F(M^0) \longrightarrow \cdots$$

We conclude by noting that $R^jF(M) = 0$ for every injective object M and every $j > 0$, as we can use the trivial short exact sequence $0 \to M \to M \to 0 \to 0$. $\square$

As an example, let us give the interpretation of group cohomology as a right derived functor.

Theorem 4.17. *Let G be a group and G-Mod be the category of G-modules. Consider the functor $F : G\text{-Mod} \to \mathrm{Ab}$ given by $F(M) = M^G$. Then the right derived functors R^iF coincide with the cohomology functors $H^i : G\text{-Mod} \to \mathrm{Ab}$.*

Proof. We already know that $H = (H^i)_{i\geq 0}$ is an exact ∂-functor extending F. It suffices to show that H is universal. By [Gro57, I, 2.2.1], it suffices to show that $H^j(M) = 0$ for every injective object $M \in G$-Mod and every $j > 0$. This is easy to see using the definition of H^j using projective resolutions (exercise). $\square$

This shows that the cohomology groups $H^i(G, M)$ can be computed using either projective or injective resolutions.

Remark 4.18. Dually, one may define *homology groups* $H_i(G, M)$ by considering the left derivatives of the right exact functor $F : G\text{-Mod} \to \mathrm{Ab}$ given by the coinvariants:

$$F(M) = M_G \stackrel{\text{def}}{=} M/\langle gm - m : m \in M, g \in G\rangle.$$

Note that the category G-Mod has enough projectives, so the left derived functor is well-defined and can be computed using projective resolutions.

Exercise 48. Let $\mathcal{C}$ be an abelian category.

1. Show that the category $\mathcal{M}$ of morphisms in $\mathcal{C}$ is abelian.

2. Show that the functor $F : \mathcal{M} \to \mathcal{C}$ defined by $F(f) = \ker(f)$ is left exact, and compute the right derived functor of F.

4.4 Definition of étale cohomology

Derived functors can be used to define cohomology of sheaves. Before doing the case of general sites, we briefly mention the case of sheaves on topological spaces (see Iversen, *Cohomology of sheaves* for more details).

Let X be a topological space and $\mathcal{S}_X$ be the category of abelian sheaves on X. The category $\mathcal{S}_X$ has enough injectives (we will prove later a more general statement for arbitrary sites). Consider the functor of global sections $\Gamma(X, \cdot) : \mathcal{S}_X \to \mathrm{Ab}$ given by $\Gamma(X, F) = F(X)$. The functor $\Gamma(X, \cdot)$ is left exact.

Definition 4.19. Let F be an abelian sheaf on X. The cohomology groups of F are defined by $H^i(X, F) = R^i\Gamma(X, F)$.

Let A be an abelian group and $\underline{A}$ be the constant sheaf on X associated to A. One can show that if X is a Hausdorff space which is paracompact and locally contractible (for example, this holds if X is a topological manifold or a CW complex), then sheaf cohomology $H^i(X, \underline{A})$ coincides with singular cohomology $H^i(X, A)$.

Let us turn to sheaves on arbitrary sites.

Lemma 4.20. *Let $\mathcal{T}$ be a site. The category $\mathcal{S}_\mathcal{T}$ of abelian sheaves on $\mathcal{T}$ has enough injectives.*

Proof. We know that $\mathcal{S}_\mathcal{T}$ has arbitrary direct sums and satisfies (Ab5) (Theorem 2.42). So it remains to prove that $\mathcal{S}_\mathcal{T}$ has a generator.

For each $U \in \mathrm{cat}(\mathcal{T})$, define a presheaf Z_U on $\mathcal{T}$ by

$$Z_U(V) = \bigoplus_{\mathrm{Hom}(V,U)} \mathbf{Z} \qquad (V \in \mathrm{cat}(\mathcal{T})).$$

For every $F \in \mathcal{P}_\mathcal{T}$, we have $F(U) \cong \mathrm{Hom}(Z_U, F)$. One can show that $Z = \bigoplus_U Z_U$ generates $\mathcal{P}_\mathcal{T}$ and that $Z^\sharp$ generates $\mathcal{S}_\mathcal{T}$ (see [Tam94]). □

Let $\mathcal{T}$ be a site. In what follows, we denote by $\mathcal{P}$ (resp. $\mathcal{S}$) the category of abelian presheaves (resp. sheaves) on $\mathcal{T}$. For each $U \in \mathcal{T}$ and each abelian presheaf F on $\mathcal{T}$, we write $\Gamma(U, F) = F(U)$ for the abelian group of sections of F over U.

Lemma 4.21. *For every $U \in \mathcal{T}$, the section functor $\Gamma(U, \cdot) : \mathcal{S} \to \mathrm{Ab}$ is left exact.*

Proof. This functor is the composition of the inclusion $i : \mathcal{S} \to \mathcal{P}$ with the functor $\Gamma(U, \cdot) : \mathcal{P} \to \mathrm{Ab}$. Both are additive, the first is left exact by Theorem 2.40, and the second is exact. □

We may thus consider the right derived functor of $\Gamma(U, \cdot)$.

Definition 4.22. Let F be an abelian sheaf on $\mathcal{T}$ and let $U \in \mathcal{T}$. For any $i \geq 0$, the *i-th cohomology group of U with coefficients in F* is defined by

$$H^i(U, F) = R^i\Gamma(U, F).$$

Note that if we were working only with presheaves, then we would get nothing interesting because the section functor $\Gamma(U, \cdot)$ is exact in this setting.

We now apply this to the étale site of a scheme X. Recall that we denote by $X_{\text{ét}}$ the (small) étale site of X.

Definition 4.23. Let F be an abelian sheaf on $X_{\text{ét}}$. For any $U \in X_{\text{ét}}$, that is any étale morphism $U \to X$, the group $H^i_{\text{ét}}(U, F) = H^i(U, F)$ is called the *i-th étale cohomology group of U with coefficients in F*.

In the case of constant sheaves, we will use the following notation.

Definition 4.24. Let A be an abelian group. For any $U \in X_{\text{ét}}$, the *i-th étale cohomology group of U with coefficients in A* is $H^i_{\text{ét}}(U, A) = H^i_{\text{ét}}(U, \underline{A})$ where $\underline{A}$ is the constant sheaf on $X_{\text{ét}}$ associated to A.

If X is a scheme, we denote by $\mathcal{S}_X^{\text{ét}}$ the category of abelian sheaves on $X_{\text{ét}}$. If $X = \operatorname{Spec} R$ is affine, we also write $\mathcal{S}_X^{\text{ét}} = \mathcal{S}_R^{\text{ét}}$ and $H^i_{\text{ét}}(R, F)$ for the étale cohomology groups.

4.5 Cohomology of a point

Let us work out these definitions in the case $X = \operatorname{Spec} k$ where k is a field. Let k^s be a separable closure of k, and let $G = \operatorname{Gal}(k^s/k)$. We see G as a profinite group, and we endow G with the profinite topology. In particular G is compact. Let us first give an explicit description of the étale site on $\operatorname{Spec} k$, which was already discussed in Section 3.6.

Theorem 4.25. *The functor $X \mapsto X(k^s)$ gives an equivalence of categories between $\acute{E}t/k$ and the category of continuous (left) G-sets.*

Recall that a G-set E is said to be *continuous* if the action map $G \times E \to E$ is continuous, where E is given the discrete topology. This amounts to say that every element of E has open stabilizer in G (in particular, every G-orbit in E is finite).

Proof of Theorem 4.25. Let f be the functor $X \mapsto X(k^s)$. Note that given a k-scheme X, an element of $X(k^s)$ is a point x of X together with a k-embedding $k(x) \to k^s$. If X is étale over k, then $k(x)$ is a finite separable extension of k. So $X(k^s)$ is indeed a continuous G-set, and the functor f is well-defined.

We first show that f has a left adjoint ${}^{\mathrm{ad}}f$, and then check that the adjoint morphisms $\mathrm{id} \to f \circ {}^{\mathrm{ad}}f$ and ${}^{\mathrm{ad}}f \circ f \to \mathrm{id}$ are isomorphisms. To show the existence of ${}^{\mathrm{ad}}f$, it suffices by Remark 1.28 to show that for every continuous G-set E, the functor

$$X \mapsto \operatorname{Hom}_G(E, X(k^s)) \tag{19}$$

is representable. Writing E as the disjoint union of its orbits, and noting that $\acute{E}t/k$ has arbitrary coproducts, we are reduced to the case $E = G/H$ where H is an open subgroup of G. Let $k' \subset k^s$ be the fixed field of H. Since the extension k'/k is finite and separable, the scheme $\operatorname{Spec} k'$ is étale over $\operatorname{Spec} k$, and for every $X \in \acute{E}t/k$, we have

$$\operatorname{Hom}_G(G/H, X(k^s)) \cong X(k^s)^H \cong X(k') = \operatorname{Hom}_k(\operatorname{Spec} k', X).$$

These isomorphisms are clearly functorial in X, so $\operatorname{Spec} k'$ represents the functor (19).

Let us compute the adjoint map $\mathrm{id} \to f \circ {}^{\mathrm{ad}}f$ (the other map is treated similarly). With the same notations, we have $f({}^{\mathrm{ad}}f(G/H)) = (\operatorname{Spec} k')(k^s) = \operatorname{Hom}_k(k', k^s)$. The map $\rho : G/H \to f({}^{\mathrm{ad}}f(G/H))$ sends the class of 1 to the inclusion $k' \subset k^s$. Note that ρ is a G-map between two transitive G-sets of the same cardinality, so it must be bijective. Since f and ${}^{\mathrm{ad}}f$ commute with coproducts, we obtain that the adjoint map is an isomorphism. $\square$

Let us now see what the étale coverings of $\operatorname{Spec} k$ look like in terms of continuous G-sets.

Lemma 4.26. *Let $f : X \to Y$ be a morphism of schemes étale over k. The map f is surjective if and only if $f(k^s) : X(k^s) \to Y(k^s)$ is surjective.*

Proof. Assume that f is surjective. Let $\bar{y} \in Y(k^s)$, in other words we have a point $y \in Y$ and a k-embedding $\iota : k(y) \to k^s$. Let $x \in X$ such that $f(x) = y$. Then $k(x)$ is an extension of $k(y)$. Since $k(x)$ is finite and separable over k, ι extends to an embedding $k(x) \to k^s$. This gives our element in $X(k^s)$ mapping to $\bar{y}$. The converse is left to the reader. $\square$

Lemma 4.26 suggests to introduce the following topology on the category $\mathcal{C}_G$ of continuous G-sets: a family of G-maps $(\varphi_i : E_i \to E)_{i \in I}$ is a covering if and only if $E = \bigcup_{i \in I} \varphi_i(E_i)$. In this way, the equivalence of categories $X \mapsto X(k^s)$ preserves the topologies.

We now investigate abelian sheaves on $(\operatorname{Spec} k)_{\acute{e}t}$. We will show that they correspond to continuous G-modules. We denote by G-Mod the category of *continuous* G-modules, in other words the (discrete) abelian groups endowed with a continuous and linear action of G.

Theorem 4.27. *The category $\mathcal{S}_k^{\text{ét}}$ of abelian sheaves on $(\operatorname{Spec} k)_{\text{ét}}$ is equivalent to the category G-Mod of continuous G-modules. The equivalence is given by $F \mapsto \varinjlim F(\operatorname{Spec} k')$, where the inductive limit is taken over the finite extensions k'/k in k^s. In the other direction, given a continuous G-module M, the associated abelian sheaf F satisfies $F(\operatorname{Spec} k') = M^{\operatorname{Gal}(k^s/k')}$ for every finite extension k'/k in k^s.*

Proof. We show that both categories are equivalent to the category $\mathcal{S}_G$ of abelian sheaves on $\mathcal{T}_G$, where $\mathcal{T}_G$ the category of continuous G-sets, endowed with the topology defined above.

By Lemma 4.26, the morphism of sites $(\operatorname{Spec} k)_{\text{ét}} \to \mathcal{T}_G$ given by $X \mapsto X(k^s)$ induces an equivalence of categories $\mathcal{S}_G \to \mathcal{S}_k^{\text{ét}}$. Explicitly, let $F \in \mathcal{S}_k^{\text{ét}}$ and $F' \in \mathcal{S}_G$ be the corresponding sheaf on $\mathcal{T}_G$. For any open subgroup H of G, with fixed field k', we have $F'(G/H) = F(\operatorname{Spec} k')$.

Let us now show that $\mathcal{S}_G$ and G-Mod are equivalent. Consider the functor

$$\begin{aligned} \Phi : \mathcal{S}_G &\to G\text{-Mod} \\ F' &\mapsto \varinjlim_H F'(G/H) \end{aligned}$$

where H runs over the normal open subgroups of G. We give $F'(G/H)$ a structure of G-module by setting $g \cdot s = F'(\mu_g)(s)$ for every $g \in G$ and $s \in F'(G/H)$, where $\mu_g : G/H \to G/H$ is right multiplication by g. By definition, H acts trivially on $F'(G/H)$, so that $\varinjlim_H F'(G/H)$ is a continuous G-module. In the other direction, define

$$\begin{aligned} \Psi : G\text{-Mod} &\to \mathcal{S}_G \\ M &\mapsto \big(E \mapsto \operatorname{Hom}_G(E, M)\big) \end{aligned}$$

Tedious computations show that $\operatorname{Hom}_G(\cdot, M)$ is indeed a sheaf on $\mathcal{T}_G$, and that Φ and Ψ are mutually quasi-inverse. The easy part is $\Phi \circ \Psi \cong \mathrm{id}$: for any continuous G-module M, we have

$$\Phi(\Psi(M)) = \varinjlim_H \operatorname{Hom}_G(G/H, M) = \varinjlim_H M^H = \bigcup_H M^H = M.$$

By Theorem 4.25 and its proof, given $F \in \mathcal{S}_k^{\text{ét}}$, the corresponding G-module is $\varinjlim F(\operatorname{Spec} k')$, where k' runs over the finite Galois extensions of k in k^s. Since every finite separable extension is contained in a finite Galois extension, this is the same as taking the limit over all finite extensions k'/k in k^s.

In the other direction, given $M \in G$-Mod, the associated abelian sheaf F satisfies

$$F(\operatorname{Spec} k') = \operatorname{Hom}_G(G/\operatorname{Gal}(k^s/k'), M) = M^{\operatorname{Gal}(k^s/k')}.$$

□

Remark 4.28. Things may look formal, but they are not. For example, consider the étale morphism $\operatorname{Spec} k' \to \operatorname{Spec} k$ where k' is a finite separable extension of k. This is an étale covering with just one (generalized) open subset. For every sheaf F on $(\operatorname{Spec} k)_{\text{ét}}$, we have a restriction map $F(\operatorname{Spec} k) \to F(\operatorname{Spec} k')$. One may think that since F is a sheaf and since there is just one open subset in the covering, any section of F over $\operatorname{Spec} k'$ extends to a section over $\operatorname{Spec} k$. However, this is not true because one must consider the (generalized) intersection of the open subset with itself. In this case, we get

$$\operatorname{Spec} k' \times_{\operatorname{Spec} k} \operatorname{Spec} k' = \operatorname{Spec}(k' \otimes_k k').$$

If k'/k is Galois, then $k' \otimes_k k' \cong (k')^{\operatorname{Gal}(k'/k)}$, so the "intersection" is a bunch of points. There are two ways to restrict to this intersection, namely the maps $k' \to k' \otimes_k k'$ given by $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$. These maps are not equal (unless $k' = k$). In fact, a section $s \in F(\operatorname{Spec} k')$ will satisfy the compatiblity condition for the gluing if and only if s is invariant under $\operatorname{Gal}(k'/k)$. This shows that the natural map $F(\operatorname{Spec} k) \to F(\operatorname{Spec} k')^{\operatorname{Gal}(k'/k)}$ is an isomorphism, which is part of the proof of $\Psi \circ \Phi \cong \mathrm{id}$ (which we have not done).

A consequence of Theorem 4.27 is that the étale cohomology groups of $\operatorname{Spec} k$ are a particular instance of Galois cohomology. More precisely:

Corollary 4.29. *Let k be a field and $G = \mathrm{Gal}(k^s/k)$. For any abelian sheaf F on $(\operatorname{Spec} k)_{ét}$ and any $i \geq 0$, we have*

$$H^i_{ét}(\operatorname{Spec} k, F) = H^i(G, M)$$

where $M = \varinjlim F(\operatorname{Spec} k')$, with k' running over the finite extensions of k in k^s.

Proof. It suffices to note that the global section functor $\Gamma(\operatorname{Spec} k, \cdot)$ on the étale side corresponds to the functor $M \mapsto M^G$ on the Galois side (this is the last statement of Theorem 4.27). $\square$

Corollary 4.30. *Let k be a separably closed field. The functor $F \mapsto F(\operatorname{Spec} k)$ is an equivalence of categories between $\mathcal{S}_k^{ét}$ and* Ab. *For any $F \in \mathcal{S}_k^{ét}$ and any $i > 0$, we have $H^i_{ét}(\operatorname{Spec} k, F) = 0$.*

In other words, a geometric point has no cohomology in degree > 0, exactly as in the classical topological case. This is false when k is not separably closed: the set $\operatorname{Spec} k$ consists only of one point but one should think of it as a bunch of points (more accurately a bunch of geometric points).

Exercise 49. Let k be a field and $G = \mathrm{Gal}(k^s/k)$. Let A be an abelian group and $\underline{A}$ be the associated constant sheaf on $(\operatorname{Spec} k)_{\text{ét}}$.

(a) Compute $\underline{A}(\operatorname{Spec} k')$ for a finite separable extension k'/k.

(b) Deduce that $H^i_{\text{ét}}(\operatorname{Spec} k, A)$ is isomorphic to $H^i(G, A)$, where the action of G on A is trivial.

Exercise 50. (More difficult) Let p be a prime and let A be an abelian group. Show that $H^i(\mathbf{Z}_p, A) \cong H^i(G, A)$ where $G = \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{ur}}/\mathbf{Q}_p)$ and $\mathbf{Q}_p^{\mathrm{ur}}$ is the maximal unramified extension of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$.

Hints: Show that the category of abelian sheaves on $\operatorname{Spec} \mathbf{Z}_p$ is equivalent to the category of triples (M, N, φ) where M is a continuous $G_{\mathbf{F}_p}$-module, N is a continuous $G_{\mathbf{Q}_p}$-module (in other words, a torsion abelian group), and $\varphi : M \to N^{I_p}$ is a $G_{\mathbf{F}_p}$-morphism, with $G_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and I_p being the inertia group of $G_{\mathbf{Q}_p}$. Use an injective resolution of A to construct an injective resolution of the constant sheaf $\underline{A}$ on $\operatorname{Spec} \mathbf{Z}_p$.

5 Properties of étale sheaves

The aim of this chapter is to investigate functoriality of étale sheaves, and thus of étale cohomology. This is part of Grothendieck's 6 functors formalism.

5.1 Stalks of étale sheaves

Definition 5.1. Let X be a scheme.

1. Let $x \in X$. An *étale neighbourhood of x in X* is an étale morphism of schemes $U \to X$ together with a point $u \in U$ mapping to x.

2. Let $\overline{x} : \operatorname{Spec} k \to X$ be a geometric point. An *étale neighbourhood of $\overline{x}$ in X* is an étale morphism of schemes $U \to X$ together with a geometric point $\overline{u} : \operatorname{Spec} k \to U$ mapping to $\overline{x}$. In other words, there is a commutative diagram

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec} k \arrow[ur, "\overline{u}"] \arrow[r, "\overline{x}"] & X. \end{tikzcd}$$

3. Morphisms of étale neighbourhoods are defined in the natural way.

Note that any Zariski open neighbourhood of $x \in X$ is in particular an étale neighbourhood. As an example of étale neighbourhood which is not Zariski, one may think of the morphism $\operatorname{Spec} k' \to \operatorname{Spec} k$ for any finite separable extension k'/k.

The étale neighbourhoods satisfy the usual properties of neighbourhoods in topology. For example, given a scheme X and a geometric point $\overline{x}$ of X, the category of étale neighbourhoods of $\overline{x}$ in X is filtered: if $(U, \overline{u})$ and $(V, \overline{v})$ are two étale neighbourhoods of $\overline{x}$, then $(U \times_X V, (\overline{u}, \overline{v}))$ is an étale neighbourhood of $\overline{x}$. Here the geometric point $(\overline{u}, \overline{v})$ is defined using the universal property of $U \times_X V$.

Definition 5.2. Let X be a scheme, and let $\overline{x}$ be a geometric point of X. Let F be an abelian sheaf on $X_{\text{ét}}$. The *stalk of F at $\overline{x}$* is the abelian group $F_{\overline{x}} = \varinjlim_{(U,\overline{u})} F(U)$.

Taking the stalk at $\overline{x}$ defines an additive functor $\mathcal{S}_X^{\text{ét}} \to \mathrm{Ab}$. As in the case of topological spaces, the exactness of a sequence of abelian sheaves on $X_{\text{ét}}$ can be checked on the stalks.

Proposition 5.3. *A sequence $F \to G \to H$ of abelian sheaves on $X_{ét}$ is exact in $\mathcal{S}_X^{ét}$ if and only if for every geometric point $\overline{x}$ of X, the sequence of abelian groups $F_{\overline{x}} \to G_{\overline{x}} \to H_{\overline{x}}$ is exact.*

Definition 5.4. Let $\overline{x}$ be a geometric point of a scheme X. The *strict localisation of X at $\overline{x}$* is the ring $\mathcal{O}_{X,\overline{x}} = \varinjlim_{(U,\overline{u})} \mathcal{O}(U)$.

Since every Zariski neighbourhood is an étale neighbourhood, there is a canonical map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\overline{x}}$. It fits in the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec} \mathcal{O}_{X,\overline{x}} \arrow[r] & \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r] & X \\
\operatorname{Spec} k(\overline{x}) \arrow[r] \arrow[u] & \operatorname{Spec} k(x) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Exercise 51. (a) Show that $\mathcal{O}_{X,\overline{x}}$ is a local ring with residue field $k(\overline{x})$.

(b) Show that $\mathcal{O}_{X,\overline{x}} = \varinjlim_{(U,\overline{u})} \mathcal{O}_{U,u}$.

(c) Show that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,\overline{u}}$ for any étale neighbourhood $(U, \overline{u})$ of $(X, \overline{x})$.

(d) Show that $U \mapsto \mathcal{O}_U(U)$ is a sheaf of rings on the étale site of X. It is called the *structural sheaf of $X_{ét}$*. Show that its stalk at $\overline{x}$ is equal to $\mathcal{O}_{X,\overline{x}}$.

We will now compare the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,\overline{x}}$. We will show that $\mathcal{O}_{X,\overline{x}}$ depends in fact only on $\mathcal{O}_{X,x}$, and can be obtained by a purely algebraic process called *strict henselization.*

Definition 5.5. Let $(R, \mathfrak{m})$ be a local ring, with residue field $k = R/\mathfrak{m}$.

1. We say that R is henselian if Hensel's lemma holds in R: for every monic $f \in R[t]$ and every $a \in k$ which is a simple root of $\overline{f} \in k[t]$, there exists a unique lift $\tilde{a} \in R$ of a such that $f(\tilde{a}) = 0$.

2. We say that R is strictly henselian if moreover k is separably closed.

For example, the ring $\mathbf{Z}_p$ is henselian. More generally, any complete discrete valuation ring is henselian (exercise).

Definition 5.6. Let R be a local ring.

1. An *extension of* R is a local ring S together with a local morphism $R \to S$ (i.e. the inverse image of the maximal ideal of S is the maximal ideal of R).

2. An *henselization of* R is an henselian extension $R \to R^h$ such that every henselian extension $R \to S$ factors through R^h.

3. A *strict henselization of* R is a strictly henselian extension $R \to R^{sh}$ such that every strictly henselian extension $R \to S$ factors through R^{sh}.

One can show that the henselization (resp. strict henselization) of R exists and is unique up to isomorphism. For example, the henselization (resp. strict henselization) of $\mathbf{Z}_{(p)}$ is given by the integral closure of $\mathbf{Z}_{(p)}$ in $\mathbf{Z}_p$ (resp. $\mathbf{Z}_p^{\mathrm{nr}}$). One may think of the henselisation as of the "algebraic part" of the completion.

In general, one constructs R^{sh} as the filtered inductive limit of all étale R-algebras. This is an abstract construction which does not lead to an explicit description of R^{sh}.

Proposition 5.7. *Let X be a scheme, and let $x \in X$. Let $k(x)^s$ be a separable closure of $k(x)$, and let $\overline{x}$ be the associated geometric point of X. Then $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of the local ring $\mathcal{O}_{X,x}$.*

Lemma 5.8. *Let $(R, \mathfrak{m})$ be a henselian local ring with residue field k. The reduction modulo $\mathfrak{m}$ establishes an equivalence of categories between finite étale R-algebras and finite étale k-algebras.*

Corollary 5.9. *Let R be a henselian local ring with residue field k. Then $\pi_1^{\acute{e}t}(\operatorname{Spec} R) \cong \pi_1^{\acute{e}t}(\operatorname{Spec} k) \cong \operatorname{Gal}(k^s/k)$.*

For example $\pi_1^{\text{ét}}(\operatorname{Spec} \mathbf{Z}_p) \cong \operatorname{Gal}(\mathbf{Q}_p^{\mathrm{nr}}/\mathbf{Q}_p)$ and $\pi_1^{\text{ét}}(\operatorname{Spec} \mathbf{F}_p) \cong \operatorname{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ are isomorphic, and the isomorphism is induced by the map $\mathbf{Z}_p^{\mathrm{nr}} \to \overline{\mathbf{F}}_p$ given by reduction modulo p.

5.2 Local systems

Let X be a scheme. Let A be an abelian group, and let A_X be the associated constant sheaf on $X_{\text{ét}}$. We are going to describe the sections of A_X.

Lemma 5.10. *For any $U \in X_{\acute{e}t}$, we have*

$$A_X(U) = \{s : U \to A \textit{ locally constant for the Zariski topology}\}.$$

In particular, if U is a connected scheme, then $A_X(U) = A$.

Proof. By definition A_X is the sheafification of the presheaf $F : U \mapsto A$ on $\text{Ét}/X$. If $\mathcal{U} = (U_i)_{i \in I}$ is an étale covering of U, then $H^0(\mathcal{U}, F) = A$ unless $\mathcal{U}$ is the empty family. Thus $F^\dagger(U) = A$ if $U \neq \emptyset$, and $F^\dagger(\emptyset) = 0$.

Now let $U \in \text{Ét}/X$, and let $\mathcal{U} = (U_i \xrightarrow{\varphi_i} U)_{i \in I}$ be an étale covering of U. We may assume the U_i are not empty. Let $(s_i)_{i \in I} \in H^0(\mathcal{U}, F^\dagger)$ with $s_i \in A$. We define a function $s : U \to A$ as follows: for any $x \in U$, pick i such that $x \in \phi_i(U_i)$, and define $s(x) = s_i$. This makes sense because if $x \in \phi_i(U_i) \cap \phi_j(U_j)$ then $U_i \times_U U_j$ is not empty, so $s_i = s_j$ by assumption. It is then clear that the function $s : U \to A$ is constant on each open set $\varphi_i(U_i)$, so that s is locally constant for the Zariski topology. Conversely, any locally constant function $s : U \to A$ belongs to $H^0(\mathcal{U}, F^\dagger)$ for some Zariski open cover of U. $\square$

Proposition 5.11. *The stalk of A_X at a geometric point $\overline{x}$ of X is equal to A.*

Proof. We construct a map $A_{X,\overline{x}} \to A$ and then show that it is an isomorphism. By definition $A_{X,\overline{x}}$ is the inductive limit of the $A_X(U)$ where $(U, \overline{u})$ runs over the étale neighbourhoods of $(X, \overline{x})$. Moreover $A_X(U)$ is the group of locally constant functions $s_U : U \to A$. Consider the map $A_X(U) \to A$ given by evaluating at $u \in U$ (the image of $\overline{u}$). When $(U, \overline{u})$ varies, these maps are compatible, so by the universal property of the inductive limit, we get a map $A_{X,\overline{x}} \to A$. This map is easily seen to be surjective, and the fact that s_U is locally constant gives the injectivity. □

We now introduce locally constant sheaves, also known as local systems. This concept originated from algebraic topology.

Definition 5.12. A *locally constant sheaf on* $X_{ét}$ (or *local system on* $X_{ét}$) is a sheaf F on $X_{\text{ét}}$ which is locally constant for the étale topology: there exists an étale covering $(U_i \to X)_{i \in I}$ such that the restriction of F to each U_i is a constant sheaf.

One should not confuse the notion of locally constant sheaf with that of a constant sheaf, whose sections are locally constant functions.

It might be useful to give a basic example from topology. Let $f : \mathbf{C} \to \mathbf{C}^\times$ be given by $f(u) = e^u$. The fiber of f above $z \in \mathbf{C}^\times$ is in bijection with $2\pi i\mathbf{Z}$ (non canonically). Let F be the sheaf of continuous sections of f. Then F is a local system of sets on $\mathbf{C}^\times$, and its stalks are given by $F_z \cong f^{-1}(z)$. Given a continuous path γ from z_0 to z_1 in $\mathbf{C}^\times$, and given an element $u_0 \in F_{z_0}$, there is a unique way to transport u_0 along γ, giving a well-defined element $u_1 \in F_{z_1}$. If γ is a closed path, then we get a map $\gamma_* : F_{z_0} \to F_{z_0}$. This induces an action of $\pi_1(\mathbf{C}^\times, z_0)$ on F_{z_0} called the *monodromy representation.* Of course, the representation here is simple to describe: using the identification $\pi_1(\mathbf{C}^\times, z_0) \cong \mathbf{Z}$, the monodromy is just the natural action of $2\pi i\mathbf{Z}$ on F_{z_0} given by addition. The sheaf F is not constant, because there is no continuous section of f over $\mathbf{C}^\times$; equivalently, the monodromy representation is not trivial.

More generally, given a local system F on a path connected topological space X, the fibers F_x are all the same (non canonically), and we get monodromy representations $\pi_1(X, x) \to \operatorname{Aut} F_x$. In fact, giving a local system on X is the same as giving the monodromy representation.

These ideas generalize well for the étale topology. Let us work out the definition in the simple case $X = \operatorname{Spec} k$ where k is a field. Let k^s be a separable closure of k, and let $G = \operatorname{Gal}(k^s/k)$. We have seen that the category of abelian sheaves on $(\operatorname{Spec} k)_{\text{ét}}$ is equivalent to the category of continuous G-modules. Let $F \in \mathcal{S}_k^{\text{ét}}$ with corresponding Galois module M. From the definitions, we see that:

- F is constant if and only if G acts trivially on M.
- F is locally constant if and only if there exists a finite separable extension k' of k inside k^s such that $\operatorname{Gal}(k^s/k')$ acts trivially on M (in other words the action of G on M factors through a finite quotient).

For example, consider the Galois module μ_n of n-th roots of unity in $\overline{k}$, where $n \geq 1$ is not divisible by $\operatorname{char}(k)$. The associated sheaf μ_n is then locally constant, because it is constant over $\operatorname{Spec} k(\mu_n)$, which is an étale covering of $\operatorname{Spec} k$. The sheaf μ_n is constant if and only if k contains the n-th roots of unity. So these notions contain interesting arithmetic information.

In general, let X be a connected scheme and F be a locally constant sheaf on $X_{\text{ét}}$.

Lemma 5.13. *The stalks of F are all (non canonically) isomorphic: there exists an abelian group A and an étale covering $(U_i \to X)_{i \in I}$ of X such that $F|_{U_i} \cong A_{U_i}$ for each $i \in I$.*

Proof. Choose an étale covering $(U_i)_{i\in I}$ of X such that $F|_{U_i}$ is constant, associated to an abelian group A_i. Observe that if $U_i \times_X U_j$ is not empty, then $A_i \cong A_j$ (consider the stalk at a point in the intersection of U_i and U_j). For each abelian group A, define $I_A = \{i \in I : A_i \cong A\}$, and let $U_A = \cup_{i\in I_A} \mathrm{Im}(U_i \to X)$. Since étale morphisms are open, U_A is open in X. Moreover, the U_A are pairwise disjoint when A varies, and they cover X. It follows that there must exist an A such that $U_A = X$. □

We say that a locally constant sheaf F on $X_{\text{ét}}$ is finite if its stalks are finite abelian groups. We have the following theorem [Sta19, Lemma 0DV5].

Theorem 5.14. *Let X be a connected scheme, and let $\overline{x}$ be a geometric point of X. There is an equivalence of categories between the finite locally constant abelian sheaves on $X_{\text{ét}}$ and the continuous $\pi_1^{\text{ét}}(X,\overline{x})$-modules.*

The proof of this theorem involves two ingredients: first, the equivalence of categories between $\text{FÉt}/X$ and the finite continuous $\pi_1^{\text{ét}}(X,\overline{x})$-sets (Galois correspondence), and second, étale descent (see the next section for an introduction to descent).

5.3 Descent

The theory of descent is concerned with the following kind of problems. Let $f : X \to Y$ be a morphism of schemes. We are given an object F on X (which may be a sheaf, but could also be any other kind of structure). We want to understand whether there exists an object G on Y such that we have an isomorphism $F \cong f^*G$, where f^*G is the pull-back of G to X. Furthermore, if F has some property (P), then does G also have property (P)?

Here is a basic example. Assume that $Y = \operatorname{Spec} R'$ and $X = \operatorname{Spec} R$ are affine, so that we have a ring map $R \to R'$. For every module M over R, we may form its base change $M_{R'} = M \otimes_R R'$ which is a module over R'. Then the two questions above become the following:

(1) Let M' be an R'-module. Does there exist an R-module M such that $M' \cong M_{R'}$?

(2) Let M be an R-module. If $M_{R'}$ has some property (P), then does M have property (P)?

Another example is when $Y = X/\Gamma$ is the quotient of X by a group Γ (assuming the quotient exists in the category), and $f : X \to X/\Gamma$ is the canonical projection. Clearly, a necessary condition for F to descend in this case is that F should be invariant under Γ. But this is not always sufficient. We want to understand the obstruction, or find sufficient conditions on f or F so that descent holds.

For a general morphism $f : X \to Y$, an obvious necessary condition for F to descend is that $p_1^*F \cong p_2^*F$, where $p_1, p_2 : X \times_Y X \to X$ are the two canonical projections. In the case of a quotient $f : X \to X/\Gamma$, this usually boils down to the invariance under Γ mentioned above.

It is possible to formalize the notion of descent. A *descent datum with respect to f* is an object F on X together with an isomorphism $\varphi : p_1^*F \cong p_2^*F$ such that the following diagram is commutative:

$$\begin{tikzcd} p_0^*F \arrow[rr, "p_{02}^*\varphi"] \arrow[dr, "p_{01}^*\varphi"'] & & p_2^*F \\ & p_1^*F \arrow[ur, "p_{12}^*\varphi"'] & \end{tikzcd} \tag{20}$$

We say that the descent datum is *effective* if there exists an object G on Y such that there exists an isomorphism $F \cong f^*G$ which is compatible with φ. The question is then whether every descent datum with respect to f is effective.

More generally, we may define a descent datum with respect to a family of morphisms $f_i : X_i \to Y$, see [Sta19, Chapter 0238]. It will involve objects F_i on each X_i and isomorphisms $\varphi_{ij} : p_1^* F_i \cong p_2^* F_j$ on $X_i \times_Y X_j$, which are required to satisfy, for each triple (i, j, k), a *cocycle condition* on $X_i \times_Y X_j \times_Y X_k$ as in (20).

Here we will only give one example of *faithfully flat descent*. We say that a morphism of schemes $f : S' \to S$ is faithfully flat if it is flat and surjective. If $S' = \operatorname{Spec} R'$ and $S = \operatorname{Spec} R$ are affine, this amounts to say that R' is a faithfully flat R-module, which means that a sequence of R-modules $N' \to N \to N''$ is exact if and only if $N' \otimes_R R' \to N \otimes_R R' \to N'' \otimes_R R'$ is exact. For example, every free R-module is faithfully flat.

Proposition 5.15. *Let R' be a faithfully flat R-algebra, and let $R'' = R' \otimes_R R'$. Consider the two maps $R' \to R''$ given by $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$. Then the diagram*

$$0 \longrightarrow R \longrightarrow R' \rightrightarrows R'' \tag{21}$$

is exact.

Proof. First assume that the morphism of rings $f : R \to R'$ has a section $h : R' \to R$ (in particular, f is injective). Let $d : R' \to R''$ be the linear map defined by $d(x) = x \otimes 1 - 1 \otimes x$. We have $d \circ f = 0$. Let $h' : R' \to R'$ be defined by $h' = f \circ h + (\mathrm{id} \otimes h) \circ d$. Then

$$h'(x) = f(h(x)) + (\mathrm{id} \otimes h)(x \otimes 1 - 1 \otimes x) = f(h(x)) + x - f(h(x)) = x$$

so that $h' = \mathrm{id}$. So for $x \in R'$, the condition $d(x) = 0$ implies $h'(x) = f(h(x))$ and thus $x = f(h(x))$.

In the general case, tensor the sequence (21) by R'. Since R' is faithfully flat, it suffices to show that the new sequence is exact. But the morphism of rings $R' \to R' \otimes_R R'$ clearly has a section, namely $x \otimes y \mapsto xy$. □

Corollary 5.16. *Let $f : S' \to S$ be a faithfully flat morphism of affine schemes, and let $S'' = S' \times_S S'$. For every affine scheme T, the sequence*

$$\mathrm{Hom}(S, T) \longrightarrow \mathrm{Hom}(S', T) \rightrightarrows \mathrm{Hom}(S'', T)$$

is exact.

More generally, we have the following result.

Theorem 5.17. *Let $f : S' \to S$ be a faithfully flat and quasi-compact*[5] *morphism of schemes. Let X and Y be S-schemes. Denote by X', Y (resp. X'', Y'') their base changes to S' (resp. $S'' = S' \times_S S'$). Then the diagram of sets*

$$\mathrm{Hom}_S(X, Y) \longrightarrow \mathrm{Hom}_{S'}(X', Y') \rightrightarrows \mathrm{Hom}_{S''}(X'', Y'')$$

is exact.

Moreover, in the faithfully flat quasi-compact setting (in other words, for the fpqc topology), many properties descend: an S-scheme X (resp. a morphism of S-schemes $f : X \to Y$) has such and such property if and only if its base change $X' = X \times_S S'$ (resp. $f' : X' \to Y'$) has the said property.

Here is an application of faithfully flat descent.

[5] This means that the inverse image of any open affine of Y is quasi-compact.

Theorem 5.18. *Let X be a scheme, and let Z be an X-scheme. Then the functor $X' \mapsto \mathrm{Hom}_X(X', Z)$ is a sheaf of sets on $X_{ét}$.*

Proof. In general, to show that a presheaf is a sheaf for the étale topology, it suffices to show the gluing property in the following cases:

(a) for the Zariski open coverings;

(b) for a single surjective morphism $V \to U$ between affine étale X-schemes.

The proof of that reduction is formal, see [Tam94, 3.1.1].

It is easy to see that the presheaf $\mathrm{Hom}_X(\cdot, Z)$ satisfies the gluing property for the Zariski open coverings. Now, let $\varphi : V \to U$ be a surjective morphism between affine étale X-schemes. Since φ is étale and surjective, it is in particular faithfully flat, so the result follows from Theorem 5.17. ☐

Definition 5.19. If Z is an X-scheme, we denote by Z_X the sheaf of sets $\mathrm{Hom}_X(\cdot, Z)$ on $X_{ét}$.

Let us look at some examples. Of course, if Z is an arbitrary X-scheme, then Z_X is only a sheaf of sets, not of abelian groups. If G is a commutative group scheme[6] over X, then for any X-scheme X', the group law on G endows the X'-valued points $G_X(X') = \mathrm{Hom}_X(X', G)$ with the structure of an abelian group. Thus G_X is a sheaf of abelian groups on $X_{ét}$.

- The *additive group* is defined by $\mathbf{G}_a = \mathrm{Spec}\,\mathbf{Z}[t]$, with multiplication law given by $\mathbf{Z}[t] \to \mathbf{Z}[u, v]; t \mapsto u + v$. For any scheme X, the base change $(\mathbf{G}_a)_X = \mathbf{G}_a \times_{\mathrm{Spec}\,\mathbf{Z}} X$ is a group scheme over X, and for every $X' \in X_{ét}$, we have
$$\begin{aligned}(\mathbf{G}_a)_X(X') &= \mathrm{Hom}_X(X', \mathrm{Spec}\,\mathbf{Z}[t] \times_{\mathrm{Spec}\,\mathbf{Z}} X) \\ &= \mathrm{Hom}(X', \mathrm{Spec}\,\mathbf{Z}[t]) \\ &= \mathrm{Hom}(\mathbf{Z}[t], \mathcal{O}(X')) = \mathcal{O}(X').\end{aligned}$$
So $(\mathbf{G}_a)_X$ is just the structural sheaf of $X_{ét}$.

- The *multiplicative group* is defined by $\mathbf{G}_m = \mathbf{G}_a \setminus \{0\} = \mathrm{Spec}\,\mathbf{Z}[t, 1/t]$, with multiplication law $t \mapsto uv$. Note that for any ring R, we have $\mathbf{G}_m(R) = R^\times$ (and not $R \setminus \{0\}$). This is because a section of $\mathbf{G}_{m,R} \to \mathrm{Spec}\,R$ is a section of $\mathbf{G}_{a,R}$ which does not cross the zero section of $\mathbf{G}_{a,R}$, so it is given by an element $s \in R$ which is not contained in any prime ideal of R, so s must be invertible. For every $X' \in X_{ét}$, we have
$$(\mathbf{G}_m)_X(X') = \mathrm{Hom}_X(X', \mathrm{Spec}\,\mathbf{Z}[t, 1/t] \times_{\mathrm{Spec}\,\mathbf{Z}} X) = \mathrm{Hom}(\mathbf{Z}[t, 1/t], \mathcal{O}(X')) = \mathcal{O}(X')^\times.$$

- The group scheme of n-th roots of unity is defined by $\mu_n = \mathrm{Spec}\,\mathbf{Z}[t]/(t^n - 1)$. Similarly as above, we have for every $X' \in X_{ét}$
$$(\mu_n)_X(X') = \{s \in \mathcal{O}(X') : s^n = 1\}.$$

- If A is an abelian group, then the constant sheaf A_X is associated to the constant group scheme $\bigsqcup_A X$. More precisely, letting $G = \bigsqcup_A X$, we have $G \times_X G = \bigsqcup_{A \times A} X$, and the group law $G \times_X G \to G$ is induced by the group law $A \times A \to A$ on A.

[6]A group scheme over X is an X-scheme G endowed with a multiplication morphism $\mu : G \times_X G \to G$ satisfying the usual group axioms, stated in the category of schemes over X.

5.4 Direct and inverse images

Let $f : X \to Y$ be a morphism of schemes. If Y' is an étale Y-scheme, then the base change $X' := X \times_Y Y'$ is an étale X-scheme. This gives a functor $\text{Ét}/Y \to \text{Ét}/X$. This functor preserves fibre products and coverings. In other words, we get a morphism of sites $f_{\text{ét}} : Y_{\text{ét}} \to X_{\text{ét}}$.

Any presheaf F on $X_{\text{ét}}$ gives rise, by composing with $f_{\text{ét}}$, to a presheaf f_*F on $Y_{\text{ét}}$. One check easily that if F is a sheaf, then f_*F is also a sheaf. Therefore, we get the *direct image functor*

$$f_* : \mathcal{S}_X^{\text{ét}} \to \mathcal{S}_Y^{\text{ét}}.$$

Conversely, let G be an abelian sheaf on $Y_{\text{ét}}$. In general, the presheaf $(f_{\text{ét}})_*(G)$ on $X_{\text{ét}}$ is not a sheaf. We define f^*G to be the sheafification of $(f_{\text{ét}})_*(G)$. In this way, we get the *inverse image functor*

$$f^* : \mathcal{S}_Y^{\text{ét}} \to \mathcal{S}_X^{\text{ét}}.$$

As in the topological case, the functor f^* is left adjoint to f_*. In particular f_* is left exact, and f^* is right exact. More generally f_* commutes with projective limits and f^* commutes with inductive limits (compare Theorem 2.17).

Exercise 52. Let k'/k be a finite separable extension of fields, and let $f : \operatorname{Spec} k' \to \operatorname{Spec} k$ be the associated morphism. Describe explicitly the functors f^* and f_* between abelian sheaves on $(\operatorname{Spec} k)_{\text{ét}}$ and $(\operatorname{Spec} k')_{\text{ét}}$, in terms of Galois modules.

We are now going to study in more detail direct images. Let $f : X \to Y$ be a morphism of schemes, and let F be an abelian sheaf on $X_{\text{ét}}$. In some cases, we can compute the stalks of f_*F.

Lemma 5.20. *Let X be a scheme and $\overline{x}$ a geometric point of X.*

1. *Let $j : U \hookrightarrow X$ be an open immersion. We have*

$$(j_*F)_{\overline{x}} = \begin{cases} F_{\overline{x}} & \textit{if } x \in U \\ ? & \textit{otherwise.} \end{cases}$$

2. *Let $i : Z \hookrightarrow X$ be a closed immersion. We have*

$$(i_*F)_{\overline{x}} = \begin{cases} F_{\overline{x}} & \textit{if } x \in Z \\ 0 & \textit{if } x \notin Z. \end{cases}$$

Proof. 1. Assume $x \in U$. Since U is Zariski open in X, the étale neighbourhoods of $(U, \overline{x})$ are cofinal in the étale neighbourhoods of $(X, \overline{x})$. Hence

$$(j_*F)_{\overline{x}} = \varinjlim_{\overline{x} \in V \subset X} F(V \cap U) = \varinjlim_{\overline{x} \in V \subset U} F(V) = F_{\overline{x}}.$$

2. Assume $x \notin Z$. Since $X \setminus Z$ is already an étale neighbourhood of $\overline{x}$, an argument similar as above shows that $(i_*F)_{\overline{x}} = 0$.

Assume now $x \in Z$. It is enough to show that any étale neighbourhood of $(Z, \overline{x})$ is the restriction to Z of an étale neighbourhood of $(X, \overline{x})$. Taking an affine open subset containing x, we may assume $X = \operatorname{Spec} R$ and $Z = \operatorname{Spec} R/I$ where I is an ideal of R. Let $R' = R/I$, and let S' be an étale R'-algebra. We may also assume S' is standard étale, since every étale neighbourhood contains a standard étale one. Write $S' = (R'[T]/f)_g$ with $f \in R'[T]$ monic and f' invertible in S'. Then $f'h = g^n$ for some $h \in R'[T]/f$ and some $n \geq 1$. Choose any monic lift $\tilde{f} \in R[T]$ of f, and any lift $\tilde{h}$ of h. Define $\tilde{g} = \tilde{f}'\tilde{h}$, so that $\tilde{g}$ lifts g^n. Then $S = (R[T]/\tilde{f})_{\tilde{g}}$ is étale over R, and $S/I \cong (R'[T]/f)_{g^n} \cong S'$.

□

The stalk of j_*F at $x \notin U$ is not necessarily 0. The following exercise gives an example.

Exercise 53. Let $X = \operatorname{Spec} \mathbf{Z}_p$ and $U = \operatorname{Spec} \mathbf{Q}_p$. Denote $j : U \to X$ the open immersion. Let $\overline{x} : \operatorname{Spec} \overline{\mathbf{F}}_p \to X$ be the closed geometric point of X.

(a) Consider the constant sheaf A_U on $U_{\text{ét}}$, where A is an abelian group. Show that the stalk of j_*A_U at $\overline{x}$ is isomorphic to A. In fact, we have $j_*A_U \cong A_X$.

(b) More generally, let F be an abelian sheaf on $U_{\text{ét}}$, corresponding to a $G_{\mathbf{Q}_p}$-module M. Show that the stalk of j_*F at $\overline{x}$ is isomorphic to M^{I_p}, where I_p is the inertia subgroup of $G_{\mathbf{Q}_p}$.

(c) Let i denote the closed immersion $\operatorname{Spec} \mathbf{F}_p \to \operatorname{Spec} \mathbf{Z}_p$. Show that the abelian sheaf i^*j_*F on $\operatorname{Spec} \mathbf{F}_p$ corresponds to the $G_{\mathbf{F}_p}$-module M^{I_p}.

If $f : X \to Y$ is an arbitrary morphism, then for every geometric point $\overline{x}$ of X, with image $\overline{y}$ in Y, we have a canonical morphism $(f_*F)_{\overline{y}} \to F_{\overline{x}}$. But it is not an isomorphism in general.

Exercise 54. Find an example where $(f_*F)_{\overline{y}} \to F_{\overline{x}}$ is not injective (resp. surjective) (you may want to consider the topological case first).

Exercise 55. Let $f : X \to Y$ be a finite morphism.

(a) Show that if F is an abelian sheaf on $X_{\text{ét}}$ then the stalks of f_*F are given by $(f_*F)_{\overline{y}} = \bigoplus_{\overline{x} \mapsto \overline{y}} F_{\overline{x}}$, where the direct sum is over the geometric fiber $f^{-1}(\overline{y})$.

(b) Show that if f is étale and F is locally constant, then f_*F is locally constant.

Let $f : X \to Y$ be a finite étale morphism. If F is constant, then f_*F is locally constant, but usually f_*F will not be constant, because of the monodromy of f. In fact, suppose that F is the constant sheaf associated to an abelian group A. By the previous exercise, we have

$$(f_*A)_{\overline{y}} \cong \bigoplus_{\overline{x} \mapsto \overline{y}} A. \tag{22}$$

But we know two things:

- f_*A is a local system, its stalk at $\overline{y}$ has an action of $\pi_1^{\text{ét}}(Y, \overline{y})$ (the monodromy representation);
- X is a finite étale cover of Y, so the geometric fiber $f^{-1}(\overline{y})$ has an action of $\pi_1^{\text{ét}}(Y, \overline{y})$.

One can check that under the isomorphism (22), the monodromy representation is simply the permutation representation associated to the action of $\pi_1^{\text{ét}}(Y, \overline{y})$ on the finite set $f^{-1}(\overline{y})$. Note that the sheaf f_*A is constant if and only if the monodromy representation is trivial. This is the case if the covering f is trivial, which means that X is isomorphic to a finite disjoint union of copies of Y.

Now let $f : X \to Y$ be an arbitrary morphism. Since the functor f_* is left exact, we may consider the right derived functor $Rf_* = (R^i f_*)_{i \geq 0}$. So for every abelian sheaf F on $X_{\text{ét}}$, we get an abelian sheaf $R^i f_* F$ on $Y_{\text{ét}}$. One way of thinking of $R^i f_* F$ is that its stalks give the cohomology (in degree i) of the fibers of f, together with a description of how these cohomology groups "vary". It is not true in general that the direct image by f of a local system on X is a local system on Y. One can show that if f is proper and of finite presentation, and F is finite, then f_*F (and in fact every $R^i f_* F$) is at least constructible, which means roughly that X is a finite union of locally closed subschemes X_i such that each $F|_{X_i}$ is locally constant.

Now let us consider inverse images.

Lemma 5.21. *Let $j : U \to X$ be an open immersion (or more generally an étale morphism). For every abelian sheaf F on $X_{ét}$, the sheaf j^*F on $U_{ét}$ is the restriction of F to U: for every $U' \in U_{ét}$, we have $(j^*F)(U') = F(U')$, where U' is seen as an étale X-scheme using the composition $U' \to U \to X$.*

Proof. To determine $(j^*F)(U')$, we must compute the inductive limit $\varinjlim_{(V,\phi)} F(V)$ where V runs over the étale X-schemes and $\phi : U' \to V \times_X U$ is a morphism in $U_{\text{ét}}$. The category of such (V, ϕ) has a terminal object, namely $V = U'$ considered as an étale X-scheme. So j^*F is the sheafification of the presheaf $U' \mapsto F(U')$. But this is just the composition of F with the natural morphism of sites $U_{\text{ét}} \to X_{\text{ét}}$, and one checks this is indeed a sheaf. $\square$

Lemma 5.22. *Let $i = \overline{x} : \operatorname{Spec} k \to X$ be a geometric point of X. For every abelian sheaf F on $X_{ét}$, we have $i^*F \cong F_{\overline{x}}$.*

Here we have identified the sheaf i^*F on $\operatorname{Spec} k$ with the corresponding abelian group.

Proof. This follows from the definition of the stalk $F_{\overline{x}}$. $\square$

Let $f : X \to Y$ be a morphism of schemes. Let G be an abelian sheaf on $Y_{\text{ét}}$.

Lemma 5.23. *For any geometric point $\overline{x}$ of X, we have $(f^*G)_{\overline{x}} \cong G_{f(\overline{x})}$.*

Proof. Consider the diagram

$$\begin{tikzcd} \overline{x} : \operatorname{Spec} k \arrow[r, "i"] \arrow[dr] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Note that $(f \circ i)^* = i^* \circ f^*$ by unicity of the left ajoint of $(f \circ i)_* = f_* \circ i_*$. Thus

$$(f^*G)_{\overline{x}} = i^*f^*G = (f \circ i)^*G = G_{f(\overline{x})}.$$

$\square$

Corollary 5.24. *The functor f^* is exact.*

Exercise 56. (a) Show that $f_* : \mathcal{S}_X^{\text{ét}} \to \mathcal{S}_Y^{\text{ét}}$ sends injectives to injectives. *Hint:* Use the adjunction (f^*, f_*) and the corollary above.

(b) Let $F \in \mathcal{S}_X^{\text{ét}}$. Show that the canonical map $F \to \prod_{\overline{x}} (i_{\overline{x}})_* i_{\overline{x}}^* F$ is a monomorphism.

(c) Deduce another proof of the fact that $\mathcal{S}_X^{\text{ét}}$ has enough injectives.

5.5 The localization sequence

We consider the following situation: X is an arbitrary scheme, $i : Z \hookrightarrow X$ is a closed immersion, $U = X \setminus Z$ and $j : U \to X$ is the corresponding open immersion. Our aim is to relate the étale cohomology of X to that of Z and U.

Lemma 5.25. *The functor $j^* : \mathcal{S}_X^{ét} \to \mathcal{S}_U^{ét}$ has a left adjoint $j_! : \mathcal{S}_U^{ét} \to \mathcal{S}_X^{ét}$.*

Proof. Let $F \in \mathcal{S}_U^{\text{ét}}$. Define a presheaf $F_!$ on $X_{\text{ét}}$ as follows: for every étale morphism $\varphi : V \to X$, put

$$F_!(V) = \begin{cases} F(V) & \text{if } \varphi(V) \subset U, \\ 0 & \text{otherwise,} \end{cases}$$

the restriction maps $F_!(V) \to F_!(V')$ being defined naturally. Using Lemma 5.21, one proves that for every abelian sheaf G on $X_{\text{ét}}$, we have

$$\operatorname{Hom}_{\mathcal{P}_X^{\text{ét}}}(F_!, G) \cong \operatorname{Hom}_{\mathcal{S}_U^{\text{ét}}}(F, j^*G).$$

Defining $j_!F$ to be the sheafification of $F_!$, we get the desired result. □

The functor $j_!$ is called the *extension by zero.* The terminology is justified by the following lemma.

Lemma 5.26. *Let $F \in \mathcal{S}_U^{ét}$. For every geometric point $\overline{x}$ of X, we have*

$$(j_!F)_{\overline{x}} = \begin{cases} F_{\overline{x}} & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It suffices to determine the stalks of the presheaf $F_!$ introduced in the proof of Lemma 5.25. In the case $x \in U$, this follows from the fact that the étale neighbourhoods of $(X, \overline{x})$ contained in U are cofinal. In the case $x \notin U$, note that an étale neighbourhood of $(X, \overline{x})$ cannot be contained in U, because it contains x. It follows that $(F_!)_{\overline{x}} = 0$. □

Corollary 5.27. *The functor $j_! : \mathcal{S}_U^{ét} \to \mathcal{S}_X^{ét}$ is exact.*

Proof. Check the stalks. □

Remark 5.28. Lemma 5.25 and Corollary 5.27 are true more generally for any étale morphism $U \to X$.

Proposition 5.29. *Let F be an abelian sheaf on $X_{ét}$. We have a short exact sequence in $\mathcal{S}_X^{ét}$:*

$$0 \longrightarrow j_!j^*F \longrightarrow F \longrightarrow i_*i^*F \longrightarrow 0 \tag{23}$$

This means roughly that we can describe an abelian sheaf F on $X_{\text{ét}}$ by two sheaves, one living over Z and one living over U. However that the extension (23) does not split globally in general.

Applying suitable ∂-functors to (23), we get long exact sequences of cohomology groups. This result is very important, as it basically reduces the computation of the cohomology of X to that of Z and U (provided we understand the boundary maps).

Proof of Proposition 5.29. The morphisms in the sequence (23) are defined using the adjunction properties, namely $\operatorname{Hom}(j_!j^*F, F) \cong \operatorname{Hom}(j^*F, j^*F)$ and $\operatorname{Hom}(F, i_*i^*F) \cong \operatorname{Hom}(i^*F, i^*F)$.

To prove the exactness of (23), it suffices to check it for the stalks. Let $\overline{x}$ be a geometric point of X. First assume $x \in U$. Then

$$(j_!j^*F)_{\overline{x}} \cong (j^*F)_{\overline{x}} \cong F_{\overline{x}}.$$

Moreover $(i_*i^*F)_{\overline{x}} = 0$ by Lemma 5.20. So the stalk of (23) reads

$$0 \longrightarrow F_{\overline{x}} \longrightarrow F_{\overline{x}} \longrightarrow 0 \longrightarrow 0$$

which is obviously exact. Similarly, in the case $x \in Z$, we get

$$0 \longrightarrow 0 \longrightarrow F_{\overline{x}} \longrightarrow F_{\overline{x}} \longrightarrow 0$$

which is also exact. □

Exercise 57. Consider $X = \operatorname{Spec} \mathbf{Z}_p$, $Z = \operatorname{Spec} \mathbf{F}_p$ and $U = \operatorname{Spec} \mathbf{Q}_p$. Recall that the category $\mathcal{S}_X^{\text{ét}}$ is equivalent to that of triples (M, N, φ), where M is a $G_{\mathbf{F}_p}$-module, N is a $G_{\mathbf{Q}_p}$-module, and $\varphi : M \to N^{I_p}$ is a morphism of $G_{\mathbf{F}_p}$-modules. Translate the exact sequence (23) in terms of such triples.

In order to state the main result (the localization sequence), we introduce *cohomology with support*.

Definition 5.30. Let F be an abelian sheaf on $X_{\text{ét}}$, and let $s \in F(X)$ be a section. We say that s *has support in* Z if $s|_U = 0$. We denote by

$$\Gamma_Z(X, F) = \ker(F(X) \to F(U))$$

the group of sections of F with support in Z.

Remark 5.31. Given an arbitrary section $s \in F(X)$, one may define its support as $\operatorname{Supp}(s) = \{x \in X : s_{\overline{x}} \neq 0\}$. This is a Zariski closed subset of X (because $s_{\overline{x}} = 0$ implies $s = 0$ on some étale neighbourhood of $(X, \overline{x})$). Then $\Gamma_Z(X, F)$ is just the set of sections $s \in F(X)$ such that $\operatorname{Supp}(s) \subset Z$.

We get an additive functor $\Gamma_Z(X, \cdot) : \mathcal{S}_X^{\text{ét}} \to \mathrm{Ab}$. Since the section functors $\Gamma(X, \cdot)$ and $\Gamma(U, \cdot)$ are left exact, it is not hard to show that $\Gamma_Z(X, \cdot)$ is also left exact.

Definition 5.32. For any $F \in \mathcal{S}_X^{\text{ét}}$, the *cohomology groups of* F *with support in* Z are defined by

$$H^r_Z(X, F) = R^r\Gamma_Z(X, F).$$

This is the analogue of the relative cohomology groups $H^r(X, U; A)$ in algebraic topology.

Theorem 5.33. *For any $F \in \mathcal{S}_X^{\text{ét}}$, we have a long exact sequence of abelian groups*

$$\cdots \longrightarrow H^r_Z(X, F) \longrightarrow H^r(X, F) \longrightarrow H^r(U, F) \xrightarrow{\ \partial\ } H^{r+1}_Z(X, F) \longrightarrow \cdots \tag{24}$$

The sequence (27) is called the *localization exact sequence* associated to $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$.

Proof. Using Proposition 5.29 with the constant sheaf $\mathbf{Z}_X$, we get

$$0 \longrightarrow j_!j^*\mathbf{Z}_X \longrightarrow \mathbf{Z}_X \longrightarrow i_*i^*\mathbf{Z}_X \longrightarrow 0. \tag{25}$$

Let G be an arbitrary abelian sheaf on $X_{\text{ét}}$. If we apply the left exact contravariant functor $\operatorname{Hom}(-, G)$ to the sequence (25), we get

$$0 \longrightarrow \operatorname{Hom}(i_*i^*\mathbf{Z}_X, G) \longrightarrow \operatorname{Hom}(\mathbf{Z}_X, G) \longrightarrow \operatorname{Hom}(j_!j^*\mathbf{Z}_X, G). \tag{26}$$

Note that $\operatorname{Hom}(\mathbf{Z}_X, G)$ is isomorphic to $G(X)$ (this follows from the definition of $\mathbf{Z}_X$ as the sheafification of the constant presheaf $X' \mapsto \mathbf{Z}$ on $X_{\text{ét}}$). Also, by adjunction

$$\operatorname{Hom}(j_!j^*\mathbf{Z}_X, G) = \operatorname{Hom}(j^*\mathbf{Z}_X, j^*G) = \operatorname{Hom}(\mathbf{Z}_U, j^*G) = (j^*G)(U) = G(U),$$

where the equality $j^*\mathbf{Z}_X = \mathbf{Z}_U$ follows from Lemmas 5.10 and 5.21. Therefore the righthand map of (26) is the restriction $G(X) \to G(U)$. It follows that

$$\operatorname{Hom}(i_*i^*\mathbf{Z}_X, G) \cong \Gamma_Z(X, G).$$

Now, let $(I^n)_{n\geq 0}$ be an injective resolution of F. Let us use (26) with $G = I^n$. Since I^n is an injective object, the sequence (26) is also exact on the right. Thus we get

$$0 \longrightarrow \Gamma_Z(X, I^n) \longrightarrow \Gamma(X, I^n) \longrightarrow \Gamma(U, I^n) \longrightarrow 0.$$

In other words, we get a short exact sequence of *complexes*

$$0 \longrightarrow \Gamma_Z(X, I^\bullet) \longrightarrow \Gamma(X, I^\bullet) \longrightarrow \Gamma(U, I^\bullet) \longrightarrow 0.$$

By a standard homological argument (essentially the snake lemma), we deduce a long exact sequence for the cohomology groups of these complexes, namely

$$\cdots \longrightarrow R^r\Gamma_Z(X,F) \longrightarrow R^r\Gamma(X,F) \longrightarrow R^r\Gamma(U,F) \xrightarrow{\partial} R^{r+1}\Gamma_Z(X,F) \longrightarrow \cdots \tag{27}$$

which is what we want. $\square$

It is possible to go further and describe explicitly the category of abelian sheaves on $X_{\text{ét}}$, in terms of sheaves on $Z_{\text{ét}}$ and $U_{\text{ét}}$.

For any abelian sheaf F on $X_{\text{ét}}$, we define $F_Z = i^*F \in \mathcal{S}_Z^{\text{ét}}$ and $F_U = j^*F \in \mathcal{S}_U^{\text{ét}}$. By adjunction, we have a canonical morphism $F \to j_*j^*F = j_*F_U$. Applying i^*, we get a map $\varphi : F_Z \to i^*j_*F_U$.

Theorem 5.34. *Let $\mathcal{T}$ be the category of triples (G, H, φ) with $G \in \mathcal{S}_Z^{ét}$, $H \in \mathcal{S}_U^{ét}$ and $\varphi : G \to i^*j_*H$, where the morphisms in $\mathcal{T}$ are defined in the natural way. The functor*

$$\begin{aligned} \mathcal{S}_X^{ét} &\to \mathcal{T} \\ F &\mapsto (F_Z, F_U, \varphi) \end{aligned}$$

is an equivalence of categories.

Proof. We first define the inverse functor. Given a triple (G, H, φ), we define the abelian sheaf F as the fibred product in the category $\mathcal{S}_X^{\text{ét}}$:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & j_*H \arrow[d] \\ i_*G \arrow[r, "i_*\varphi"] & i_*i^*j_*H \end{tikzcd}$$

Applying the exact functor j^* to this cartesian square, and noting that $j^*j_* = \mathrm{id}$ and $j^*i_* = 0$, we get an isomorphism $F_Z \cong H$. Similarly, applying the exact functor i^* and using $i^*i_* = \mathrm{id}$, we get a cartesian square

$$\begin{tikzcd} i^*F \arrow[r] \arrow[d] & i^*j_*H \arrow[d, equal] \\ G \arrow[r, "\varphi"] & i^*j_*H \end{tikzcd}$$

which implies that $F_U \cong G$ and also that the canonical map $F_Z \to i^*j_*F_U$ is identified with φ.

Now let us start with $F \in \mathcal{S}_X^{\text{ét}}$. We have to show that the diagram of sheaves

$$\begin{tikzcd} F \arrow[r] \arrow[d] & j_*F_U \arrow[d] \\ i_*F_Z \arrow[r, "i_*\varphi"] & i_*i^*j_*F_U \end{tikzcd}$$

is cartesian. For this it is enough to check the stalks. For $x \in U$, we obtain

$$\begin{tikzcd} F_{\overline{x}} \arrow[r] \arrow[d] & F_{\overline{x}} \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

while for $x \in Z$, we obtain

$$\begin{tikzcd} F_{\overline{x}} \arrow[r] \arrow[d] & (j_* F_U)_{\overline{x}} \arrow[d] \\ F_{\overline{x}} \arrow[r] & (j_* F_U)_{\overline{x}}. \end{tikzcd}$$

Both squares are clearly cartesian. □

Exercise 58. (a) Describe explicitly the functors i_*, j_* and $j_!$ in terms of triples in $\mathcal{T}$ as above.

(b) In the case $X = \operatorname{Spec} \mathbf{Z}_p$, $Z = \operatorname{Spec} \mathbf{F}_p$ and $U = \operatorname{Spec} \mathbf{Q}_p$, show that the category of abelian sheaves on $\operatorname{Spec} \mathbf{Z}_p$ is equivalent to the category of triples (M, N, φ) where M is a continuous $G_{\mathbf{F}_p}$-module, N is a continuous $G_{\mathbf{Q}_p}$-module, and $\varphi : M \to N^{I_p}$ is a $G_{\mathbf{F}_p}$-morphism.

Consider our favorite example We are now going to define a fourth functor $i^! : \mathcal{S}_X^{\text{ét}} \to \mathcal{S}_Z^{\text{ét}}$.

Definition 5.35. Using the identification $\mathcal{S}_X^{\text{ét}} \cong \mathcal{T}$ from Proposition 5.34, we define

$$\begin{aligned} i^! : \mathcal{S}_X^{\text{ét}} &\to \mathcal{S}_Z^{\text{ét}} \\ (G, H, \varphi) &\mapsto \ker(\varphi). \end{aligned}$$

Since $\varphi : G \to i^* j_* H$ is a morphism of sheaves on Z, we have that $\ker(\varphi)$ is a subsheaf of G.

Definition 5.36. Let F be an abelian sheaf on $X_{\text{ét}}$. We say that F *has support in* Z if $j^* F = 0$ (equivalently, $F_{\overline{x}} = 0$ for every $x \in U$).

Lemma 5.37. *The functor i_* identifies $\mathcal{S}_Z^{\text{ét}}$ with the full subcategory of $\mathcal{S}_X^{\text{ét}}$ consisting of sheaves with support in Z.*

Proof. Since $j^* i_* = 0$, we have one inclusion. Conversely, let $F \in \mathcal{S}_X^{\text{ét}}$ with support in Z. Writing F as a triple (G, H, φ), we have $H = j^* F = 0$. Thus $F = i_* G$. □

By means of i_*, we may view $i^! F$ as the subsheaf of F of sections with support in Z.

Proposition 5.38. *The functor $i^!$ is right adjoint to i_*. In particular, $i^!$ is left exact.*

Proof. We have to show that for every abelian sheaf G on $Z_{\text{ét}}$ and every abelian sheaf F' on $X_{\text{ét}}$, we have

$$\operatorname{Hom}_X(i_* G, F') \cong \operatorname{Hom}_Z(G, i^! F').$$

Write F' as a triple (G', H', φ'). Then $i^! F' = \ker(\varphi')$ so the right hand side is $\operatorname{Hom}(G, \ker(\varphi'))$. Moreover $i_* G = (G, 0, 0)$, so giving a morphism $i_* G \to F'$ is the same as giving a morphism $G \to G'$ fitting in the commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & 0 \arrow[d] \\ G' \arrow[r, "\varphi'"] & i^* j_* H' \end{tikzcd}$$

This means exactly that $G \to G'$ factors through $\ker(\varphi')$. □

To sum, we have constructed functors i^*, i_*, $i^!$ and j^*, j_*, $j_!$. These functors fit into the following diagram of adjunctions:

$$\begin{tikzcd}
\mathcal{S}_Z^{\text{ét}} \arrow[d, "i_*"] \\
\mathcal{S}_X^{\text{ét}} \arrow[u, bend left=60, "i^*"] \arrow[u, bend right=60, "i^!"'] \arrow[d, "j^*"] \\
\mathcal{S}_U^{\text{ét}} \arrow[u, bend left=60, "j_!"] \arrow[u, bend right=60, "j_*"']
\end{tikzcd}$$

In this diagram, if a map f is immediately to the left of g, then f is a left adjoint to g. Moreover, in each column the composition of the arrows is 0, e.g. $i^! j_* = 0$.

Let us also summarize the exactness properties. The following properties can be obtained from the diagram above by noting that a left (resp. right) adjoint is always right (resp. left) exact, or by using the description in terms of triples:

- i^*, j^*, i_*, $j_!$ are exact.
- j_*, $i^!$ are left exact.

In particular, we may consider the derived functors Rj_* and $Ri^!$. These can be determined in terms of triples as an exercise.

5.6 Some words about the 6 functors

The functors $j^!$ and $i_!$ generalize to a much more general setting, called the Grothendieck 6 functors formalism. For any morphism of schemes $f : X \to Y$ which is separated and of finite type, there are functors $Rf_!$ (direct image with compact support) and $Rf^!$ (exceptional inverse image). We will not say anything about how to construct them, but rather explain a little bit the properties they satisfy.

Contrary to what the notation suggests, these functors are not the derived functors of some functor $f_!$ or $f^!$. They are defined only at the level of the derived categories. These are the good framework for defining and using derived functors. Given an abelian category $\mathcal{C}$, the derived category $D^+(\mathcal{C})$ of $\mathcal{C}$ is a category containing $\mathcal{C}$ as a full subcategory, and endowed with functors $H^i : D^+(\mathcal{C}) \to \mathcal{C}$ for every $i \geq 0$. Very roughly, $D^+(\mathcal{C})$ is made out of complexes in $\mathcal{C}$ (modulo some equivalence), and H^i is the functors giving the cohomology of the complex in degree i. The important property is that every left exact additive functor $F : \mathcal{C} \to \mathcal{C}'$ between abelian categories extends uniquely to a derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$. The functors $R^iF : \mathcal{C} \to \mathcal{C}'$ we have defined can then be recovered as $R^iF = H^i \circ RF$.

Let Λ be a torsion ring (e.g. $\Lambda = \mathbf{Z}/n\mathbf{Z}$). For any scheme X, we denote by $\mathcal{S}_{X,\Lambda}^{\text{ét}}$ the category of sheaves of Λ-modules on $X_{\text{ét}}$. For example, if $\Lambda = \mathbf{Z}/n\mathbf{Z}$ then $\mathcal{S}_{X,\Lambda}^{\text{ét}}$ is the subcategory of $\mathcal{S}_X^{\text{ét}}$ consisting of those abelian sheaves F which are killed by n, i.e. the map $F \xrightarrow{n} F$ given by multiplication by n is zero. Let $D^+(X, \Lambda)$ be the derived category of $\mathcal{S}_{X,\Lambda}^{\text{ét}}$.

Let us go back to our morphism $f : X \to Y$ (separated and of finite type). In general, there does not exist a functor $f_! : \mathcal{S}_{X,\Lambda}^{\text{ét}} \to \mathcal{S}_{Y,\Lambda}^{\text{ét}}$, but there exists a functor $Rf_! : D^+(X, \Lambda) \to D^+(Y, \Lambda)$. The functor $Rf_!$ has a right adjoint $Rf^! : D^+(Y, \Lambda) \to D^+(X, \Lambda)$. From this, we get functors for usual Λ-sheaves $R^if_!$ and $R^if^!$.

- If f is proper, then $Rf_* : D^+(X, \Lambda) \to D^+(Y, \Lambda)$ commutes with arbitrary base change.

This means the following. Let $g : Y' \to Y$ be the base change. Let $X' = X \times_Y Y'$. We have a cartesian square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then for every sheaf of Λ-modules F on $X_{\text{ét}}$, the canonical morphism $g^*(R^i f_* F) \to R^i f'_*(g'^* F)$ is an isomorphism of sheaves on $Y'_{\text{ét}}$.

The following particular case is useful. Let $g = \overline{y} : \operatorname{Spec} k \to Y$ be a geometric point of Y. By Lemma 5.22, the abelian group $g^*(R^i f_* F)$ is simply the stalk $(R^i f_* F)_{\overline{y}}$. Thus we get an isomorphism

$$(R^i f_* F)_{\overline{y}} \cong H^i_{\text{ét}}(X_{\overline{y}}, F_{\overline{y}})$$

where $F_{\overline{y}}$ is the pull-back of F to the geometric fiber $X_{\overline{y}}$. Thus *the stalks of $R^i f_* F$ give the cohomology of the fibers of f*. In the case of constant sheaves, we can write with very loose notations $(R^i f_* A)_y \cong H^i(f^{-1}(y), A)$. Note that $R^i f_* A$ is a sheaf, so typically we have additional structure, like the monodromy action of the fundamental group $\pi_1(Y, y)$ on $H^i(f^{-1}(y), A)$, that we explained earlier. This is important because if $R^i f_* A$ is, say, a local system on Y, then knowing only the stalks is not a big information (the stalks are all isomorphic). What is interesting in having a sheaf $R^i f_* A$ is that it gives precise sense to "how the stalks vary".

- Rf_* commutes with smooth base change.
- $Rf_!$ commutes with arbitrary base change.

  In the particular case where $\overline{y}$ is a geometric point of Y as above, this gives

  $$(R^i f_! F)_{\overline{y}} \cong R^i f'_!(F|_{X_{\overline{y}}}).$$

  Let us assume that f is *compactifiable*, namely there exists an open immersion $j : X \to \overline{X}$ and a proper morphism $\bar{f} : \overline{X} \to Y$ such that $f = \bar{f} \circ j$. Then we may define $Rf_!$ as $Rf_! F = R\bar{f}_*(j_! F)$. So we get

  $$(R^i f_! F)_{\overline{y}} \cong R^i \bar{f}_*(j_! F|_{\overline{y}}) \cong H^i_{\text{ét}}(\overline{X}_{\overline{y}}, j_! F|_{\overline{y}}).$$

  This last group is the *étale cohomology with compact support* $H^i_c(X_{\overline{y}}, F_{\overline{y}})$. So again, we have a reasonable interpretation of the stalks of $Rf_!$.

- There exists a natural transformation $Rf_* \to Rf_!$ which is an isomorphism if f is proper.
- $Rf^!$ commutes with arbitrary base change.
- From now on, assume $\Lambda = \mathbf{Z}/n\mathbf{Z}$. If $f : X \to Y$ is smooth separated of relative dimension d, we have an isomorphism $(R)f^* \cong Rf^!(-d)[-2d]$.

  This means the following. For any scheme S, we denote by $\mathbf{Z}/n\mathbf{Z}(1)$ the n-torsion sheaf $(\mu_n)_S$ on $S_{\text{ét}}$. For every $m \in \mathbf{Z}$, the Tate twist $\mathbf{Z}/n\mathbf{Z}(m)$ is defined as $(\mathbf{Z}/n\mathbf{Z}(1))^{\otimes m}$, and for any n-torsion sheaf F on $S_{\text{ét}}$, we define $F(m) = F \otimes \mathbf{Z}/n\mathbf{Z}(m)$. Then the above statement says that for any n-torsion sheaf F on $Y_{\text{ét}}$, we have

  $$R^{2d} f^! F \cong f^* F(d)$$

  and all the other sheaves $R^i f^!$ with $i \neq 2d$ vanish. In particular, if f is an open immersion, this simply says that $j^! = j^*$.

- The fact that $(Rf_!, Rf^!)$ are adjoint is known as Verdier duality.

The two remaining functors are the tensor product $\otimes$ and the internal Hom. They satisfy various axioms; in particular it is possible to formulate duality using dualizing objects.

5.7 Étale cohomology of curves

Again, we only give statements here, see [Tam94, II.10] for more details.

Theorem 5.39. *For any scheme X, we have isomorphisms*

$$H^1_{ét}(X, (\mathbf{G}_m)_X) \cong H^1_{Zar}(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X)$$

where $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X (i.e. invertible $\mathcal{O}_X$-modules).

Let X be a scheme, and let $n \geq 1$ be an integer invertible on X. The Kummer sequence is the following exact sequence of abelian sheaves on $X_{\text{ét}}$ (see [Tam94, II.4.4.1] for the proof)

$$0 \longrightarrow (\mu_n)_X \longrightarrow (\mathbf{G}_m)_X \xrightarrow{x \mapsto x^n} (\mathbf{G}_m)_X \longrightarrow 0.$$

Taking cohomology, we get the following long exact sequence

$$\begin{aligned}
&0 \to \mu_n(X) \to \mathcal{O}(X)^\times \xrightarrow{n} \mathcal{O}(X)^\times \\
&\quad \to H^1(X, \mu_n) \to \mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X) \\
&\quad \to H^2(X, \mu_n) \to H^2(X, \mathbf{G}_m) \xrightarrow{n} H^2(X, \mathbf{G}_m) \to \cdots
\end{aligned}$$

Note that if $X = \operatorname{Spec} k$ with k a field, the étale cohomology group $H^2(\operatorname{Spec} k, \mathbf{G}_m)$ is isomorphic to the Brauer group of k. In general $H^2(X, \mathbf{G}_m)$ is closely related to the Brauer group $\mathrm{Br}(X)$ of X, defined using equivalence classes of Azumaya algebras on X. There is a canonical injective map $\delta : \mathrm{Br}(X) \to H^2(X, \mathbf{G}_m)$. Gabber has shown that if X is quasi-projective over a commutative ring, then the image of δ is the torsion subgroup of $H^2(X, \mathbf{G}_m)$. If futhermore X is regular, then $H^2(X, \mathbf{G}_m)$ is torsion, so in this case is isomorphic to $\mathrm{Br}(X)$.

Now let us assume that X is an algebraic curve over a separably closed field k (so we assume $n \neq 0$ in k).

Theorem 5.40. *If $\mathrm{char}(k) = 0$ then $H^q(X, \mathbf{G}_m) = 0$ for every $q \geq 2$.*
If $\mathrm{char}(k) = p > 0$, then $H^q(X, \mathbf{G}_m)$ is a p-power torsion abelian group for every $q \geq 2$.

Using this theorem, the above long exact sequence becomes

$$0 \to \mu_n(X) \to \mathcal{O}(X)^\times \xrightarrow{n} \mathcal{O}(X)^\times \to H^1(X, \mu_n) \to \mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X) \to H^2(X, \mu_n) \to 0$$

and we also deduce $H^q(X, \mu_n) = 0$ for every $q > 2$.

If moreover X is connected and projective, then $\mathcal{O}(X) = k$, and the map $k^\times \xrightarrow{n} k^\times$ is surjective. We deduce the following theorem.

Theorem 5.41. *Let X be a connected projective curve over a separably closed field k, and let $n \neq 0$ in k. Then*

$$\begin{aligned}
&H^0(X, \mu_n) \cong \mu_n(k) \\
&H^1(X, \mu_n) \cong \mathrm{Pic}(X)[n] \text{ (the n-torsion subgroup of } \mathrm{Pic}(X)) \\
&H^2(X, \mu_n) \cong \mathrm{Pic}(X)/n\,\mathrm{Pic}(X) \\
&H^q(X, \mu_n) = 0 \text{ for } q > 2.
\end{aligned}$$

If X is a smooth connected projective curve over k, then $\mathrm{Pic}(X) \cong \mathbf{Z} \oplus J$ where J is the Jacobian variety of X. It is known that $J[n] \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$ where g is the genus of X. So in this case, we get $H^1(X, \mu_n) \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$ and $H^2(X, \mu_n) \cong \mathbf{Z}/n\mathbf{Z}$. Note that since k is separably closed, we have an isomorphism of abelian sheaves $\mu_n \cong \mathbf{Z}/n\mathbf{Z}$ on X, so this also gives the étale cohomology of X with trivial coefficients $\mathbf{Z}/n\mathbf{Z}$.

To compute the étale cohomology of higher-dimensional varieties, a standard technique is *dévissage*. For example, let S be a surface. Assume that we have a fibration $f : S \to C$ over a curve C. Given a fibration, the *Leray spectral sequence* is a way to compute the cohomology of the total space in terms of the cohomology of the fibers and the cohomology of the base. In our case, the Leray spectral sequence is traditionally written as follows

$$H^p_{\text{ét}}(X, R^q f_* F) \Rightarrow H^{p+q}_{\text{ét}}(S, F)$$

where F is any abelian sheaf on $S_{\text{ét}}$. This means (very roughly) that the group $H^r_{\text{ét}}(S, F)$ is approximated by the direct sum of the groups $H^p_{\text{ét}}(X, R^q f_* F)$ with $p + q = r$. So in some sense, we are reduced to compute the cohomology of the fibers of f and then the cohomology of the base C. The fibers of f are curves, and the base C is a curve, so in some sense we are back to a "known" case. For example, the étale cohomology group $H^2_{\text{ét}}(S, F)$ will involve the groups $H^p_{\text{ét}}(C, R^q f_* F)$ with $p + q = 2$. The most important term is that for $p = q = 1$, because the cohomology of a curve is interesting only in degree 1, by the previous results.

6 An application: defining L-functions

In this section we define L-functions associated to smooth projective varieties over $\mathbf{Q}$. Very roughly, the picture is as follows:

$$\begin{tikzcd}
\text{Algebraic varieties over a number field } K \arrow[d, "\text{étale cohomology}"'] \arrow[dr, dashed] & \\
& L\text{-functions} \\
\text{Galois representations of } G_K = \mathrm{Gal}(\overline{K}/K) \arrow[ur] &
\end{tikzcd}$$

Let X be a smooth connected projective algebraic variety defined over $\mathbf{Q}$. We denote by d the dimension of X, and by $f : X \to \operatorname{Spec} \mathbf{Q}$ the structural morphism.

Since f is smooth, the set of complex points $X(\mathbf{C})$ is a complex analytic manifold of dimension d, and we can look at its singular cohomology groups $H^i(X(\mathbf{C}), \mathbf{Z})$, where $0 \le i \le 2d$ is an integer. What is the analogue in étale cohomology? Choose a prime number ℓ. For every integer n, we look at the abelian sheaf $R^i f_*(\mathbf{Z}/\ell^n\mathbf{Z})$ on $\operatorname{Spec} \mathbf{Q}$. As explained in Theorem 4.27, it can be seen as a $G_{\mathbf{Q}}$-module, where $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is the absolute Galois group of $\mathbf{Q}$. In other words, we have a representation of $G_{\mathbf{Q}}$ with coefficients in $\mathbf{Z}/\ell^n\mathbf{Z}$. In fact, by the base change theorem explained above, we have

$$R^i f_*(\mathbf{Z}/\ell^n\mathbf{Z}) \cong H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Z}/\ell^n\mathbf{Z})$$

where $X_{\overline{\mathbf{Q}}}$ the base change of X to $\overline{\mathbf{Q}}$. The *ℓ-adic cohomology groups of* X are then defined as

$$H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Z}_\ell) = \varprojlim_{n \ge 1} H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Z}/\ell^n\mathbf{Z}),$$
$$H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell) = H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

A deep theorem in étale cohomology [Tam94, II.11.1] asserts that there is an isomorphism

$$H^i(X(\mathbf{C}), \mathbf{Z}/\ell^n\mathbf{Z}) \cong H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Z}/\ell^n\mathbf{Z}) \tag{28}$$

depending on the choice of an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. In particular $V_\ell = H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$ is a finite-dimensional $\mathbf{Q}_\ell$-vector space, its dimension being given by the topological Betti number $b_i = \dim_{\mathbf{Q}} H^1(X(\mathbf{C}), \mathbf{Q})$. This already gives a non-trivial link between the arithmetic and the geometry of X. But we have more: the space V_ℓ is endowed with a continuous action of $G_{\mathbf{Q}}$ (here continuous means with respect to the ℓ-adic topology on V_ℓ, not the discrete topology).

We are going to define *the L-function associated to $H^i(X)$* as a Euler product. Let p be a prime, let $D_p \subset G_{\mathbf{Q}}$ be a decomposition group at p, and let $I_p \subset D_p$ be the inertia subgroup, so that $D_p/I_p \cong \operatorname{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$. Let $\operatorname{Frob}_p \in D_p/I_p$ be the Frobenius at p. We define

$$L_p(H^i(X), t) = \det(1 - \operatorname{Frob}_p^{-1} \cdot t | V_\ell^{I_p}),$$

where ℓ is any prime $\neq p$. This is a priori a polynomial in t with coefficients in $\mathbf{Q}_\ell$.

If the variety X has good reduction at p (see below), then Deligne's proof of the Weil conjectures implies that this polynomial has coefficients in $\mathbf{Z}$ and is independent of $\ell \neq p$. In fact, in this case $V_\ell^{I_p}$ is isomorphic as a Galois module to the étale cohomology $H^i_{\text{ét}}(X_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell)$ where $X_{\mathbf{F}_p}$ is the reduction modulo p of a smooth proper model of X at p. Deligne's result is about the latter group: he shows in particular that

$$\exp\Big(\sum_{n \geq 1} \frac{|X_{\mathbf{F}_p}(\mathbf{F}_{p^n})|}{n} t^n\Big) = \prod_{i=0}^{2d} L_p(H^i(X), t)^{(-1)^{i+1}}, \tag{29}$$

where $|X_{\mathbf{F}_p}(\mathbf{F}_{p^n})|$ is the number of points of $X_{\mathbf{F}_p}$ over $\mathbf{F}_{p^n}$. This is an analogue of the Lefschetz fixed point formula for étale cohomology: we are looking at the fixed points of (powers of) the Frobenius acting on $X_{\mathbf{F}_p}(\overline{\mathbf{F}}_p)$. In a word, the polynomials $L_p(H^i(X), t)$ are essentially generating series for the numbers $|X_{\mathbf{F}_p}(\mathbf{F}_{p^n})|$.

It is conjectured that the independence of ℓ is also true if X has bad reduction at p.

Remark 6.1. Following Grothendieck, this independence of ℓ and the isomorphism (28) suggest that there should be some kind of universal cohomology theory $H^i(X)$ behind these various cohomology theories. This is the philosophy of *motives*. For smooth projective varieties, there is a well-defined abelian category of so-called Chow motives; for example $H^i(X)$ is a Chow motive which is pure of weight i. There has been much work and progress on establishing a rigorous theory of motives for arbitrary varieties or schemes, but in general the "category of motives" is still a dream which is largely conjectural.

The L-function $L(H^i(X), s)$ is then defined, for $s \in \mathbf{C}$ with $\Re(s) \gg 0$, by the Euler product

$$L(H^i(X), s) = \prod_{p \text{ prime}} \frac{1}{L_p(H^i(X), p^{-s})}.$$

So we may view the L-function of $H^i(X)$ as a way of packaging the local informations about the number of points of $X_{\mathbf{F}_p}$ for every prime p.

The Weil conjectures (more precisely, the Riemann hypothesis for the various $X_{\mathbf{F}_p}$) imply that if we remove the bad primes, then the infinite product defining $L(H^i(X), s)$ converges for $\Re(s) > i/2 + 1$.

Example 6.2. If $X = \operatorname{Spec} \mathbf{Q}$ is a point, then $L(H^0(X), s)$ is the Riemann zeta function $\zeta(s)$, because in this case the ℓ-adic cohomology V_ℓ is just $\mathbf{Q}_\ell$ with trivial Galois action.

Let's consider the next simplest example, namely the curve $\mathbf{P}^1$ over $\mathbf{Q}$. There is no cohomology in degree 1 because $\mathbf{P}^1(\mathbf{C})$ is simply connected. So let's look in degree 2. By Theorem 5.41, we have $H^2(\mathbf{P}^1_{\overline{\mathbf{Q}}}, \mu_{\ell^n}) \cong \mathbf{Z}/\ell^n\mathbf{Z}$ and thus $H^2(\mathbf{P}^1_{\overline{\mathbf{Q}}}, \mathbf{Z}/\ell^n\mathbf{Z}) \cong \mathbf{Z}/\ell^n\mathbf{Z}(-1)$. Moreover, this isomorphism is compatible with the Galois action. The Frobenius Frob_p acts as $\zeta \mapsto \zeta^p$ on the ℓ^n-th roots of unity, and thus $\mathrm{Frob}_p^{-1} = p$ on $H^2_{\text{ét}}(\mathbf{P}^1_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$. In this way, we get

$$L(H^2(\mathbf{P}^1), s) = \zeta(s-1).$$

If X is a curve defined over $\mathbf{Q}$, then by Section 5.7, the only interesting cohomology is $H^1(X)$. If $X = E$ is an elliptic curve defined over $\mathbf{Q}$, then $L(H^1(E), s)$ is the L-function $L(E, s)$ previously defined by Hasse and Weil. In general $L(H^1(X), s)$ is the Hasse-Weil zeta function $L(J, s)$ where J is the Jacobian variety of X (it is an abelian variety defined over $\mathbf{Q}$).

Conjecture 6.3. Let X be a smooth projective variety over $\mathbf{Q}$. The function $L(H^i(X), s)$ has a meromorphic continuation to $\mathbf{C}$ and satisfies a functional equation relating the values at s and $i + 1 - s$.

More precisely, one can define a completed L-function

$$\Lambda(H^i(X), s) = L_\infty(H^i(X), s) L(H^i(X), s)$$

by including an Euler factor at the archimedean prime. Then the conjectural functional equation takes the form

$$\Lambda(H^i(X), s) = \varepsilon(H^i(X), s) \cdot \Lambda(H^i(X), i + 1 - s)$$

where $\varepsilon(s)$ is of the form $a \cdot b^s$ and is called the *espilon factor of* $H^i(X)$.

This conjecture is true for Hasse-Weil zeta functions $L(E, s)$ as a consequence of the modularity theorem of Wiles, Taylor–Wiles, Breuil–Conrad–Diamond–Taylor. More precisely, take an elliptic curve E defined over $\mathbf{Q}$. Write $L(E, s)$ as a Dirichlet series $L(E, s) = \sum_{n \geq 1} a_n/n^s$. Since the polynomials L_p have integral coefficients, the a_n are integers (and there is a simple recipe to compute them given an equation of E, using (29)). Since the a_n are of arithmetic nature, $L(E, s)$ is an arithmetic object and there is a priori no obvious reason why it should have an analytic continuation. A general result about Dirichlet series tells us that we can always write them as *Mellin transforms*

$$(2\pi)^{-s}\Gamma(s)L(E, s) = \int_0^\infty f(iy) y^s \frac{dy}{y} \tag{30}$$

where $f(z) = \sum_{n=1}^\infty a_n e^{2\pi i n z}$ is an holomorphic function on the upper-half plane $\mathcal{H}$. Now the amazing (and very deep) theorem mentioned above is that f is a *modular form*. It has weight 2 and level N, where N is the conductor of E (an integer divisible exactly by the primes of bad reduction for E). The modular properties of f are enough to ensure that the integral (30) has an analytic (in fact holomorphic) continuation to the whole complex plane.

This modularity result, and thus the Conjecture above, have been generalized recently to abelian surfaces over totally real fields (and thus curves of genus 2, through their Jacobian varieties) by Boxer, Calegari, Gee and Pilloni. This involves automorphic forms on the more complicated reductive group GSp_4.

Finally, we discuss briefly Grothendieck's monodromy theorem about ℓ-adic representations attached to algebraic varieties.

Let K be a finite extension of $\mathbf{Q}_p$, and let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K. Let ℓ be a prime number $\neq p$. An *ℓ-adic representation of G_K* is a finite-dimensional $\mathbf{Q}_\ell$-vector space V endowed with a continuous action of G_K.

Definition 6.4. We say that

- V has *good reduction* if the inertia group I_K acts trivially on V;
- V has *potential good reduction* of there exists a finite extension L/K such that $V|_{G_L}$ has good reduction;
- V is *semistable* if the image of I_K in $\mathrm{GL}(V)$ is unipotent (equivalently, the semisimplification of V has good reduction);
- V is *potentially semistable* if there exists a finite extension L/K such that $V|_{G_L}$ is semistable.

The following is Grothendieck's monodromy theorem, see SGA 7.I.

Theorem 6.5. *Let X be a smooth projective algebraic variety over K. Then the ℓ-adic representations $H^i_{ét}(X_{\overline{K}}, \mathbf{Q}_\ell)$ are potentially semistable.*

There is a corresponding notion of good or semistable reduction for algebraic varieties over K. Let $\mathcal{O}_K$ be the ring of integers of K, and let k be the residue field. Let X be an algebraic variety over K. A *model of X over $\mathcal{O}_K$* is a scheme $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ whose generic fiber $\mathcal{X} \otimes K$ is isomorphic to X. If X is affine (resp. projective), then affine (resp. projective) models of X over $\mathcal{O}_K$ always exist. Namely, take equations $P_1 = \cdots = P_r = 0$ for X in $\mathbf{A}^n_K$ (resp. $\mathbf{P}^n_K$), and clear out the denominators. This defines an affine (resp. projective) scheme over $\mathcal{O}_K$ and the process doesn't alter the generic fiber. We can even take $\mathcal{X}$ to be the schematic closure[7] of X in $\mathbf{A}^n_{\mathcal{O}_K}$ (resp. $\mathbf{P}^n_{\mathcal{O}_K}$). However, the special fiber $\mathcal{X} \otimes k$ may look ugly, e.g. it may be singular, even if X is non-singular. (Exercise: find an example.)

Let X be a smooth proper (e.g. projective) algebraic variety over K. We say that:

- X has *good reduction* if X admits a smooth proper model $\mathcal{X}$ over $\mathcal{O}_K$ (in other words, we ask the special fiber $\mathcal{X} \otimes k$ to be a non-singular variety over k);
- X has *semistable reduction* if X admits a proper and flat model $\mathcal{X}$ over $\mathcal{O}_K$ which is regular and such that the special fiber $\mathcal{X} \otimes k$ is a reduced divisor with normal crossings in $\mathcal{X}$.

If X has good reduction, then the associated ℓ-adic representations $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_\ell)$ have good reduction. It is known (but much more difficult) that if X is semistable then the associated Galois representations are semistable. Several proofs are available: by the Japanese school (see Tsuji's survey in Astérisque 279), Faltings, Niziol, Beilinson.

The converse, namely whether good (resp. semistable) reduction for the ℓ-adic cohomology groups implies good (resp. semistable) reduction for the variety, is false in general. For example, if E is an elliptic curve over $\mathbf{Q}_p$ with good reduction, and X is a twisted form[8] of E with no $\mathbf{Q}_p$-rational point, then the Galois representations attached to X and E are conjugate, but X does not have good reduction.

It is expected, however, that every smooth proper variety over K has potentially semistable reduction. This is a very difficult problem in general, known only in special cases like curves (by Deligne-Mumford, see Romagny, *Models of curves*, Progress in Math. 2013) or abelian varieties (by Coleman–Iovita, Breuil).

[7]This means that $\mathcal{X}$ is defined by $I \cap \mathcal{O}_K[x_1, \ldots, x_n]$ (resp. $I \cap \mathcal{O}_K[x_0, \ldots, x_n]$), where I is the ideal defining X. In general, the schematic closure of the image of a morphism of schemes $f : X \to Y$ is the closed subscheme of Y defined by the sheaf of ideals $\ker(\mathcal{O}_Y \to f_* \mathcal{O}_X)$.

[8]This means that X and E are isomorphic over $\overline{\mathbf{Q}}_p$.

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