ON THE BORISOV–GUNNELLS RELATIONS FOR PRODUCTS OF EISENSTEIN SERIES

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Abstract. Borisov and Gunnells have proved that certain linear combinations of products of Eisenstein series are Eisenstein series themselves, in analogy with the Manin relations for modular symbols. We devise a new method to determine and prove such relations, by differentiating with respect to the parameters of the Eisenstein series.

1. Introduction

We mainly consider in this article the following Eisenstein series. Let $\mathcal{H} = \{\tau \in \mathbf{C},\ \mathrm{Im}(\tau) > 0\}$ denote the upper half-plane. For integers $k, N \geqslant 1$ and parameters $x_1, x_2 \in \mathbf{Z}/N\mathbf{Z}$, we define

$$E^{(k;N)}_{(x_1,x_2)}(\tau) = -\frac{(k-1)!}{(-2\pi i)^k} \sum_{\substack{m,n\in\mathbf{Z}\\(m,n)\neq(0,0)}} \frac{e^{\frac{2\pi i}{N}(mx_2-nx_1)}}{(m\tau+n)^k|m\tau+n|^s}\bigg|_{s=0} \qquad (\tau\in\mathcal{H}).$$

This series converges for $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 2-k$, and $\cdot|_{s=0}$ denotes the analytic continuation to $s = 0$. The series $E^{(k;N)}_{(x_1,x_2)}$ is a modular form of weight k on $\Gamma(N)$, except in the case $k = 2$ and $x_1 = x_2 = 0$ (where it is modular, but not holomorphic).

For any integers $r, s \geqslant 0$ and $a, b \in (\mathbf{Z}/N\mathbf{Z})^2$, we introduce the following symbol

$$X^rY^s[a,b] := E^{(r+1;N)}_a E^{(s+1;N)}_b. \tag{1}$$

For a homogeneous polynomial $P = \sum c_{r,s} X^rY^s$ in $\mathbf{C}[X,Y]$, we define by linearity

$$P[a,b] = \sum c_{r,s} \cdot X^rY^s[a,b]. \tag{2}$$

Note that if P is homogeneous of degree ℓ and $a, b \neq (0,0)$, then $P[a,b]$ is a modular form of weight $\ell+2$ on $\Gamma(N)$. Our main result is that these modular forms satisfy the following linear dependence relations.

Theorem 1. *For any weight $k \geqslant 2$, any integers $r, s \geqslant 0$ such that $r+s = k-2$, and any $a, b, c \in (\mathbf{Z}/N\mathbf{Z})^2$ such that $a+b+c = 0$ and $a, b, c \neq 0$, we have*

$$\begin{aligned} &X^rY^s[a,b] + (-X-Y)^rX^s[b,c] + Y^r(-X-Y)^s[c,a] \\ &\quad = \frac{(-1)^{s+1}}{s+1}E^{(k;N)}_a + \frac{(-1)^{r+1}}{r+1}E^{(k;N)}_b + (-1)^{r+s+1}\frac{r!s!}{(r+s+1)!}E^{(k;N)}_c. \end{aligned} \tag{3}$$

Note that the product of two Eisenstein series is not an Eisenstein series in general, so that (3) gives non-trivial relations in the full space of modular forms $M_k(\Gamma(N))$. In fact, under certain conditions, such products span the space of modular forms [2, 3, 8]. This has algorithmic applications to computing Fourier expansions of modular forms at the cusps [5, 6].

The relations (3) have been proved by Borisov and Gunnells [3, Theorem 6.2] in the special case of Eisenstein series on $\Gamma_1(N)$, without an explicit right-hand side. Similar relations for different Eisenstein series have been proved by Paşol [10, Theorem 3.1], and Khuri-Makdisi and Raji [9, Theorem 2.8]. We refer to Section 2 for the comparison between our Eisenstein series and those of [3, 10, 9].

Date: April 11, 2024.

The author was supported by the research project "Motivic homotopy, quadratic invariants and diagonal classes" (ANR-21-CE40-0015) operated by the French National Research Agency (ANR).

The Eisenstein series $E^{(k;N)}_{(x_1,x_2)}$ are in fact specialisations of the more general Eisenstein-Kronecker function. For any $z \in \mathbf{C}$, we denote $e(z) := \exp(2\pi i z)$. Let $k \geqslant 1$ be an integer, and $x, x_0, s \in \mathbf{C}$. In the notations of Weil [11, VII, §12], we have the Kronecker double series

$$K_k(x,x_0,s) = \sum_{\substack{w \in \mathbf{Z}+\tau\mathbf{Z} \\ w \neq -x}} e\left(\frac{w\overline{x}_0 - \overline{w}x_0}{\tau - \overline{\tau}}\right) \frac{(\overline{w}+\overline{x})^k}{|w+x|^{2s}} \qquad (\tau \in \mathcal{H}), \tag{4}$$

where the sum is extended to all $w \in \mathbf{Z}+\tau\mathbf{Z}$, except $w = -x$ if $x \in \mathbf{Z}+\tau\mathbf{Z}$. The series $K_k(x,x_0,s)$ converges for $\mathrm{Re}(s) > 1 + \frac{k}{2}$ and extends to a holomorphic function on $\mathbf{C}$ [11, VII, §13].

For an integer $k \geqslant 1$, we define the Eisenstein series $E^{(k)}_x$ with parameter $x \in \mathbf{C}$ by

$$E^{(k)}_x(\tau) = -\frac{(k-1)!}{(-2\pi i)^k} K_k(0,x,k).$$

As a function of x, the series $E^{(k)}_x$ is periodic with respect to $\mathbf{Z}+\tau\mathbf{Z}$, so that x may be considered in the complex torus $\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$. We also define symbols $X^rY^s[a,b]$ and $P[a,b]$ for $a,b \in \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ as in (1) and (2). We recover the Eisenstein series $E^{(k;N)}_{(x_1,x_2)}$ as $E^{(k)}_{(x_1\tau+x_2)/N}$ for any choice of lifts of x_1, x_2 to $\mathbf{Z}$. Theorem 1 is then a consequence of the following more general result.

Theorem 2. *For any weight $k \geqslant 2$, any integers $r,s \geqslant 0$ such that $r+s = k-2$, and any $a,b,c \in \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ such that $a+b+c = 0$ and $a,b,c \neq 0$, we have*

$$\begin{aligned} &X^rY^s[a,b] + (-X-Y)^rX^s[b,c] + Y^r(-X-Y)^s[c,a] \\ &\quad = \frac{(-1)^{s+1}}{s+1}E^{(k)}_a + \frac{(-1)^{r+1}}{r+1}E^{(k)}_b + (-1)^{r+s+1}\frac{r!s!}{(r+s+1)!}E^{(k)}_c. \end{aligned} \tag{5}$$

In the case of weight $k = 2$, this identity was proved by Zhang [12, Corollary 1.4.9] using a different method. In weight $k = 3$, the identity (5) was proved with a different method in [4], where it was crucially used to compute the regulator of elements in the K_4 group of modular curves.

The proof of Theorem 2 proceeds by induction on the weight k, using differential properties of the series $E^{(k)}_x$ explained in Section 3. Our method actually provides a way to determine the precise form of the identity (5), without knowing it beforehand.

The idea of differentiating Eisenstein series with respect to their parameters dates back to the work of Eisenstein [7, p. 223] and appears in [3, Remark 6.4] and [10]. The process here is a bit different in that we actually take primitives to get from weight k to weight $k+1$.

The Eisenstein series considered in [10, 9] are obtained via another specialisation $K_k(x,0,k)$ of the Eisenstein-Kronecker function. It might be possible to recover the identities in [10, 9] from Theorem 1 by applying a discrete Fourier transform with respect to the parameters of the Eisenstein series (see Section 2). Said differently, the explicit identities in [10] might lead to another proof of Theorem 2.

It would be interesting to try to generalise Theorem 2 to products of more than two Eisenstein series, as suggested in [3, Remark 7.15].

Acknowledgements. I am grateful to Pierre Charollois, Kamal Khuri-Makdisi and Wadim Zudilin for useful feedback on this paper.

2. Eisenstein series with continuous parameters

For $k \geqslant 1$ and $x \in \mathbf{C}$, we have the relation $K_k(0,-x,k) = (-1)^k K_k(0,x,k)$, by changing w to $-w$ in the sum over the lattice. It follows that

$$E^{(k)}_{-x} = (-1)^k E^{(k)}_x. \tag{6}$$

Writing the complex parameter x as $x_1\tau + x_2$ with $x_1, x_2 \in \mathbf{R}$, and identifying x with the row vector (x_1, x_2), we have the following modularity property [4, Lemma 35]:

$$E_x^{(k)}|_k\,\gamma = E_{x\gamma}^{(k)} \qquad (k \geqslant 1,\ \gamma \in \mathrm{SL}_2(\mathbf{Z})), \tag{7}$$

where $(f|_k\,\gamma)(\tau) = (c\tau+d)^{-k} f(\gamma\tau)$ denotes the usual weight k action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{Z})$. The identity (7) implies that $E_x^{(k)}$ is a modular form of weight k on $\Gamma(N)$ when x is in $\frac{1}{N}(\mathbf{Z} + \tau\mathbf{Z})$, except in the case $k = 2$ and $x \in \mathbf{Z} + \tau\mathbf{Z}$. The identity (7) also directly implies

$$P[a,b]|_k\,\gamma = P[a\gamma, b\gamma] \qquad (a, b \in \mathbf{C},\ \gamma \in \mathrm{SL}_2(\mathbf{Z})) \tag{8}$$

for any homogeneous polynomial $P(X,Y)$ of degree $k-2$.

We will also need the Fourier expansion of $E_x^{(k)}$ with respect to τ (see [4, Lemma 34]). For any $\tau \in \mathcal{H}$ and $\alpha \in \mathbf{R}$, we write $q^\alpha = e(\alpha\tau)$.

Lemma 1. *Let $k \geqslant 1$ be an integer, and $x = x_1\tau + x_2 \in \mathbf{C}$ with $x_1, x_2 \in \mathbf{R}$. Assume that $x \notin \mathbf{Z} + \tau\mathbf{Z}$ in the case $k = 2$. We have*

$$E_x^{(k)}(\tau) = a_0(E_x^{(k)}) - \sum_{\substack{m \geqslant 1 \\ n \in \mathbf{R}_{>0} \\ n \equiv x_1 \bmod 1}} e(mx_2) n^{k-1} q^{mn} + (-1)^{k+1} \sum_{\substack{m \geqslant 1 \\ n \in \mathbf{R}_{>0} \\ n \equiv -x_1 \bmod 1}} e(-mx_2) n^{k-1} q^{mn}, \tag{9}$$

with

$$a_0(E_x^{(1)}) = \begin{cases} 0 & \text{if } x_1 = x_2 = 0, \\ -\frac{1}{2}\frac{1+e(x_2)}{1-e(x_2)} & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ \{x_1\} - \frac{1}{2} & \text{if } x_1 \neq 0, \end{cases}$$

$$a_0(E_x^{(k)}) = \frac{B_k(\{x_1\})}{k} \qquad (k \geqslant 2),$$

where $B_k(t)$ is the k-th Bernoulli polynomial and $\{\cdot\}$ stands for the fractional part.

Finally, let us compare our Eisenstein series with those found in [3, 10, 9]. For integers $k, N \geqslant 1$ and $0 \leqslant a \leqslant N-1$, with $a \neq 0$ in the case $k = 2$, Borisov and Gunnells define an Eisenstein series $\tilde{s}_{a/N}^{(k)}$ on $\Gamma_1(N)$ in [3, Section 2.1]. From Lemma 1, we obtain

$$\tilde{s}_{a/N}^{(k)}(\tau) = -N^{k-1} E_{a,0}^{(k;N)}(N\tau).$$

Note that [3, Theorem 6.2] is then a special case of Theorem 1.

For an integer $k \geqslant 1$ and $a = (a_1, a_2) \in \mathbf{Q}^2 \smallsetminus \mathbf{Z}^2$, Paşol considers in [10, Section 2.1] the Eisenstein series $E_{k,a}$ defined by

$$E_{k,a}(\tau) = K_k(a_1\tau + a_2, 0, k).$$

It is a different specialisation of the Eisenstein-Kronecker function. It is related to our $E_x^{(k;N)}$ by a discrete Fourier transform:

$$E_{k,a} = -\frac{(2\pi i)^k N^{k-2}}{(k-1)!} \sum_{x_1, x_2 = 0}^{N-1} e(a_1x_2 - a_2x_1) E_{(x_1,x_2)}^{(k;N)} \qquad \Big(a \in \frac{1}{N}\mathbf{Z}^2 \smallsetminus \mathbf{Z}^2\Big).$$

This follows from the definition (4) of $K_k(x, x_0, k)$.

The Eisenstein series considered by Khuri-Makdisi and Raji in [9] are the same as in [10].

3. A differential relation for Eisenstein series

One crucial ingredient is the following differential property of $E_x^{(k)}$ as a function of $x \in \mathbf{C}$.

Lemma 2. *For any $k \geqslant 1$, the function $x \mapsto E_x^{(k)}(\tau)$ is smooth on $\mathbf{C} \smallsetminus (\mathbf{Z} + \tau\mathbf{Z})$. On this domain, we have*

$$-(\tau - \overline{\tau})\frac{\partial}{\partial \overline{x}} E_x^{(1)} = 1, \qquad -(\tau - \overline{\tau})\frac{\partial}{\partial \overline{x}} E_x^{(k)} = (k-1)E_x^{(k-1)} \qquad (k \geqslant 2). \tag{10}$$

Proof. The function $x \mapsto E_x^{(k)}(\tau)$ is smooth away from $\mathbf{Z}+\tau\mathbf{Z}$ by [4, Lemma 38]. Assume first $k \geqslant 2$. By [1, Lemma 1.4], the Eisenstein-Kronecker function satisfies the differential property

$$\frac{\partial}{\partial \overline{x}} K_k(0,x,s) = \frac{\pi}{\mathrm{Im}(\tau)} K_{k-1}(0,x,s-1).$$

Taking $s=k$, this implies the second formula of (10).

For $k=1$, we will argue by using the Fourier expansion of $E_x^{(1)}$ given in Lemma 1. For $x = x_1\tau + x_2$ and $0 < x_1 < 1$, we have

$$E_x^{(1)} = x_1 - \frac{1}{2} - \sum_{\substack{m \geqslant 1 \\ n \geqslant 0}} e(mx) q^{mn} + \sum_{\substack{m \geqslant 1 \\ n \geqslant 1}} e(-mx) q^{mn}.$$

It follows that

$$\frac{\partial}{\partial \overline{x}} E_x^{(1)} = \frac{\partial x_1}{\partial \overline{x}} = \frac{\partial}{\partial \overline{x}} \Big(\frac{x-\overline{x}}{\tau - \overline{\tau}} \Big) = -\frac{1}{\tau - \overline{\tau}}.$$

The general case $x \in \mathbf{C} \smallsetminus (\mathbf{Z}+\tau\mathbf{Z})$ follows by continuity. □

We now study the derivatives of the symbols $P[a,b]$. Let $k \geqslant 2$ be an integer, and V_{k-2} be the space of homogeneous polynomials in X, Y of degree $k-2$ with coefficients in $\mathbf{Q}$. Recall that for any polynomial $P = \sum c_{r,s} X^r Y^s$ in V_{k-2}, we have defined the symbol

$$P(X,Y)[a,b] = \sum_{r,s} c_{r,s} \cdot X^r Y^s [a,b] = \sum_{r,s} c_{r,s} E_a^{(r+1)} E_b^{(s+1)}.$$

Lemma 3. *For every $P \in V_{k-2}$ and every $a, b \in \mathbf{C} \smallsetminus (\mathbf{Z}+\tau\mathbf{Z})$, we have*

$$-(\tau-\overline{\tau}) \frac{\partial}{\partial \overline{a}} P[a,b] = \frac{\partial P}{\partial X}[a,b] + P(0,1) E_b^{(k-1)}$$
$$-(\tau-\overline{\tau}) \frac{\partial}{\partial \overline{b}} P[a,b] = \frac{\partial P}{\partial Y}[a,b] + P(1,0) E_a^{(k-1)}$$

Proof. The formula to be proved is linear in P, so it suffices to prove it for $P = X^r Y^s$. In the case $r \geqslant 1$, we have by Lemma 2

$$-(\tau-\overline{\tau}) \frac{\partial}{\partial \overline{a}} E_a^{(r+1)} E_b^{(s+1)} = r E_a^{(r)} E_b^{(s+1)} = r X^{r-1} Y^s [a,b] = \frac{\partial P}{\partial X}[a,b].$$

This concludes since in this case $P(0,1) = 0$. For $P = Y^{k-2}$, we have

$$-(\tau-\overline{\tau}) \frac{\partial}{\partial \overline{a}} E_a^{(1)} E_b^{(k-1)} = E_b^{(k-1)},$$

which is what we want since $\partial P / \partial X = 0$ and $P(0,1) = 1$. The formula for the derivative of $P[a,b]$ with respect to $\overline{b}$ is proved similarly. □

4. Determining the explicit form of the relations

We postulate the following shape of the Borisov-Gunnells 3-term relations:

$$P_{r,s}[a,b] + Q_{r,s}[b,-a-b] + R_{r,s}[-a-b,a] = \alpha_{r,s} E_a^{(k)} + \beta_{r,s} E_b^{(k)} + \gamma_{r,s} E_{-a-b}^{(k)} \qquad (r+s=k-2) \tag{11}$$

with $P_{r,s} = X^r Y^s$, for some polynomials $Q_{r,s}, R_{r,s}$ in V_{k-2} and some constants $\alpha_{r,s}, \beta_{r,s}, \gamma_{r,s}$ in $\mathbf{Q}$.

Differentiating (11) with respect to $\overline{a}$ and using Lemma 3, we have

$$\begin{aligned} &\frac{\partial P_{r,s}}{\partial X}[a,b] + P_{r,s}(0,1) E_b^{(k-1)} - \frac{\partial Q_{r,s}}{\partial Y}[b,-a-b] - Q_{r,s}(1,0) E_b^{(k-1)} \\ &\quad - \frac{\partial R_{r,s}}{\partial X}[-a-b,a] - R_{r,s}(0,1) E_a^{(k-1)} + \frac{\partial R_{r,s}}{\partial Y}[-a-b,a] + R_{r,s}(1,0) E_{-a-b}^{(k-1)} \\ &= (k-1)\alpha_{r,s} E_a^{(k-1)} - (k-1)\gamma_{r,s} E_{-a-b}^{(k-1)}. \end{aligned} \tag{12}$$

Note that $\partial P_{r,s}/\partial X = rX^{r-1}Y^s = rP_{r-1,s}$. Here we use the convention that any symbol $P_{r,s}, Q_{r,s}, R_{r,s}, \alpha_{r,s}, \beta_{r,s}, \gamma_{r,s}$ is zero whenever one of the indices r, s is equal to -1. This will not affect the validity of the identities below.

So in order for the identity (12) to match the one corresponding to the indices $(r-1, s)$, we should have

$$\begin{aligned} -\frac{\partial Q_{r,s}}{\partial Y} &= rQ_{r-1,s} & \Big(\frac{\partial}{\partial Y} - \frac{\partial}{\partial X}\Big)R_{r,s} &= rR_{r-1,s} \\ (k-1)\alpha_{r,s} + R_{r,s}(0,1) &= r\alpha_{r-1,s} & Q_{r,s}(1,0) - P_{r,s}(0,1) &= r\beta_{r-1,s} \\ -(k-1)\gamma_{r,s} - R_{r,s}(1,0) &= r\gamma_{r-1,s}. \end{aligned} \tag{13}$$

Similarly, differentiating (11) with respect to $\bar{b}$, we should have

$$\begin{aligned} \Big(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y}\Big)Q_{r,s} &= sQ_{r,s-1} & -\frac{\partial R_{r,s}}{\partial X} &= sR_{r,s-1} \\ R_{r,s}(0,1) - P_{r,s}(1,0) &= s\alpha_{r,s-1} & (k-1)\beta_{r,s} + Q_{r,s}(1,0) &= s\beta_{r,s-1} \\ -(k-1)\gamma_{r,s} - Q_{r,s}(0,1) &= s\gamma_{r,s-1}. \end{aligned} \tag{14}$$

We put $Q_{0,0} = R_{0,0} = 1$ and $\alpha_{0,0} = \beta_{0,0} = \gamma_{0,0} = -1$. For these values, the identities (13) and (14) are then all satisfied for $r = s = 0$. Furthermore, note that a polynomial in V_{k-2} with $k > 2$ is uniquely determined by its two partial derivatives. By induction, the polynomials $Q_{r,s}$ and $R_{r,s}$ with $(r,s) \neq (0,0)$ are therefore uniquely determined by the conditions (13) and (14). We find that the following polynomials fulfill these conditions:

$$P_{r,s} = X^r Y^s, \qquad Q_{r,s} = (-X-Y)^r X^s, \qquad R_{r,s} = Y^r(-X-Y)^s.$$

In the same way, the numbers $\alpha_{r,s}$, $\beta_{r,s}$, $\gamma_{r,s}$ with $(r,s) \neq (0,0)$ are uniquely determined by (13) and (14), and we find the following solutions:

$$\alpha_{r,s} = \frac{(-1)^{s+1}}{s+1}, \qquad \beta_{r,s} = \frac{(-1)^{r+1}}{r+1}, \qquad \gamma_{r,s} = (-1)^{r+s+1}\frac{r!s!}{(r+s+1)!}.$$

We now have the precise form of the identity to be proved; it is stated in Theorem 2.

5. Proof of the main theorem

We prove Theorem 2 by induction on the weight. For integers $r, s \geqslant 0$ and complex numbers $a, b \in \mathbf{C}$, denote by $H_{r,s}(a,b)$ the difference of the left-hand side and the right-hand side of (5). It is a $(\mathbf{Z} + \tau\mathbf{Z})$-periodic function of a and b. Fix an integer $k \geqslant 2$, and assume that Theorem 2 holds for all weights less than k. Let $r, s \geqslant 0$ such that $r + s = k - 2$. Fix $b \in \mathbf{C}$, $b \notin \mathbf{Z} + \tau\mathbf{Z}$. The computation in Section 4 and the induction hypothesis show that $H_{r,s}(a,b)$ is holomorphic in a in the domain $a \not\equiv 0, -b \bmod \mathbf{Z} + \tau\mathbf{Z}$ (this also holds in the case $r = s = 0$).

We are going to show that $a \mapsto H_{r,s}(a,b)$ extends to an holomorphic function on $\mathbf{C}$, and therefore is constant. To this end, we need to control the behaviour of $E_x^{(k)}$ as the parameter $x \in \mathbf{C}$ tends to a point of the lattice $\mathbf{Z} + \tau\mathbf{Z}$. By periodicity, it suffices to look at what happens as $x \to 0$.

Lemma 4. *If $k \geqslant 3$, then $x \mapsto E_x^{(k)}(\tau)$ is continuous at the points of $\mathbf{Z} + \tau\mathbf{Z}$.*

If $k = 2$, then for $x \to 0$, $x \neq 0$, with $x = x_1\tau + x_2$, we have

$$E_x^{(2)}(\tau) = -2G_2(\tau) + \frac{x_1}{2\pi i x} + O_{x\to 0}(x)$$

where $G_2(\tau) = -\dfrac{1}{24} + \sum_{m,n\geqslant 1} nq^{mn}$. In particular $E_x^{(2)}(\tau)$ is bounded as $x \to 0$ (but is not continuous at $x = 0$).

If $k = 1$, then for $x \to 0$, $x \neq 0$, we have

$$E_x^{(1)}(\tau) = \frac{1}{2\pi i x} + O_{x\to 0}(x).$$

Proof. If $k \geqslant 3$, then the series defining $K_k(0,x,k)$ is normally convergent, hence defines a continuous function of x on $\mathbf{C}$.

Assume $k=2$. Since $E^{(2)}_{-x} = E^{(2)}_x$ by (6), it suffices to prove the asymptotics as $x = x_1\tau + x_2$ tends to 0 with $x_1 \geqslant 0$. We assume $x_1 \in [0, \frac{1}{2}]$ in what follows. By Lemma 1, we have

$$E^{(2)}_x(\tau) = A(x_1,x_2) - x_1 \sum_{m\geqslant 1} e(mx_2) q^{mx_1}, \tag{15}$$

where

$$A(x_1,x_2) = \frac{B_2(x_1)}{2} - \sum_{m,n\geqslant 1} e(mx_2)(n+x_1)q^{m(n+x_1)} - \sum_{m,n\geqslant 1} e(-mx_2)(n-x_1)q^{m(n-x_1)}$$

is a C^∞ function of x_1, x_2, whose value at $x_1 = x_2 = 0$ is $-2G_2(\tau)$. The second term of (15), to be considered only when $x_1 > 0$, is

$$-x_1 \frac{e(x)}{1-e(x)} = -x_1\Big(-\frac{1}{2\pi i x} + O_{x\to 0}(1)\Big) = \frac{x_1}{2\pi i x} + O_{x\to 0}(x).$$

Assume $k=1$. Since $E^{(1)}_{-x} = -E^{(1)}_x$ by (6), it suffices to prove the asymptotics as $x = x_1\tau + x_2$ tends to 0 with $x_1 \geqslant 0$. We again assume $x_1 \in [0, \frac{1}{2}]$. We use the Fourier expansion of $E^{(1)}_x(\tau)$ from Lemma 1, distinguishing the cases $x_1 > 0$ and $x_1 = 0$. If $x_1 > 0$, we have

$$E^{(1)}_x(\tau) = x_1 - \frac{1}{2} - \sum_{\substack{m\geqslant 1\\ n\geqslant 0}} e(mx_2) q^{m(n+x_1)} + \sum_{\substack{m\geqslant 1\\ n\geqslant 1}} e(-mx_2) q^{m(n-x_1)}.$$

The terms of the series with $m,n \geqslant 1$ define a C^∞ function of x_1, x_2, whose value at $x_1 = x_2 = 0$ is zero. The remaining terms are

$$x_1 - \frac{1}{2} - \sum_{m\geqslant 1} e(mx) = x_1 - \frac{1}{2} - \frac{e(x)}{1-e(x)} = \frac{1}{2\pi i x} + O_{x\to 0}(x).$$

In the case $x_1 = 0$, $x_2 \notin \mathbf{Z}$, we have

$$E^{(1)}_x(\tau) = -\frac{1}{2}\cdot\frac{1+e(x_2)}{1-e(x_2)} - \sum_{m,n\geqslant 1} e(mx_2)q^{mn} + \sum_{m,n\geqslant 1} e(-mx_2) q^{mn}.$$

Again, the series over $m,n \geqslant 1$ define a C^∞ function of x_1, x_2 whose value at $x_1 = x_2 = 0$ is zero. The remaining term is

$$-\frac{1}{2}\cdot\frac{1+e(x_2)}{1-e(x_2)} = \frac{1}{2\pi i x_2} + O_{x\to 0}(x_2) = \frac{1}{2\pi i x} + O_{x\to 0}(x). \qquad \square$$

Let us go back to $H_{r,s}(a,b)$ and its behaviour as a tends to 0. Recall that $b \in \mathbf{C}$ is fixed, with $b \notin \mathbf{Z} + \tau\mathbf{Z}$. By Lemma 2, the function $a \mapsto E^{(\ell)}_{-a-b}$ is bounded as $a \to 0$ for any $\ell \geqslant 1$. So is the function $a \mapsto E^{(\ell)}_a$ for any $\ell \geqslant 2$, by Lemma 4. Therefore, it suffices to look at the contributions of $E^{(1)}_a$ in the expression of $H_{r,s}(a,b)$. This series appears only when $r = 0$, and the contribution in this case is

$$\begin{aligned} Y^{k-2}[a,b] + (-X)^{k-2}[-a-b,a] &= E^{(1)}_a E^{(k-1)}_b + (-1)^{k-2} E^{(k-1)}_{-a-b} E^{(1)}_a \\ &= E^{(1)}_a \big(E^{(k-1)}_b - E^{(k-1)}_{a+b}\big), \end{aligned} \tag{16}$$

where we use the identity $E^{(k-1)}_{-x} = (-1)^{k-1} E^{(k-1)}_x$ from (6). Using the estimate $E^{(1)}_a = O(\frac{1}{a})$ from Lemma 4 and the fact that $x \mapsto E^{(k-1)}_x$ is smooth at $x = b$ by Lemma 2, we see that the expression (16) is bounded as $a \to 0$.

The same analysis can be carried out when a tends to $-b$. The only contributing terms in $H_{r,s}(a,b)$ are those involving $E^{(1)}_{-a-b}$, namely

$$\begin{aligned} (-X)^r X^s [b,-a-b] + Y^r(-Y)^s[-a-b,a] &= (-1)^r E^{(k-1)}_b E^{(1)}_{-a-b} + (-1)^s E^{(1)}_{-a-b} E^{(k-1)}_a \\ &= (-1)^s E^{(1)}_{-a-b}\big(E^{(k-1)}_a - E^{(k-1)}_{-b}\big), \end{aligned}$$

which is again bounded as $a \to -b$.

Since a bounded elliptic function is constant, we deduce that $H_{r,s}(a,b)$ does not depend on a. Because of the symmetry $P(X,Y)[a,b] = P(Y,X)[b,a]$, we have $H_{r,s}(a,b) = H_{s,r}(b,a)$, so that $H_{r,s}(a,b)$ does not depend on b either. Let us write $f_{r,s}(\tau) = H_{r,s}(a,b)$. By the modularity property (8), we have

$$f_{r,s}|_k \gamma = H_{r,s}(a,b)|_k \gamma = H_{r,s}(a\gamma, b\gamma) = f_{r,s} \qquad (\gamma \in \mathrm{SL}_2(\mathbf{Z})),$$

so that $f_{r,s}$ is invariant under the weight k action of $\mathrm{SL}_2(\mathbf{Z})$. Moreover, when we specialise the parameters a, b to nonzero N-torsion points of $\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$, the function $P[a,b]$ is a modular form of weight k on $\Gamma(N)$. For any $N \geqslant 2$, we may take N-torsion points $a, b \neq 0$ such that $a+b \neq 0$. It follows that $f_{r,s} = H_{r,s}(a,b)$ is a modular form of weight k on $\Gamma(N)$, and therefore also on $\mathrm{SL}_2(\mathbf{Z})$.

We are finally going to specialise the parameters a, b to points of infinite order in the torus. We see from the q-expansion (9) of $E_x^{(k)}$ that except the constant term, all the exponents of q are of the form $mx_1 + n$ with $m \in \mathbf{Z} \smallsetminus \{0\}$ and $n \in \mathbf{Z}$. Now fix $a, b \in \mathbf{C}$ with $0 < a_1, b_1 < 1$ such that $(1, a_1, b_1)$ are linearly independent over $\mathbf{Q}$. Except the constant term, all the exponents of q in $P[a,b]$ are of the form $ma_1 + m'b_1 + n$ with $(m,m') \neq (0,0)$. This is still true for $H_{r,s}(a,b)$, since $c_1 \equiv -a_1 - b_1 \bmod \mathbf{Z}$. By assumption, these exponents can never be integral. But the Fourier expansion of the modular form $f_{r,s}$ involves only integer powers of q. This implies that $f_{r,s}$ is constant, and in fact $f_{r,s} = 0$ since the weight is $\geqslant 2$. This finishes the proof of Theorem 2.

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