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On the modularity of endomorphism algebras

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Summary. We show that any homomorphism between Jacobians of modular curves arises from a linear combination of Hecke modular correspondences. The proof uses the adelic language and is based on a study of the actions of GL_2 and Galois on the étale cohomology of the tower of modular curves. We also make this result explicit for Ribet's twisting operators on modular abelian varieties.

It is natural to ask whether the endomorphism algebra of the Jacobian of a modular curve is generated by the Hecke operators. Ribet [6] showed that if N is prime, the algebra $(\mathrm{End}\, J_0(N)) \otimes \mathbf{Q}$ is generated by the Hecke operators T_n with n prime to N, answering positively a question of Shimura. Mazur [5] subsequently showed an integral refinement of Ribet's result, namely that the algebra $\mathrm{End}\, J_0(N)$ is generated by the Hecke operators T_p with p prime, $p \neq N$, and by the Atkin–Lehner involution w_N.

For general N, the obvious generalisation of Ribet's result does not hold, since the Hecke operators commute with each other, while the algebra $\mathrm{End}\, J_0(N)$ is not commutative in general. The reason behind the non-commutativity is the existence of certain degeneracy operators, giving rise to old modular forms (which do not exist in the case of weight 2 and prime level). For arbitrary N, Kani [3] showed that if Γ is a congruence subgroup intermediate between $\Gamma_1(N)$ and $\Gamma_0(N)$, and J_Γ is the Jacobian of the modular curve associated to Γ, then the algebra $\mathrm{End}(J_\Gamma) \otimes \mathbf{Q}$ is generated by the Hecke operators together with explicit degeneracy operators.

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The purpose of this note is to develop an alternative approach to these questions using the adelic language. We show that after tensoring with $\mathbf{Q}$, any homomorphism between Jacobians of modular curves arises from a linear combination of Hecke modular correspondences. The cost of our abstract approach is that our results are less explicit in nature: the Hecke double coset algebra is a complicated object whose structure is not known in general. On the other hand, our results are stronger in that we consider homomorphisms instead of endomorphisms (Theorem 1), and in that we allow for homomorphisms defined over abelian number fields (Theorem 2). I hope to convince the reader that the adelic language provides a convenient point of view for studying these questions.

Ribet [7] showed that the endomorphism algebra of a modular abelian variety A_f is generated over the Hecke field of f by a finite set of endomorphisms coming from the inner twists of f. In the last section of this paper, we explain how to write these endomorphisms in terms of Hecke correspondences, giving thus some substance to Theorem 2.

1. Statement of the main result. Let $\mathbf{A}_f$ denote the ring of finite adèles of $\mathbf{Q}$, and let $G = \mathrm{GL}_2(\mathbf{A}_f)$. For any compact open subgroup K of G, let M_K denote the open modular curve over $\mathbf{Q}$ associated to K, and let $\overline{M}_K$ denote the compactification of M_K. If M_K is geometrically connected, we denote by J_K the Jacobian variety of $\overline{M}_K$.

Let K, K' be compact open subgroups of G. We denote by $\tilde{\mathbf{T}}_{K,K'}$ the free abelian group $\mathbf{Z}[K\backslash G/K']$ on $K\backslash G/K'$. In the case M_K and $M_{K'}$ are geometrically connected, we have a canonical map $\rho_J : \tilde{\mathbf{T}}_{K,K'} \to \mathrm{Hom}_{\mathbf{Q}}(J_K, J_{K'})$ (see Section 2).

THEOREM 1. *Let K, K' be compact open subgroups of G such that the modular curves M_K and $M_{K'}$ are geometrically connected. Then*

$$\rho_J(\tilde{\mathbf{T}}_{K,K'}) \otimes \mathbf{Q} = \mathrm{Hom}_{\mathbf{Q}}(J_K, J_{K'}) \otimes \mathbf{Q}.$$

REMARK 1. According to the Langlands correspondence, the Galois representations associated to algebraic varieties are expected to be automorphic. In fact, this conjectural correspondence should be functorial: not only the Galois representations, but also the morphisms between them should have an automorphic explanation. Theorem 1 can be seen as a very simple case of this principle.

REMARK 2. Let J be the Jacobian of a smooth projective curve X. It is known that every endomorphism of J arises from an effective linear combination of correspondences on X. If X is a modular curve, a general correspondence on X could arise from a cover associated to a noncongruence subgroup. Our result says that congruence subgroups are enough to generate the endomorphism algebra.

2. Modular curves in the adelic setting. Let K be a compact open subgroup of $G = \mathrm{GL}_2(\mathbf{A}_f)$. The complex points of the modular curve M_K are given by

$$M_K(\mathbf{C}) = \mathrm{GL}_2^+(\mathbf{Q})\backslash(\mathcal{H} \times G)/K$$

where $\mathrm{GL}_2^+(\mathbf{Q})$ acts on $\mathcal{H} \times G$ by $\gamma \cdot (\tau, g) = (\gamma(\tau), \gamma g)$, and K acts on G by right multiplication.

The set of connected components of $M_K(\mathbf{C})$ is in bijection with the quotient $\hat{\mathbf{Z}}^\times/\det(K)$. More precisely, let $\chi : \mathrm{Gal}(\mathbf{Q}^{\mathrm{ab}}/\mathbf{Q}) \xrightarrow{\cong} \hat{\mathbf{Z}}^\times$ denote the cyclotomic character, and let F be the finite abelian extension of $\mathbf{Q}$ associated to $\chi^{-1}(\det(K))$. Then the structural morphism $M_K \to \operatorname{Spec} \mathbf{Q}$ factors through $\operatorname{Spec} F$, and the curve M_K over $\operatorname{Spec} F$ is geometrically connected. We refer to F as the *base field* of M_K.

Let K, K' be compact open subgroups of G, and let $g \in G$. We define a correspondence $\tilde{T}(g)$ between M_K and $M_{K'}$ by the diagram

$$\begin{tikzcd} & M_{K\cap gK'g^{-1}} \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ M_K \arrow[rr, dashed, "\tilde{T}(g)"] & & M_{K'} \end{tikzcd}$$

given on the complex points by $\alpha([\tau, h]) = [\tau, h]$ and $\alpha'([\tau, h]) = [\tau, hg]$. The correspondence $\tilde{T}(g)$ extends to the compactifications and induces a map

$$T(g) = \alpha'_* \circ \alpha^* : \Omega^1(\overline{M}_K) \to \Omega^1(\overline{M}_{K'}).$$

This map depends only on the double coset KgK'. It gives rise to a map

$$\rho_\Omega : \tilde{\mathbf{T}}_{K,K'} \to \mathrm{Hom}_{\mathbf{Q}}(\Omega^1(\overline{M}_K), \Omega^1(\overline{M}_{K'}))$$

sending KgK' to $T(g)$. We let $\mathbf{T}_{K,K'} = \rho_\Omega(\tilde{\mathbf{T}}_{K,K'})$.

Assume that M_K and $M_{K'}$ are geometrically connected. For any $g \in G$, we define similarly $T(g) = \alpha'_* \circ \alpha^* : J_K \to J_{K'}$. Note that the homomorphism $T(g)$ is a priori defined over the base field of $M_{K\cap gK'g^{-1}}$, but its differential at the origin maps the tangent space $\Omega^1(\overline{M}_K)$ into $\Omega^1(\overline{M}_{K'})$, hence it is defined over $\mathbf{Q}$. We therefore get a map $\rho_J : \tilde{\mathbf{T}}_{K,K'} \to \mathrm{Hom}(J_K, J_{K'})$. Since $\mathrm{Hom}(J_K, J_{K'})$ acts faithfully on the tangent spaces, the map ρ_J factors through $\mathbf{T}_{K,K'}$. Summing up, we have a commutative diagram

$$\begin{tikzcd} \tilde{\mathbf{T}}_{K,K'} \arrow[r] \arrow[rrr, bend left, "\rho_\Omega"] & \mathbf{T}_{K,K'} \arrow[r, "\rho_J"] & \mathrm{Hom}(J_K, J_{K'}) \arrow[r, "\lambda"] & \mathrm{Hom}_{\mathbf{Q}}(\Omega^1(\overline{M}_K), \Omega^1(\overline{M}_{K'})) \end{tikzcd}$$

where λ denotes the differential at the origin.

In the case $K = K'$, we define $\tilde{\mathbf{T}}_K = \tilde{\mathbf{T}}_{K,K}$ and $\mathbf{T}_K = \mathbf{T}_{K,K}$. The convolution product endows $\tilde{\mathbf{T}}_K$ with the structure of a unitary ring. We refer to $\tilde{\mathbf{T}}_K$ as the *Hecke double coset algebra*. Note that $\tilde{\mathbf{T}}_{K,K'}$ has the structure of a $(\tilde{\mathbf{T}}_K, \tilde{\mathbf{T}}_{K'})$-bimodule.

3. Proof of Theorem 1. Consider the space

$$\Omega = \varinjlim_{K} \Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}},$$

where the direct limit is taken with respect to the pull-back maps. The space Ω is endowed with an action of G and the subspace Ω^K of K-invariants coincides with $\Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}}$. According to the multiplicity 1 theorem, the space Ω decomposes as a direct sum of distinct irreducible admissible representations of G:

$$\Omega = \bigoplus_{\pi \in \Pi} \Omega(\pi).$$

Let $\Pi(K)$ be the set of those representations $\pi \in \Pi$ satisfying $\Omega(\pi)^K \neq 0$. We have a direct sum decomposition

$$\Omega^1(\overline{M}_K) \otimes \overline{\mathbf{Q}} = \bigoplus_{\pi \in \Pi(K)} \Omega(\pi)^K,$$

and the spaces $\Omega(\pi)^K$ are pairwise non-isomorphic simple $\mathbf{T}_K \otimes \overline{\mathbf{Q}}$-modules [4, p. 393].

LEMMA 1. *The canonical map*

$$\rho_K : \mathbf{T}_K \otimes \overline{\mathbf{Q}} \to \prod_{\pi \in \Pi(K)} \mathrm{End}_{\overline{\mathbf{Q}}}(\Omega(\pi)^K)$$

is an isomorphism.

Proof. The map ρ_K is injective by definition of $\mathbf{T}_K$. The surjectivity follows from Burnside's Theorem [1, §5, N°3, Cor. 1 of Prop. 4, p. 79]. ∎

LEMMA 2. *Let K, K' be compact open subgroups of G. For any $\pi \in \Pi$, the bimodule $\mathbf{T}_{K,K'}$ maps $\Omega(\pi)^K$ into $\Omega(\pi)^{K'}$. Let $\mathcal{R} = \Pi(K) \cap \Pi(K')$. The map*

$$\rho_{K,K'} : \mathbf{T}_{K,K'} \otimes \overline{\mathbf{Q}} \to \prod_{\pi \in \mathcal{R}} \mathrm{Hom}_{\overline{\mathbf{Q}}}(\Omega(\pi)^K, \Omega(\pi)^{K'})$$

is an isomorphism of $(\mathbf{T}_K, \mathbf{T}_{K'})$-bimodules.

Proof. The map $\rho_{K,K'}$ is injective by definition of $\mathbf{T}_{K,K'}$. For the surjectivity, let K'' be a compact open subgroup of G such that $\mathcal{R} \subset \Pi(K'')$. We have a commutative diagram

$$\begin{array}{ccc}
\mathbf{T}_{K,K''} \otimes \mathbf{T}_{K'',K'} \otimes \overline{\mathbf{Q}} & \longrightarrow & \mathrm{Hom}(\Omega(\pi)^K, \Omega(\pi)^{K''}) \otimes \mathrm{Hom}(\Omega(\pi)^{K''}, \Omega(\pi)^{K'}) \\
\downarrow & & \downarrow \\
\mathbf{T}_{K,K'} \otimes \overline{\mathbf{Q}} & \longrightarrow & \mathrm{Hom}(\Omega(\pi)^K, \Omega(\pi)^{K'})
\end{array}$$

Since the right-hand map is surjective, it suffices to show that the maps $\rho_{K,K''}$ and $\rho_{K'',K'}$ are surjective. Choosing $K'' = K \cap K'$, we are reduced to showing the lemma in the cases $K' \subset K$ and $K \subset K'$. Moreover, since $\mathbf{T}_{K,K'}$ is a $(\mathbf{T}_K, \mathbf{T}_{K'})$-bimodule, and thanks to Lemma 1, it suffices to show that for any $\pi \in \mathcal{R}$, the map

$$\rho_\pi : \mathbf{T}_{K,K'} \otimes \overline{\mathbf{Q}} \to \mathrm{Hom}_{\overline{\mathbf{Q}}}(\Omega(\pi)^K, \Omega(\pi)^{K'})$$

is non-zero.

In the case $K' \subset K$, the image of the double coset $K \cdot 1 \cdot K' = K$ under ρ_π is the inclusion map of $\Omega(\pi)^K$ into $\Omega(\pi)^{K'}$, which is non-zero.

In the case $K \subset K'$, the image of the double coset $K \cdot 1 \cdot K' = K'$ under ρ_π is the trace map from $\Omega(\pi)^K$ to $\Omega(\pi)^{K'}$. Since the restriction of the trace map to $\Omega(\pi)^{K'}$ is multiplication by the index $(K' : K)$, the trace map is non-zero as required. ■

Now let us consider the direct limit of the étale cohomology groups of $\overline{M}_K$:

$$H = \varinjlim_{K} H^1_{\text{ét}}(\overline{M}_K \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Z}_\ell) \otimes \overline{\mathbf{Q}}_\ell.$$

The space H is endowed with two commuting actions of G and $\Gamma_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, and we have

$$H^K = H^1_{\text{ét}}(\overline{M}_K \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Z}_\ell) \otimes \overline{\mathbf{Q}}_\ell.$$

We will now see how to "separate" these two actions. Let us fix an embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_\ell$.

Definition 3. For any $\pi \in \Pi$, let $V_\pi = \mathrm{Hom}_G(\Omega(\pi), H)$.

Note that V_π is a $\overline{\mathbf{Q}}_\ell$-vector space endowed with an action of $\Gamma_{\mathbf{Q}}$.

Lemma 4. *The Galois representation V_π is 2-dimensional, and we have a $G \times \Gamma_{\mathbf{Q}}$-equivariant isomorphism*

$$H \cong \bigoplus_{\pi \in \Pi} \Omega(\pi) \otimes_{\overline{\mathbf{Q}}} V_\pi.$$

In particular, for any compact open subgroup K of G, we have a $\mathbf{T}_K[\Gamma_{\mathbf{Q}}]$-equivariant isomorphism

$$H^K \cong \bigoplus_{\pi \in \Pi(K)} \Omega(\pi)^K \otimes_{\overline{\mathbf{Q}}} V_\pi.$$

Proof. Let us fix an isomorphism $\overline{\mathbf{Q}}_\ell \cong \mathbf{C}$. By the comparison theorem between Betti and étale cohomology, we have

$$H \cong \varinjlim_{K} H^1_B(\overline{M}_K(\mathbf{C}), \overline{\mathbf{Q}}_\ell).$$

On the other hand, we have a $\mathbf{C}$-linear isomorphism

$$\begin{aligned}\Omega^1(\overline{M}_K(\mathbf{C})) \oplus \Omega^1(\overline{M}_K(\mathbf{C})) &\xrightarrow{\cong} H^1_B(\overline{M}_K(\mathbf{C}), \mathbf{C}),\\ (\omega, \omega') &\mapsto [\omega + c^*\omega'],\end{aligned}$$

where c denotes the complex conjugation on $\overline{M}_K(\mathbf{C})$. It follows that $H \cong (\Omega \oplus \Omega) \otimes \overline{\mathbf{Q}}_\ell$. Since

$$\mathrm{Hom}_G(\Omega(\pi), \Omega(\pi')) = \begin{cases} \overline{\mathbf{Q}} & \text{if } \pi = \pi', \\ 0 & \text{if } \pi \neq \pi', \end{cases}$$

we deduce that V_π has dimension 2. Finally, there is a canonical map $\Omega(\pi) \otimes V_\pi \to H$, and the space H decomposes as the direct sum of the images of these maps. ∎

Remark 3. The isomorphisms in Lemma 4 have motivic origin: see [9, 2.2.4, 2.2.5].

The following lemma is well-known (see the proof of [7, Thm. 4.4]).

Lemma 5. *The representation V_π is irreducible, and we have*

$$\mathrm{Hom}_{\Gamma_{\mathbf{Q}}}(V_\pi, V_{\pi'}) = \begin{cases} \overline{\mathbf{Q}}_\ell & \text{if } \pi = \pi', \\ 0 & \text{if } \pi \neq \pi'. \end{cases}$$

Proof of the main theorem. Let K, K' be compact open subgroups of G such that M_K and $M_{K'}$ are geometrically connected. Consider the composite map

$$\begin{tikzcd} \mathbf{T}_{K,K'} \otimes \overline{\mathbf{Q}}_\ell \arrow[r, "\rho_J \otimes 1"] \arrow[rr, bend left, "\rho_{\text{ét}}"] & \mathrm{Hom}(J_K, J_{K'}) \otimes \overline{\mathbf{Q}}_\ell \arrow[r, "\mu"] & \mathrm{Hom}_{\Gamma_{\mathbf{Q}}}(H^K, H^{K'}) \end{tikzcd}$$

Since these maps are injective, it suffices to show that $\rho_{\text{ét}}$ is surjective, and for this it is enough to compare the dimensions. Let $\mathcal{R} = \Pi(K) \cap \Pi(K')$. By Lemma 2, we have

$$\dim \mathbf{T}_{K,K'} = \sum_{\pi \in \mathcal{R}} (\dim \Omega(\pi)^K)(\dim \Omega(\pi)^{K'}).$$

On the other hand, using Lemmas 4 and 5, we get

$$\mathrm{Hom}_{\Gamma_{\mathbf{Q}}}(H^K, H^{K'}) = \bigoplus_{\pi \in \mathcal{R}} \mathrm{Hom}(\Omega(\pi)^K, \Omega(\pi)^{K'}) \otimes \overline{\mathbf{Q}}_\ell,$$

and thus $\dim \mathrm{Hom}_{\Gamma_{\mathbf{Q}}}(H^K, H^{K'}) = \dim \mathbf{T}_{K,K'}$ as desired. ∎

4. Generalisation to abelian extensions. Let K be a compact open subgroup of G, and let F be the base field of M_K. Let F' be a finite abelian extension of $\mathbf{Q}$ containing F, and let $U_{F'}$ be the subgroup of $U = \det(K)$

defined by $U_{F'} = \chi(\mathrm{Gal}(\mathbf{Q}^{\mathrm{ab}}/F'))$. Then we have a canonical isomorphism $M_K \otimes_F F' \cong M_{K_{F'}}$ where $K_{F'}$ is the subgroup of K defined by

$$K_{F'} = \{g \in K : \det(g) \in U_{F'}\}.$$

Let K, K' be compact open subgroups of G such that the base fields of M_K and $M_{K'}$ are equal to a fixed finite abelian extension F of $\mathbf{Q}$.

Definition 6. Let $T = (X, \alpha, \alpha')$ be a finite correspondence between $\overline{M}_K$ and $\overline{M}_{K'}$, seen as curves over $\mathbf{Q}$. We say that T *is defined over* F if the following diagram commutes:

$$\begin{tikzcd}
& X \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\
\overline{M}_K \arrow[rr, dashed, "T"] \arrow[dr, "\delta"'] & & \overline{M}_{K'} \arrow[dl, "\delta'"] \\
& \operatorname{Spec} F &
\end{tikzcd}$$

Lemma 7. *Let* $U_F = \chi(\mathrm{Gal}(\mathbf{Q}^{\mathrm{ab}}/F))$ *and let* $g \in G$. *The correspondence* $\tilde{T}(g) = KgK'$ *is defined over* F *if and only if* $\det(g) \in \mathbf{Q}_{>0} \cdot U_F$.

We denote by $\mathbf{T}_{K,K';F}$ the subgroup of $\mathbf{T}_{K,K'}$ generated by those correspondences $T(g)$ which are defined over F. Note that we have a canonical map $\rho_J : \mathbf{T}_{K,K';F} \to \mathrm{Hom}_F(J_K, J_{K'})$ where J_K (resp. $J_{K'}$) denotes the Jacobian variety of $\overline{M}_K$ (resp. $\overline{M}_{K'}$) over F.

Theorem 2. *Let* K, K' *be compact open subgroups of* $\mathrm{GL}_2(\mathbf{A}_f)$, *and let* F *be a finite abelian extension of* $\mathbf{Q}$ *containing the base fields of* M_K *and* $M_{K'}$. *Then the canonical map*

$$\rho_J : \mathbf{T}_{K_F, K'_F; F} \otimes \mathbf{Q} \to \mathrm{Hom}_F(J_K, J_{K'}) \otimes \mathbf{Q}$$

is an isomorphism.

Proof. By the above discussion, it is sufficient to prove the theorem in the case when the base fields of M_K and $M_{K'}$ are equal to F. Let $\Gamma = \mathrm{Gal}(F/\mathbf{Q})$. For any $\gamma \in \Gamma$, let $\mathbf{T}_{K,K';\gamma}$ denote the subgroup of $\mathbf{T}_{K,K'}$ generated by those correspondences $T(g)$ satisfying $\det(g) \in \mathbf{Q}_{>0} \cdot (\hat{\gamma} U_F)$, where $\hat{\gamma} \in \hat{\mathbf{Z}}^\times$ is any element satisfying $\chi^{-1}(\hat{\gamma})|_F = \gamma$. Since the elements of $\mathbf{T}_{K,K';\gamma}$ are γ-linear, we have a direct sum decomposition

$$\mathbf{T}_{K,K'} = \bigoplus_{\gamma \in \Gamma} \mathbf{T}_{K,K';\gamma}.$$

By the proof of Theorem 1, we have an isomorphism

$$\rho_{\text{ét}} : \mathbf{T}_{K,K'} \otimes \overline{\mathbf{Q}}_\ell \xrightarrow{\cong} \mathrm{Hom}_{\Gamma_{\mathbf{Q}}}(H^K, H^{K'}). \tag{1}$$

We now wish to identify those elements of $\mathrm{Hom}_{\Gamma_{\mathbf{Q}}}(H^K, H^{K'})$ which come from $\mathbf{T}_{K,K';\gamma}$. Let Σ denote the set of embeddings of F into $\overline{\mathbf{Q}}$. We have

$$\overline{M}_K \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} = \bigsqcup_{\sigma\in\Sigma} \overline{M}_K \otimes_{F,\sigma} \overline{\mathbf{Q}},$$

inducing a direct sum decomposition $H^K = \bigoplus_{\sigma\in\Sigma} H^K_\sigma$ with

$$H^K_\sigma = H^1_{\text{ét}}(\overline{M}_K \otimes_{F,\sigma} \overline{\mathbf{Q}}, \mathbf{Z}_\ell) \otimes \overline{\mathbf{Q}}_\ell.$$

Note that the action of $\Gamma_{\mathbf{Q}}$ on H^K permutes the components H^K_σ according to the rule $\gamma \cdot H^K_\sigma = H^K_{\gamma\sigma}$ for any $\gamma \in \Gamma_{\mathbf{Q}}$. Fixing an element $\sigma_0 \in \Sigma$, we have an isomorphism

$$\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbf{Q}}} H^K_{\sigma_0} \cong H^K$$

where $\Gamma_F = \mathrm{Gal}(\overline{\mathbf{Q}}/F)$. By Frobenius reciprocity, we have

$$\mathrm{Hom}_{\Gamma_{\mathbf{Q}}}(H^K, H^{K'}) \cong \mathrm{Hom}_{\Gamma_F}(H^K_{\sigma_0}, H^{K'}). \tag{2}$$

Moreover, $\mathbf{T}_{K,K';\gamma}$ maps H^K_σ into $H^{K'}_{\sigma\gamma}$. Combining the isomorphisms (1) and (2), we get

$$\mathbf{T}_{K,K';\gamma} \otimes \overline{\mathbf{Q}}_\ell \cong \mathrm{Hom}_{\Gamma_F}(H^K_{\sigma_0}, H^{K'}_{\sigma_0\gamma}) \qquad (\gamma \in G).$$

Taking $\gamma = 1$, we get a commutative diagram

$$\begin{tikzcd} \mathbf{T}_{K,K';F} \otimes \overline{\mathbf{Q}}_\ell \arrow[r, "\rho_J\otimes 1"] \arrow[rr, bend left, "\rho_{\text{ét}}"] & \mathrm{Hom}_F(J_K, J_{K'}) \otimes \overline{\mathbf{Q}}_\ell \arrow[r, "\mu"] & \mathrm{Hom}_{\Gamma_F}(H^K_{\sigma_0}, H^{K'}_{\sigma_0}) \end{tikzcd}$$

where $\rho_{\text{ét}}$ is an isomorphism. We conclude as in the proof of Theorem 1. ∎

We now give some consequences for modular abelian varieties. The following corollary shows that every endomorphism of a modular abelian variety defined over an abelian number field arises from the Hecke double coset algebra.

COROLLARY 1. *Let K be a compact open subgroup of G. Let F be a finite abelian extension of $\mathbf{Q}$ containing the base field of M_K.*

(1) *Let A/F be an abelian subvariety of J_K/F. Define*

$$\mathbf{T}_A = \{T \in \mathbf{T}_{K_F;F} : \rho_J(T) \textit{ leaves } A \textit{ stable}\}.$$

Then the canonical map $\mathbf{T}_A \otimes \mathbf{Q} \to \mathrm{End}_F(A) \otimes \mathbf{Q}$ is surjective.

(2) *Let A/F be an abelian variety which is a quotient of J_K/F. Define*

$$\mathbf{T}^A = \{T \in \mathbf{T}_{K_F;F} : \rho_J(T) \textit{ factors through } A\}.$$

Then the canonical map $\mathbf{T}^A \otimes \mathbf{Q} \to \mathrm{End}_F(A) \otimes \mathbf{Q}$ is surjective.

Proof. Let us prove (1). Let $\iota : A \to J_K$ denote the inclusion map. Let $p : J_K \to A$ be a homomorphism such that $p \circ \iota = [n]_A$ for some integer

$n \neq 0$. Let $\phi \in \mathrm{End}_F(A)$. Define $\psi = \iota \circ \phi \circ p \in \mathrm{End}_F(J_K)$. By Theorem 2, there exists $T \in \mathbf{T}_{K_F;F} \otimes \mathbf{Q}$ such that $\rho_J(T) = \psi$. Note that ψ leaves A stable, so that $T \in \mathbf{T}_A \otimes \mathbf{Q}$, and we have $\psi|_A = [n]\phi$.

The proof of (2) is similar. ∎

We emphasise that Corollary 1 is true even for elliptic curves with complex multiplication, as long as the endomorphisms we consider are defined over an abelian extension of $\mathbf{Q}$. For example, the elliptic curve $E = X_0(32)$ has complex multiplication by $\mathbf{Z}[i]$ defined over $\mathbf{Q}(i)$. Let

$$K = K_0(32)_{\mathbf{Q}(i)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbf{Z}}) : c \equiv 0 \ (32),\ ad \equiv 1 \ (4) \right\}.$$

The matrix $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 8 & 1 \end{smallmatrix}\right)$ normalises K, and the canonical map

$$\mathbf{T}_{K;\mathbf{Q}(i)} \to \mathrm{End}_{\mathbf{Q}(i)} E \cong \mathbf{Z}[i]$$

maps the coset $K\gamma K = K\gamma$ to the element $-i$. Indeed, if $f_{32} \in S_2(\Gamma_0(32))$ is the newform corresponding to E, we have $f_{32}|\gamma = -if_{32}$ [2, p. 141]. Hence $\mathbf{T}_{K;\mathbf{Q}(i)} \cong \mathrm{End}_{\mathbf{Q}(i)} E$.

To conclude this section, let me mention some open questions.

Questions.

(1) Do Theorems 1 and 2 hold integrally?
(2) Do Theorems 1 and 2 hold for modular curves in positive characteristic?
(3) The analogue of J_K in weight $k > 2$ is the (Chow) motive associated to the space of cusp forms of weight k and level K [8]. Do the results presented here extend to these motives? Do they extend to automorphic forms on more general groups?

5. Comparison with Ribet's result. Let $f = \sum_{n \geq 1} a_n q^n$ be a newform of weight 2 on $\Gamma_1(N)$, and let $A_f/\mathbf{Q}$ be the modular abelian variety associated to f. The abelian variety A_f is simple over $\mathbf{Q}$ and the algebra $\mathrm{End}_{\mathbf{Q}}(A_f) \otimes \mathbf{Q}$ is isomorphic to the Hecke field K_f of f. Ribet [7] determined the structure of the endomorphism algebra $\mathrm{End}_{\overline{\mathbf{Q}}}(A_f) \otimes \mathbf{Q}$. In particular, he proved that this algebra is generated over K_f by finitely many endomorphisms coming from the inner twists of f. Our goal in this section is to write these endomorphisms of A_f in terms of Hecke correspondences, thus making Corollary 1 explicit for these endomorphisms.

Let us first recall Ribet's construction [7, §5]. We assume that f does not have complex multiplication. Let Γ denote the set of automorphisms γ of K_f such that $f^\gamma = f \otimes \chi_\gamma$ for some Dirichlet character χ_γ. Let m denote the least common multiple of N and the conductors of the characters χ_γ. Then $h := \sum_{(n,N)=1} a_n q^n$ is an eigenform on the group $\Gamma_0(m^2) \cap \Gamma_1(m)$. Let J denote the Jacobian variety of the modular curve associated to this group. By Shimura's construction [7, §2], there exists an optimal quotient $\nu : J \to A_h$ associated to h. The abelian varieties A_f and A_h are isogenous. In particular, their endomorphism algebras are isomorphic.

For every $\gamma \in \Gamma$, Ribet constructs an endomorphism η_γ of A_h as follows. Write $f^\gamma = f \otimes \chi$. Let r denote the conductor of χ. For every $u \in \mathbf{Z}$, there is an endomorphism $\alpha_{u/r}$ of J acting on the space of cusp forms as $g \mapsto g(z + u/r)$. Define

$$\tilde{\eta}_\gamma = \sum_{u \in \mathbf{Z}/r\mathbf{Z}} \chi^{-1}(u) \circ \nu \circ \alpha_{u/r} \in \mathrm{Hom}(J, A_h) \otimes \mathbf{Q}$$

where $\chi^{-1}(u) \in K_f$ is seen as an element of $\mathrm{End}_{\mathbf{Q}}(A_h) \otimes \mathbf{Q}$. Then $\tilde{\eta}_\gamma$ factors through ν and induces an endomorphism η_γ of A_h. Since $\alpha_{u/r}$ is defined over $\mathbf{Q}(\zeta_r)$, we have $\eta_\gamma \in \mathrm{End}_{\mathbf{Q}(\zeta_r)}(A_h) \otimes \mathbf{Q}$.

Let us now turn to the adelic language. Consider the group

$$K = K_0(m^2) \cap K_1(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbf{Z}}) : c \equiv 0 \ (m^2),\ d \equiv 1 \ (m) \right\}$$

and its subgroup $K' = K_{\mathbf{Q}(\zeta_r)}$. The modular curve $\overline{M}_{K'}$ and its Jacobian $J' = J_{K'}$ are defined over the field $\mathbf{Q}(\zeta_r)$, and we have a canonical isomorphism $J' \cong J_{\mathbf{Q}(\zeta_r)}$.

Since the elements $\chi^{-1}(u)$ belong to K_f, they certainly come from the Hecke algebra of K. Therefore, there exist elements λ_u in $\mathbf{T}_{K';\mathbf{Q}(\zeta_r)} \otimes \mathbf{Q}$ such that $\rho_{J'}(\lambda_u)$ factors through A_h and induces $\chi^{-1}(u)$ on A_h.

Lemma 8. *For every $u \in \mathbf{Z}$, we have $\alpha_{u/r} = \rho_{J'}\big(T\big(\begin{smallmatrix} 1 & u/r \\ 0 & 1 \end{smallmatrix}\big)\big)$.*

Proof. By Lemma 7, the correspondence $\tilde{T}\big(\begin{smallmatrix} 1 & u/r \\ 0 & 1 \end{smallmatrix}\big)$ is defined over $\mathbf{Q}(\zeta_r)$. Moreover, $\big(\begin{smallmatrix} 1 & u/r \\ 0 & 1 \end{smallmatrix}\big)$ normalises K', so that $T\big(\begin{smallmatrix} 1 & u/r \\ 0 & 1 \end{smallmatrix}\big)$ acts on $\Omega^1(\overline{M}_{K'})$ by sending a cusp form g to $g(z + u/r)$. It follows that $\alpha^*_{u/r} = \rho_\Omega\big(\tilde{T}\big(\begin{smallmatrix} 1 & u/r \\ 0 & 1 \end{smallmatrix}\big)\big)$, proving the lemma. ∎

We now define

$$X_\gamma = \sum_{u=0}^{r-1} \lambda_u \cdot T\begin{pmatrix} 1 & u/r \\ 0 & 1 \end{pmatrix} \in \mathbf{T}_{K';\mathbf{Q}(\zeta_r)} \otimes \mathbf{Q}.$$

Proposition 9. *The endomorphism $\rho_{J'}(X_\gamma)$ factors through A_h and induces the endomorphism η_γ on A_h.*

Proof. This follows from the definition of λ_u and Lemma 8. ∎

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References

[1] N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitre 8. Modules et anneaux semi-simples*, 2nd rev. ed. of the 1958 edition, Springer, Berlin, 2012.
[2] M. Dickson and M. Neururer, *Products of Eisenstein series and Fourier expansions of modular forms at cusps*, J. Number Theory 188 (2018), 137–164.
[3] E. Kani, *Endomorphisms of Jacobians of modular curves*, Arch. Math. (Basel) 91 (2008), 226–237.
[4] R. P. Langlands, *Modular forms and ℓ-adic representations*, in: Modular Functions of One Variable, II (Antwerp, 1972), Lecture Notes in Math. 349, Springer, Berlin, 1973, 361–500.
[5] B. Mazur, *Modular curves and the Eisenstein ideal*, Publ. Math. Inst. Hautes Études Sci. 47 (1977), 33–186.
[6] K. A. Ribet, *Endomorphisms of semi-stable abelian varieties over number fields*, Ann. of Math. (2) 101 (1975), 555–562.
[7] K. A. Ribet, *Twists of modular forms and endomorphisms of abelian varieties*, Math. Ann. 253 (1980), 43–62.
[8] A. J. Scholl, *Motives for modular forms*, Invent. Math. 100 (1990),419–430.
[9] A. J. Scholl, *Integral elements in K-theory and products of modular curves*, in: The Arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci. 548, Kluwer, Dordrecht, 2000, 467–489.

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