COMPARING ELEMENTS IN K_2 OF ELLIPTIC CURVES

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In this appendix, we use results of Goncharov and Levin [5] to compare the Asakura element and the Beilinson–Kato element in K_2 of the elliptic curve $X_0(24)$. This comparison is used in the proof of Theorem 5.2.

1. Describing K_2 of elliptic curves

Goncharov and Levin gave an explicit description of K_2 of an elliptic curve using the so-called elliptic Bloch group. In this section we recall this construction and state the result needed later to compare the two elements in K_2 (see Theorem 5).

Let E be an elliptic curve defined over a number field k. To describe Quillen's K-group $K_2(E)$, our starting point is the localisation map $K_2(E) \to K_2(k(E))$, where $k(E)$ is the function field of E. The group $K_2(k(E))$ can be described using Matsumoto's theorem: for any field F, we have an isomorphism

$$K_2(F) \cong \frac{F^\times \otimes_{\mathbf{Z}} F^\times}{\langle x \otimes (1-x) : x \in F \smallsetminus \{0,1\}\rangle}.$$

The class of $x \otimes y$ in $K_2(F)$ is denoted by $\{x,y\}$ and is called a Milnor symbol. The relations $\{x, 1-x\} = 0$ are called the Steinberg relations.

Let $\mathbf{Z}[E(\overline{k})]$ be the group algebra of $E(\overline{k})$. Consider the Bloch map

$$\begin{aligned} \beta : \overline{k}(E)^\times \times \overline{k}(E)^\times &\longrightarrow \mathbf{Z}[E(\overline{k})] \\ (f,g) &\longmapsto \sum_{i,j} m_i n_j (p_i - q_j), \end{aligned}$$

where $\operatorname{div}(f) = \sum_i m_i(p_i)$ and $\operatorname{div}(g) = \sum_j n_j(q_j)$ are the divisors of f and g. The map β is bilinear, so it induces a map

$$\overline{k}(E)^\times \otimes_{\mathbf{Z}} \overline{k}(E)^\times \longrightarrow \mathbf{Z}[E(\overline{k})],$$

which we still denote by β.

Let I be the augmentation ideal of $\mathbf{Z}[E(\overline{k})]$. The group P of principal divisors on $E/\overline{k}$ is generated by the divisors

$$(p+q) - (p) - (q) + (0) = \big((p) - (0)\big) \cdot \big((q) - (0)\big) \qquad \text{with } p, q \in E(\overline{k}),$$

so we have $P = I^2$ and $I/I^2 \cong E(\overline{k})$. It follows that β takes values in I^4, and the image of β generates I^4. Following Goncharov and Levin, we now define the elliptic Bloch group of E.

Definition 1. Let $R_3^*(E/\overline{k})$ be the subgroup of $\mathbf{Z}[E(\overline{k})]$ generated by the divisors $\beta(f, 1-f)$ with $f \in \overline{k}(E)$, $f \neq 0, 1$. The elliptic Bloch group of $E/\overline{k}$ is

$$B_3^*(E/\overline{k}) = \frac{I^4}{R_3^*(E/\overline{k})}.$$

The elliptic Bloch group of E/k is

$$B_3^*(E) = B_3^*(E/\overline{k})^{\mathrm{Gal}(\overline{k}/k)}.$$

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By Matsumoto's theorem and the definition of the elliptic Bloch group, the map β gives rise to a commutative diagram

$$\begin{tikzcd} K_2(\overline{k}(E)) \arrow[r] & B_3^*(E/\overline{k}) \\ K_2(k(E)) \arrow[r] \arrow[u] & B_3^*(E). \arrow[u] \end{tikzcd}$$

Goncharov and Levin [5] proved the following fundamental result.

Theorem 2. *The composite map*

$$K_2(E) \otimes \mathbf{Q} \longrightarrow K_2(k(E)) \otimes \mathbf{Q} \longrightarrow B_3^*(E) \otimes \mathbf{Q}$$

is injective.

In fact, Goncharov and Levin showed that modulo torsion, $K_2(E)$ is isomorphic to the kernel of an explicit map

$$\delta_3 : B_3^*(E) \longrightarrow \left(\overline{k}^\times \otimes E(\overline{k})\right)^{\mathrm{Gal}(\overline{k}/k)}.$$

Proof of Theorem 2. By Quillen's localisation theorem, there is a long exact sequence

$$\cdots \longrightarrow \bigoplus_{p\in E} K_2(k(p)) \longrightarrow K_2(E) \longrightarrow K_2(k(E)) \xrightarrow{\partial} \bigoplus_{p\in E} k(p)^\times \longrightarrow \cdots,$$

where p runs over the closed points of E; see the exact sequence in the proof of [7, Theorem 5.4], with $i=2$ and $p=0$. Tensoring with $\mathbf{Q}$ and using the fact that K_2 of a number field is a torsion group [4], we get an isomorphism $K_2(E)\otimes\mathbf{Q} \cong \ker(\partial)\otimes\mathbf{Q}$.

By [5, Theorem 3.8], the natural map $K_2(k(E)) \to B_3^*(E)$ induces an isomorphism

$$\left(\frac{H^0(E,\mathcal{K}_2)}{\mathrm{Tor}(k^\times, E(k)) + K_2(k)}\right) \otimes \mathbf{Q} \cong \ker(\delta_3)\otimes\mathbf{Q},$$

where $H^0(E,\mathcal{K}_2) = \ker(\partial)$. Since $\mathrm{Tor}(k^\times, E(k))$ is a torsion group [9, Proposition 3.1.2(a)] and $K_2(k)$ is also torsion, we get the result. □

We will need to work with the full group of divisors $\mathbf{Z}[E(\overline{k})]$, using (a modification of) the group $B_3(E)$ introduced in [5, Definition 3.1]. The difference is that we use the m-distribution relations only for $m=-1$.

Definition 3. Let $R_3(E/\overline{k})$ be the subgroup of $\mathbf{Z}[E(\overline{k})]$ generated by the divisors $\beta(f,1-f)$ with $f\in\overline{k}(E)$, $f\neq 0,1$ as well as the divisors $(p)+(-p)$ with $p\in E(\overline{k})$. We define

$$B_3(E/\overline{k}) = \frac{\mathbf{Z}[E(\overline{k})]}{R_3(E/\overline{k})}, \qquad \text{and} \qquad B_3(E) = B_3(E/\overline{k})^{\mathrm{Gal}(\overline{k}/k)}.$$

Goncharov and Levin proved the following result (compare [5, Proposition 3.19(a)]).

Proposition 4. *The canonical map $B_3^*(E)\otimes\mathbf{Q} \to B_3(E)\otimes\mathbf{Q}$ is injective.*

Proof. It suffices to establish the result for $E/\overline{k}$. Let $D=\sum n_i(p_i) \in I^4$ be a divisor belonging to $R_3(E/\overline{k})$. Write $D = D' + D''$ with $D' \in R_3^*(E/\overline{k})$ and $D'' = \sum_j m_j((q_j)+(-q_j))$. The divisor D'' belongs to I^4 and is invariant under the elliptic involution $\sigma : p \to -p$ on E. Thus we can write

$$2D'' = D'' + \sigma(D'') = \beta\Big(\sum_\ell (f_\ell\otimes g_\ell) + \sigma^*(f_\ell\otimes g_\ell)\Big)$$

for some rational functions f_ℓ, g_ℓ. By [5, Lemma 3.21], for any rational functions f,g on $E/\overline{k}$, we have $\sigma^*\{f,g\} = -\{f,g\}$ in $K_2(\overline{k}(E))/\{\overline{k}^\times, \overline{k}(E)^\times\}$. It follows that $(f\otimes g)+\sigma^*(f\otimes g)$ is a linear combination of Steinberg relations and of elements $\lambda\otimes h$ with $\lambda\in\overline{k}^\times$ and $h\in\overline{k}(E)^\times$. Applying the map β and noting that $\beta(\lambda\otimes h)=0$, we get $2D''\in R_3^*(E/\overline{k})$ as desired. □

Putting together Theorem 2 and Proposition 4, we get the following result.

Theorem 5. *The composite map*

$$\overline{\beta}: \quad K_2(E)\otimes \mathbf{Q} \longrightarrow K_2(k(E))\otimes \mathbf{Q} \longrightarrow B_3(E)\otimes \mathbf{Q},$$

sending an element $\sum_i \{f_i, g_i\}$ *to the class of the divisor* $\sum_i \beta(f_i, g_i)$*, is injective.*

Thanks to Theorem 5, any equality in $K_2(E)\otimes\mathbf{Q}$ can be proved by applying the map $\overline{\beta}$ and comparing the divisors. Of course, the difficult part is to find the necessary Steinberg relations. In the following sections, we use this strategy to compare an "elliptic" and a "modular" element in K_2 of the elliptic curve $X_0(24)$.

2. Special elements in K_2 of $X_0(24)$

2.1. **The minimal model.** The curve E_4 in the Legendre family is given by the equation $y^2 = x(1-x)(1+3x)$. A minimal Weierstrass equation is

$$E : Y^2 = X^3 - X^2 - 4X + 4,$$

obtained with the change of variables $(X,Y) = (1-3x, -3y)$. This is the curve $24a1$ in the Cremona database [3]. The Néron differential is (up to sign)

$$\omega_E = \frac{dX}{2Y} = \frac{dx}{2y}.$$

The torsion subgroup of E is isomorphic to $\mathbf{Z}/4\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$, generated by the points $p_1 = (0,2)$ of order 4, and $p_2 = (1,0)$ of order 2.

2.2. **The modular parametrisation.** The curve E is in fact isomorphic to the modular curve $X_0(24)$. We denote by

$$\varphi : X_0(24) \to E$$

the modular parametrisation, normalised so that $\varphi(\infty) = 0$ and $\varphi^*(\omega_E) = \omega_f = 2\pi i f(z)dz$, where f is the unique newform of weight 2 and level $\Gamma_0(24)$.

The modular curve $X_0(24)$ has 8 cusps: $\infty, 0, \frac12, \frac13, \frac14, \frac16, \frac18, \frac1{12}$. They are all rational, and therefore correspond via φ to the 8 rationals points on E. We now make explicit this bijection. Let $\Lambda\subset\mathbf{C}$ be the lattice of periods of $\omega_E = dX/(2Y)$. We have a canonical isomorphism

$$\gamma : \mathbf{C}/\Lambda \xrightarrow{\cong} E(\mathbf{C})$$

such that $\gamma^*(\omega_E) = dz$ (so that $\gamma^{-1}(p) = \int_0^p \omega_E$). The idea is now the following: given a point $\tau\in\mathcal{H}\cup\mathbf{P}^1(\mathbf{Q})$, we have

$$\varphi(\tau) = \gamma\Big(\int_0^{\varphi(\tau)} \omega_E\Big) = \gamma\Big(\int_\infty^{\tau} \omega_f\Big).$$

The last integral, as well as the map γ, can be computed using Pari/GP [8]. Note that although the computation is numerical, we know that if τ is a cusp, then $\int_\infty^\tau \omega_f$ belongs to the lattice $\frac14\Lambda$, hence its value can be ascertained.

We used the following Pari/GP code to compute the images of the cusps of $X_0(24)$ by φ.

```
E = ellinit("24a1");
mf = mfinit([24, 2]);
f = mfeigenbasis(mf)[1];
symb = mfsymbol(mf, f);
phiE(c) = ellztopoint(E, polcoef(mfsymboleval(symb, [oo, c])*2*Pi*I, 0));
apply(phiE, [oo, 0, 1/2, 1/3, 1/4, 1/6, 1/8, 1/12])
```

The results are shown in the following table.

c	∞	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{12}$
$\varphi(c)$	0	$3p_1+p_2$	p_1+p_2	$3p_1$	$2p_1+p_2$	p_1	p_2	$2p_1$
(X,Y)	$(0:1:0)$	$(4,-6)$	$(4,6)$	$(0,-2)$	$(-2,0)$	$(0,2)$	$(1,0)$	$(2,0)$

The sign of the functional equation of the L-function $L(E,s)$ is $+1$, hence the Atkin-Lehner involution $W_{24}:\tau\to -1/(24\tau)$ satisfies $W_{24}(f)=-f$. This implies that the map $W_{24}:E\to E$ has the form $p\to p_0-p$ for some rational point p_0 on E, and the table above gives $p_0=3p_1+p_2$.

2.3. **The Beilinson–Kato element.** Recall the definition of the Beilinson–Kato element z_E in $K_2(E)\otimes\mathbf{Q}$ (see [2, Définition 9.3]):

$$z_E=\varphi_*\Big(\frac{1}{2}\{u_{24},W_{24}(u_{24})\}'\Big),$$

where u_N, for any integer N, is the following product of Siegel units

$$u_N=\prod_{\substack{a\in(\mathbf{Z}/N\mathbf{Z})^\times\\ b\in\mathbf{Z}/N\mathbf{Z}}} g_{a,b},$$

and the superscript $'$ means addition of Milnor symbols $\{\lambda,v\}$ with $\lambda\in\mathbf{Q}^\times$ and $v\in\mathcal{O}(Y_0(24))^\times$ in order to obtain an element of $K_2(X_0(24))\otimes\mathbf{Q}$. Since the symbols $\{\lambda,v\}$ are killed by β, we can safely ignore them in the computation.

We wish to compute the divisor of the modular unit u_{24}. Let us work more generally with u_N. From the definition of Siegel units as infinite products, we know that the order of vanishing of $g_{a,b}$ at the cusp ∞ of $X(N)$ is equal to $NB_2(\{\frac{a}{N}\})/2$, where $B_2(x)=x^2-x+1/6$ is the second Bernoulli polynomial and $\{t\}=t-\lfloor t\rfloor$ is the fractional part of t. Moreover, we have the transformation formula $g_{a,b}\circ\alpha=g_{(a,b)\alpha}$ in $\mathcal{O}(Y(N))^\times\otimes\mathbf{Q}$ for any $\alpha\in\mathrm{SL}_2(\mathbf{Z})$. Using this, we can compute the order of vanishing of $g_{a,b}$ at any cusp.

Since we are working with $X_0(N)$ instead of $X(N)$, we need to take into account the widths of the cusps of $X_0(N)$. The width of the cusp $1/d\in X_0(N)$ is

$$w(1/d)=\frac{N}{d\cdot\gcd(d,N/d)}.$$

Since $1/d=\left(\begin{smallmatrix}1&0\\ d&1\end{smallmatrix}\right)\infty$, we have

$$\begin{aligned}\mathrm{ord}_{1/d}(u_N)&=\sum_{\substack{a\in(\mathbf{Z}/N\mathbf{Z})^\times\\ b\in\mathbf{Z}/N\mathbf{Z}}}\mathrm{ord}_{1/d}(g_{a,b})\\ &=\sum_{\substack{a\in(\mathbf{Z}/N\mathbf{Z})^\times\\ b\in\mathbf{Z}/N\mathbf{Z}}} w(1/d)\,\mathrm{ord}_\infty(g_{a+db,b})\\ &=\frac{w(1/d)}{2}\sum_{\substack{a\in(\mathbf{Z}/N\mathbf{Z})^\times\\ b\in\mathbf{Z}/N\mathbf{Z}}} B_2\Big(\Big\{\frac{a+db}{N}\Big\}\Big)\\ &=\frac{d\varphi(N)}{2\gcd(d,N/d)\varphi(d)}\sum_{a\in(\mathbf{Z}/d\mathbf{Z})^\times} B_2\Big(\Big\{\frac{a}{d}\Big\}\Big).\end{aligned}$$

Here we used the distribution relation for the periodic Bernoulli polynomials,

$$B_n(\{mt\})=m^{n-1}\sum_{k=0}^{m-1}B_n\Big(\Big\{t+\frac{k}{m}\Big\}\Big)\qquad(m\geq 1).$$

We deduce the order of vanishing of u_{24} at each cusp of $X_0(24)$, and therefore its divisor:

$$\mathrm{div}(u_{24})=\frac{1}{6}(\infty)+\frac{2}{3}(0)-\frac{1}{3}(1/2)-\frac{2}{3}(1/3)-\frac{1}{6}(1/4)+\frac{1}{3}(1/6)-\frac{1}{6}(1/8)+\frac{1}{6}(1/12).$$

The fractions appearing here mean that u_{24} is only an element of $\mathcal{O}(Y_0(24))^\times \otimes \mathbf{Q}$, in other words some power of u_{24} is a modular unit.

Applying the modular parametrisation φ, we get

$$\operatorname{div}(u_{24}) = \frac{1}{6}\big((0)+4(3p_1+p_2)-2(p_1+p_2)-4(3p_1)+2(p_1)-(2p_1+p_2)-(p_2)+(2p_1)\big). \tag{1}$$

Applying the Atkin-Lehner involution, we also have

$$\operatorname{div}(W_{24}(u_{24})) = \frac{1}{6}\big((3p_1+p_2)+4(0)-2(2p_1)-4(p_2)+2(2p_1+p_2)-(p_1)-(3p_1)+(p_1+p_2)\big). \tag{2}$$

2.4. **The Asakura element.** It is defined by

$$\xi = \{f,g\} \in K_2(E_4)\otimes\mathbf{Q}, \qquad f = \frac{y-x+1}{y+x-1}, \qquad g = -\frac{(x-1)^2}{4x^2}. \tag{3}$$

Using MAGMA [1], we can find the divisors of f and g. Here is the code:

```
A2<x,y> := AffinePlane(Rationals());
C := Curve(A2, y^2-x*(1-x)*(1+3*x));
Cbar := ProjectiveClosure(C);
E, phi := EllipticCurve(Cbar);
Emin, psi := MinimalModel(E);
F<x,y> := FunctionField(Cbar);
f := (y-x+1)/(y+x-1);
g := -(x-1)^2/(4*x^2);
div_f := Decomposition(Divisor(f));
div_g := Decomposition(Divisor(g));
print "div(f) =", [<p[2], psi(phi(RepresentativePoint(p[1])))> : p in div_f];
print "div(g) =", [<p[2], psi(phi(RepresentativePoint(p[1])))> : p in div_g];
```

We obtain

$$\begin{aligned} \operatorname{div}(f) &= -(3p_1)+(p_1)-(3p_1+p_2)+(p_1+p_2) \\ \operatorname{div}(g) &= 4(2p_1+p_2)-4(p_2). \end{aligned} \tag{4}$$

3. Comparing the divisors

We are now going to apply $\overline{\beta}$ to the two elements in $K_2(E)\otimes\mathbf{Q}$, and compare the results.

For the Beilinson–Kato element, we find using (1) and (2) that

$$\begin{aligned} \beta(u_{24}, W_{24}(u_{24})) = \frac{1}{36}\big(&8(0)-8(p_2)+28(p_1)-28(p_1+p_2) \\ &+8(2p_1)-8(2p_1+p_2)-44(3p_1)+44(3p_1+p_2)\big). \end{aligned}$$

In the group $B_3(E)\otimes\mathbf{Q}$, we have the relation $(p)+(-p)=0$ for any point p, hence $(p)=0$ if p is 2-torsion. So we can remove the 2-torsion points from the above divisor. In fact, we can express everything in terms of p_1 and p_1+p_2 alone. We find

$$\overline{\beta}\big(\{u_{24}, W_{24}(u_{24})\}\big) = 2(p_1)-2(p_1+p_2),$$

and thus

$$\overline{\beta}(z_E) = (p_1)-(p_1+p_2). \tag{5}$$

We proceed similarly for the Asakura element. Using (4), we compute

$$\beta(f,g) = -8(p_1)-8(p_1+p_2)+8(3p_1)+8(3p_1+p_2),$$

which gives

$$\overline{\beta}(\xi) = -16(p_1)-16(p_1+p_2). \tag{6}$$

The divisors $\overline{\beta}(z_E)$ and $\overline{\beta}(\xi)$ are not proportional, which suggests that there should be a non-trivial relation involving p_1 and p_1+p_2. We can determine it experimentally by computing the elliptic dilogarithm of these points. Let us denote by $D_E : E(\mathbf{C}) \to \mathbf{R}$ the Bloch elliptic dilogarithm. Using Pari/GP, we find numerically

$$5D_E(p_1) + 3D_E(p_1 + p_2) \approx 0. \tag{7}$$

This means that we should have $5(p_1) + 3(p_1 + p_2) = 0$ in the group $B_3(E) \otimes \mathbf{Q}$. We will prove that this is indeed the case, by exhibiting a Steinberg relation.

We search for a rational function h on E such that the zeros and poles of both h and $1 - h$ are among the 8 torsion points of E. To do this, we use Mellit's technique of *incident lines* [6]; see also [5, Proof of Lemma 3.29].

We view E as a non-singular plane cubic. We generate all the lines passing only through the 8 torsion points of E. Say we have found three distinct lines ℓ_1, ℓ_2, ℓ_3 satisfying this condition and which, moreover, meet at a point p_0 of $\mathbf{P}^2$. We may choose equations f_1, f_2, f_3 for these lines satisfying $f_1 + f_2 = f_3$. Then $h = f_1/f_3$ has the property that the divisors of h and $1 - h$ are supported at the torsion points. In particular $\beta(h, 1 - h)$ is also supported at the torsion points, which gives a relation in $B_3(E) \otimes \mathbf{Q}$.

If the intersection point p_0 lies on the curve, then the above relation is trivial: it is a linear combination of divisors of the form $(p) + (-p)$. If, however, p_0 does not lie on the curve, then we usually get something interesting. It turns out that this method of incident lines works remarkably well in practice.

Using a computer, it is possible to search for all incident lines, and determine the associated Steinberg relations. In the present situation, we find the lines ℓ_1, ℓ_2 defined by the equations

$$f_1 = -\frac{1}{4}(X + Y - 2) \qquad f_2 = \frac{1}{4}(X + Y + 2).$$

We have $f_1 + f_2 = 1$, so that the lines are parallel (taking ℓ_3 to be the line at infinity, the lines ℓ_1, ℓ_2, ℓ_3 are incident, so this is a particular case of the situation above). The divisors of these functions are given by

$$\begin{aligned} \operatorname{div}(f_1) &= 2(p_1) + (2p_1) - 3(0) \\ \operatorname{div}(f_2) &= (3p_1) + (2p_1 + p_2) + (3p_1 + p_2) - 3(0) \end{aligned}$$

and the associated Steinberg relation is

$$\begin{aligned} \beta(f_1, f_2) &= 9(0) + (p_2) - 9(p_1) - 3(p_1 + p_2) - (2p_1) - (2p_1 + p_2) + (3p_1) + 3(3p_1 + p_2) \\ &\equiv -10(p_1) - 6(p_1 + p_2) \qquad \text{in } B_3(E) \otimes \mathbf{Q}. \end{aligned}$$

This shows that indeed $5(p_1) + 3(p_1 + p_2) = 0$ in $B_3(E) \otimes \mathbf{Q}$. Thus (5) and (6) simplify:

$$\overline{\beta}(z_E) = (p_1) + \frac{5}{3}(p_1) = \frac{8}{3}(p_1)$$

and

$$\overline{\beta}(\xi) = -16(p_1) - 16 \times -\frac{5}{3}(p_1) = \frac{32}{3}(p_1).$$

Using Theorem 5, we deduce that $\xi = 4z_E$ in $K_2(E) \otimes \mathbf{Q}$.

References

[1] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[2] F. Brunault. Régulateurs p-adiques explicites pour le K_2 des courbes elliptiques. In *Actes de la Conférence "Fonctions L et Arithmétique"*, volume 2010 of *Publ. Math. Besançon Algèbre Théorie Nr.*, pages 29–57. Lab. Math. Besançon, Besançon, 2010.

[3] J. E. Cremona. *Algorithms for modular elliptic curves.* Cambridge University Press, Cambridge, second edition, 1997.

[4] H. Garland. A finiteness theorem for K_2 of a number field. *Ann. of Math. (2)*, 94:534–548, 1971.

[5] A. B. Goncharov and A. M. Levin. Zagier's conjecture on $L(E, 2)$. *Invent. Math.*, 132(2):393–432, 1998.

[6] A. Mellit. Elliptic dilogarithms and parallel lines. *J. Number Theory*, 204:1–24, 2019.

[7] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[8] The PARI Group, Univ. Bordeaux. *PARI/GP version* `2.11.3`, 2020. available from `http://pari.math.u-bordeaux.fr/`.

[9] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

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