MODULAR REGULATORS AND MULTIPLE EISENSTEIN VALUES

FRANÇOIS BRUNAULT AND WADIM ZUDILIN

In memoriam: Professor Yuri Ivanovich Manin

Abstract. We compute explicitly the Goncharov regulator integral associated to K_4 classes on modular curves in terms of L-values of modular forms. We use this expression to connect it with the Beilinson regulator integral.

1. Introduction

The principal goal of this paper is to give an explicit relation between the integrals of two regulators defined on the K-group $H^2_{\mathcal{M}}(Y(N),\mathbf{Q}(n)) \cong K^{(n)}_{2n-2}(Y(N))$ in the motivic cohomology of the modular curve $Y(N)$ of full level N, in the case $n = 3$.

In the recent work [8], Brunault constructs explicit motivic cohomology classes $\xi(\boldsymbol{a},\boldsymbol{b})$ in $H^2_{\mathcal{M}}(Y(N),\mathbf{Q}(3))$ enumerated by $\boldsymbol{a},\boldsymbol{b} \in (\mathbf{Z}/N\mathbf{Z})^2$; the classes are the images of degree 2 cocycles $\tilde{\xi}(\boldsymbol{a},\boldsymbol{b})$ in the Goncharov polylogarithmic complex $\Gamma(Y(N),3)$ under De Jeu's map [13, 14, 15]. The construction uses the so-called Siegel units $g_{\boldsymbol{x}} \in \mathcal{O}(Y(N))^\times \otimes \mathbf{Q}$, $\boldsymbol{x} \in (\mathbf{Z}/N\mathbf{Z})^2$, and certain relations analogous to modular symbols involving Milnor symbols $\{g_{\boldsymbol{x}}, g_{\boldsymbol{y}}\}$ in $K_2(Y(N)) \otimes \mathbf{Q}$.

The nontriviality of $\xi(\boldsymbol{a},\boldsymbol{b})$ for small values of N is shown in [8] via computing numerically their images under the Goncharov regulator map $r_3(2)$ defined in [12]. It is harder to compute the integral of $r_3(2)(\tilde{\xi}(\boldsymbol{a},\boldsymbol{b}))$ theoretically; the existing literature lacks any such explicit calculations for the weight 3 polylogarithmic complex of curves. At the same time — and this serves as a natural motivation for these calculations — such integrals are related to (longstanding conjectural evaluations of) the Mahler measure of three-variable polynomials [10, Chapter 6]. As an example, Lalín [19] has made an explicit connection between the Mahler measure of $(1+x)(1+y)+z$ and the Goncharov regulator for the elliptic curve $(1+x)(1+y)(1+\frac{1}{x})(1+\frac{1}{y}) = 1$.

Another motivation for computing the Goncharov regulator integrals comes from a conjecture of the first author [8, Conjecture 9.3] predicting the proportionality of the Goncharov type elements $\xi((0,a),(0,b))$ and the Beilinson elements [2] in the motivic cohomology of the modular curve $Y_1(N)$. This conjecture is based on numerical computations of the associated regulator integrals. This suggests, more generally, the existence of a relation between the two integrals in the case of $Y(N)$, not just $Y_1(N)$, and possibly at the level of cocycles, not just cohomology classes. It is this task that we perform in the present paper.

In order to compare the Goncharov and Beilinson regulator integrals

$$\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) = \int_0^\infty r_3(2)(\tilde{\xi}(\boldsymbol{a},\boldsymbol{b})) \quad \text{and} \quad \mathcal{B}(\boldsymbol{a},\boldsymbol{b}) = \int_0^\infty \mathrm{Eis}^{0,0,1}_{\mathcal{D}}(\boldsymbol{a},\boldsymbol{b}),$$

where $\boldsymbol{a},\boldsymbol{b} \in (\mathbf{Z}/N\mathbf{Z})^2$, we first express $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ in terms of multiple (in fact, triple) modular values (MMV) — more specifically, multiple Eisenstein values (MEV). This step requires defining the latter objects and the corresponding regularisation of integrals along the imaginary axis $]0,i\infty[$, and setting up numerous properties and rules for MMVs. This part follows closely Brown's expositions [4, 5] which we complement with our needs in Sections 2 and 3; Section 4 serves a toy model for expressing the regulator integral on $K^{(2)}_2(Y(N))$ as a double modular value (a fact that seems to escape the literature). The MMV expression for the regulator integral $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ is computed in Section 5 for generic $\boldsymbol{a},\boldsymbol{b} \in (\mathbf{Z}/N\mathbf{Z})^2$; the result can be

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interpreted in terms of interpolated Eisenstein series, when each $\boldsymbol{a} \in (\mathbf{Z}/N\mathbf{Z})^2$ is rescaled to $\boldsymbol{a}/N \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2$ and the latter interpolates to a function of $\boldsymbol{a}$ on $(\mathbf{R}/\mathbf{Z})^2$. This line famously settled by A. Weil in [24] allows us to differentiate with respect to the (real) *elliptic* parameters $\boldsymbol{a}, \boldsymbol{b}$; more specifically, we choose to differentiate with respect to a_2. The differentiation of the Goncharov regulator integral in Section 7 is preceded, in Section 6, by derivation of auxiliary Borisov–Gunnells relations for pairwise products of Eisenstein series, and followed by reduction, in Section 8, of the resulting expression of $\frac{\partial}{\partial a_2}\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ using the Rogers–Zudilin method. Note that our proof of the Borisov–Gunnells relations requires the level N structure to be used, so that we make several switches between interpolated and non-interpolated Eisenstein series. Finally, in Section 9 we deduce an L-value expression for $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ by integrating its a_2-derivative; this brings us to the comparison of $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ with $\mathcal{B}(\boldsymbol{a},\boldsymbol{b})$ in Section 10.

Our main results can be stated precisely as follows. We need the following Eisenstein series. Given a level $N \geq 1$, a weight $k \geq 1$ and an elliptic parameter $\boldsymbol{x} = (x_1, x_2)$ in $(\mathbf{Z}/N\mathbf{Z})^2$, we define as in [10, Section 10.4]

$$(1) \qquad G_{\boldsymbol{x}}^{(k);N}(\tau) = a_0(G_{\boldsymbol{x}}^{(k);N}) + \sum_{\substack{m,n\geq 1 \\ (m,n)\equiv \boldsymbol{x} \bmod N}} m^{k-1}q^{mn/N} + (-1)^k \sum_{\substack{m,n\geq 1 \\ (m,n)\equiv -\boldsymbol{x} \bmod N}} m^{k-1}q^{mn/N},$$

where the constant term is given by

$$a_0(G_{\boldsymbol{x}}^{(1);N}) = \begin{cases} -B_1(\{\frac{x_2}{N}\}) & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ -B_1(\{\frac{x_1}{N}\}) & \text{if } x_1 \neq 0 \text{ and } x_2 = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for $k \geq 2$,

$$a_0(G_{\boldsymbol{x}}^{(k);N}) = \begin{cases} -N^{k-1}B_k(\{\frac{x_1}{N}\})/k & \text{if } x_2 = 0, \\ 0 & \text{if } x_2 \neq 0. \end{cases}$$

Here $B_k(t)$ is the k-th Bernoulli polynomial (in particular $B_1(t) = t - \frac{1}{2}$), and $\{\cdot\}$ stands for the fractional part. The function $G_{\boldsymbol{x}}^{(k);N}$ is an Eisenstein series of weight k and level $\Gamma(N)$, except for the case $k = 2$ and $x_1 = 0$. Given a modular form $f = \sum_{n\geq 0} a_n q^{n/N}$ on $\Gamma(N)$, we write $L(f,s) = \sum_{n\geq 1} a_n (n/N)^{-s}$ for (the analytic continuation of) the L-function of f.

Theorem 1. *For any* $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{b} = (b_1, b_2)$ *in* $(\mathbf{Z}/N\mathbf{Z})^2$ *such that the coordinates of* $\boldsymbol{a}$, $\boldsymbol{b}$ *and* $\boldsymbol{a}+\boldsymbol{b}$ *are non-zero, we have*

$$\begin{aligned} \mathcal{G}(\boldsymbol{a},\boldsymbol{b}) = \frac{3\pi^2}{N} L'\big(G_{a_1,b_2}^{(1);N} G_{b_1,-a_2}^{(1);N} + G_{a_1,-b_2}^{(1);N} G_{b_1,a_2}^{(1);N}, -1\big) \\ - \frac{\zeta(3)}{4}\big(B_2(\{\tfrac{a_1}{N}\}) + B_2(\{\tfrac{b_1}{N}\}) + 4B_1(\{\tfrac{a_1}{N}\})B_1(\{\tfrac{b_1}{N}\}) \\ - B_2(\{\tfrac{a_2}{N}\}) - B_2(\{\tfrac{b_2}{N}\}) - 4B_1(\{\tfrac{a_2}{N}\})B_1(\{\tfrac{b_2}{N}\})\big). \end{aligned}$$

In his PhD thesis, Weijia Wang has made explicit Beilinson's theorem, by computing $\mathcal{B}(\boldsymbol{a},\boldsymbol{b})$ using the Rogers–Zudilin method [23, Théorème 0.1.3]. The resulting L-value turns out to match the one in Theorem 1. We deduce our second main result, which is an explicit connection between $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ and $\mathcal{B}(\boldsymbol{a},\boldsymbol{b})$.

Theorem 2. *For any* $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{b} = (b_1, b_2)$ *in* $(\mathbf{Z}/N\mathbf{Z})^2$ *such that the coordinates of* $\boldsymbol{a}$, $\boldsymbol{b}$ *and* $\boldsymbol{a}+\boldsymbol{b}$ *are non-zero, we have*

$$\begin{aligned} \mathcal{G}(\boldsymbol{a},\boldsymbol{b}) = \frac{N^2}{6}\mathcal{B}(\boldsymbol{a},\boldsymbol{b}) - \frac{\zeta(3)}{4}\big(B_2(\{\tfrac{a_1}{N}\}) + B_2(\{\tfrac{b_1}{N}\}) + 4B_1(\{\tfrac{a_1}{N}\})B_1(\{\tfrac{b_1}{N}\}) \\ - B_2(\{\tfrac{a_2}{N}\}) - B_2(\{\tfrac{b_2}{N}\}) - 4B_1(\{\tfrac{a_2}{N}\})B_1(\{\tfrac{b_2}{N}\})\big). \end{aligned}$$

This gives some evidence for [8, Conjecture 9.3] asserting the proportionality of the motivic cohomology classes $\xi(\boldsymbol{a},\boldsymbol{b})$ and $\mathrm{Eis}^{0,0,1}(\boldsymbol{a},\boldsymbol{b})$ — this was formulated for $Y_1(N)$, but we expect it to hold also for $Y(N)$. The discrepancy appearing with the rational multiple of $\zeta(3)$ may come from the particular choices of representatives of the Deligne–Beilinson cohomology classes, since $]0,i\infty[$ is not a closed path in $Y(N)(\mathbf{C})$.

Our strategy and its execution reveal several interesting arithmetic phenomena and prospects for the general K-groups $K_{2n-2}^{(n)}(Y(N))$ with $n \geq 2$. First of all, we find the theory of multiple modular values developed by Brown [4, 5], specifically of multiple Eisenstein values (MEVs), intrinsic to dealing with both the L-values $L(E,n)$ of modular elliptic curves E and regulators of Beilinson and Goncharov types. One may hope that if E has conductor N, then $L(E,n)$ can be always written as a $\mathbf{Q}$-linear combination of length n MEVs with Eisenstein series of weight 2 and level N. This should be explained by a relation between the Goncharov regulator $r_n(2)$ and iterated integrals of length n.

In contrast, the Beilinson regulator produces MEVs of length 2, with weights of Eisenstein series depending on n, and this corresponds to a representation of $L(E,n)$ as a $\mathbf{Q}$-linear combination of length 2 MEVs. The difference in production from the two regulators suggests the existence of intermediate regulators in the case $n \geq 4$, to cover the entire spectrum of possibilities of MEVs. At the moment we can only speculate in this direction. Notice that representativeness of $L(E,n)$ by different length MEVs seems to be part of some general structure; this indicates existence of possible 'length drops' for MMVs themselves. Our calculation of $\frac{\partial}{\partial a_2}\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ in Sections 6–8 gives an example of such a length drop by 1. Are there identities of MMVs in which the length drops by 2 or more? Does a general theory for length reduction exist? Answering such questions will help to understand the cases with $n \geq 4$.

Most of our results in Sections 6–8 are limited to the situations required for dealing with the Goncharov regulator $r_n(2)$ when $n = 3$ but can be potentially generalised. Our Theorem 61 below is already more general than needed in this paper but can be generalised further; the Borisov–Gunnells relations exist in arbitrary weight. Differentiation of such relations with respect to elliptic parameters was already used by Borisov and Gunnells in [3, Section 3], though with no connection to computing regulators or MMVs.

Our final remark is that writing $r_3(2)$ in terms of MMVs provides one with an efficient way for computing the Goncharov regulator, which is faster when compared with the method used in [8].

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Iterated integrals of modular forms appear intrinsically in the study of modular regulators and we feel appropriate to dedicate our work to Yuri Manin, who pioneered this topic in [20, 21]. We would benefit from discussing our results with him. But he passed away unexpectedly, full of many ideas that our mathematics world could have grown further on.

2. Regularised iterated integrals

2.1. Admissible functions. We define the class of functions and differential forms that we wish to integrate. Let $\mathcal{H} = \{\tau \in \mathbf{C} : \mathrm{Im}(\tau) > 0\}$ be the upper half-plane, and $]0,i\infty[= \{iy : y > 0\}$ the imaginary axis.

Definition 3 (Admissibility at infinity)**.** A C^∞ function $f\colon]0,i\infty[\to \mathbf{C}$ is called *admissible at* ∞ if it can be written $f(\tau) = f^\infty(\tau) + f^0(\tau)$, where $f^\infty(\tau) \in \mathbf{C}[\tau]$ is a polynomial, and $f^0(\tau)$ has exponential decay as $\mathrm{Im}(\tau) \to +\infty$: there exists $0 < c < 1$ such that $f^0(\tau) = O_{\tau\to\infty}(c^{\mathrm{Im}(\tau)})$. In this case, the *regularised value of* f *at infinity*, denoted by $f(\infty)$, is defined as the constant term of the polynomial f^∞.

Note that the decomposition $f = f^\infty + f^0$ is unique, hence $f(\infty)$ is well defined.

Definition 4. A C^∞ differential form $\omega = f(\tau)\,d\tau$ on $]0, i\infty[$ is called *admissible at* ∞ if f is admissible at ∞. We then write $\omega = \omega^\infty + \omega^0$ with $\omega^\infty = f^\infty(\tau)\,d\tau$ and $\omega^0 = f^0(\tau)\,d\tau$.

As an example, if f is a modular form of weight $k \geq 1$ on some finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$, then $\omega = f(\tau)\tau^m\,d\tau$ is admissible at ∞ for any integer $m \geq 0$. Note that if a form ω is admissible at ∞, then so are $\mathrm{Re}(\omega) = \frac{1}{2}(\omega + \bar{\omega})$ and $\mathrm{Im}(\omega) = \frac{1}{2i}(\omega - \bar{\omega})$.

Lemma 5. *If a function f and a form ω on $]0, i\infty[$ are admissible at ∞, then so is $f\omega$.*

2.2. **Regularisation at infinity.** We now come to regularisation of iterated integrals. We follow Brown's definition [4, Section 4.1] and show how it can be expressed via successive one-variable regularisations.

Let us first consider the case of a single integral from τ to ∞. Let ω be a differential form on $]0, i\infty[$ which is admissible at ∞. Brown's definition translates to

$$\int_\tau^\infty \omega := \lim_{p\to\infty} \int_\tau^p \omega + \int_p^0 \omega^\infty \qquad (p = iy,\ y \to +\infty). \tag{2}$$

We can actually get rid of the limit in (2).

Lemma 6. *Let ω be a differential form on $]0, i\infty[$ which is admissible at ∞. The limit in (2) exists, and we have*

$$\int_\tau^\infty \omega = \int_\tau^\infty \omega^0 + \int_\tau^0 \omega^\infty. \tag{3}$$

Moreover, the error term in the convergence of (2) is $O_{p\to\infty}(c^{\mathrm{Im}(p)})$ with $0 < c < 1$, the constant c being uniform with respect to τ on domains of the form $\{\mathrm{Im}(\tau) \geq y_0 > 0\}$.

Proof. Indeed,

$$\int_\tau^p \omega + \int_p^0 \omega^\infty = \int_\tau^p \omega^0 + \int_\tau^p \omega^\infty + \int_p^0 \omega^\infty = \int_\tau^p \omega^0 + \int_\tau^0 \omega^\infty. \qquad \square$$

We refer to the right-hand side of (3) as the practical regularised integral. Note that the regularised integral recovers the classical integral in the case ω is integrable on $[\tau, i\infty[$ (which happens if and only if $\omega^\infty = 0$). Lemma 6 has the following consequence.

Lemma 7. *Let ω be a differential form on $]0, i\infty[$ which is admissible at ∞. Then the function $F(\tau) = -\int_\tau^\infty \omega$ is admissible at ∞. Moreover, F is the unique primitive of ω whose regularised value at ∞ is zero.*

In particular, if a form ω is admissible at ∞, then any primitive of ω is again admissible at ∞. On the other hand, the differential of an admissible function f need not be admissible, because there is no control on the derivative of f^0.

Lemma 8. *Let $f\colon]0, i\infty[\to \mathbf{C}$ be a function such that df is admissible at ∞. Then f is admissible at ∞ and $\int_\tau^\infty df = f(\infty) - f(\tau)$, where $f(\infty)$ is the regularised value at ∞ as in Definition 3.*

Proof. This follows from Lemma 7 applied to $\omega = df$. $\square$

One should be careful that in general $\int_\tau^\infty \omega$ does not converge to zero as $\tau \to \infty$: this can be seen from (3). For example, if $f(\tau) = \sum_{n\geq 0} a_n q^n$ is a modular form, then

$$\int_\tau^\infty f(\tau_1)\,d\tau_1 = -\frac{1}{2\pi i}\sum_{n\geq 1}\frac{a_n}{n}q^n - a_0\tau.$$

One outcome of Lemma 8 is the following formula for integration by parts: if the forms df and dg are admissible, then f and g are admissible as well, and

$$\int_\tau^\infty \frac{df}{d\tau}(\tau_1)g(\tau_1)\,d\tau_1 = f(\infty)g(\infty) - f(\tau)g(\tau) - \int_\tau^\infty f(\tau_1)\frac{dg}{d\tau}(\tau_1)\,d\tau_1. \tag{4}$$

Once again, here $f(\infty)$ and $g(\infty)$ are the regularised values at ∞ as in Definition 3.

Now let us consider the case of iterated integrals. Brown's definition [4, Section 4.1] uses a tangential base point at ∞. This intrinsic definition has the advantage of giving naturally the shuffle relations for the regularised iterated integrals. Unraveling Brown's definition gives:

Definition 9. Let $\omega_1, \dots, \omega_n$ be differential forms on $]0, i\infty[$ which are admissible at ∞. Define

$$\int_\tau^\infty \omega_1 \dots \omega_n := \lim_{p \to \infty} \sum_{k=0}^{n} \int_\tau^p \omega_1 \dots \omega_k \times \int_p^0 \omega_{k+1}^\infty \dots \omega_n^\infty. \tag{5}$$

We will justify below the convergence in (5). For certain computations, we will need to express the regularised iterated integral as a succession of one-variable regularised integrals. We introduce the following 'naïve' regularisation:

$$\int_\tau^{\infty,*} \omega_1 \dots \omega_n := \int_\tau^\infty \omega_1(\tau_1) \int_{\tau_1}^\infty \omega_2(\tau_2) \cdots \int_{\tau_{n-1}}^\infty \omega_n(\tau_n), \tag{6}$$

where the right-hand integrals are understood as (3).

Lemma 10. *The naïve regularised integral $\int_\tau^{\infty,*} \omega_1 \dots \omega_n$ is well-defined and is admissible at ∞ as a function of τ. Its regularised value at ∞ is zero.*

Proof. This follows from inductive application of Lemmas 5 and 7. □

Proposition 11. *Let $\omega_1, \dots, \omega_n$ be differential forms which are admissible at ∞. Then*

$$\int_\tau^\infty \omega_1 \dots \omega_n = \int_\tau^{\infty,*} \omega_1 \dots \omega_n.$$

To prove this, we need the following lemma.

Lemma 12. *The polynomial part of the naïve regularised integral is given by*

$$\left(\int_\tau^{\infty,*} \omega_1 \dots \omega_n \right)^\infty = \int_\tau^0 \omega_1^\infty \dots \omega_n^\infty,$$

where the right-hand side is the usual (absolutely convergent) iterated integral.

Proof. We proceed by induction on n. The case $n = 1$ follows from Lemma 6. For $n \geq 2$, we have

$$\int_\tau^{\infty,*} \omega_1 \dots \omega_n = \int_\tau^{\infty,*} \omega_1(\tau_1) \int_{\tau_1}^{\infty,*} \omega_2 \dots \omega_n.$$

By the induction hypothesis applied to $\omega_2 \dots \omega_n$, we have

$$\left(\omega_1(\tau_1) \int_{\tau_1}^{\infty,*} \omega_2 \dots \omega_n \right)^\infty = \omega_1^\infty(\tau_1) \left(\int_{\tau_1}^{\infty,*} \omega_2 \dots \omega_n \right)^\infty = \omega_1^\infty(\tau_1) \int_{\tau_1}^0 \omega_2^\infty \dots \omega_n^\infty.$$

Therefore, using Lemma 6,

$$\int_\tau^{\infty,*} \omega_1 \dots \omega_n = \int_\tau^\infty \left(\omega_1(\tau_1) \int_{\tau_1}^{\infty,*} \omega_2 \dots \omega_n \right)^0 + \int_\tau^0 \omega_1^\infty(\tau_1) \int_{\tau_1}^0 \omega_2^\infty \dots \omega_n^\infty. \tag{7}$$

The first term in (7) decays exponentially as $\tau \to \infty$, and the second term is a polynomial in τ, which finishes the proof. □

Proposition 11 is now a consequence of the following finer result, which controls the convergence as $p \to \infty$.

Proposition 13. *We have*

$$\sum_{k=0}^{n} \int_\tau^p \omega_1 \dots \omega_k \times \int_p^0 \omega_{k+1}^\infty \dots \omega_n^\infty = \int_\tau^{\infty,*} \omega_1 \dots \omega_n + O_{p \to \infty}(c^{\mathrm{Im}(p)}) \qquad (0 < c < 1), \tag{8}$$

the constant c being uniform with respect to τ on domains of the form $\{\mathrm{Im}(\tau) \geq y_0 > 0\}$.

Proof. We proceed by induction on n. The case $n=1$ follows from Lemma 6. Let $n \geq 2$. Using the induction hypothesis to $\omega_2 \dots \omega_n$, the left-hand side of (8) can be written as

$$\begin{aligned}
&\int_\tau^p \omega_1(\tau_1)\Big(\sum_{k=1}^n \int_{\tau_1}^p \omega_2 \dots \omega_k \times \int_p^0 \omega_{k+1}^\infty \dots \omega_n^\infty\Big) + \int_p^0 \omega_1^\infty \dots \omega_n^\infty \\
&= \int_\tau^p \omega_1(\tau_1)\Big(\int_{\tau_1}^{\infty,*} \omega_2 \dots \omega_n + O_{p\to\infty}(c^{\mathrm{Im}(p)})\Big) + \int_p^0 \omega_1^\infty \dots \omega_n^\infty \\
&= \int_\tau^p \omega_1(\tau_1) \int_{\tau_1}^{\infty,*} \omega_2 \dots \omega_n + \Big(\int_\tau^p \omega_1(\tau_1)\Big) O_{p\to\infty}(c^{\mathrm{Im}(p)}) + \int_p^0 \omega_1^\infty \dots \omega_n^\infty \\
&= \int_\tau^p \omega_1(\tau_1) \int_{\tau_1}^{\infty,*} \omega_2 \dots \omega_n + \int_p^0 \omega_1^\infty \dots \omega_n^\infty + O_{p\to\infty}(c_2^{\mathrm{Im}(p)}). \qquad (9)
\end{aligned}$$

Consider the differential form

$$\alpha(\tau_1) = \omega_1(\tau_1) \int_{\tau_1}^{\infty,*} \omega_2 \dots \omega_n.$$

Applying Lemma 12 to $\omega_2 \dots \omega_n$, the polynomial part of α is

$$\alpha^\infty(\tau_1) = \omega_1^\infty(\tau_1) \int_{\tau_1}^0 \omega_2^\infty \dots \omega_n^\infty.$$

Therefore,

$$\begin{aligned}
(9) &= \int_\tau^p \alpha(\tau_1) + \int_p^0 \omega_1^\infty(\tau_1) \int_{\tau_1}^0 \omega_2^\infty \dots \omega_n^\infty + O_{p\to\infty}(c_2^{\mathrm{Im}(p)}) \\
&= \int_\tau^p \alpha(\tau_1) + \int_p^0 \alpha^\infty(\tau_1) + O_{p\to\infty}(c_2^{\mathrm{Im}(p)}) \\
&= \int_\tau^{\infty,*} \alpha + O_{p\to\infty}(c_3^{\mathrm{Im}(p)}).
\end{aligned}$$

□

Proposition 11 and Lemma 7 have the following consequence.

Lemma 14. *For any differential forms $\omega_1, \dots, \omega_n$ which are admissible at ∞, we have*

$$d\Big(\int_\tau^\infty \omega_1 \dots \omega_n\Big) = -\omega_1(\tau) \int_\tau^\infty \omega_2 \dots \omega_n.$$

2.3. **Regularisation at zero.** The matrix $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ acts on $\mathcal{H}$ by $\tau \mapsto -1/\tau$. For a differential form ω on $]0, i\infty[$, we write $\omega^\sigma = \sigma^*\omega$.

Definition 15 (Admissibility at 0)**.** A C^∞ function $f\colon]0, i\infty[\to \mathbf{C}$ is called *admissible at* 0 if the function $g(\tau) = f(-1/\tau)$ is admissible at ∞. In this case, the *regularised value of f at* 0 is defined as $f(0) = g(\infty)$.

A C^∞ differential form ω on $]0, i\infty[$ is called *admissible at* 0 if ω^σ is admissible at ∞.

Definition 16 (Admissibility)**.** A function or differential form on $]0, i\infty[$ is called admissible if it is admissible at both 0 and ∞.

Example 17.
- The only polynomials in τ which are admissible are the constants.
- If f is a modular form of weight $k \geq 2$ on a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$, then $\omega = f(\tau)\,d\tau$ is admissible. In fact $f(\tau)\tau^{m-1} d\tau$ is admissible for any $m \in \{1, \dots, k-1\}$. If f is a cusp form, then $f(\tau)\tau^{m-1} d\tau$ is admissible for any $m \in \mathbf{Z}$.

Lemmas 5 and 7 also hold for admissibility at 0, and thus for admissibility:

Lemma 18. *Let ω be an admissible differential form on $]0, i\infty[$, and $f(\tau)$ any primitive of ω. Then f is admissible.*

We now want to define the regularised iterated integral from 0 to τ of differential forms $\omega_1, \dots, \omega_n$ which are admissible at 0. We begin with the case $n = 1$. Formal considerations lead to the following definition.

Definition 19. Let ω be a differential form on $]0, i\infty[$ which is admissible at 0. We set

$$\int_0^\tau \omega := -\int_{-1/\tau}^{\infty} \omega^\sigma,$$

which is well-defined since ω^σ is admissible at ∞.

Lemma 20. *Let ω be a differential form on $]0, i\infty[$ which is admissible at 0. Then $\int_0^\tau \omega$ is the unique primitive of ω whose regularised value at 0 is zero.*

Proof. By Lemma 7 applied to ω^σ, we know that $d\left(\int_\tau^\infty \omega^\sigma\right) = -\omega^\sigma$. Pulling back by $\sigma\colon \tau \mapsto -1/\tau$ gives the desired identity. The statement about the regularised value at 0 follows from the definition and Lemma 7. □

Now we proceed to the iterated case. Let $\omega_1, \dots, \omega_n$ be differential forms on $]0, i\infty[$ which are admissible at 0. We want to set

$$\int_0^\tau \omega_1 \dots \omega_n = \int_\infty^{-1/\tau} \omega_1^\sigma \dots \omega_n^\sigma.$$

The right-hand side can be given a meaning using the reversal of paths formula

$$\int_a^b \omega_1 \dots \omega_n = (-1)^n \int_b^a \omega_n \dots \omega_1.$$

This leads to:

Definition 21. For any forms $\omega_1, \dots, \omega_n$ on $]0, i\infty[$ which are admissible at 0, we define

$$\int_0^\tau \omega_1 \dots \omega_n := (-1)^n \int_{-1/\tau}^\infty \omega_n^\sigma \dots \omega_1^\sigma.$$

We have the following analogues of Lemmas 10 and 14.

Lemma 22. *The integral $\int_0^\tau \omega_1 \dots \omega_n$ is admissible at 0 as a function of τ, and its regularised value at 0 is zero.*

Proof. This follows from Lemma 10 applied to $\omega_n^\sigma \dots \omega_1^\sigma$. □

Lemma 23. *We have*

$$d\left(\int_0^\tau \omega_1 \dots \omega_n\right) = \omega_n(\tau) \int_0^\tau \omega_1 \dots \omega_{n-1}.$$

Proof. By Lemma 14, we have

$$d\left(\int_\tau^\infty \omega_n^\sigma \dots \omega_1^\sigma\right) = -\omega_n^\sigma(\tau) \int_\tau^\infty \omega_{n-1}^\sigma \dots \omega_1^\sigma.$$

Applying σ^* to this identity gives

$$d\left(\int_{-1/\tau}^\infty \omega_n^\sigma \dots \omega_1^\sigma\right) = -\omega_n(\tau) \int_{-1/\tau}^\infty \omega_{n-1}^\sigma \dots \omega_1^\sigma = (-1)^n \omega_n(\tau) \int_0^\tau \omega_1 \dots \omega_{n-1}.$$ □

2.4. **Regularisation from zero to infinity.** Note that if ω is admissible, then the integral $\int_0^\infty \omega := \int_0^\tau \omega + \int_\tau^\infty \omega$ is well-defined and independent of τ by Lemmas 7 and 20. Moreover, if ω is integrable, then this definition coincides with the usual (convergent) integral of ω on $]0, i\infty[$.

In the iterated case, the composition of paths formula forces the following definition.

Definition 24. Let $\omega_1, \dots, \omega_n$ be admissible differential forms on $]0, i\infty[$. We define

$$\int_0^\infty \omega_1 \dots \omega_n = \sum_{k=0}^n \int_0^\tau \omega_1 \dots \omega_k \times \int_\tau^\infty \omega_{k+1} \dots \omega_n. \tag{10}$$

Lemma 25. *The definition (10) does not depend on τ.*

Proof. Using Lemmas 14 and 23, the differential of the right-hand side of (10) is

$$\sum_{k=0}^{n-1} \int_0^\tau \omega_1 \dots \omega_k \times \big(-\omega_{k+1}(\tau)\big) \int_\tau^\infty \omega_{k+2} \dots\dots \omega_n$$
$$+ \sum_{k=1}^{n} \omega_k(\tau) \int_0^\tau \omega_1 \dots \omega_{k-1} \times \int_\tau^\infty \omega_{k+1} \dots\dots \omega_n$$

which vanishes by changing $k \to k+1$ in the second sum. □

The last lemma naturally brings us to a statement which will be important for expressing the Goncharov regulator integral in terms of iterated integrals.

Proposition 26. *Let $\omega_1, \dots, \omega_n$ be admissible differential forms on $]0, i\infty[$. Then*

$$\int_\tau^\infty \omega_1 \dots \omega_n$$

is admissible at 0 as a function of τ, and its regularised value at 0 is $\int_0^\infty \omega_1 \dots \omega_n$. Moreover,

$$\int_0^\infty \omega_1 \dots \omega_n = \int_0^\infty \omega_1(\tau_1) \int_{\tau_1}^\infty \omega_2(\tau_2) \cdots \int_{\tau_{n-1}}^\infty \omega_n(\tau_n), \tag{11}$$

where the right-hand side of (11) is understood as successive one-variable regularisations.

Proof. For the first part of the proposition, we proceed by induction on n. The case $n = 1$ follows from $\int_0^\infty \omega_1 = \int_0^\tau \omega_1 + \int_\tau^\infty \omega_1$ and Lemma 20. For $n \geq 2$, we can write

$$\int_\tau^\infty \omega_1 \dots \omega_n = \int_0^\infty \omega_1 \dots \omega_n - \sum_{k=1}^{n} \int_0^\tau \omega_1 \dots \omega_k \times \int_\tau^\infty \omega_{k+1} \dots \omega_n.$$

By the induction hypothesis and Lemma 22, the right-hand side is admissible at 0. Moreover, the regularised value at 0 of the product

$$\int_0^\tau \omega_1 \dots \omega_k \times \int_\tau^\infty \omega_{k+1} \dots \omega_n$$

is the product of the regularised values, hence it is zero by Lemma 22.

Finally, (11) follows formally by using the case $n = 1$ with the form $\omega_1(\tau_1) \int_{\tau_1}^\infty \omega_2 \dots \omega_n$. □

2.5. **Shuffle relations of iterated integrals.** An important feature of all the regularisations we have discussed, $\int_0^\infty$ as well as $\int_0^\tau$ and $\int_\tau^\infty$, is that they satisfy the shuffle relations. Let V be the $\mathbf{C}$-vector space of admissible differential 1-forms on $]0, i\infty[$. Consider the functional $I_0^\infty : V \to \mathbf{C}$ sending ω to the regularised integral $\int_0^\infty \omega$. Then the regularised iterated integrals of Section 2.4 provide a natural extension of I_0^∞ to the tensor algebra $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$,

$$I_0^\infty : T(V) \to \mathbf{C}, \quad \omega_1 \otimes \dots \otimes \omega_n \mapsto \int_0^\infty \omega_1 \dots \omega_n \qquad (\omega_i \in V).$$

The algebra $T(V)$ has a structure of Hopf algebra, called the shuffle algebra, with the multiplication $T(V) \otimes T(V) \to T(V)$ given by the shuffle product

$$\omega_1 \dots \omega_p \sqcup\!\sqcup\, \omega_{p+1} \dots \omega_n = \sum_{\sigma \in S_{p,n-p}} \omega_{\sigma^{-1}(1)} \dots \omega_{\sigma^{-1}(n)},$$

where the sum is over the $(p, n-p)$-shuffles.

More generally, one may integrate over a path γ which is either a finite interval in $]0, i\infty[$, or a path in the 'tangent space of $\mathcal{H}$ at 0 or ∞' involving tangential base points $\vec{1}_0$ or $\vec{1}_\infty$, as defined in [4, Section 4]. For such a path γ, there is an associated functional $I_\gamma : T(V) \to \mathbf{C}$. The important point is that, as I_γ is essentially an ordinary iterated integral, it satisfies the shuffle relations; in other words, I_γ is a morphism of algebras. Moreover, regularised integrals on $]0, i\infty[$ are defined by formally concatenating the paths $\vec{1}_0 \to i/y \to iy \to \vec{1}_\infty$ (with $y \to \infty$). Formal considerations using the Hopf algebra structure on $T(V)$ lead to the following proposition.

Proposition 27. *The functional $I_0^\infty : T(V) \to \mathbf{C}$ satisfies the shuffle relations; in other words,*

$$\int_0^\infty \omega_1 \dots \omega_p \times \int_0^\infty \omega_{p+1} \dots \omega_n = \int_0^\infty \omega_1 \dots \omega_p ⧢ \omega_{p+1} \dots \omega_n \qquad (\omega_i \in V)$$

for any choice of $p \in \{1, \dots, n\}$.

For more details, we refer the reader to [4, Section 4].

2.6. **The Newton–Leibniz formula and integration by parts.** We now want to generalise Lemma 8 in the form of formula (4) for integration by parts to iterated integrals $\int_\tau^\infty \omega_1 \dots \omega_n$. 'with respect to a particular form' $\omega_p(\tau) = f(\tau)\, d\tau$, assuming that the 1-forms $\omega_1, \dots, \omega_n$ are admissible. As we already know from the lemma, f is an admissible function; we keep the notation $f(\infty)$ and $f(0)$ for its regularised values at ∞ and 0 as in Definition 3.

If $p = 1$ we get, using (4),

$$\begin{aligned} &\int_\tau^\infty \frac{df}{d\tau}(\tau_1)\, d\tau_1\, \omega_2(\tau_2) \dots \omega_n(\tau_n) \\ &\quad = \int_\tau^\infty \frac{df}{d\tau}(\tau_1)\, d\tau_1 \int_{\tau_1}^\infty \omega_2(\tau_2) \dots \omega_n(\tau_n) \\ &\quad = -f(\tau) \int_\tau^\infty \omega_2(\tau_2) \dots \omega_n(\tau_n) + \int_\tau^\infty f(\tau_2)\omega_2(\tau_2) \dots \omega_n(\tau_n). \end{aligned}$$

For $p > 1$, we write

$$\begin{aligned} &\int_\tau^\infty \omega_1(\tau_1) \dots \frac{df}{d\tau}(\tau_p)\, d\tau_l \dots \omega_n(\tau_n) \\ &= \int_\tau^\infty \omega_1(\tau_1) \dots \int_{\tau_{p-3}}^\infty \omega_{p-2}(\tau_{p-2}) \int_{\tau_{p-2}}^\infty \omega_{p-1}(\tau_{p-1}) \int_{\tau_{p-1}}^\infty \frac{df}{d\tau}(\tau_p)\, d\tau_p\, \omega_{p+1}(\tau_{p+1}) \dots \omega_n(\tau_n) \end{aligned}$$

and use the above derivation to conclude that this is

$$\begin{aligned} &= -\int_\tau^\infty \omega_1(\tau_1) \dots \omega_{p-2}(\tau_{p-2})\, f(\tau_{p-1})\omega_{p-1}(\tau_{p-1})\, \omega_{p+1}(\tau_{p+1}) \dots \omega_n(\tau_n) \\ &\qquad + \int_\tau^\infty \omega_1(\tau_1) \dots \omega_{p-1}(\tau_{p-1})\, f(\tau_{p+1})\omega_{p+1}(\tau_{p+1})\, \omega_{p+2}(\tau_{p+2}) \dots \omega_n(\tau_n). \end{aligned}$$

Taking the regularised value as $\tau \to 0$ and using Proposition 26, we get

$$\begin{aligned} &\int_0^\infty \omega_1(\tau_1) \dots \frac{df}{d\tau}(\tau_p)\, d\tau_p \dots \omega_n(\tau_n) \\ &\quad = \int_0^\infty \omega_1(\tau_1) \dots \omega_{p-1}(\tau_{p-1})\, f(\tau_{p+1})\omega_{p+1}(\tau_{p+1})\, \omega_{p+2}(\tau_{p+2}) \dots \omega_n(\tau_n) \\ &\qquad - \int_0^\infty \omega_1(\tau_1) \dots \omega_{p-2}(\tau_{p-2})\, f(\tau_{p-1})\omega_{p-1}(\tau_{p-1})\, \omega_{p+1}(\tau_{p+1}) \dots \omega_n(\tau_n), \end{aligned} \tag{12}$$

where the first summand is interpreted as

$$f(\infty) \int_0^\infty \omega_1(\tau_1) \dots \omega_{n-1}(\tau_{n-1})$$

when $p = n$, while the second summand is

$$-f(0) \int_0^\infty \omega_2(\tau_2) \dots \omega_n(\tau_n)$$

when $p = 1$.

In the particular case $p = n = 1$, formula (12) extends Lemma 8 to regularised integrals from 0 to ∞:

Lemma 28. *Let $f :]0, i\infty[\;\to \mathbf{C}$ be a C^∞ function such that df is admissible. Then f is admissible and $\int_0^\infty df = f(\infty) - f(0)$.*

2.7. **Iterated integrals with parameters.** In this part we record our needs for differentiating the regularised (iterated) integral when a differential form depends smoothly on a *real* parameter.

Proposition 29. *Let $(\omega_a)_a$ be a family of differential forms on $]0,i\infty[$ admissible at ∞ depending on a single real parameter a. Write $\omega_a = f_a(\tau)\,d\tau$, and assume that:*

(i) *the polynomial $f_a^\infty(\tau)$ has degree bounded independently of a, and its coefficients are differentiable functions of a;*
(ii) *$f_a^0(\tau)$ is differentiable as a function of a;*
(iii) *locally on a, there exists a constant $0<c<1$ such that $\frac{d}{da}f_a^0(\tau) = O_{\tau\to\infty}(c^{\mathrm{Im}(\tau)})$, where the implied constant does not depend on a.*

Then the function $a \mapsto \int_\tau^\infty \omega_a$ is differentiable, and we have

$$\frac{d}{da}\Big(\int_\tau^\infty \omega_a\Big) = \int_\tau^\infty \frac{d}{da}\omega_a.$$

Proof. Note that the assumptions imply that for every a, the form $\frac{d}{da}\omega_a = \frac{d}{da}f_a(\tau)\,d\tau$ is admissible, with $(\frac{d}{da}\omega_a)^\infty = \frac{d}{da}\omega_a^\infty$ and $(\frac{d}{da}\omega_a)^0 = \frac{d}{da}\omega_a^0$. We have:

$$\int_\tau^\infty \omega_a = \int_\tau^\infty \omega_a^0 + \int_\tau^0 \omega_a^\infty.$$

This shows that $a \mapsto \int_\tau^\infty \omega_a$ is differentiable, and we can differentiate inside the integral:

$$\frac{d}{da}\Big(\int_\tau^\infty \omega_a\Big) = \int_\tau^\infty \Big(\frac{d}{da}\omega_a\Big)^0 + \int_\tau^0 \Big(\frac{d}{da}\omega_a\Big)^\infty = \int_\tau^\infty \frac{d}{da}\omega_a. \qquad \square$$

Proposition 29 motivates calling a real-parameter family $(\omega_a)_a$ of admissible differential forms on $]0,i\infty[$ differentially admissible at ∞ if they are subject to conditions (i)–(iii) above, written as $\omega_a = f_a(\tau)\,d\tau$. Furthermore, we call a real-parameter family $(\omega_a)_a$ differentially admissible at 0 if the family $(\omega_a^\sigma)_a$ is differentially admissible at ∞; see Definition 15. With these definitions in mind, we apply Proposition 29 twice to deduce the following statement.

Proposition 30. *Let $(\omega_a)_a$ be a family of differentially admissible at 0 and ∞ differential forms depending on a single real parameter a. Then the function $a \mapsto \int_0^\infty \omega_a$ is differentiable, and we have*

$$\frac{d}{da}\Big(\int_0^\infty \omega_a\Big) = \int_0^\infty \frac{d}{da}\omega_a.$$

Observe that Propositions 29 and 30 cover the iterated integral situation as well, since $f_a(\tau)$ themselves may come as iterated integrals of admissible forms. For this, we simply apply the propositions inductively using Proposition 26.

2.8. **Mellin transforms.** A powerful analytic tool to compute regularised integrals is the theory of Mellin transforms. Since we consider admissible forms on $]0,i\infty[$ with possible poles at 0 and ∞, we will need generalised Mellin transforms as described in [11, Section 3.4]. We use notably this theory in Section 8 to compute integrals of products of two Eisenstein series using the Rogers–Zudilin method.

We enlarge a bit our setting by considering functions $f\colon]0,i\infty[\to \mathbf{C}$ of the form $f(\tau) = f^\infty(\tau) + f^0(\tau)$, where $f^\infty(\tau) \in \mathbf{C}[\tau,\tau^{-1}]$ is a Laurent polynomial, and f^0 is a C^∞ function with exponential decay at $i\infty$. Moreover, we assume that $f\circ\sigma(\tau) = f(-1/\tau)$ is also of this form. For such a function f, the (generalised) Mellin transform is defined as

$$\mathcal{M}(f,s) = \int_0^\infty f(iy)y^s \frac{dy}{y} \qquad (s\in\mathbf{C}).$$

In general, this integral may not converge at any $s\in\mathbf{C}$. However, splitting the integral as $\int_0^1 + \int_1^\infty$, and analytically continuing each term, it is possible to make sense of $\mathcal{M}(f,s)$ as a meromorphic function of $s\in\mathbf{C}$, with at most simple poles at finitely many integers. A pole of $\mathcal{M}(f,s)$ can occur at $n_0\in\mathbf{Z}$ only if $-n_0$ arises as an exponent in the polynomial f^∞, or n_0

arises as an exponent in $(f\circ\sigma)^\infty$. As a remark, $\mathcal{M}(f,s)$ is identically zero if f is a polynomial. Therefore, we can always reduce to the situation where $f^\infty = 0$.

For any $s_0 \in \mathbf{C}$, we denote by $\mathcal{M}^*(f,s_0)$ the constant term of the Laurent expansion of $\mathcal{M}(f,s)$ at $s = s_0$.

From Lemma 28, we get the following computational tool.

Proposition 31 ([11, Section 3.4])**.** *Let $\omega = f(\tau)\,d\tau$ be an admissible differential form on $]0,i\infty[$. Then $\mathcal{M}(f,s)$ is holomorphic at $s=1$, and we have $\int_0^\infty \omega = i\mathcal{M}(f,1)$.*

3. Multiple modular values

We use the notation $e(z) = e^{2\pi i z}$ for $z \in \mathbf{C}$, so that $q = e(\tau)$ for $\tau \in \mathcal{H}$. For any $\alpha \in \mathbf{R}$, write also $q^\alpha = e(\alpha\tau)$. Introduce the differential operators

$$\delta = \delta_\tau := \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq} \quad \text{and} \quad \delta_a = \frac{1}{2\pi i}\frac{d}{da}$$

if a is a real variable.

Recall the Hurwitz zeta function

$$\zeta(y,s) = \sum_{\substack{n>0 \\ n\equiv y \bmod 1}} n^{-s} \qquad (y \in \mathbf{R}/\mathbf{Z},\ \mathrm{Re}(s) > 1),$$

and the periodic zeta function

$$\hat{\zeta}(y,s) = \sum_{n\geq 1} e(ny)n^{-s} \qquad (y \in \mathbf{R}/\mathbf{Z},\ \mathrm{Re}(s) > 1).$$

Some properties of these functions can be found in [7, Section 2]. Let us point out that the relation [7, eq. (11)] is incorrect in the case $n = 1$. Indeed, for $x \in \mathbf{R}/\mathbf{Z}$, we have

$$\hat{\zeta}(x,0) = \begin{cases} \frac{e(x)}{1-e(x)} & \text{if } x \neq 0, \\ -\frac{1}{2} & \text{if } x = 0. \end{cases} \tag{13}$$

This can be shown by differentiating the relation

$$\hat{\zeta}(x,1) = \sum_{n=1}^\infty \frac{e(nx)}{n} = -\log(1-e(x)).$$

3.1. Eisenstein series.

It will be essential to us to view Eisenstein series not only as functions of the modular variable $\tau \in \mathcal{H}$, but also as functions of the elliptic variable $z \in \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$. To this end, we recall the Eisenstein-Kronecker function, in the notations of Weil [24, VII, §12].

Let L be a lattice in $\mathbf{C}$, and let (ω_1,ω_2) be a basis of L such that $\mathrm{Im}(\omega_2/\omega_1) > 0$. Then $A(L) := (2\pi i)^{-1}(\overline{\omega}_1\omega_2 - \omega_1\overline{\omega}_2)$ is a positive real number which does not depend on the choice of (ω_1,ω_2).

Definition 32. For an integer $a \geq 0$ and $x, x_0, s \in \mathbf{C}$, introduce the *Kronecker double series*

$$K_a(x,x_0,s;L) = \sum_{\substack{w\in L \\ w\neq -x}} \exp\left(A(L)^{-1}(w\overline{x}_0 - \overline{w}x_0)\right)\frac{(\overline{w}+\overline{x})^a}{|w+x|^{2s}},$$

where the sum is extended to all $\omega \in L$, except $\omega = -x$ if $x \in L$. In the case $L = \mathbf{Z}+\tau\mathbf{Z}$ with $\tau \in \mathcal{H}$, we write $K_a(x,x_0,s;\tau)$ or simply $K_a(x,x_0,s)$ when the context is clear.

The series $K_a(x,x_0,s;L)$ converges for $\mathrm{Re}(s) > 1+\frac{a}{2}$. For $a \geq 1$, the function $s \mapsto K_a(x,x_0,s;L)$ extends to a holomorphic function on $\mathbf{C}$ [24, VII, §13]. Moreover, the functions $x \mapsto K_a(x,0,s;L)$ and $x \mapsto K_a(0,x,s;L)$ are periodic with respect to L, which justifies the following definition.

Definition 33. Let $k \geq 1$ be an integer. For $\boldsymbol{x} = (x_1,x_2) \in (\mathbf{R}/\mathbf{Z})^2$, we define

$$E_{\boldsymbol{x}}^{(k)}(\tau) = -\frac{(k-1)!}{(-2\pi i)^k}K_k(0,x_1\tau+x_2,k) \qquad \hat{E}_{\boldsymbol{x}}^{(k)}(\tau) = \frac{(k-1)!}{(-2\pi i)^k}K_k(x_1\tau+x_2,0,k).$$

Kato has given in [16, Section 3] an algebraic interpretation of $\hat{E}_{\boldsymbol{x}}^{(k)}$ in the case $\boldsymbol{x} \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2$.

We are particularly interested in the series $E_{\boldsymbol{x}}^{(k)}$. We will determine its Fourier expansion with respect to τ, and then examine its behaviour with respect to the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathcal{H}$. Finally, we will give a differential property of $E_{\boldsymbol{x}}^{(k)}$ with respect to the elliptic variable.

Lemma 34. *Let $k \geq 1$ be an integer, and $\boldsymbol{x} = (x_1, x_2) \in (\mathbf{R}/\mathbf{Z})^2$, with $\boldsymbol{x} \neq \mathbf{0}$ in the case $k = 2$. We have*

$$E_{\boldsymbol{x}}^{(k)}(\tau) = a_0(E_{\boldsymbol{x}}^{(k)}) - \sum_{\substack{m \geq 1 \\ n \in \mathbf{R}_{>0} \\ n \equiv x_1 \bmod 1}} e(mx_2) n^{k-1} q^{mn} + (-1)^{k+1} \sum_{\substack{m \geq 1 \\ n \in \mathbf{R}_{>0} \\ n \equiv -x_1 \bmod 1}} e(-mx_2) n^{k-1} q^{mn}, \tag{14}$$

with

$$a_0(E_{\boldsymbol{x}}^{(1)}) = \begin{cases} 0 & \text{if } x_1 = x_2 = 0, \\ -\frac{1}{2}\frac{1+e(x_2)}{1-e(x_2)} & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ \{x_1\} - \frac{1}{2} & \text{if } x_1 \neq 0, \end{cases}$$

$$a_0(E_{\boldsymbol{x}}^{(k)}) = \frac{B_k(\{x_1\})}{k} \qquad (k \geq 2),$$

where $B_k(t)$ is the k-th Bernoulli polynomial and $\{\cdot\}$ stands for the fractional part.

Proof. In the case $\boldsymbol{x}$ is an N-torsion point in $(\mathbf{R}/\mathbf{Z})^2$, the Fourier expansions of $E_{\boldsymbol{x}}^{(k)}$ and $\hat{E}_{\boldsymbol{x}}^{(k)}$ can be found in [16, Proposition 3.10]. The general case can be handled as in [22, VII], we only sketch the details in the case $k \geq 3$. We have

$$\sum_{\substack{(m,n) \in \mathbf{Z}^2 \\ (m,n) \neq (0,0)}} \frac{e(mx_2 - nx_1)}{(m\tau + n)^k} = \sum_{n \neq 0} \frac{e(-nx_1)}{n^k} + \sum_{m \geq 1} e(mx_2) S(x_1; m\tau) + (-1)^k \sum_{m \geq 1} e(-mx_2) S(-x_1; m\tau)$$

with $S(x;\tau) = \sum_{n \in \mathbf{Z}} e(-nx)(\tau + n)^{-k}$. By [7, Section 2], the first term is

$$\sum_{n \neq 0} \frac{e(-nx_1)}{n^k} = \hat{\zeta}(-x_1, k) + (-1)^k \hat{\zeta}(x_1, k) = (-1)^{k+1} \frac{(2\pi i)^k}{k!} B_k(\{x_1\}).$$

For any $x \in \mathbf{R}$, the function $e(-\tau x) S(x;\tau)$ is invariant under $\tau \mapsto \tau + 1$, hence has a Fourier expansion

$$e(-\tau x) S(x;\tau) = \sum_{r \in \mathbf{Z}} c_r(x) e(r\tau).$$

The Fourier coefficients $c_r(x)$ can be computed as in [22, VII] using the Poisson summation formula and the residue theorem, leading to (14). □

Lemma 35. *Let $k \geq 1$ be an integer, and $\boldsymbol{x} \in (\mathbf{R}/\mathbf{Z})^2$. For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$, we have*

$$E_{\boldsymbol{x}}^{(k)}(\gamma\tau) = (c\tau + d)^k E_{\boldsymbol{x}\gamma}^{(k)}(\tau),$$

where $\boldsymbol{x}\gamma$ means the right multiplication by γ on the row vector $\boldsymbol{x}$.

Proof. Putting $x_0 = x_1\tau + x_2$ and $\alpha = c\tau + d$, this follows from the identity $K_k(0, x_0, k; L) = \alpha^k K_k(0, \alpha x_0, k; \alpha L)$, valid for any lattice L in $\mathbf{C}$. □

Taking $\gamma = -I_2$ in Lemma 35, we see that $E_{-\boldsymbol{x}}^{(k)} = (-1)^k E_{\boldsymbol{x}}^{(k)}$. Lemma 35 also shows that if $\boldsymbol{x}$ is N-torsion in $(\mathbf{R}/\mathbf{Z})^2$, then $E_{\boldsymbol{x}}^{(k)}$ is a modular form of weight k on $\Gamma(N)$, except when $k = 2$ and $\boldsymbol{x} = \mathbf{0}$ (in which case $E_{\mathbf{0}}^{(2)}$ is not holomorphic).

Using Lemmas 34 and 35 with $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we obtain the following admissibility property.

Lemma 36. *For any $k \geq 2$ and $\boldsymbol{x} \in (\mathbf{R}/\mathbf{Z})^2$, with $\boldsymbol{x} \neq \mathbf{0}$ in the case $k = 2$, the differential form $E_{\boldsymbol{x}}^{(k)}(\tau)\tau^{m-1} d\tau$ is admissible on $]0, i\infty[$ for any integer $1 \leq m \leq k-1$.*

The Eisenstein series $E^{(2)}_{\boldsymbol{x}}$ are related to the so-called Siegel units as follows. For $\boldsymbol{x} = (x_1, x_2) \in (\mathbf{R}/\mathbf{Z})^2$, $\boldsymbol{x} \neq \mathbf{0}$, consider the following function on $\mathcal{H}$:

$$g_{\boldsymbol{x}}(\tau) = q^{B_2(\{x_1\})/2} \prod_{\substack{n \in \mathbf{R}_{\geq 0} \\ n \equiv x_1 \bmod 1}} (1 - q^n e(x_2)) \prod_{\substack{n \in \mathbf{R}_{>0} \\ n \equiv -x_1 \bmod 1}} (1 - q^n e(-x_2)). \tag{15}$$

For $a, b \in \mathbf{Z}$, $(a,b) \not\equiv (0,0) \bmod N$, the function $g_{a/N, b/N}$ is none other than the classical Siegel unit $g_{\overline{a},\overline{b}}$ [16, Section 1]. This function is a $(12N)$-th root of a unit on the modular curve $Y(N)$ over $\mathbf{Q}$, thus defining an element of $\mathcal{O}(Y(N))^\times \otimes \mathbf{Q}$.

We also define, for $\boldsymbol{x} \in (\mathbf{R}/\mathbf{Z})^2$, $\boldsymbol{x} \neq \mathbf{0}$, the logarithm of $g_{\boldsymbol{x}}$ by taking the logarithm of the infinite product (15) and specifying the branch:

$$\log g_{\boldsymbol{x}}(\tau) = \pi i B_2(\{x_1\})\tau + \log(1 - e(x_2)) \cdot \mathbf{1}_{x_1 = 0} - \sum_{\substack{m \geq 1 \\ n \in \mathbf{R}_{>0} \\ n \equiv x_1 \bmod 1}} \frac{e(mx_2)}{m} q^{mn} - \sum_{\substack{m \geq 1 \\ n \in \mathbf{R}_{>0} \\ n \equiv -x_1 \bmod 1}} \frac{e(-mx_2)}{m} q^{mn}, \tag{16}$$

where

$$\log(1 - e(x_2)) = -\hat{\zeta}(x_2, 1) = \log|1 - e(x_2)| + \pi i\Big(\{x_2\} - \frac{1}{2}\Big).$$

Lemma 37. *For any $\boldsymbol{x} \in (\mathbf{R}/\mathbf{Z})^2$, $\boldsymbol{x} \neq \mathbf{0}$, we have $\mathrm{dlog}\, g_{\boldsymbol{x}}(\tau) = 2\pi i E^{(2)}_{\boldsymbol{x}}(\tau)\, d\tau$.*

Proof. This follows from comparing the Fourier expansions (14) and (16). ☐

The Kronecker double series $K_a(x, x_0, s; L)$ satisfies differential equations with respect to the elliptic parameters x and x_0 [1, Lemma 1.4]. Similarly, the series $E^{(k)}_{\boldsymbol{x}}$ satisfies a differential relation with respect to both elliptic and modular parameters, which will be especially important.

Lemma 38. *For $k \geq 1$, the function $\boldsymbol{x} \mapsto E^{(k)}_{\boldsymbol{x}}(\tau)$ is smooth on the domain $(\mathbf{R}/\mathbf{Z})^2 \smallsetminus \{\mathbf{0}\}$. Moreover, we have*

$$\delta_{x_2} E^{(k+1)}_{\boldsymbol{x}}(\tau) = \delta_\tau E^{(k)}_{\boldsymbol{x}}(\tau). \tag{17}$$

Proof. The Fourier expansion (14) shows that $\boldsymbol{x} \mapsto E^{(k)}_{\boldsymbol{x}}(\tau)$ is smooth on the domain $\{x_1 \neq 0\}$. Using Lemma 35 with $\gamma = \sigma$, the function is also smooth on $\{x_2 \neq 0\}$, whence the claim.

The identity (17) follows either by inspecting the Fourier expansions of both sides (using Lemma 34), or directly from Definition 33. ☐

We now introduce an interpolated version of the Eisenstein series $G^{(k);N}_{\boldsymbol{x}}$ defined in (1).

Definition 39. For an integer $k \geq 1$ and $\boldsymbol{x} = (x_1, x_2) \in (\mathbf{R}/\mathbf{Z})^2$, define

$$G^{(k)}_{\boldsymbol{x}}(\tau) = a_0(G^{(k)}_{\boldsymbol{x}}) + \Bigg(\sum_{\substack{m,n \in \mathbf{R}_{>0} \\ (m,n) \equiv \boldsymbol{x} \bmod 1}} + (-1)^k \sum_{\substack{m,n \in \mathbf{R}_{>0} \\ (m,n) \equiv -\boldsymbol{x} \bmod 1}} \Bigg) m^{k-1} q^{mn}$$

with

$$a_0(G^{(1)}_{\boldsymbol{x}}) = \begin{cases} -B_1(\{x_2\}) & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ -B_1(\{x_1\}) & \text{if } x_1 \neq 0 \text{ and } x_2 = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$(k \geq 2) \qquad a_0(G^{(k)}_{\boldsymbol{x}}) = \begin{cases} -\frac{B_k(\{x_1\})}{k} & \text{if } x_2 = 0, \\ 0 & \text{if } x_2 \neq 0. \end{cases}$$

The relation with $G^{(k);N}_{\boldsymbol{x}}$ is as follows. If $\boldsymbol{x} = (a/N, b/N)$ is an N-torsion point in $(\mathbf{R}/\mathbf{Z})^2$, then

$$G^{(k)}_{\boldsymbol{x}}(N\tau) = N^{1-k} G^{(k);N}_{\overline{a},\overline{b}}(\tau). \tag{18}$$

Lemma 40. *For $k \geq 1$, the function $\boldsymbol{x} \mapsto G_{\boldsymbol{x}}^{(k)}(\tau)$ is smooth on the domain $(\mathbf{R}/\mathbf{Z} \smallsetminus \{0\})^2$, and we have*

$$\delta_{x_2} G_{\boldsymbol{x}}^{(k)}(\tau) = \tau G_{\boldsymbol{x}}^{(k+1)}(\tau).$$

Proof. It suffices to consider the domain $0 < x_1, x_2 < 1$. There $G_{\boldsymbol{x}}^{(k)}$ can be written as

$$G_{\boldsymbol{x}}^{(k)}(\tau) = \sum_{m,n \geq 0} (m+x_1)^{k-1} q^{(m+x_1)(n+x_2)} + (-1)^k \sum_{m,n \geq 1} (m-x_1)^{k-1} q^{(m-x_1)(n-x_2)}.$$

Therefore

$$\delta_{x_2} G_{\boldsymbol{x}}^{(k)}(\tau) = \sum_{m,n \geq 0} (m+x_1)^k \tau q^{(m+x_1)(n+x_2)} + (-1)^k \sum_{m,n \geq 1} -(m-x_1)^k \tau q^{(m-x_1)(n-x_2)} = \tau G_{\boldsymbol{x}}^{(k+1)}(\tau). \qquad \square$$

To end this section, we give an explicit formula for the Mellin transform of the Eisenstein series of type $E^{(k)}$ and $G^{(k)}$.

Lemma 41. *For any integer $k \geq 1$ and $\boldsymbol{x} = (x_1, x_2) \in (\mathbf{R}/\mathbf{Z})^2$, with $\boldsymbol{x} \neq \mathbf{0}$ in the case $k = 2$, we have*

$$\mathcal{M}(E_{\boldsymbol{x}}^{(k)}, s) = (2\pi)^{-s}\Gamma(s)\big(-\zeta(x_1, s-k+1)\hat{\zeta}(x_2, s) + (-1)^{k+1}\zeta(-x_1, s-k+1)\hat{\zeta}(-x_2, s)\big) \tag{19}$$

$$\mathcal{M}(G_{\boldsymbol{x}}^{(k)}, s) = (2\pi)^{-s}\Gamma(s)\big(\zeta(x_1, s-k+1)\zeta(x_2, s) + (-1)^k\zeta(-x_1, s-k+1)\zeta(-x_2, s)\big). \tag{20}$$

Proof. We give the proof for $G_{\boldsymbol{x}}^{(k)}$, the other case being similar. Writing $G_{\boldsymbol{x}}^{(k)}(\tau) = \sum_{n \in \mathbf{R}_{\geq 0}} c_n q^n$, we have for $\mathrm{Re}(s)$ large enough:

$$\begin{aligned}\mathcal{M}(G_{\boldsymbol{x}}^{(k)}, s) &= (2\pi)^{-s}\Gamma(s) \sum_{n \in \mathbf{R}_{>0}} \frac{c_n}{n^s} \\ &= (2\pi)^{-s}\Gamma(s)\Bigg(\sum_{\substack{m_1, m_2 \in \mathbf{R}_{>0} \\ (m_1,m_2) \equiv (x_1,x_2) \bmod 1}} + (-1)^k \sum_{\substack{m_1, m_2 \in \mathbf{R}_{>0} \\ (m_1,m_2) \equiv (-x_1,-x_2) \bmod 1}}\Bigg) \frac{1}{m_1^{s-k+1} m_2^s}. \qquad \square\end{aligned}$$

From the description of the poles of the Mellin transform in Section 2.8, one can show that the only possible poles of $\mathcal{M}(E_{\boldsymbol{x}}^{(k)}, s)$ and $\mathcal{M}(G_{\boldsymbol{x}}^{(k)}, s)$ are located at $s = 0$ and $s = k$. For $E_{\boldsymbol{x}}^{(k)}$ this follows from using Lemma 35 with $\gamma = \sigma$, while for $G_{\boldsymbol{x}}^{(k)}$ this follows from Lemma 60 below.

3.2. **Multiple modular values.** Recall that if f is a modular form of weight $k \geq 2$ on some finite index subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$, the differential form $f(\tau)\tau^{m-1}d\tau$ is admissible on $]0, i\infty[$ for any $1 \leq m \leq k-1$ (see Example 17). For any modular forms $f_1, \ldots, f_n$ of respective weights $k_1, \ldots, k_n \geq 2$, and any integers $m_1, \ldots, m_n$ with $1 \leq m_i \leq k_i - 1$, the regularised iterated integral

(21)

$$\begin{aligned}\Lambda(f_1, \ldots, f_n; m_1, \ldots, m_n) &= \int_0^\infty f_1(\tau)\tau^{m_1-1}d\tau \ldots f_n(\tau)\tau^{m_n-1}d\tau \\ &= \int_0^\infty f_1(\tau_1)\tau_1^{m_1-1}d\tau_1 \int_{\tau_1}^\infty f_2(\tau_2)\tau_2^{m_2-1}d\tau_2 \cdots \int_{\tau_{n-1}}^\infty f_n(\tau_n)\tau_n^{m_n-1}d\tau_n\end{aligned}$$

is called a totally holomorphic multiple modular value (MMV) [5, Section 5]. In the case all m_i are equal to 1, we simply write $\Lambda(f_1, \ldots, f_n) = \Lambda(f_1, \ldots, f_n; 1, \ldots, 1)$.

In the case $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, the multiple modular values are periods of the relative completion of the fundamental group of $\mathcal{M}_{1,1}$ [4, 5]. In this article, we are particularly interested in the case Γ is the principal congruence subgroup $\Gamma(N)$, and all f_i are Eisenstein series of weight ≥ 2 on $\Gamma(N)$. In this case (21) is called a multiple Eisenstein value.

Example 42. When the Eisenstein series in question are $E_{\boldsymbol{x}_i}^{(k_i)}(\tau)$ with $k_i \geq 2$, all $m_i = 1$, and allowing continuous parameters $\boldsymbol{x}_i \in (\mathbf{R}/\mathbf{Z})^2$, we can view the MEV as a function

$$(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \mapsto \Lambda(E_{\boldsymbol{x}_1}^{(k_1)}, \ldots, E_{\boldsymbol{x}_n}^{(k_n)})$$

which has partial derivatives with respect all elliptic parameters x_{p1}, x_{p2} restricted to the interval $(0,1)$ (or to any shift of it by an integer) where $1 \le p \le n$. This follows from viewing the MEV as the iterated integral of a family of differential forms that depend on each such parameter, the forms differentially admissible at both 0 and ∞ as defined in Section 2.7. The diffentiation of the MEV with respect to the x_2-component of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is particularly simple. When an index p in the range $1 \le p \le n$ is fixed, we can apply Proposition 30 to the corresponding parameter x_{p2} and then combine the result with Lemma 38 and formula (12) to obtain

$$\frac{\partial}{\partial x_{p2}} \Lambda(E^{(k_1)}_{\boldsymbol{x}_1}, \ldots, E^{(k_p)}_{\boldsymbol{x}_p}, \ldots, E^{(k_n)}_{\boldsymbol{x}_n}) = \Lambda(E^{(k_1)}_{\boldsymbol{x}_1}, \ldots, E^{(k_{p-1})}_{\boldsymbol{x}_{p-1}}, E^{(k_p-1)}_{\boldsymbol{x}_p} E^{(k_{p+1})}_{\boldsymbol{x}_{p+1}}, \ldots, E^{(k_n)}_{\boldsymbol{x}_n}) - \Lambda(E^{(k_1)}_{\boldsymbol{x}_1}, \ldots, E^{(k_{p-1})}_{\boldsymbol{x}_{p-1}} E^{(k_p-1)}_{\boldsymbol{x}_p}, E^{(k_{p+1})}_{\boldsymbol{x}_{p+1}}, \ldots, E^{(k_n)}_{\boldsymbol{x}_n}), \tag{22}$$

with the first term interpreted as $a_0(E^{(k_n-1)}_{\boldsymbol{x}_n})\Lambda(E^{(k_1)}_{\boldsymbol{x}_1}, \ldots, E^{(k_{n-1})}_{\boldsymbol{x}_{n-1}})$ if $p = n$, while the second term is discarded if $p = 1$. This formula means that differentiation of $\Lambda(E^{(k_1)}_{\boldsymbol{x}_1}, \ldots, E^{(k_n)}_{\boldsymbol{x}_n})$ with respect to the elliptic parameter x_{p2} reduces the length of the MEV by 1.

Definition 43. For $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in (\mathbf{R}/\mathbf{Z})^2$, we define

$$\begin{aligned}\Lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) &= (2\pi i)^n \Lambda(E^{(2)}_{\boldsymbol{x}_1}, \ldots, E^{(2)}_{\boldsymbol{x}_n}) \\ &= (2\pi i)^n \int_0^\infty E^{(2)}_{\boldsymbol{x}_1}(\tau_1)d\tau_1 \int_{\tau_1}^\infty E^{(2)}_{\boldsymbol{x}_2}(\tau_2)d\tau_2 \cdots \int_{\tau_{n-1}}^\infty E^{(2)}_{\boldsymbol{x}_n}(\tau_n)d\tau_n.\end{aligned}$$

We call $\Lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ a (totally holomorphic) *multiple Eisenstein value* (MEV) of length n. In general, we expect $\Lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ to be a period only when the parameters $\boldsymbol{x}_i$ belong to $(\mathbf{Q}/\mathbf{Z})^2$. In the sequel, we implicitly identify $(\mathbf{Z}/N\mathbf{Z})^2$ with a subgroup of $(\mathbf{R}/\mathbf{Z})^2$ by mapping a pair $(\overline{x}_1, \overline{x}_2)$ to the class of $(x_1/N, x_2/N)$. In this way $\Lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ makes sense for arguments $\boldsymbol{x}_i$ in $(\mathbf{Z}/N\mathbf{Z})^2$.

Since $\operatorname{dlog} g_{\boldsymbol{x}} = 2\pi i E^{(2)}_{\boldsymbol{x}}(\tau)\, d\tau$, the multiple Eisenstein value can also be written

$$\Lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \int_0^\infty \operatorname{dlog} g_{\boldsymbol{x}_1} \operatorname{dlog} g_{\boldsymbol{x}_2} \ldots \operatorname{dlog} g_{\boldsymbol{x}_n}.$$

Recall that $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ acts on $\mathcal{H}$. If $\boldsymbol{x} \in (\mathbf{Z}/N\mathbf{Z})^2$, $\boldsymbol{x} \neq \mathbf{0}$, then $\sigma^*(\operatorname{dlog} g_{\boldsymbol{x}}) = \operatorname{dlog} g_{\boldsymbol{x}\sigma}$ by [16, Lemma 1.7(1)]. By continuity, this identity holds for arbitrary $\boldsymbol{x} \neq \mathbf{0}$. Since σ reverses the path $]0, i\infty[$, the path reversal formula for iterated integrals gives

$$\Lambda(\boldsymbol{x}_1\sigma, \ldots, \boldsymbol{x}_n\sigma) = (-1)^n \Lambda(\boldsymbol{x}_n, \ldots, \boldsymbol{x}_1). \tag{23}$$

The *single* modular values are essentially the critical L-values of a modular form. In the particular case of an Eisenstein series, these values are computed classically in terms of Bernoulli polynomials.

Proposition 44. *For any* $\boldsymbol{x} = (x_1, x_2) \in (\mathbf{R}/\mathbf{Z})^2 \setminus \{\mathbf{0}\}$, *we have*

$$\Lambda(\boldsymbol{x}) = \begin{cases} 2\pi i \left(\{x_1\} - \frac{1}{2}\right)\left(\{x_2\} - \frac{1}{2}\right) & \text{if } x_1, x_2 \neq 0, \\ \log|1 - e(x_2)| & \text{if } x_1 = 0,\ x_2 \neq 0, \\ -\log|1 - e(x_1)| & \text{if } x_1 \neq 0,\ x_2 = 0. \end{cases}$$

Note that the function $\boldsymbol{x} \mapsto \Lambda(\boldsymbol{x})$ has discontinuities at $\{x_1 = 0\} \cup \{x_2 = 0\}$.

Proof. Assume first $x_2 \neq 0$. By Proposition 31 and Lemma 41, we have

$$\Lambda(\boldsymbol{x}) = -2\pi \mathcal{M}(E^{(2)}_{\boldsymbol{x}}, 1) = \hat{\zeta}(x_2, 1)\zeta(x_1, 0) + \hat{\zeta}(-x_2, 1)\zeta(-x_1, 0).$$

It remains to apply the identities [7, Section 2]

$$\zeta(x_1, 0) = \begin{cases} \frac{1}{2} - \{x_1\} & \text{if } x_1 \neq 0, \\ -\frac{1}{2} & \text{if } x_1 = 0, \end{cases}$$

$$\hat{\zeta}(x_2, 1) = \sum_{n \ge 1} \frac{e(nx_2)}{n} = -\log(1 - e(x_2)).$$

The case $x_2 = 0$ follows by noting that $\Lambda((x_1,0)) = -\Lambda((0,x_1))$ thanks to (23). □

With the same method in mind, one can show that for $k \geq 2$ and $m \in \{1,\dots,k-1\}$, we have

$$\Lambda(E_{\boldsymbol{x}}^{(k)};m) = (-1)^{m+1}\frac{B_{k-m}(x_1)B_m(x_2)}{(k-m)m} \qquad (0 < x_1,x_2 < 1).$$

We will also need the above iterated integrals with the Eisenstein series replaced by their real or imaginary parts. For $\boldsymbol{x} \in (\mathbf{R}/\mathbf{Z})^2 \smallsetminus \{\mathbf{0}\}$, write

$$\omega_{\boldsymbol{x}}^+ = \operatorname{Re}(\operatorname{dlog} g_{\boldsymbol{x}}) = \operatorname{dlog}|g_{\boldsymbol{x}}|, \qquad \omega_{\boldsymbol{x}}^- = \operatorname{Im}(\operatorname{dlog} g_{\boldsymbol{x}}) = \operatorname{darg}(g_{\boldsymbol{x}}).$$

Then for any $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n \in (\mathbf{R}/\mathbf{Z})^2 \smallsetminus \{\mathbf{0}\}$ and any sequence of signs $\varepsilon_1,\dots,\varepsilon_n \in \{\pm\}$, consider the regularised iterated integral

$$\Lambda^{\varepsilon_1\cdots\varepsilon_n}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n) = \int_0^\infty \omega_{\boldsymbol{x}_1}^{\varepsilon_1}\dots\omega_{\boldsymbol{x}_n}^{\varepsilon_n}.$$

For example, taking the real and imaginary parts in Proposition 44, we get

$$\Lambda^+(\boldsymbol{x}) = 0 \quad\text{and}\quad \Lambda^-(\boldsymbol{x}) = 2\pi\Big(x_1-\frac{1}{2}\Big)\Big(x_2-\frac{1}{2}\Big) \qquad (0 < x_1,x_2 < 1). \tag{24}$$

As discussed in Example 42, the function $(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n) \mapsto \Lambda(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)$ is differentiable on the domain $(\mathbf{R}/\mathbf{Z} \smallsetminus \{0\})^{2n}$ and its partial derivatives with respect to the x_2-components of indices $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n$ can be explicitly computed using equation (22); we make use of this differentiation in Section 7.

4. A baby case: The K_2 regulator and double modular values

Let $Y(N)$ be the modular curve over $\mathbf{Q}$ of level $N \geq 1$. The cup-products $\{g_{\boldsymbol{a}}, g_{\boldsymbol{b}}\}$ of two Siegel units $g_{\boldsymbol{a}}$ and $g_{\boldsymbol{b}}$ provide important elements in the K-group $K_2^{(2)}(Y(N))$. Let us consider their images under the Beilinson regulator map [16, 2.10]

$$K_2^{(2)}(Y(N)) \longrightarrow H^1(Y(N)(\mathbf{C}),\mathbf{R}\cdot i).$$

The regulator of $\{g_{\boldsymbol{a}}, g_{\boldsymbol{b}}\}$ is represented by the differential form $i\,\eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}})$ on $Y(N)(\mathbf{C})$, where

$$\eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}}) = \log|g_{\boldsymbol{a}}|\operatorname{darg} g_{\boldsymbol{b}} - \log|g_{\boldsymbol{b}}|\operatorname{darg} g_{\boldsymbol{a}}.$$

The regulator integral of $\eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}})$ along the modular symbol $\{0,i\infty\}$ can be computed in terms of L-values at $s=0$ of modular forms of weight 2 and level $\Gamma(N)$ [6]. Here we show that this regulator integral can be expressed in terms of double Eisenstein values.

Proposition 45. *Let $\boldsymbol{a},\boldsymbol{b} \in (\mathbf{Z}/N\mathbf{Z})^2 \smallsetminus \{\mathbf{0}\}$. We have*

$$\int_0^\infty \eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}}) = \operatorname{Im}\Lambda(\boldsymbol{a},\boldsymbol{b}) - \Lambda^+(\boldsymbol{a})\Lambda^-(\boldsymbol{b}) + R_{\boldsymbol{a}}\Lambda^-(\boldsymbol{b}) - R_{\boldsymbol{b}}\Lambda^-(\boldsymbol{a}),$$

where $R_{\boldsymbol{x}}$ is the regularised value of $\log|g_{\boldsymbol{x}}|$ at ∞, obtained from (16) by taking the real part of the constant term. In the case the coordinates of $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero, this simplifies to

$$\int_0^\infty \eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}}) = \operatorname{Im}\Lambda(\boldsymbol{a},\boldsymbol{b}).$$

Proof. Recall that $\operatorname{dlog}|g_{\boldsymbol{x}}|$ and $\operatorname{darg} g_{\boldsymbol{x}}$ are admissible by Lemmas 36 and 37, and note that $\log|g_{\boldsymbol{x}}(\tau)| = R_{\boldsymbol{x}} - \int_\tau^\infty \operatorname{dlog}|g_{\boldsymbol{x}}|$ by Lemma 8. Then

$$\eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}})(\tau) = \Big(R_{\boldsymbol{a}} - \int_\tau^\infty \operatorname{dlog}|g_{\boldsymbol{a}}|\Big)\operatorname{darg} g_{\boldsymbol{b}}(\tau) - \Big(R_{\boldsymbol{b}} - \int_\tau^\infty \operatorname{dlog}|g_{\boldsymbol{b}}|\Big)\operatorname{darg} g_{\boldsymbol{a}}(\tau).$$

This expression shows that the form $\eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}})$ is admissible at ∞. It is also admissible at 0 since $\sigma^*(g_{\boldsymbol{x}}) = g_{\boldsymbol{x}\sigma}$ in $\mathcal{O}(Y(N))^\times \otimes \mathbf{Q}$ by [16, Lemma 1.7(1)]. Integrating from 0 to ∞ and using Proposition 26, this gives

$$\int_0^\infty \eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}}) = R_{\boldsymbol{a}}\Lambda^-(\boldsymbol{b}) - \Lambda^{-+}(\boldsymbol{b},\boldsymbol{a}) - R_{\boldsymbol{b}}\Lambda^-(\boldsymbol{a}) + \Lambda^{-+}(\boldsymbol{a},\boldsymbol{b}).$$

Using the shuffle relation $\Lambda^-(\boldsymbol{b})\Lambda^+(\boldsymbol{a}) = \Lambda^{-+}(\boldsymbol{b},\boldsymbol{a}) + \Lambda^{+-}(\boldsymbol{a},\boldsymbol{b})$, we arrive at

$$\begin{aligned}\int_0^\infty \eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}}) &= R_{\boldsymbol{a}}\Lambda^-(\boldsymbol{b}) - \Lambda^-(\boldsymbol{b})\Lambda^+(\boldsymbol{a}) + \Lambda^{+-}(\boldsymbol{a},\boldsymbol{b}) - R_{\boldsymbol{b}}\Lambda^-(\boldsymbol{a}) + \Lambda^{-+}(\boldsymbol{a},\boldsymbol{b})\\ &= \operatorname{Im}\Lambda(\boldsymbol{a},\boldsymbol{b}) - \Lambda^+(\boldsymbol{a})\Lambda^-(\boldsymbol{b}) + R_{\boldsymbol{a}}\Lambda^-(\boldsymbol{b}) - R_{\boldsymbol{b}}\Lambda^-(\boldsymbol{a}). \qquad \square\end{aligned}$$

Proposition 45 could be refined by considering the *integral* regulator of $\{g_{\boldsymbol{a}}, g_{\boldsymbol{b}}\}$, which is a class in $H^1(Y(N)(\mathbf{C}), \mathbf{C}/(2\pi i)^2\mathbf{Q})$; for the definition of this regulator map see [10, Exercise 7.10, p. 93]. The associated regulator integral should then involve the real part of $\Lambda(\boldsymbol{a},\boldsymbol{b})$.

5. The Goncharov regulator in terms of triple modular values

In [8] the first author constructed classes $\xi(\boldsymbol{a},\boldsymbol{b})$ in $K_4^{(3)}(Y(N))$, for $\boldsymbol{a},\boldsymbol{b} \in (\mathbf{Z}/N\mathbf{Z})^2$. Our aim in this section is to express the regulator of $\xi(\boldsymbol{a},\boldsymbol{b})$ in terms of triple Eisenstein values. Since we integrate from 0 to ∞, the regulator integral depends on a choice of representative $\tilde{\xi}(\boldsymbol{a},\boldsymbol{b})$ of $\xi(\boldsymbol{a},\boldsymbol{b})$ in the Goncharov complex $\Gamma(Y(N),3)$. We choose the one given in [8, Construction 6.1]. One consequence of our main formula (Theorem 54) is that the regulator integral interpolates as a function of $\boldsymbol{a},\boldsymbol{b} \in (\mathbf{R}/\mathbf{Z})^2$, at least in the domain where the coordinates of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}+\boldsymbol{b}$ are non-zero.

Let us recall the construction of $\tilde{\xi}(\boldsymbol{a},\boldsymbol{b})$. Let $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in (\mathbf{Z}/N\mathbf{Z})^2$ be such that $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c} = \mathbf{0}$. From now on, *we assume that all the coordinates of* $\boldsymbol{a}$, $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *are non-zero*. This considerably simplifies the expressions with multiple modular values below.

According to [8, Section 4], there is a triangulation

$$g_{\boldsymbol{a}} \wedge g_{\boldsymbol{b}} + g_{\boldsymbol{b}} \wedge g_{\boldsymbol{c}} + g_{\boldsymbol{c}} \wedge g_{\boldsymbol{a}} = \sum_i m_i \cdot u_i \wedge (1-u_i) \quad \text{in } \Lambda^2\mathcal{O}(Y(N))^\times \otimes \mathbf{Q}, \tag{25}$$

where u_i and $1-u_i$ are certain modular units, and $m_i \in \mathbf{Q}$. Then our cocycle is

$$\tilde{\xi}(\boldsymbol{a},\boldsymbol{b}) := \sum_i m_i\{u_i\}_2 \otimes \frac{g_{\boldsymbol{b}}}{g_{\boldsymbol{a}}} \in B_2(\mathbf{Q}(Y(N))) \otimes \mathcal{O}(Y(N))^\times \otimes \mathbf{Q}.$$

For the definition of the group $B_2(F)$ of a field F, see [12, Section 2.2].

Recall the expression of Goncharov's explicit regulator map $r_3(2)$. Let $D\colon \mathbf{P}^1(\mathbf{C}) \to \mathbf{R}$ be the Bloch–Wigner dilogarithm. For any two functions f,g on a Riemann surface, define the 1-form

$$r_3(2)(\{f\}_2 \otimes g) = -D(f)\cdot \operatorname{darg} g - \frac{1}{3}\log|g|\cdot \alpha((1-f)\wedge f), \tag{26}$$

where

$$\alpha(f_1 \wedge f_2) = -\log|f_1|\operatorname{dlog}|f_2| + \log|f_2|\operatorname{dlog}|f_1|.$$

By linearity using (26), the regulator 1-form associated to $\tilde{\xi}(\boldsymbol{a},\boldsymbol{b})$ is

$$\begin{aligned} r_3(2)(\tilde{\xi}(\boldsymbol{a},\boldsymbol{b})) &= \sum_i m_i\Big(-D(u_i)\cdot \operatorname{darg}(g_{\boldsymbol{b}}/g_{\boldsymbol{a}}) - \frac{1}{3}\log|g_{\boldsymbol{b}}/g_{\boldsymbol{a}}|\cdot\alpha((1-u_i)\wedge u_i)\Big)\\ &= -\Big(\sum_i m_i D(u_i)\Big)\operatorname{darg}(g_{\boldsymbol{b}}/g_{\boldsymbol{a}}) + \frac{1}{3}\log|g_{\boldsymbol{b}}/g_{\boldsymbol{a}}|\cdot\alpha(g_{\boldsymbol{a}}\wedge g_{\boldsymbol{b}} + g_{\boldsymbol{b}}\wedge g_{\boldsymbol{c}} + g_{\boldsymbol{c}}\wedge g_{\boldsymbol{a}}).\end{aligned}$$

Let us introduce the following notation for the regulator integral:

$$\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) = \int_0^\infty r_3(2)(\tilde{\xi}(\boldsymbol{a},\boldsymbol{b})).$$

By [8, Corollary 7.3], this integral is absolutely convergent. To express $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ as a triple iterated integral, a key idea is to cast the Bloch–Wigner function $D(z)$ as a primitive:

$$d(D(z)) = \eta(z\wedge(1-z)), \qquad \text{where} \quad \eta(f\wedge g) = \log|f|\operatorname{darg}(g) - \log|g|\operatorname{darg}(f).$$

Then using (25) we can write

$$d\Big(\sum_i m_i D(u_i)\Big) = \sum_i m_i\,\eta(u_i\wedge(1-u_i)) = \eta(g_{\boldsymbol{a}}\wedge g_{\boldsymbol{b}} + g_{\boldsymbol{b}}\wedge g_{\boldsymbol{c}} + g_{\boldsymbol{c}}\wedge g_{\boldsymbol{a}}). \tag{27}$$

As we saw in the proof of Proposition 45, the right-hand side of (27) is an admissible form on $]0, i\infty[$. Moreover, if u is a modular unit such that $1-u$ is also a modular unit, then $\eta(u \wedge (1-u))$ is admissible and Lemma 18 implies that $D(u)$ is admissible. Actually $D(u(\tau))$ converges as $\tau \to \infty$ since D is continuous on $\mathbf{P}^1(\mathbf{C})$. So the regularised value of $D(u)$ at ∞ is simply $D(u(\infty))$, and Lemma 8 tells us that

$$D(u(\tau)) = D(u(\infty)) - \int_\tau^\infty \eta(u, 1-u).$$

Note that the form $D(u)\,\mathrm{darg}(g_{\boldsymbol{b}}/g_{\boldsymbol{a}})$ is then admissible. Therefore, using (27) the regulator integral can be written

$$\mathcal{G}(\boldsymbol{a}, \boldsymbol{b}) = A_1 + A_2 + A_3,$$

where

$$A_1 = -\sum_i m_i D(u_i(\infty)) \int_0^\infty \mathrm{darg}(g_{\boldsymbol{b}}/g_{\boldsymbol{a}}), \tag{28}$$

$$A_2 = \int_0^\infty \mathrm{darg}(g_{\boldsymbol{b}}/g_{\boldsymbol{a}})(\tau) \int_\tau^\infty \eta(g_{\boldsymbol{a}} \wedge g_{\boldsymbol{b}} + g_{\boldsymbol{b}} \wedge g_{\boldsymbol{c}} + g_{\boldsymbol{c}} \wedge g_{\boldsymbol{a}}), \tag{29}$$

$$A_3 = \frac{1}{3}\int_0^\infty \log|g_{\boldsymbol{b}}/g_{\boldsymbol{a}}| \cdot \alpha(g_{\boldsymbol{a}} \wedge g_{\boldsymbol{b}} + g_{\boldsymbol{b}} \wedge g_{\boldsymbol{c}} + g_{\boldsymbol{c}} \wedge g_{\boldsymbol{a}}). \tag{30}$$

Similar arguments show that the integrand of A_3 is admissible on $]0, i\infty[$.

5.1. **The A_1 term.** The explicit form of the triangulation (25) is given by [8, Theorem 4.3]:

$$\sum_i m_i \{u_i\}_2 = \frac{1}{N^2} \sum_{\boldsymbol{x} \in (\mathbf{Z}/N\mathbf{Z})^2} \{u(\boldsymbol{0}, \boldsymbol{x}, \boldsymbol{a}-\boldsymbol{x}, \boldsymbol{b}+\boldsymbol{x})\}_2$$
$$- \frac{1}{4N^4} \sum_{\boldsymbol{x},\boldsymbol{y} \in (\mathbf{Z}/N\mathbf{Z})^2} \big(\{u(\boldsymbol{0}, \boldsymbol{a}, \boldsymbol{c}+2\boldsymbol{x}, \boldsymbol{y})\}_2 + \{u(\boldsymbol{0}, \boldsymbol{c}, \boldsymbol{b}+2\boldsymbol{x}, \boldsymbol{y})\}_2 + \{u(\boldsymbol{0}, \boldsymbol{b}, \boldsymbol{a}+2\boldsymbol{x}, \boldsymbol{y})\}_2\big),$$

which simplifies to

$$\sum_i m_i \{u_i\}_2 = \frac{1}{N^2} \sum_{\boldsymbol{x} \in (\mathbf{Z}/N\mathbf{Z})^2} \{u(\boldsymbol{0}, \boldsymbol{x}, \boldsymbol{a}-\boldsymbol{x}, \boldsymbol{b}+\boldsymbol{x})\}_2$$

in the case N is odd. By convention, in the above sums we keep only those terms $u(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$ for which $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}$ are distinct in $(\mathbf{Z}/N\mathbf{Z})^2/\pm 1$. The same convention takes place below.

Lemma 46. *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in (\mathbf{Z}/N\mathbf{Z})^2$ such that $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c} = \boldsymbol{0}$. Assume that all the coordinates of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are non-zero. Then*

$$\sum_{\boldsymbol{x} \in (\mathbf{Z}/N\mathbf{Z})^2} D\big(u(\boldsymbol{0}, \boldsymbol{x}, \boldsymbol{a}-\boldsymbol{x}, \boldsymbol{b}+\boldsymbol{x})(\infty)\big) = 0.$$

Proof. We write $\hat{x}$ for the representative of x/N, where $x \in \mathbf{Z}/N\mathbf{Z}$, on the interval $[0,1)$, so that $\hat{x} \in \frac{1}{N}\mathbf{Z} \cap [0,1)$. According to [8, Lemma 3.4] we have

$$u(\boldsymbol{0}, \boldsymbol{x}, \boldsymbol{a}-\boldsymbol{x}, \boldsymbol{b}+\boldsymbol{x}) = \frac{\Delta^2_{\hat{\boldsymbol{a}}-\hat{\boldsymbol{x}}}}{\Delta_{\hat{\boldsymbol{a}}}\Delta_{\hat{\boldsymbol{a}}-2\hat{\boldsymbol{x}}}} \, \frac{\Delta_{\hat{\boldsymbol{b}}}\Delta_{\hat{\boldsymbol{b}}+2\hat{\boldsymbol{x}}}}{\Delta^2_{\hat{\boldsymbol{b}}+\hat{\boldsymbol{x}}}}, \tag{31}$$

where

$$\Delta_{u,v} = (-e(-v))^{\lfloor u \rfloor} q^{B_2(\{u\})/2}\big(1 - e(v)\mathbf{1}_{u \in \mathbf{Z}} + O(q^{1/N})\big) \quad \text{as } q \to 0.$$

We now collect relevant information for determining when the unit (31) has order 0 at ∞ and what is the corresponding constant term in the latter case.

For $0 \le \hat{a}_1 < 1$ and $0 \le \hat{x}_1 < 1$ we have

$$\mathrm{ord}_q \frac{\Delta^2_{\hat{\boldsymbol{a}}-\hat{\boldsymbol{x}}}}{\Delta_{\hat{\boldsymbol{a}}}\Delta_{\hat{\boldsymbol{a}}-2\hat{\boldsymbol{x}}}} = \begin{cases} -\hat{x}_1^2 & \text{if } 0 \le \hat{x}_1 \le \frac{1}{2}\hat{a}_1, \\ -(1-\hat{x}_1)^2 + 1 - \hat{a}_1 & \text{if } \frac{1}{2}\hat{a}_1 < \hat{x}_1 \le \hat{a}_1, \\ -\hat{x}_1^2 + \hat{a}_1 & \text{if } \hat{a}_1 < \hat{x}_1 \le \frac{1}{2} + \frac{1}{2}\hat{a}_1, \\ -(1-\hat{x}_1)^2 & \text{if } \frac{1}{2} + \frac{1}{2}\hat{a}_1 < \hat{x}_1 < 1. \end{cases}$$

If moreover $\hat{a}_1, \hat{a}_1 - \hat{x}_1, \hat{a}_1 - 2\hat{x}_1 \notin \mathbf{Z}$, we find

$$\frac{(-e(-\hat{a}_2+\hat{x}_2))^{2\lfloor \hat{a}_1-\hat{x}_1\rfloor}}{(-e(-\hat{a}_2))^{\lfloor \hat{a}_1\rfloor}(-e(-\hat{a}_2+2\hat{x}_2))^{\lfloor \hat{a}_1-2\hat{x}_1\rfloor}} = \begin{cases} 1 & \text{if } 0 \le \hat{x}_1 \le \frac{1}{2}\hat{a}_1, \\ -e(-\hat{a}_2+2\hat{x}_2) & \text{if } \frac{1}{2}\hat{a}_1 < \hat{x}_1 \le \hat{a}_1, \\ -e(\hat{a}_2) & \text{if } \hat{a}_1 < \hat{x}_1 \le \frac{1}{2}+\frac{1}{2}\hat{a}_1, \\ e(2\hat{x}_2) & \text{if } \frac{1}{2}+\frac{1}{2}\hat{a}_1 < \hat{x}_1 < 1. \end{cases}$$

Similarly, for $0 \le \hat{b}_1 < 1$ and $0 \le \hat{x}_1 < 1$ we have

$$\operatorname{ord}_q \frac{\Delta_{\hat{\boldsymbol{b}}}\Delta_{\hat{\boldsymbol{b}}+2\hat{\boldsymbol{x}}}}{\Delta^2_{\hat{\boldsymbol{b}}+\hat{\boldsymbol{x}}}} = \begin{cases} \hat{x}_1^2 & \text{if } 0 \le \hat{x}_1 < \frac{1}{2}-\frac{1}{2}\hat{b}_1, \\ (1-\hat{x}_1)^2 - \hat{b}_1 & \text{if } \frac{1}{2}-\frac{1}{2}\hat{b}_1 \le \hat{x}_1 < 1-\hat{b}_1, \\ \hat{x}_1^2 - (1-\hat{b}_1) & \text{if } 1-\hat{b}_1 \le \hat{x}_1 < 1-\frac{1}{2}\hat{b}_1, \\ (1-\hat{x}_1)^2 & \text{if } 1-\frac{1}{2}\hat{b}_1 \le \hat{x}_1 < 1. \end{cases}$$

If moreover $\hat{b}_1, \hat{b}_1 + \hat{x}_1, \hat{b}_1 + 2\hat{x}_1 \notin \mathbf{Z}$, we obtain

$$\frac{(-e(-\hat{b}_2))^{\lfloor \hat{b}_1\rfloor}(-e(-\hat{b}_2-2\hat{x}_2))^{\lfloor \hat{b}_1+2\hat{x}_1\rfloor}}{(-e(-\hat{b}_2-\hat{x}_2))^{2\lfloor \hat{b}_1+\hat{x}_1\rfloor}} = \begin{cases} 1 & \text{if } 0 \le \hat{x}_1 < \frac{1}{2}-\frac{1}{2}\hat{b}_1, \\ -e(-\hat{b}_2-2\hat{x}_2) & \text{if } \frac{1}{2}-\frac{1}{2}\hat{b}_1 \le \hat{x}_1 < 1-\hat{b}_1, \\ -e(\hat{b}_2) & \text{if } 1-\hat{b}_1 \le \hat{x}_1 < 1-\frac{1}{2}\hat{b}_1, \\ e(-2\hat{x}_2) & \text{if } 1-\frac{1}{2}\hat{b}_1 \le \hat{x}_1 < 1. \end{cases}$$

Our sum of interest is

$$\Sigma(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = \sum_{\boldsymbol{x}\in(\mathbf{Z}/N\mathbf{Z})^2} D\big(u(\mathbf{0},\boldsymbol{x},\boldsymbol{a}-\boldsymbol{x},\boldsymbol{b}+\boldsymbol{x})(\infty)\big).$$

Notice the following symmetries of the sum: it is invariant under $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \mapsto (-\boldsymbol{a},-\boldsymbol{b},-\boldsymbol{c})$ (as $u(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})$ is defined for indices in $(\mathbf{Z}/N\mathbf{Z})^2/\pm1$) and it is cyclic invariant. The latter follows from changing the summation for $u(\mathbf{0},\boldsymbol{x},\boldsymbol{a}-\boldsymbol{x},\boldsymbol{b}+\boldsymbol{x}) = u(\mathbf{0},-\boldsymbol{x},-\boldsymbol{a}+\boldsymbol{x},\boldsymbol{b}+\boldsymbol{x})$ to the one over $\boldsymbol{y}=\boldsymbol{b}+\boldsymbol{x}$ and using the definition of $u(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})$ as the cross-ratio of Weierstrass $\wp$-functions:

$$\Sigma(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = \sum_{\boldsymbol{y}\in(\mathbf{Z}/N\mathbf{Z})^2} D\big(u(\mathbf{0},\boldsymbol{b}-\boldsymbol{y},\boldsymbol{c}+\boldsymbol{y},\boldsymbol{y})(\infty)\big) = \sum_{\boldsymbol{y}\in(\mathbf{Z}/N\mathbf{Z})^2} D\big(u(\mathbf{0},\boldsymbol{y},\boldsymbol{b}-\boldsymbol{y},\boldsymbol{c}+\boldsymbol{y})(\infty)\big) = \Sigma(\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}).$$

For similar reasons $\Sigma(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ is antisymmetric under transpositions:

$$\begin{aligned} \Sigma(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) &= \sum_{\boldsymbol{x}\in(\mathbf{Z}/N\mathbf{Z})^2} D\big(1-u(\mathbf{0},\boldsymbol{a}-\boldsymbol{x},\boldsymbol{x},\boldsymbol{b}+\boldsymbol{x})(\infty)\big) = -\sum_{\boldsymbol{x}\in(\mathbf{Z}/N\mathbf{Z})^2} D\big(u(\mathbf{0},\boldsymbol{a}-\boldsymbol{x},\boldsymbol{x},-\boldsymbol{b}-\boldsymbol{x})(\infty)\big) \\ &= -\sum_{\boldsymbol{y}\in(\mathbf{Z}/N\mathbf{Z})^2} D\big(u(\mathbf{0},\boldsymbol{y},\boldsymbol{a}-\boldsymbol{y},\boldsymbol{c}+\boldsymbol{y})(\infty)\big) = -\Sigma(\boldsymbol{a},\boldsymbol{c},\boldsymbol{b}). \end{aligned}$$

Recall that $\hat{a}_1, \hat{b}_1, \hat{c}_1$ are the representatives of $a_1/N, b_1/N, c_1/N$ in the interval $(0,1)$. After possibly replacing $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ by $(-\boldsymbol{a},-\boldsymbol{b},-\boldsymbol{c})$ we may assume that $\hat{a}_1+\hat{b}_1+\hat{c}_1 = 1$; furthermore, since our goal is to demonstrate that $\Sigma(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = 0$, after possibly permuting $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ we may assume that $0 < \hat{a}_1 \le \hat{b}_1 \le \hat{c}_1 < 1$. Then we get

$$0 < \tfrac{1}{2}\hat{a}_1 < \hat{a}_1 \le \tfrac{1}{2}-\tfrac{1}{2}\hat{b}_1 < \tfrac{1}{2}+\tfrac{1}{2}\hat{a}_1 \le 1-\hat{b}_1 < 1-\tfrac{1}{2}\hat{b}_1 < 1,$$

so that

$$\operatorname{ord}_q u(\mathbf{0},\boldsymbol{x},\boldsymbol{a}-\boldsymbol{x},\boldsymbol{b}+\boldsymbol{x}) = \begin{cases} 0 & \text{if } 0 \le \hat{x}_1 \le \frac{1}{2}\hat{a}_1, \\ 2\hat{x}_1 - \hat{a}_1 \ne 0 & \text{if } \frac{1}{2}\hat{a}_1 < \hat{x}_1 \le \hat{a}_1, \\ \hat{a}_1 \ne 0 & \text{if } \hat{a}_1 < \hat{x}_1 < \frac{1}{2}-\frac{1}{2}\hat{b}_1, \\ \hat{a}_1 - \hat{b}_1 + 1 - 2\hat{x}_1 & \text{if } \frac{1}{2}-\frac{1}{2}\hat{b}_1 \le \hat{x}_1 \le \frac{1}{2}+\frac{1}{2}\hat{a}_1, \\ -\hat{b}_1 \ne 0 & \text{if } \frac{1}{2}+\frac{1}{2}\hat{a}_1 < \hat{x}_1 < 1-\hat{b}_1, \\ \hat{b}_1 - 2(1-\hat{x}_1) \ne 0 & \text{if } 1-\hat{b}_1 \le \hat{x}_1 < 1-\frac{1}{2}\hat{b}_1, \\ 0 & \text{if } 1-\frac{1}{2}\hat{b}_1 \le \hat{x}_1 < 1. \end{cases}$$

This means that $\operatorname{ord}_q u(\mathbf{0},\boldsymbol{x},\boldsymbol{a}-\boldsymbol{x},\boldsymbol{b}+\boldsymbol{x})=0$ iff $\hat{x}_1\in[0,\frac12\hat{a}_1]\cup\{\frac12(\hat{a}_1-\hat{b}_1+1)\}\cup[1-\frac12\hat{b}_1,1)$. Furthermore, the constant term of $u(\mathbf{0},\boldsymbol{x},\boldsymbol{a}-\boldsymbol{x},\boldsymbol{b}+\boldsymbol{x})$ is equal to 1 for $\hat{x}_1\in[0,\frac12\hat{a}_1)\cup(1-\frac12\hat{b}_1,1)$, and it is

$$\begin{cases}1/(1-e(\hat{a}_2-2\hat{x}_2)) & \text{if } \hat{x}_1=\frac12\hat{a}_1,\\ e(\hat{a}_2-\hat{b}_2-2\hat{x}_2) & \text{if } \hat{x}_1=\frac12(\hat{a}_1-\hat{b}_1+1),\\ 1-e(\hat{b}_2+2\hat{x}_2) & \text{if } \hat{x}_1=1-\frac12\hat{b}_1.\end{cases}$$

No matter whether these values of $\hat{x}_1$ are in $\frac1N\mathbf{Z}$ or not, using the relations $D(1-x)=D(1/x)=-D(x)$ we see that the resulting sums over $\hat{x}_2\in\frac1N\mathbf{Z}/\mathbf{Z}$ vanish. For example,

$$\sum_{\hat{x}_2\in\frac1N\mathbf{Z}/\mathbf{Z}} D\big(1/(1-e(\hat{a}_2-2\hat{x}_2))\big)=\sum_{\hat{x}_2\in\frac1N\mathbf{Z}/\mathbf{Z}} D\big(e(\hat{a}_2-2\hat{x}_2)\big)=0,$$

because the latter sum involves pairs of complex conjugate roots of unity, apart from possibly ±1, and $D(\overline{x})=-D(x)$. Therefore, $\Sigma(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})=0$. □

Lemma 47. *Let $N>1$ be an integer, and let $u\neq1$ be an N-th root of unity. Then*

$$\sum_{v:v^N=1} D\Big(\frac{1-v}{1-u}\Big)=\frac{N}{2}D(u).$$

Proof. We use the 5-term relation for the Bloch–Wigner dilogarithm with the quintuple $(\infty,0,1,v,u)$:

$$D(v)+D\Big(\frac{1-u^{-1}}{1-v^{-1}}\Big)+D\Big(\frac{u-1}{u-v}\Big)+D\Big(\frac{u}{v}\Big)+D\Big(\frac{-u}{1-u}\Big)=0.$$

Using the relations $D(1-x)=D(1/x)=D(\bar{x})=-D(x)$ as well as $u^{-1}=\bar{u}$ and $v^{-1}=\bar{v}$, this can be written

$$D(v)+D\Big(\frac{1-v}{1-u}\Big)+D\Big(\frac{1-v}{1-u}\Big)+D\Big(\frac{u}{v}\Big)-D(u)=0.$$

Summing over $v\neq1,u$ and using the relation $\sum_{v:v^N=1}D(v)=0$, we deduce the required result. □

Lemma 48. *Let $\boldsymbol{a},\boldsymbol{c}\in(\mathbf{Z}/N\mathbf{Z})^2$ and the coordinates of $\boldsymbol{a}$ non-zero. Then the double sum*

$$\sum_{\boldsymbol{x},\boldsymbol{y}\in(\mathbf{Z}/N\mathbf{Z})^2} D\big(u(\mathbf{0},\boldsymbol{a},\boldsymbol{c}+2\boldsymbol{x},\boldsymbol{y})(\infty)\big) \tag{32}$$

vanishes.

Proof. To compute the double sum (32) notice that

$$u(\mathbf{0},\boldsymbol{a},\boldsymbol{z},\boldsymbol{y})=\frac{\mathcal{E}(\boldsymbol{z},\boldsymbol{a})}{\mathcal{E}(\boldsymbol{y},\boldsymbol{a})}$$

where $\mathcal{E}(\boldsymbol{z},\boldsymbol{a})=\Delta_{\hat{\boldsymbol{z}}}^2/(\Delta_{\hat{\boldsymbol{z}}+\hat{\boldsymbol{a}}}\Delta_{\hat{\boldsymbol{z}}-\hat{\boldsymbol{a}}})$, and the sum can be rearranged to run over $\boldsymbol{z},\boldsymbol{y}$. Notice that this rearrangement affects the summation on $\boldsymbol{z}=(z_1,z_2)$ in the case of even N, because it becomes 4 times a sum over $\boldsymbol{z}\in(\mathbf{Z}/N\mathbf{Z})^2$ subject to the congruence conditions $z_1\equiv c_1$, $z_2\equiv c_2 \bmod 2$. Changing $\boldsymbol{a}$ into $-\boldsymbol{a}$ does not change the modular unit $u(\mathbf{0},\boldsymbol{a},\boldsymbol{c}+2\boldsymbol{x},\boldsymbol{y})$, hence we can assume that the representative $\hat{a}_1$ of a_1/N satisfies $0<\hat{a}_1\le\frac12\le1-\hat{a}_1<1$.

With $0\le\hat{z}_1<1$ we obtain

$$\operatorname{ord}_q\mathcal{E}(\boldsymbol{z},\boldsymbol{a})=\hat{a}_1(1-\hat{a}_1)-\min\{\hat{a}_1,1-\hat{a}_1,\hat{z}_1,1-\hat{z}_1\}=\hat{a}_1(1-\hat{a}_1)-\min\{\hat{a}_1,\hat{z}_1,1-\hat{z}_1\},$$

while the leading coefficient of $\mathcal{E}(\boldsymbol{z},\boldsymbol{a})$ is equal to

$$\begin{cases} \text{(a)} \ -(1-e(\hat{z}_2))^2/e(\hat{z}_2-\hat{a}_2) = 4e(\hat{a}_2)\sin^2(\pi\hat{z}_2) & \text{if } \hat{z}_1 = 0, \\ \text{(b)} \ -1/e(\hat{z}_2-\hat{a}_2) = -e(\hat{a}_2)e(-\hat{z}_2) & \text{if } 0 < \hat{z}_1 < \hat{a}_1, \\ \text{(c)} \ -1/e(-\hat{z}_2-\hat{a}_2) = -e(\hat{a}_2)e(\hat{z}_2) & \text{if } 0 < 1-\hat{z}_1 < \hat{a}_1, \\ \text{(d)} \ 1/(1-e(\hat{z}_2-\hat{a}_2)) & \text{if } \hat{z}_1 = \hat{a}_1 < 1-\hat{a}_1 = 1-\hat{z}_1, \\ \text{(e)} \ 1/(1-e(-\hat{z}_2-\hat{a}_2)) & \text{if } 1-\hat{z}_1 = \hat{a}_1 < 1-\hat{a}_1 = \hat{z}_1, \\ \text{(f)} \ 1 & \text{if } \hat{a}_1 < \min\{\hat{z}_1, 1-\hat{z}_1\}, \\ \text{(g)} \ \frac{1}{2}e(\hat{a}_2)/(\cos(2\pi\hat{a}_2)-\cos(2\pi\hat{z}_2)) & \text{if } \hat{a}_1 = \hat{z}_1 = \frac{1}{2} \end{cases}$$

(the case $1-\hat{a}_1 < \min\{\hat{z}_1, 1-\hat{z}_1\}$ is excluded from the consideration because $\hat{a}_1 \le \frac{1}{2}$). We now want to control when the terms $\mathcal{E}(\boldsymbol{z},\boldsymbol{a})/\mathcal{E}(\boldsymbol{y},\boldsymbol{a})$ in the sum (32) have constant terms, that is, when

$$\min\{\hat{a}_1, \hat{z}_1, 1-\hat{z}_1\} = \min\{\hat{a}_1, \hat{y}_1, 1-\hat{y}_1\}. \tag{33}$$

For each of these situations, call them $(\mathrm{r}_z)\times(\mathrm{s}_y)$ with $\mathrm{r},\mathrm{s} \in \{\mathrm{a},\dots,\mathrm{g}\}$, we want to compute a related sum of the dilogarithms of the products of corresponding constant terms over $\hat{z}_2, \hat{y}_2 \in \frac{1}{N}\mathbf{Z}/\mathbf{Z}$. Because of condition (33), the case (a_z) occurs if only (a_y) occurs, and vice versa; the corresponding sum of $D(\sin^2(\pi\hat{z}_2)/\sin^2(\pi\hat{y}_2))$ over $\hat{z}_2, \hat{y}_2 \in \frac{1}{N}\mathbf{Z}/\mathbf{Z}$ vanishes, because each term is zero (as $D(x) = 0$ for $x \in \mathbf{R}$). Similarly, the case (g_z) exclusively pairs up with (g_y), and the dilogarithm arguments are real-valued for this combination as well, leading to the zero value for the sum in question. The case (f_z) may only pair up with (f_y), in which case the sum of $D(1)$ terms is void, or with (d_y) or (e_y). If one of the latter situations occur, for instance (d_y), we can write our sum as

$$\sum_{\hat{z}_2,\hat{y}_2\in\frac{1}{N}\mathbf{Z}/\mathbf{Z}}' D(1-e(\hat{y}_2-\hat{a}_2)) = -\sum_{\hat{z}_2,\hat{y}_2\in\frac{1}{N}\mathbf{Z}/\mathbf{Z}}' D(e(\hat{y}_2-\hat{a}_2)) = -\sum_{\hat{z}_2\in\frac{1}{N}\mathbf{Z}/\mathbf{Z}}' \sum_{\hat{t}\in\frac{1}{N}\mathbf{Z}/\mathbf{Z}} D(e(\hat{t})),$$

where $\sum'_{\hat{z}_2}$ means that we sum over $\hat{z}_2$ under the constraint $z_2 \equiv c_2 \bmod 2$ if N is even. The double sum then vanishes because the sum over $\hat{t}$ does. Similarly, the case (f_y) pairs up with (f_z) (which we already discussed), or with (d_z) or (e_z), and we argue as above using the summation

$$\sum_{\hat{z}_2\in\frac{1}{N}\mathbf{Z}/\mathbf{Z}}' D(e(\pm\hat{z}_2-\hat{a}_2)) = 0$$

followed from

$$\sum_{\hat{t}\in\frac{2}{N}\mathbf{Z}/\mathbf{Z}} D(e(\hat{t})) = \sum_{\hat{t}\in\frac{1}{N}+\frac{2}{N}\mathbf{Z}/\mathbf{Z}} D(e(\hat{t})) = 0 \tag{34}$$

in the case of even N (because conjugate roots of unity $e(\hat{t})$ and $e(-\hat{t})$, when different from $\pm1 \in \mathbf{R}$, combine). Furthermore, the situations $(\mathrm{d}_z)\times(\mathrm{d}_y)$, $(\mathrm{d}_z)\times(\mathrm{e}_y)$, $(\mathrm{e}_z)\times(\mathrm{d}_y)$ and $(\mathrm{e}_z)\times(\mathrm{e}_y)$ are all treated with the help of Lemma 47 applied to the summation over $\hat{y}_2 \in \frac{1}{N}\mathbf{Z}/\mathbf{Z}$ and the external summation $\sum'_{\hat{z}_2\in\frac{1}{N}\mathbf{Z}/\mathbf{Z}}$ is performed on the basis of (34) if N is even. Finally, the cases (b_z), (c_z) may only pair up with (b_y), (c_y) in view of condition (33), and we obtain the sum

$$\sum_{\hat{z}_2,\hat{y}_2\in\frac{1}{N}\mathbf{Z}/\mathbf{Z}}' D(e(\pm\hat{z}_2\pm\hat{y}_2))$$

for an appropriate choice of both '±', again a vanishing sum. □

Consequently, Lemmas 46 and 48 imply the following.

Proposition 49. *We have* $A_1 = 0$.

Though proving that A_1 vanishes is surprisingly involved, we do not exclude intrinsic reasons behind this degeneracy.

5.2. **The A_2 term.** We now deal with the A_2 term (29).

Lemma 50. *If the coordinates of $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in(\mathbf{Z}/N\mathbf{Z})^2$ are non-zero, then*

$$\int_0^\infty \operatorname{darg} g_{\boldsymbol{x}}(\tau)\int_\tau^\infty \eta(g_{\boldsymbol{y}},g_{\boldsymbol{z}}) = \Lambda^{--+}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})-\Lambda^{--+}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{y}). \tag{35}$$

Proof. We expand $\eta(g_{\boldsymbol{y}},g_{\boldsymbol{z}})$ using just the definition. The form $\operatorname{dlog}|g_{\boldsymbol{y}}|$ is admissible, so Lemma 8 implies

$$\log|g_{\boldsymbol{y}}(\tau_1)| = \log|g_{\boldsymbol{y}}|(\infty)-\int_{\tau_1}^\infty \operatorname{dlog}|g_{\boldsymbol{y}}|.$$

Noting that $\log|g_{\boldsymbol{y}}|(\infty)=0$ here, this leads to

$$\eta(g_{\boldsymbol{y}},g_{\boldsymbol{z}})(\tau_1) = -\operatorname{darg} g_{\boldsymbol{z}}(\tau_1)\int_{\tau_1}^\infty \operatorname{dlog}|g_{\boldsymbol{y}}| + \operatorname{darg} g_{\boldsymbol{y}}(\tau_1)\int_{\tau_1}^\infty \operatorname{dlog}|g_{\boldsymbol{z}}|,$$

which implies (35). □

Expanding A_2 in (29) using Lemma 50, we get $A_2 = I_1+\cdots+I_6$ with

$$\begin{aligned} I_1 &= \Lambda^{--+}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{b})-\Lambda^{--+}(\boldsymbol{b},\boldsymbol{b},\boldsymbol{a}), & I_2 &= \Lambda^{--+}(\boldsymbol{b},\boldsymbol{b},\boldsymbol{c})-\Lambda^{--+}(\boldsymbol{b},\boldsymbol{c},\boldsymbol{b}),\\ I_3 &= \Lambda^{--+}(\boldsymbol{b},\boldsymbol{c},\boldsymbol{a})-\Lambda^{--+}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{c}), & I_4 &= -\Lambda^{--+}(\boldsymbol{a},\boldsymbol{a},\boldsymbol{b})+\Lambda^{--+}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{a}),\\ I_5 &= -\Lambda^{--+}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})+\Lambda^{--+}(\boldsymbol{a},\boldsymbol{c},\boldsymbol{b}), & I_6 &= -\Lambda^{--+}(\boldsymbol{a},\boldsymbol{c},\boldsymbol{a})+\Lambda^{--+}(\boldsymbol{a},\boldsymbol{a},\boldsymbol{c}). \end{aligned}$$

To simplify this expression for A_2, we use shuffle relations between iterated integrals $\Lambda^{\varepsilon_1\varepsilon_2\varepsilon_3}$ with $\varepsilon_1,\varepsilon_2,\varepsilon_3\in\{\pm1\}$. Consider the relation

$$\Lambda^-(\boldsymbol{x})\Lambda^{-+}(\boldsymbol{y},\boldsymbol{z}) = \Lambda^{--+}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})+\Lambda^{--+}(\boldsymbol{y},\boldsymbol{x},\boldsymbol{z})+\Lambda^{-+-}(\boldsymbol{y},\boldsymbol{z},\boldsymbol{x}).$$

Specialising to $\boldsymbol{z}=\boldsymbol{x}$ and $\boldsymbol{y}=\boldsymbol{x}$ respectively, we get

$$\Lambda^{--+}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{x}) = -\Lambda^{-+-}(\boldsymbol{y},\boldsymbol{x},\boldsymbol{x})-\Lambda^{--+}(\boldsymbol{y},\boldsymbol{x},\boldsymbol{x})+\Lambda^-(\boldsymbol{x})\Lambda^{-+}(\boldsymbol{y},\boldsymbol{x}), \tag{36}$$

$$\Lambda^{-+-}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{x}) = -2\Lambda^{--+}(\boldsymbol{x},\boldsymbol{x},\boldsymbol{z})+\Lambda^-(\boldsymbol{x})\Lambda^{-+}(\boldsymbol{x},\boldsymbol{z}). \tag{37}$$

Similarly,

$$\Lambda^-(\boldsymbol{x})\Lambda^{+-}(\boldsymbol{z},\boldsymbol{y}) = \Lambda^{-+-}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{y})+\Lambda^{+--}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{y})+\Lambda^{+--}(\boldsymbol{z},\boldsymbol{y},\boldsymbol{x})$$

which, taking $\boldsymbol{y}=\boldsymbol{x}$, specialises to

$$\Lambda^{-+-}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{x}) = -2\Lambda^{+--}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x})+\Lambda^-(\boldsymbol{x})\Lambda^{+-}(\boldsymbol{z},\boldsymbol{x}). \tag{38}$$

Equating the right-hand sides of (38) and (37) gives

$$\Lambda^{--+}(\boldsymbol{x},\boldsymbol{x},\boldsymbol{z}) = \Lambda^{+--}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x})+\frac12\Lambda^-(\boldsymbol{x})(\Lambda^{-+}(\boldsymbol{x},\boldsymbol{z})-\Lambda^{+-}(\boldsymbol{z},\boldsymbol{x})). \tag{39}$$

We now simplify $I_1,\dots,I_6$. We introduce the shortcut

$$\Lambda_1(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) := (\Lambda^{+--}+\Lambda^{-+-}+\Lambda^{--+})(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}).$$

Note that in this way,

$$\operatorname{Re}\Lambda(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = -\Lambda_1(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})+\Lambda^{+++}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \tag{40}$$

for any $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$. Using (39) and (36), we have

$$\begin{aligned} I_1 &= \Lambda^{--+}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{b})-\Lambda^{--+}(\boldsymbol{b},\boldsymbol{b},\boldsymbol{a})\\ &= \Lambda^{--+}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{b})-\Lambda^{+--}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b})-\frac12\Lambda^-(\boldsymbol{b})(\Lambda^{-+}(\boldsymbol{b},\boldsymbol{a})-\Lambda^{+-}(\boldsymbol{a},\boldsymbol{b}))\\ &= -\Lambda^{-+-}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b})-\Lambda^{--+}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b})+\Lambda^-(\boldsymbol{b})\Lambda^{-+}(\boldsymbol{a},\boldsymbol{b})-\Lambda^{+--}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b})\\ &\qquad -\frac12\Lambda^-(\boldsymbol{b})(\Lambda^{-+}(\boldsymbol{b},\boldsymbol{a})-\Lambda^{+-}(\boldsymbol{a},\boldsymbol{b}))\\ &= -\Lambda_1(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b})+\Lambda^-(\boldsymbol{b})\Big(\Lambda^{-+}(\boldsymbol{a},\boldsymbol{b})-\frac12\Lambda^{-+}(\boldsymbol{b},\boldsymbol{a})+\frac12\Lambda^{+-}(\boldsymbol{a},\boldsymbol{b})\Big). \end{aligned}$$

The integrals I_2, I_4 and I_6 are obtained from I_1 by simply rearranging the letters:

$$\begin{aligned}
I_2 &= +\Lambda_1(\boldsymbol{c},\boldsymbol{b},\boldsymbol{b}) - \Lambda^-(\boldsymbol{b})\Big(\Lambda^{-+}(\boldsymbol{c},\boldsymbol{b}) - \frac{1}{2}\Lambda^{-+}(\boldsymbol{b},\boldsymbol{c}) + \frac{1}{2}\Lambda^{+-}(\boldsymbol{c},\boldsymbol{b})\Big),\\
I_4 &= -\Lambda_1(\boldsymbol{b},\boldsymbol{a},\boldsymbol{a}) + \Lambda^-(\boldsymbol{a})\Big(\Lambda^{-+}(\boldsymbol{b},\boldsymbol{a}) - \frac{1}{2}\Lambda^{-+}(\boldsymbol{a},\boldsymbol{b}) + \frac{1}{2}\Lambda^{+-}(\boldsymbol{b},\boldsymbol{a})\Big),\\
I_6 &= +\Lambda_1(\boldsymbol{c},\boldsymbol{a},\boldsymbol{a}) - \Lambda^-(\boldsymbol{a})\Big(\Lambda^{-+}(\boldsymbol{c},\boldsymbol{a}) - \frac{1}{2}\Lambda^{-+}(\boldsymbol{a},\boldsymbol{c}) + \frac{1}{2}\Lambda^{+-}(\boldsymbol{c},\boldsymbol{a})\Big).
\end{aligned}$$

It remains to treat the terms I_3 and I_5, involving permutations of $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$. By the shuffle relations, we have

$$\Lambda^{--+}(\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}) = \Lambda^-(\boldsymbol{b})\Lambda^{-+}(\boldsymbol{c},\boldsymbol{a}) - \Lambda^{--+}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}) - \Lambda^{-+-}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}), \tag{41}$$

$$\Lambda^{--+}(\boldsymbol{a},\boldsymbol{c},\boldsymbol{b}) = \Lambda^-(\boldsymbol{a})\Lambda^{-+}(\boldsymbol{c},\boldsymbol{b}) - \Lambda^{--+}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}) - \Lambda^{-+-}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}). \tag{42}$$

We also have

$$\begin{aligned}
\Lambda^-(\boldsymbol{b})\Lambda^{-+}(\boldsymbol{a},\boldsymbol{c}) &= \Lambda^{--+}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{c}) + \Lambda^{--+}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) + \Lambda^{-+-}(\boldsymbol{a},\boldsymbol{c},\boldsymbol{b}),\\
\Lambda^-(\boldsymbol{a})\Lambda^{+-}(\boldsymbol{c},\boldsymbol{b}) &= \Lambda^{-+-}(\boldsymbol{a},\boldsymbol{c},\boldsymbol{b}) + \Lambda^{+--}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}) + \Lambda^{+--}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}),
\end{aligned}$$

and thus

$$\Lambda^{--+}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{c}) + \Lambda^{--+}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = \Lambda^-(\boldsymbol{b})\Lambda^{-+}(\boldsymbol{a},\boldsymbol{c}) - \Lambda^-(\boldsymbol{a})\Lambda^{+-}(\boldsymbol{c},\boldsymbol{b}) + \Lambda^{+--}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}) + \Lambda^{+--}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}). \tag{43}$$

Therefore,

$$\begin{aligned}
I_3 + I_5 &= (41) + (42) - (43)\\
&= \Lambda^-(\boldsymbol{b})\Lambda^{-+}(\boldsymbol{c},\boldsymbol{a}) - \Lambda^{--+}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}) - \Lambda^{-+-}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b})\\
&\quad + \Lambda^-(\boldsymbol{a})\Lambda^{-+}(\boldsymbol{c},\boldsymbol{b}) - \Lambda^{--+}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}) - \Lambda^{-+-}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a})\\
&\quad - \Lambda^-(\boldsymbol{b})\Lambda^{-+}(\boldsymbol{a},\boldsymbol{c}) + \Lambda^-(\boldsymbol{a})\Lambda^{+-}(\boldsymbol{c},\boldsymbol{b}) - \Lambda^{+--}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}) - \Lambda^{+--}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a})\\
&= -\Lambda_1(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}) - \Lambda_1(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b})\\
&\quad + \Lambda^-(\boldsymbol{b})\Lambda^{-+}(\boldsymbol{c},\boldsymbol{a}) + \Lambda^-(\boldsymbol{a})\Lambda^{-+}(\boldsymbol{c},\boldsymbol{b}) - \Lambda^-(\boldsymbol{b})\Lambda^{-+}(\boldsymbol{a},\boldsymbol{c}) + \Lambda^-(\boldsymbol{a})\Lambda^{+-}(\boldsymbol{c},\boldsymbol{b}).
\end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}
A_2 = &-\Lambda_1(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b}) + \Lambda_1(\boldsymbol{c},\boldsymbol{b},\boldsymbol{b}) - \Lambda_1(\boldsymbol{b},\boldsymbol{a},\boldsymbol{a}) + \Lambda_1(\boldsymbol{c},\boldsymbol{a},\boldsymbol{a}) - \Lambda_1(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}) - \Lambda_1(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b})\\
&+ \Lambda^-(\boldsymbol{b})\Big(\Lambda^{-+}(\boldsymbol{a},\boldsymbol{b}) + \Lambda^{-+}(\boldsymbol{c},\boldsymbol{a}) - \Lambda^{-+}(\boldsymbol{a},\boldsymbol{c}) - \Lambda^{-+}(\boldsymbol{c},\boldsymbol{b})\\
&\qquad - \frac{1}{2}\Lambda^{-+}(\boldsymbol{b},\boldsymbol{a}) + \frac{1}{2}\Lambda^{+-}(\boldsymbol{a},\boldsymbol{b}) + \frac{1}{2}\Lambda^{-+}(\boldsymbol{b},\boldsymbol{c}) - \frac{1}{2}\Lambda^{+-}(\boldsymbol{c},\boldsymbol{b})\Big)\\
&+ \Lambda^-(\boldsymbol{a})\Big(\Lambda^{-+}(\boldsymbol{b},\boldsymbol{a}) + \Lambda^{-+}(\boldsymbol{c},\boldsymbol{b}) + \Lambda^{+-}(\boldsymbol{c},\boldsymbol{b}) - \Lambda^{-+}(\boldsymbol{c},\boldsymbol{a})\\
&\qquad + \frac{1}{2}\Lambda^{+-}(\boldsymbol{b},\boldsymbol{a}) - \frac{1}{2}\Lambda^{-+}(\boldsymbol{a},\boldsymbol{b}) + \frac{1}{2}\Lambda^{-+}(\boldsymbol{a},\boldsymbol{c}) - \frac{1}{2}\Lambda^{+-}(\boldsymbol{c},\boldsymbol{a})\Big).
\end{aligned} \tag{44}$$

The terms involving double modular values can be rewritten using the shuffle relations. In our generic situation when the coordinates of the vectors are non-zero, we have $\Lambda^{-+}(\boldsymbol{x},\boldsymbol{y}) + \Lambda^{+-}(\boldsymbol{y},\boldsymbol{x}) = \Lambda^-(\boldsymbol{x})\Lambda^+(\boldsymbol{y}) = 0$ by (24). Therefore,

$$\begin{aligned}
&\Lambda^{-+}(\boldsymbol{a},\boldsymbol{b}) + \Lambda^{-+}(\boldsymbol{c},\boldsymbol{a}) - \Lambda^{-+}(\boldsymbol{a},\boldsymbol{c}) - \Lambda^{-+}(\boldsymbol{c},\boldsymbol{b})\\
&\quad - \frac{1}{2}\Lambda^{-+}(\boldsymbol{b},\boldsymbol{a}) + \frac{1}{2}\Lambda^{+-}(\boldsymbol{a},\boldsymbol{b}) + \frac{1}{2}\Lambda^{-+}(\boldsymbol{b},\boldsymbol{c}) - \frac{1}{2}\Lambda^{+-}(\boldsymbol{c},\boldsymbol{b})\\
&= (\Lambda^{-+}(\boldsymbol{a},\boldsymbol{b}) + \Lambda^{+-}(\boldsymbol{a},\boldsymbol{b})) + (\Lambda^{-+}(\boldsymbol{b},\boldsymbol{c}) + \Lambda^{+-}(\boldsymbol{b},\boldsymbol{c})) + (\Lambda^{-+}(\boldsymbol{c},\boldsymbol{a}) + \Lambda^{+-}(\boldsymbol{c},\boldsymbol{a}))\\
&= \operatorname{Im}\Lambda(\boldsymbol{a},\boldsymbol{b}) + \operatorname{Im}\Lambda(\boldsymbol{b},\boldsymbol{c}) + \operatorname{Im}\Lambda(\boldsymbol{c},\boldsymbol{a}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\Lambda^{-+}(\boldsymbol{b},\boldsymbol{a}) + \Lambda^{-+}(\boldsymbol{c},\boldsymbol{b}) + \Lambda^{+-}(\boldsymbol{c},\boldsymbol{b}) - \Lambda^{-+}(\boldsymbol{c},\boldsymbol{a}) \\
&\quad + \frac{1}{2}\Lambda^{+-}(\boldsymbol{b},\boldsymbol{a}) - \frac{1}{2}\Lambda^{-+}(\boldsymbol{a},\boldsymbol{b}) + \frac{1}{2}\Lambda^{-+}(\boldsymbol{a},\boldsymbol{c}) - \frac{1}{2}\Lambda^{+-}(\boldsymbol{c},\boldsymbol{a}) \\
&= -\operatorname{Im}\Lambda(\boldsymbol{a},\boldsymbol{b}) - \operatorname{Im}\Lambda(\boldsymbol{b},\boldsymbol{c}) - \operatorname{Im}\Lambda(\boldsymbol{c},\boldsymbol{a}).
\end{aligned}$$

Copying into (44) gives the following proposition.

Proposition 51. *Let $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in (\mathbf{Z}/N\mathbf{Z})^2$ such that $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\boldsymbol{0}$, with all the coordinates of $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ non-zero. Then*

$$\begin{aligned}
A_2 &= -\Lambda_1(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b}) + \Lambda_1(\boldsymbol{c},\boldsymbol{b},\boldsymbol{b}) - \Lambda_1(\boldsymbol{b},\boldsymbol{a},\boldsymbol{a}) + \Lambda_1(\boldsymbol{c},\boldsymbol{a},\boldsymbol{a}) - \Lambda_1(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}) - \Lambda_1(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}) \\
&\quad + (\Lambda^-(\boldsymbol{b}) - \Lambda^-(\boldsymbol{a}))\operatorname{Im}\big(\Lambda(\boldsymbol{a},\boldsymbol{b}) + \Lambda(\boldsymbol{b},\boldsymbol{c}) + \Lambda(\boldsymbol{c},\boldsymbol{a})\big).
\end{aligned}$$

5.3. **The A_3 term.** Finally, we treat the A_3 term (30). We leave the required admissibility properties of the differential forms to the reader.

Lemma 52. *If the coordinates of $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in (\mathbf{Z}/N\mathbf{Z})^2$ are non-zero, then*

$$\int_0^\infty \log|g_{\boldsymbol{x}}|\,\alpha(g_{\boldsymbol{y}},g_{\boldsymbol{z}}) = -\Lambda^{+++}(\boldsymbol{z},\boldsymbol{y},\boldsymbol{x}) - \Lambda^{+++}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{y}) + \Lambda^{+++}(\boldsymbol{y},\boldsymbol{z},\boldsymbol{x}) + \Lambda^{+++}(\boldsymbol{y},\boldsymbol{x},\boldsymbol{z}). \tag{45}$$

Proof. By definition

$$\int_0^\infty \log|g_{\boldsymbol{x}}|\,\alpha(g_{\boldsymbol{y}},g_{\boldsymbol{z}}) = \int_0^\infty -\operatorname{dlog}|g_{\boldsymbol{z}}|\cdot\log|g_{\boldsymbol{x}}|\log|g_{\boldsymbol{y}}| + \operatorname{dlog}|g_{\boldsymbol{y}}|\cdot\log|g_{\boldsymbol{x}}|\log|g_{\boldsymbol{z}}|. \tag{46}$$

Recall that the regularised value of the various $\log|g_{\boldsymbol{x}}|$ at ∞ is zero. Therefore

$$\begin{aligned}
\log|g_{\boldsymbol{x}}(\tau)|\log|g_{\boldsymbol{y}}(\tau)| &= (\log|g_{\boldsymbol{x}}|\log|g_{\boldsymbol{y}}|)(\infty) - \int_\tau^\infty d(\log|g_{\boldsymbol{x}}|\log|g_{\boldsymbol{y}}|) \\
&= -\int_\tau^\infty \operatorname{dlog}|g_{\boldsymbol{y}}|\cdot\log|g_{\boldsymbol{x}}| + \operatorname{dlog}|g_{\boldsymbol{x}}|\cdot\log|g_{\boldsymbol{y}}| \\
&= \int_\tau^\infty \operatorname{dlog}|g_{\boldsymbol{y}}|\operatorname{dlog}|g_{\boldsymbol{x}}| + \operatorname{dlog}|g_{\boldsymbol{x}}|\operatorname{dlog}|g_{\boldsymbol{y}}|,
\end{aligned}$$

so that (46) continues as

$$\begin{aligned}
\int_0^\infty \log|g_{\boldsymbol{x}}|\,\alpha(g_{\boldsymbol{y}},g_{\boldsymbol{z}}) &= -\int_0^\infty \operatorname{dlog}|g_{\boldsymbol{z}}(\tau)|\int_\tau^\infty \operatorname{dlog}|g_{\boldsymbol{y}}|\operatorname{dlog}|g_{\boldsymbol{x}}| + \operatorname{dlog}|g_{\boldsymbol{x}}|\operatorname{dlog}|g_{\boldsymbol{y}}| \\
&\quad + \int_0^\infty \operatorname{dlog}|g_{\boldsymbol{y}}(\tau)|\int_\tau^\infty \operatorname{dlog}|g_{\boldsymbol{z}}|\operatorname{dlog}|g_{\boldsymbol{x}}| + \operatorname{dlog}|g_{\boldsymbol{x}}|\operatorname{dlog}|g_{\boldsymbol{z}}|. \qquad \square
\end{aligned}$$

Using Lemma 52, the term A_3 can be written as a sum of six expressions of type (45):

$$\begin{aligned}
3A_3 &= -\Lambda^{+++}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{b}) - \Lambda^{+++}(\boldsymbol{b},\boldsymbol{b},\boldsymbol{a}) + \Lambda^{+++}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b}) + \Lambda^{+++}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b}) \\
&\quad - \Lambda^{+++}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{b}) - \Lambda^{+++}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{b}) + \Lambda^{+++}(\boldsymbol{b},\boldsymbol{c},\boldsymbol{b}) + \Lambda^{+++}(\boldsymbol{b},\boldsymbol{b},\boldsymbol{c}) \\
&\quad - \Lambda^{+++}(\boldsymbol{a},\boldsymbol{c},\boldsymbol{b}) - \Lambda^{+++}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) + \Lambda^{+++}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}) + \Lambda^{+++}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}) \\
&\quad + \Lambda^{+++}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{a}) + \Lambda^{+++}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{a}) - \Lambda^{+++}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{a}) - \Lambda^{+++}(\boldsymbol{a},\boldsymbol{a},\boldsymbol{b}) \\
&\quad + \Lambda^{+++}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}) + \Lambda^{+++}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}) - \Lambda^{+++}(\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}) - \Lambda^{+++}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{c}) \\
&\quad + \Lambda^{+++}(\boldsymbol{a},\boldsymbol{c},\boldsymbol{a}) + \Lambda^{+++}(\boldsymbol{a},\boldsymbol{a},\boldsymbol{c}) - \Lambda^{+++}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{a}) - \Lambda^{+++}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{a}).
\end{aligned} \tag{47}$$

Using the shuffle relations

$$0 = \Lambda^+(\boldsymbol{x})\Lambda^{++}(\boldsymbol{y},\boldsymbol{z}) = \Lambda^{+++}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) + \Lambda^{+++}(\boldsymbol{y},\boldsymbol{x},\boldsymbol{z}) + \Lambda^{+++}(\boldsymbol{y},\boldsymbol{z},\boldsymbol{x}),$$

the six lines in (47) can be simplified, respectively, to

$$\begin{array}{lll}
3\Lambda^{+++}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b}), & -3\Lambda^{+++}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{b}), & 2\Lambda^{+++}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}) + \Lambda^{+++}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}), \\
3\Lambda^{+++}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{a}), & 2\Lambda^{+++}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}) + \Lambda^{+++}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}), & -3\Lambda^{+++}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{a}).
\end{array}$$

In this way we obtain the following expression for A_3.

Proposition 53. *Let $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in(\mathbf{Z}/N\mathbf{Z})^2$ such that $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\mathbf{0}$, with all the coordinates of $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ non-zero. Then*

$$A_3=\Lambda^{+++}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b})-\Lambda^{+++}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{b})+\Lambda^{+++}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b})+\Lambda^{+++}(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a})+\Lambda^{+++}(\boldsymbol{b},\boldsymbol{a},\boldsymbol{a})-\Lambda^{+++}(\boldsymbol{c},\boldsymbol{a},\boldsymbol{a}).$$

Putting together Propositions 49, 51 and 53, we obtain an expression for $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$. The terms of type Λ_1 and Λ^{+++} collect thanks to (40). This results in the following final formula.

Theorem 54. *Let $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in(\mathbf{Z}/N\mathbf{Z})^2$ such that $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\mathbf{0}$. Assume that all the coordinates of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are non-zero. Then*

$$\begin{aligned}\mathcal{G}(\boldsymbol{a},\boldsymbol{b})=\operatorname{Re}\big(&\Lambda(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b})-\Lambda(\boldsymbol{c},\boldsymbol{b},\boldsymbol{b})+\Lambda(\boldsymbol{b},\boldsymbol{a},\boldsymbol{a})-\Lambda(\boldsymbol{c},\boldsymbol{a},\boldsymbol{a})+\Lambda(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a})+\Lambda(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b})\\&-(\Lambda(\boldsymbol{b})-\Lambda(\boldsymbol{a}))(\Lambda(\boldsymbol{a},\boldsymbol{b})+\Lambda(\boldsymbol{b},\boldsymbol{c})+\Lambda(\boldsymbol{c},\boldsymbol{a}))\big).\end{aligned}$$

Corollary 55. *The Goncharov regulator $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ interpolates as a differentiable function of $\boldsymbol{a},\boldsymbol{b}$ in the domain $\{(\boldsymbol{a},\boldsymbol{b})\in(\mathbf{R}/\mathbf{Z})^4 : a_k,b_k,a_k+b_k\neq 0 \text{ for } k=1,2\}$.*

6. The Borisov–Gunnells relations

Borisov and Gunnells [3] have shown that certain pairwise products of Eisenstein series on the group $\Gamma_1(N)$ satisfy linear dependence relations, which strikingly resemble the Manin 3-term relations for modular symbols. We show in Theorem 56 below an explicit version of the result of Borisov and Gunnells [3, Theorem 6.2] in weight 3 and for Eisenstein series on $\Gamma(N)$. We then deduce in Theorems 58 and 59 similar relations for Eisenstein series with rational Fourier coefficients.

Theorem 56. *Let $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in(\mathbf{R}/\mathbf{Z})^2\smallsetminus\{\mathbf{0}\}$ such that $\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}=\mathbf{0}$. Then*

$$E^{(1)}_{\boldsymbol{z}}E^{(2)}_{\boldsymbol{y}}-E^{(1)}_{\boldsymbol{y}}E^{(2)}_{\boldsymbol{x}}-E^{(1)}_{\boldsymbol{z}}E^{(2)}_{\boldsymbol{x}}+E^{(1)}_{\boldsymbol{y}}E^{(2)}_{\boldsymbol{z}}=E^{(3)}_{\boldsymbol{x}}-\frac12E^{(3)}_{\boldsymbol{y}}-\frac12E^{(3)}_{\boldsymbol{z}}. \tag{48}$$

Proof. Our starting point is an addition formula due to Weil [24, IV, §2, eq. (10)]:

$$(E_2^*(x)-E_2^*(x'))(E_1^*(x+x')-E_1^*(x)-E_1^*(x'))+E_3^*(x)-E_3^*(x')=0 \qquad (x,x'\in\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z}),\ x,x',x+x'\neq0), \tag{49}$$

where, in Weil's notations, $E_k^*(x)=K_k(x,0,k;\tau)$. (Weil states the identity for series denoted by $E_k(x)$, but they can be expressed in terms of $E_k^*(x)$ [24, VI, §2].) In terms of $\hat{E}_{\boldsymbol{x}}$ (see Definition 33), the identity (49) can be rewritten

$$(\hat{E}^{(1)}_{\boldsymbol{a}}+\hat{E}^{(1)}_{\boldsymbol{b}}+\hat{E}^{(1)}_{-\boldsymbol{a}-\boldsymbol{b}})(\hat{E}^{(2)}_{\boldsymbol{a}}-\hat{E}^{(2)}_{-\boldsymbol{a}-\boldsymbol{b}})=\frac12(\hat{E}^{(3)}_{\boldsymbol{a}}-\hat{E}^{(3)}_{-\boldsymbol{a}-\boldsymbol{b}}) \qquad (\boldsymbol{a},\boldsymbol{b}\in(\mathbf{R}/\mathbf{Z})^2,\ \boldsymbol{a},\boldsymbol{b},\boldsymbol{a}+\boldsymbol{b}\neq\mathbf{0}). \tag{50}$$

Our original source of (49) was a nice geometric interpretation given by Khuri-Makdisi [17, eq. (4.39)]: this identity expresses the slope of the line passing through 3 points P,Q,R on $\mathcal{E}_\tau=\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$, where $P+Q+R=0$. Another proof is given in [18, p. 177–178].

Now, after restricting to N-torsion points, the Eisenstein series $E^{(k)}_{x_1,x_2}$ is essentially the discrete Fourier transform of $\hat{E}^{(k)}_{a_1,a_2}$. In the sequel, we implicitly identify $(\mathbf{Z}/N\mathbf{Z})^2$ with a subset of $(\mathbf{R}/\mathbf{Z})^2$ by mapping a pair $(\overline{x}_1,\overline{x}_2)$ to the class of $(x_1/N,x_2/N)$. Moreover, let us introduce the Weil pairing on $\mathcal{E}_\tau[N]\cong(\mathbf{Z}/N\mathbf{Z})^2$:

$$e_N\colon(\mathbf{Z}/N\mathbf{Z})^2\times(\mathbf{Z}/N\mathbf{Z})^2\to\mathbf{C}^\times, \qquad (\boldsymbol{a},\boldsymbol{x})\mapsto e\left(\frac{a_2x_1-a_1x_2}{N}\right).$$

For $k\geq1$, the relation between $E^{(k)}$ and $\hat{E}^{(k)}$ is

$$\sum_{\boldsymbol{a}\in(\mathbf{Z}/N\mathbf{Z})^2}e_N(\boldsymbol{a},\boldsymbol{x})\hat{E}^{(k)}_{\boldsymbol{a}}=(-1)^{k+1}N^kE^{(k)}_{\boldsymbol{x}} \qquad \big(\boldsymbol{x}\in(\mathbf{Z}/N\mathbf{Z})^2\big).$$

This can be proved directly from the definitions of $E^{(k)}_{\boldsymbol{a}}$ and $\hat{E}^{(k)}_{\boldsymbol{x}}$ (Definition 33).

This leads us to taking the Fourier transform of (50) with respect to both $\boldsymbol{a}$ and $\boldsymbol{b}$. However, it is important to note that (50) only holds when $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}+\boldsymbol{b}$ are non-zero. For example, when $\boldsymbol{a} = \boldsymbol{0}$, the left-hand side of (50) is zero while the right-hand side may not be. We thus take the Fourier transform of both sides of (50) separately, and then use the inclusion-exclusion principle:

$$\sum_{\substack{\boldsymbol{a},\boldsymbol{b}\in(\mathbf{Z}/N\mathbf{Z})^2\\ \boldsymbol{a},\boldsymbol{b},\boldsymbol{a}+\boldsymbol{b}\neq\boldsymbol{0}}} = \sum_{\boldsymbol{a},\boldsymbol{b}\in(\mathbf{Z}/N\mathbf{Z})^2} - \sum_{\substack{\boldsymbol{a}=\boldsymbol{0}\\ \boldsymbol{b}\in(\mathbf{Z}/N\mathbf{Z})^2}} - \sum_{\substack{\boldsymbol{a}\in(\mathbf{Z}/N\mathbf{Z})^2\\ \boldsymbol{b}=\boldsymbol{0}}} - \sum_{\substack{\boldsymbol{a}\in(\mathbf{Z}/N\mathbf{Z})^2\\ \boldsymbol{b}=-\boldsymbol{a}}} + 2\sum_{\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{0}}. \tag{51}$$

We will denote by $\mathcal{L}_{\boldsymbol{a},\boldsymbol{b}}$ the left-hand side of (50), and by $\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}$ its right-hand side. Let $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in (\mathbf{Z}/N\mathbf{Z})^2$ be as in the statement of Theorem 56. Noting that $\mathcal{L}_{\boldsymbol{a},\boldsymbol{b}}$ is zero when $\boldsymbol{a}$, $\boldsymbol{b}$ or $\boldsymbol{a}+\boldsymbol{b}$ is zero, we have

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{a},\boldsymbol{b}\in(\mathbf{Z}/N\mathbf{Z})^2\\ \boldsymbol{a},\boldsymbol{b},\boldsymbol{a}+\boldsymbol{b}\neq\boldsymbol{0}}} e_N(\boldsymbol{a},\boldsymbol{x})e_N(-\boldsymbol{b},\boldsymbol{y}) \times \mathcal{L}_{\boldsymbol{a},\boldsymbol{b}} \\
&= \sum_{\boldsymbol{a},\boldsymbol{b}\in(\mathbf{Z}/N\mathbf{Z})^2} e_N(\boldsymbol{a},\boldsymbol{x})e_N(-\boldsymbol{b},\boldsymbol{y})(\hat{E}^{(1)}_{\boldsymbol{a}} + \hat{E}^{(1)}_{\boldsymbol{b}} + \hat{E}^{(1)}_{-\boldsymbol{a}-\boldsymbol{b}})(\hat{E}^{(2)}_{\boldsymbol{a}} - \hat{E}^{(2)}_{-\boldsymbol{a}-\boldsymbol{b}}) \\
&= \sum_{\boldsymbol{a},\boldsymbol{b}\in(\mathbf{Z}/N\mathbf{Z})^2} e_N(\boldsymbol{a},\boldsymbol{x})e_N(-\boldsymbol{b},\boldsymbol{y})(-\hat{E}^{(1)}_{\boldsymbol{a}}\hat{E}^{(2)}_{-\boldsymbol{a}-\boldsymbol{b}} + \hat{E}^{(1)}_{\boldsymbol{b}}\hat{E}^{(2)}_{\boldsymbol{a}} - \hat{E}^{(1)}_{\boldsymbol{b}}\hat{E}^{(2)}_{-\boldsymbol{a}-\boldsymbol{b}} + \hat{E}^{(1)}_{-\boldsymbol{a}-\boldsymbol{b}}\hat{E}^{(2)}_{\boldsymbol{a}}) \\
&= N^3\left(E^{(1)}_{\boldsymbol{x}+\boldsymbol{y}}E^{(2)}_{\boldsymbol{y}} + E^{(1)}_{\boldsymbol{y}}E^{(2)}_{\boldsymbol{x}} - E^{(1)}_{\boldsymbol{x}+\boldsymbol{y}}E^{(2)}_{\boldsymbol{x}} - E^{(1)}_{\boldsymbol{y}}E^{(2)}_{\boldsymbol{x}+\boldsymbol{y}}\right).
\end{aligned} \tag{52}$$

We compute the Fourier transform of $\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}$ similarly, keeping in mind the correction terms (51):

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{a},\boldsymbol{b}\in(\mathbf{Z}/N\mathbf{Z})^2\\ \boldsymbol{a},\boldsymbol{b},\boldsymbol{a}+\boldsymbol{b}\neq\boldsymbol{0}}} e_N(\boldsymbol{a},\boldsymbol{x})e_N(-\boldsymbol{b},\boldsymbol{y}) \times \mathcal{R}_{\boldsymbol{a},\boldsymbol{b}} \\
&= -\frac{1}{2}\left(\sum_{\substack{\boldsymbol{a}=\boldsymbol{0}\\ \boldsymbol{b}\in(\mathbf{Z}/N\mathbf{Z})^2}} + \sum_{\substack{\boldsymbol{a}\in(\mathbf{Z}/N\mathbf{Z})^2\\ \boldsymbol{b}=\boldsymbol{0}}} + \sum_{\substack{\boldsymbol{a}\in(\mathbf{Z}/N\mathbf{Z})^2\\ \boldsymbol{b}=-\boldsymbol{a}}}\right) e_N(\boldsymbol{a},\boldsymbol{x})e_N(-\boldsymbol{b},\boldsymbol{y})(\hat{E}^{(3)}_{\boldsymbol{a}} - \hat{E}^{(3)}_{-\boldsymbol{a}-\boldsymbol{b}}) \\
&= N^3\left(-E^{(3)}_{\boldsymbol{x}} + \frac{1}{2}E^{(3)}_{\boldsymbol{y}} - \frac{1}{2}E^{(3)}_{\boldsymbol{x}+\boldsymbol{y}}\right).
\end{aligned} \tag{53}$$

The identity (48) now follows from (52), (53) and the relation $E^{(k)}_{\boldsymbol{x}+\boldsymbol{y}} = (-1)^k E^{(k)}_{\boldsymbol{z}}$.

So far we have established the result for N-torsion points. Since both sides of the identity are continuous in $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$ by Lemma 38, the result is true in general. □

Let us now consider Eisenstein series with rational Fourier coefficients, and investigate the Borisov–Gunnell type relations for them. As the following lemma shows, $G^{(k);N}_{\boldsymbol{x}}$ is essentially the partial Fourier transform of $E^{(k)}_{\boldsymbol{x}}$ with respect to the second parameter.

Lemma 57. *For $x_1, u \in \mathbf{Z}/N\mathbf{Z}$, we have*

$$\sum_{x_2\in\mathbf{Z}/N\mathbf{Z}} e\left(-\frac{ux_2}{N}\right) E^{(k)}_{x_1,x_2} = -N^{2-k} G^{(k);N}_{x_1,u}.$$

Proof. This is a direct computation using the q-expansions (1) and (14). □

The Borisov–Gunnells relation for $G^{(k)}_{\boldsymbol{x}}$ is as follows. We first state the case when the first coordinates are non-zero.

Theorem 58. *Let $x_1, y_1, u_2, v_2 \in (\mathbf{R}/\mathbf{Z}) \smallsetminus \{0\}$ such that $x_1 + y_1, u_2 - v_2 \neq 0$. Then*

$$G^{(1)}_{x_1+y_1,u_2}G^{(2)}_{y_1,v_2-u_2} + G^{(1)}_{y_1,v_2}G^{(2)}_{x_1,u_2} - G^{(1)}_{x_1+y_1,v_2}G^{(2)}_{x_1,u_2-v_2} - G^{(1)}_{y_1,v_2-u_2}G^{(2)}_{x_1+y_1,u_2} = 0.$$

Proof. As for Theorem 56, it suffices to treat the case of N-torsion points. In this case, the identity takes the form

$$G^{(1);N}_{x_1+y_1,u_2}G^{(2);N}_{y_1,v_2-u_2} + G^{(1);N}_{y_1,v_2}G^{(2);N}_{x_1,u_2} - G^{(1);N}_{x_1+y_1,v_2}G^{(2);N}_{x_1,u_2-v_2} - G^{(1);N}_{y_1,v_2-u_2}G^{(2);N}_{x_1+y_1,u_2} = 0$$

with $x_1, y_1, u_2, v_2 \in (\mathbf{Z}/N\mathbf{Z}) \smallsetminus \{0\}$ such that $x_1 + y_1, u_2 - v_2 \neq 0$. Now the idea is to apply the partial Fourier transform to the identity (48). Using Lemma 57, the transform of the left-hand side $\mathcal{L}_{x_2,y_2}$ of (48) can be computed as

$$\begin{aligned}(54)\qquad &\sum_{x_2,y_2\in\mathbf{Z}/N\mathbf{Z}} e\left(-\frac{u_2x_2+v_2y_2}{N}\right)\times\mathcal{L}_{x_2,y_2}\\ &= -N\left(G^{(1);N}_{x_1+y_1,u_2}G^{(2);N}_{y_1,v_2-u_2} + G^{(1);N}_{y_1,v_2}G^{(2);N}_{x_1,u_2} - G^{(1);N}_{x_1+y_1,v_2}G^{(2);N}_{x_1,u_2-v_2} - G^{(1);N}_{y_1,v_2-u_2}G^{(2);N}_{x_1+y_1,u_2}\right).\end{aligned}$$

Moreover, the transform of the right-hand side vanishes, as for example

$$\sum_{x_2,y_2\in\mathbf{Z}/N\mathbf{Z}} e\left(-\frac{u_2x_2+v_2y_2}{N}\right)E^{(3)}_{z_1,-x_2-y_2} \overset{t=-x_2-y_2}{=} \sum_{x_2,t\in\mathbf{Z}/N\mathbf{Z}} e\left(-\frac{(u_2-v_2)x_2-v_2t}{N}\right)E^{(3)}_{z_1,t} = 0$$

thanks to our assumption $u_2 - v_2 \neq 0$. □

The case when the first coordinate of $\boldsymbol{x}$, $\boldsymbol{y}$ or $\boldsymbol{z}$ is zero requires special care. We will not state it in general, but content ourselves with the following result.

Theorem 59. *Let $u_1, u_2 \in \mathbf{R}/\mathbf{Z}$ with $u_1 \neq 0$. Then*

$$G^{(1)}_{u_1,u_2}G^{(2)}_{u_1,-u_2} - G^{(1)}_{u_1,-u_2}G^{(2)}_{u_1,u_2} = G^{(3)}_{0,u_2}.$$

Proof. Again, it suffices to show that

$$G^{(1);N}_{u_1,u_2}G^{(2);N}_{u_1,-u_2} - G^{(1);N}_{u_1,-u_2}G^{(2);N}_{u_1,u_2} = \frac{1}{N}G^{(3);N}_{0,u_2} \qquad (u_1,u_2\in\mathbf{Z}/N\mathbf{Z},\ u_1\neq 0).$$

We use (48) with $x_1 = 0$, $y_1 = u_1$ and $x_2 \neq 0$. The left-hand side of (48) is

$$\mathcal{L}_{x_2,y_2} = -E^{(1)}_{u_1,x_2+y_2}E^{(2)}_{u_1,y_2} + (E^{(1)}_{u_1,x_2+y_2} - E^{(1)}_{u_1,y_2})E^{(2)}_{0,x_2} + E^{(1)}_{u_1,y_2}E^{(2)}_{u_1,x_2+y_2}.$$

Note that $\mathcal{L}_{x_2,y_2}$ is zero when $x_2 = 0$. Using (54) with $v_2 = 0$, we have

$$\begin{aligned}\sum_{\substack{x_2\neq 0\\ y_2\in\mathbf{Z}/N\mathbf{Z}}} e\left(-\frac{u_2x_2}{N}\right)\times\mathcal{L}_{x_2,y_2} &= \sum_{x_2,y_2\in\mathbf{Z}/N\mathbf{Z}} e\left(-\frac{u_2x_2}{N}\right)\times\mathcal{L}_{x_2,y_2}\\ &= -N\left(G^{(1);N}_{u_1,u_2}G^{(2);N}_{u_1,-u_2} + G^{(1);N}_{u_1,0}G^{(2);N}_{0,u_2} - G^{(1);N}_{u_1,0}G^{(2);N}_{0,u_2} - G^{(1);N}_{u_1,-u_2}G^{(2);N}_{u_1,u_2}\right).\end{aligned}$$

A similar computation gives

$$\sum_{\substack{x_2\neq 0\\ y_2\in\mathbf{Z}/N\mathbf{Z}}} e\left(-\frac{u_2x_2}{N}\right)\times\mathcal{R}_{x_2,y_2} = -G^{(3);N}_{0,u_2}.$$ □

7. Differentiating the Goncharov regulator

All elliptic parameters $\boldsymbol{x} = (x_1, x_2)$ etc., $\boldsymbol{a} = (a_1, a_2)$ etc. considered below are generic, not hitting the integers. Apart from the already established

$$\frac{\partial}{\partial x_2}\Lambda(\boldsymbol{x}) = 2\pi i(\{x_1\} - \tfrac{1}{2}) = 2\pi i\,E^{(1)}_{\boldsymbol{x}}(\infty)$$

we need to consider similar partial derivatives for the regularised multiple integrals

$$\Lambda(\boldsymbol{x},\boldsymbol{y}) = (2\pi i)^2\int_0^\infty \omega^{(2)}_{\boldsymbol{x}}(\tau_1)\omega^{(2)}_{\boldsymbol{y}}(\tau_2)$$

and

$$\Lambda(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = (2\pi i)^3\int_0^\infty \omega^{(2)}_{\boldsymbol{x}}(\tau_1)\omega^{(2)}_{\boldsymbol{y}}(\tau_2)\omega^{(2)}_{\boldsymbol{z}}(\tau_3),$$

where from now on we set $\omega_{\boldsymbol{x}}^{(k)}(\tau) = E_{\boldsymbol{x}}^{(k)}(\tau)\,d\tau$, $\omega_{\boldsymbol{x};\boldsymbol{y}}^{(k;m)}(\tau) = E_{\boldsymbol{x}}^{(k)}(\tau)E_{\boldsymbol{y}}^{(m)}(\tau)\,d\tau$, etc. Using (22) and $E_{\boldsymbol{x}}^{(1)}(0) = 0$, which follows from Lemma 35 with $k = 1$ and $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial x_2}\Lambda(\boldsymbol{x},\boldsymbol{y}) &= (2\pi i)^2 \int_0^\infty \omega_{\boldsymbol{x};\boldsymbol{y}}^{(1;2)}(\tau_1),\\
\frac{\partial}{\partial y_2}\Lambda(\boldsymbol{x},\boldsymbol{y}) &= 2\pi i\,\Lambda(\boldsymbol{x})E_{\boldsymbol{y}}^{(1)}(\infty) - (2\pi i)^2 \int_0^\infty \omega_{\boldsymbol{x};\boldsymbol{y}}^{(2;1)}(\tau_1),\\
\frac{\partial}{\partial x_2}\Lambda(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) &= (2\pi i)^3 \int_0^\infty \omega_{\boldsymbol{x};\boldsymbol{y}}^{(1;2)}(\tau_1)\omega_{\boldsymbol{z}}^{(2)}(\tau_2),\\
\frac{\partial}{\partial y_2}\Lambda(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) &= (2\pi i)^3 \int_0^\infty \omega_{\boldsymbol{x}}^{(2)}(\tau_1)\omega_{\boldsymbol{y};\boldsymbol{z}}^{(1;2)}(\tau_2) - (2\pi i)^3 \int_0^\infty \omega_{\boldsymbol{x};\boldsymbol{y}}^{(2;1)}(\tau_1)\omega_{\boldsymbol{z}}^{(2)}(\tau_2),\\
\frac{\partial}{\partial z_2}\Lambda(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) &= 2\pi i\,\Lambda(\boldsymbol{x},\boldsymbol{y})E_{\boldsymbol{z}}^{(1)}(\infty) - (2\pi i)^3 \int_0^\infty \omega_{\boldsymbol{x}}^{(2)}(\tau_1)\omega_{\boldsymbol{y};\boldsymbol{z}}^{(2;1)}(\tau_2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\delta_{a_2}\big((\Lambda(\boldsymbol{a})-\Lambda(\boldsymbol{b}))(\Lambda(\boldsymbol{a},\boldsymbol{b})+\Lambda(\boldsymbol{b},\boldsymbol{c})+\Lambda(\boldsymbol{c},\boldsymbol{a}))\big)\\
&\quad= E_{\boldsymbol{a}}^{(1)}(\infty)(\Lambda(\boldsymbol{a},\boldsymbol{b})+\Lambda(\boldsymbol{b},\boldsymbol{c})+\Lambda(\boldsymbol{c},\boldsymbol{a}))\big)\\
&\qquad+ (\Lambda(\boldsymbol{a})-\Lambda(\boldsymbol{b}))\cdot\Bigg(2\pi i\int_0^\infty \omega_{\boldsymbol{a};\boldsymbol{b}}^{(1;2)} - \Lambda(\boldsymbol{b})E_{\boldsymbol{c}}^{(1)}(\infty) + 2\pi i\int_0^\infty \omega_{\boldsymbol{b};\boldsymbol{c}}^{(2;1)}\\
&\qquad\qquad - 2\pi i\int_0^\infty \omega_{\boldsymbol{c};\boldsymbol{a}}^{(1;2)} + \Lambda(\boldsymbol{c})E_{\boldsymbol{a}}^{(1)}(\infty) - 2\pi i\int_0^\infty \omega_{\boldsymbol{c};\boldsymbol{a}}^{(2;1)}\Bigg)\\
&\quad= E_{\boldsymbol{a}}^{(1)}(\infty)\big(\Lambda(\boldsymbol{a},\boldsymbol{b})+\Lambda(\boldsymbol{b},\boldsymbol{c})+\Lambda(\boldsymbol{c},\boldsymbol{a})+(\Lambda(\boldsymbol{a})-\Lambda(\boldsymbol{b}))\Lambda(\boldsymbol{c})\big)\\
&\qquad- E_{\boldsymbol{c}}^{(1)}(\infty)(\Lambda(\boldsymbol{a})-\Lambda(\boldsymbol{b}))\Lambda(\boldsymbol{b})\\
&\qquad- 2\pi i(\Lambda(\boldsymbol{a})-\Lambda(\boldsymbol{b}))\int_0^\infty(\omega_{\boldsymbol{a};\boldsymbol{c}}^{(1;2)}+\omega_{\boldsymbol{c};\boldsymbol{a}}^{(1;2)}-\omega_{\boldsymbol{a};\boldsymbol{b}}^{(1;2)}-\omega_{\boldsymbol{c};\boldsymbol{b}}^{(1;2)})
\end{aligned}$$

where $\boldsymbol{c} = -(\boldsymbol{a}+\boldsymbol{b})$, while the δ_{a_2}-derivative of

$$\Lambda(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b})-\Lambda(\boldsymbol{c},\boldsymbol{b},\boldsymbol{b})+\Lambda(\boldsymbol{b},\boldsymbol{a},\boldsymbol{a})-\Lambda(\boldsymbol{c},\boldsymbol{a},\boldsymbol{a})+\Lambda(\boldsymbol{c},\boldsymbol{b},\boldsymbol{a})+\Lambda(\boldsymbol{c},\boldsymbol{a},\boldsymbol{b})$$

is as follows:

$$\begin{aligned}
&(2\pi i)^2\int_0^\infty \omega_{\boldsymbol{a};\boldsymbol{b}}^{(1;2)}\omega_{\boldsymbol{b}}^{(2)} + (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{c};\boldsymbol{b}}^{(1;2)}\omega_{\boldsymbol{b}}^{(2)}\\
&\quad+ (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{b}}^{(2)}\omega_{\boldsymbol{a};\boldsymbol{a}}^{(1;2)} - (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{b};\boldsymbol{a}}^{(2;1)}\omega_{\boldsymbol{a}}^{(2)}\\
&\quad+ \Lambda(\boldsymbol{b},\boldsymbol{a})E_{\boldsymbol{a}}^{(1)}(\infty) - (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{b}}^{(2)}\omega_{\boldsymbol{a};\boldsymbol{a}}^{(2;1)}\\
&\quad+ (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{c};\boldsymbol{a}}^{(1;2)}\omega_{\boldsymbol{a}}^{(2)} - (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{c}}^{(2)}\omega_{\boldsymbol{a};\boldsymbol{a}}^{(1;2)}\\
&\quad+ (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{c};\boldsymbol{a}}^{(2;1)}\omega_{\boldsymbol{a}}^{(2)} - \Lambda(\boldsymbol{c},\boldsymbol{a})E_{\boldsymbol{a}}^{(1)}(\infty) + (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{c}}^{(2)}\omega_{\boldsymbol{a};\boldsymbol{a}}^{(2;1)}\\
&\quad- (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{c};\boldsymbol{b}}^{(1;2)}\omega_{\boldsymbol{a}}^{(2)} + \Lambda(\boldsymbol{c},\boldsymbol{b})E_{\boldsymbol{a}}^{(1)}(\infty) - (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{c}}^{(2)}\omega_{\boldsymbol{b};\boldsymbol{a}}^{(2;1)}\\
&\quad- (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{c};\boldsymbol{a}}^{(1;2)}\omega_{\boldsymbol{b}}^{(2)} + (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{c}}^{(2)}\omega_{\boldsymbol{a};\boldsymbol{b}}^{(1;2)} - (2\pi i)^2\int_0^\infty \omega_{\boldsymbol{c};\boldsymbol{a}}^{(2;1)}\omega_{\boldsymbol{b}}^{(2)}\\
&= E_{\boldsymbol{a}}^{(1)}(\infty)\,\big(\Lambda(\boldsymbol{b},\boldsymbol{a})-\Lambda(\boldsymbol{c},\boldsymbol{a})+\Lambda(\boldsymbol{c},\boldsymbol{b})\big)\\
&\quad+ (2\pi i)^2\int_0^\infty(\omega_{\boldsymbol{a};\boldsymbol{c}}^{(1;2)}+\omega_{\boldsymbol{c};\boldsymbol{a}}^{(1;2)}-\omega_{\boldsymbol{a};\boldsymbol{b}}^{(1;2)}-\omega_{\boldsymbol{c};\boldsymbol{b}}^{(1;2)})(\omega_{\boldsymbol{a}}^{(2)}-\omega_{\boldsymbol{b}}^{(2)}).
\end{aligned}$$

Recall that

$$\omega_{\boldsymbol{a};\boldsymbol{c}}^{(1;2)}+\omega_{\boldsymbol{c};\boldsymbol{a}}^{(1;2)}-\omega_{\boldsymbol{a};\boldsymbol{b}}^{(1;2)}-\omega_{\boldsymbol{c};\boldsymbol{b}}^{(1;2)} = \big(E_{\boldsymbol{a}}^{(1)}E_{\boldsymbol{c}}^{(2)}+E_{\boldsymbol{c}}^{(1)}E_{\boldsymbol{a}}^{(2)}-(E_{\boldsymbol{a}}^{(1)}+E_{\boldsymbol{c}}^{(1)})E_{\boldsymbol{b}}^{(2)}\big)\,d\tau.$$

The latter expression is subject to the Borisov–Gunnells type relation in weight 3,

$$E_{\boldsymbol{a}}^{(1)}E_{\boldsymbol{c}}^{(2)} + E_{\boldsymbol{c}}^{(1)}E_{\boldsymbol{a}}^{(2)} - (E_{\boldsymbol{a}}^{(1)} + E_{\boldsymbol{c}}^{(1)})E_{\boldsymbol{b}}^{(2)} = E_{\boldsymbol{b}}^{(3)} - \tfrac{1}{2}E_{\boldsymbol{a}}^{(3)} - \tfrac{1}{2}E_{\boldsymbol{c}}^{(3)},$$

implying

$$\omega_{\boldsymbol{a};\boldsymbol{c}}^{(1;2)} + \omega_{\boldsymbol{c};\boldsymbol{a}}^{(1;2)} - \omega_{\boldsymbol{a};\boldsymbol{b}}^{(1;2)} - \omega_{\boldsymbol{c};\boldsymbol{b}}^{(1;2)} = \omega_{\boldsymbol{b}}^{(3)} - \tfrac{1}{2}\omega_{\boldsymbol{a}}^{(3)} - \tfrac{1}{2}\omega_{\boldsymbol{c}}^{(3)}.$$

In addition, the shuffle relations imply

$$\begin{aligned}\Lambda(\boldsymbol{a},\boldsymbol{b}) + \Lambda(\boldsymbol{b},\boldsymbol{c}) + \Lambda(\boldsymbol{c},\boldsymbol{a}) + (\Lambda(\boldsymbol{a}) - \Lambda(\boldsymbol{b}))\Lambda(\boldsymbol{c}) + (\Lambda(\boldsymbol{b},\boldsymbol{a}) - \Lambda(\boldsymbol{c},\boldsymbol{a}) + \Lambda(\boldsymbol{c},\boldsymbol{b}))\\ = \Lambda(\boldsymbol{a})\Lambda(\boldsymbol{b}) + \Lambda(\boldsymbol{b})\Lambda(\boldsymbol{c}) + (\Lambda(\boldsymbol{a}) - \Lambda(\boldsymbol{b}))\Lambda(\boldsymbol{c}) = \Lambda(\boldsymbol{a})(\Lambda(\boldsymbol{b}) + \Lambda(\boldsymbol{c})).\end{aligned}$$

Combining the above derivations and using the fact that the quantities

$$\begin{aligned}E_{\boldsymbol{a}}^{(1)}(\infty)\Lambda(\boldsymbol{a})(\Lambda(\boldsymbol{b}) + \Lambda(\boldsymbol{c})) &= (2\pi i)^2(\{a_1\} - \tfrac{1}{2})^2(\{a_2\} - \tfrac{1}{2})\\ &\quad\times\big((\{b_1\} - \tfrac{1}{2})(\{b_2\} - \tfrac{1}{2}) + (\{c_1\} - \tfrac{1}{2})(\{c_2\} - \tfrac{1}{2})\big)\end{aligned}$$

and

$$\begin{aligned}E_{\boldsymbol{c}}^{(1)}(\infty)(\Lambda(\boldsymbol{a}) - \Lambda(\boldsymbol{b}))\Lambda(\boldsymbol{b}) &= (2\pi i)^2(\{b_1\} - \tfrac{1}{2})(\{b_2\} - \tfrac{1}{2})(\{c_1\} - \tfrac{1}{2})\\ &\quad\times\big((\{a_1\} - \tfrac{1}{2})(\{a_2\} - \tfrac{1}{2}) - (\{b_1\} - \tfrac{1}{2})(\{b_2\} - \tfrac{1}{2})\big)\end{aligned}$$

are purely real, we finally arrive at

$$\begin{aligned}\frac{1}{2\pi}\frac{\partial}{\partial a_2}\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) &= \operatorname{Im}\Big(2\pi i(\Lambda(\boldsymbol{a}) - \Lambda(\boldsymbol{b}))\int_0^\infty(\omega_{\boldsymbol{b}}^{(3)} - \tfrac{1}{2}\omega_{\boldsymbol{a}}^{(3)} - \tfrac{1}{2}\omega_{\boldsymbol{c}}^{(3)})\\ &\qquad - (2\pi i)^2\int_0^\infty(\omega_{\boldsymbol{b}}^{(3)} - \tfrac{1}{2}\omega_{\boldsymbol{a}}^{(3)} - \tfrac{1}{2}\omega_{\boldsymbol{c}}^{(3)})(\omega_{\boldsymbol{a}}^{(2)} - \omega_{\boldsymbol{b}}^{(2)})\Big)\\ &= -4\pi^2\operatorname{Im}\Big(\int_0^\infty(\omega_{\boldsymbol{a}}^{(2)} - \omega_{\boldsymbol{b}}^{(2)})(\omega_{\boldsymbol{b}}^{(3)} - \tfrac{1}{2}\omega_{\boldsymbol{a}}^{(3)} - \tfrac{1}{2}\omega_{\boldsymbol{c}}^{(3)})\Big);\end{aligned}\tag{55}$$

in the final step we applied the shuffle relations again.

8. Using the Rogers–Zudilin method

To handle the integrals $\int_0^\infty \omega_{\boldsymbol{u}}^{(2)}\omega_{\boldsymbol{v}}^{(3)}$ in (55), we use the Rogers–Zudilin method.

8.1. The setup. For weights $\ell \ge k \ge 2$, we want to work out the integral

$$I_{\boldsymbol{u},\boldsymbol{v}}^{(k,\ell)} = \int_0^\infty E_{\boldsymbol{u}}^{(k)}(iy)\widetilde{E}_{\boldsymbol{v}}^{(\ell)}(iy)\,dy \qquad (\boldsymbol{u},\boldsymbol{v} \in (\mathbf{R}/\mathbf{Z})^2)$$

in terms of L-values. Here $\widetilde{E}_{\boldsymbol{v}}^{(\ell)}$ denotes the Eichler integral of $E_{\boldsymbol{v}}^{(\ell)}$, that is, the unique primitive of $2\pi i E_{\boldsymbol{v}}^{(\ell)}(\tau)\,d\tau$ whose regularised value at ∞ is zero. The function $E_{\boldsymbol{u}}^{(k)}(\tau)\widetilde{E}_{\boldsymbol{v}}^{(\ell)}(\tau)$ is admissible, so that $I_{\boldsymbol{u},\boldsymbol{v}}^{(k,\ell)}$ is well-defined.

Recall the modularity with respect to $\sigma = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ (Lemma 35):

$$E_{\boldsymbol{u}}^{(k)}(iy) = (E_{\boldsymbol{u}\sigma^{-1}}^{(k)}|_k\sigma)(iy) = (iy)^{-k}E_{\boldsymbol{u}\sigma^{-1}}^{(k)}\Big(\frac{i}{y}\Big) = (-i)^k y^{-k}E_{-\boldsymbol{u}\sigma}^{(k)}\Big(\frac{i}{y}\Big) = i^k y^{-k}E_{\boldsymbol{u}\sigma}^{(k)}\Big(\frac{i}{y}\Big).\tag{56}$$

8.2. The computation. We have

$$I_{\boldsymbol{u},\boldsymbol{v}}^{(k,\ell)} = i^k\int_0^\infty E_{\boldsymbol{u}\sigma}^{(k)}\Big(\frac{i}{y}\Big)\widetilde{E}_{\boldsymbol{v}}^{(\ell)}(iy)\,\frac{dy}{y^k}.$$

Write $E^{(k)}_{\boldsymbol{u}\sigma}(i/y) = C_1 + S_1(y)$ and $\widetilde{E}^{(\ell)}_{\boldsymbol{v}}(iy) = C_2 y + S_2(y)$, where $S_1(y)$, respectively $S_2(y)$, decays exponentially as $y \to 0^+$, respectively $y \to +\infty$. Explicitly,

$$\begin{aligned}
C_1 &= a_0(E^{(k)}_{\boldsymbol{u}\sigma}),\\
C_2 &= -2\pi a_0(E^{(\ell)}_{\boldsymbol{v}}),\\
S_1(y) &= \sum_{\substack{m_1\ge 1\\ n_1\in\mathbf{R}_{>0}}} (a(m_1)b(n_1) + (-1)^k a(-m_1)b(-n_1)) n_1^{k-1} e^{-2\pi m_1 n_1/y},\\
S_2(y) &= \sum_{\substack{m_2\ge 1\\ n_2\in\mathbf{R}_{>0}}} (c(m_2)d(n_2) + (-1)^\ell c(-m_2)d(-n_2)) \frac{n_2^{\ell-2}}{m_2} e^{-2\pi m_2 n_2 y},
\end{aligned}$$

where the functions $a, b, c, d\colon \mathbf{R} \to \mathbf{C}$ are defined by

$$\begin{aligned}
a(m) &= -e(-mu_1), & c(m) &= -e(mv_2),\\
b(n) &= \mathbf{1}_{n\equiv u_2 \bmod 1}, & d(n) &= \mathbf{1}_{n\equiv v_1 \bmod 1}.
\end{aligned}$$

We can write $I^{(k,\ell)}_{\boldsymbol{u},\boldsymbol{v}} = T_1 + T_2 + T_3$ with

$$\begin{aligned}
T_1 &= i^k \int_0^\infty S_1(y) S_2(y) \frac{dy}{y^k},\\
T_2 &= i^k C_1 \int_0^\infty \widetilde{E}^{(\ell)}_{\boldsymbol{v}}(iy) \frac{dy}{y^k},\\
T_3 &= i^k C_2 \int_0^\infty E^{(k)}_{\boldsymbol{u}\sigma}\Big(\frac{i}{y}\Big) \frac{dy}{y^{k-1}},
\end{aligned}$$

where each term T_i is understood as the regularised value of the corresponding Mellin transform (actually the integral T_1 converges exponentially at 0 and ∞). The terms T_2 and T_3 essentially boil down to L-values of Eisenstein series, and will be dealt with later.

We compute T_1 using the Rogers–Zudilin method. We first consider the terms $a(m_1)b(n_1)$ and $c(m_2)d(n_2)$ inside the series S_1 and S_2 respectively:

$$\begin{aligned}
&\int_0^\infty \Bigg(\sum_{\substack{m_1\ge 1\\ n_1\in\mathbf{R}_{>0}}} a(m_1)b(n_1) n_1^{k-1} e^{-2\pi m_1 n_1/y}\Bigg)\Bigg(\sum_{\substack{m_2\ge 1\\ n_2\in\mathbf{R}_{>0}}} c(m_2)d(n_2)\frac{n_2^{\ell-2}}{m_2} e^{-2\pi m_2 n_2 y}\Bigg)\frac{dy}{y^k}\\
&= \sum_{\substack{m_1\ge 1\\ n_1\in\mathbf{R}_{>0}}}\sum_{\substack{m_2\ge 1\\ n_2\in\mathbf{R}_{>0}}} a(m_1)b(n_1)c(m_2)d(n_2) n_1^{k-1}\cdot\frac{n_2^{\ell-2}}{m_2}\int_0^\infty e^{-2\pi(m_2 n_2 y + \frac{m_1 n_1}{y})}\frac{dy}{y^k}\\
&\overset{y\to\frac{n_1}{m_2}\cdot y}{=} \sum_{\substack{m_1\ge 1\\ n_1\in\mathbf{R}_{>0}}}\sum_{\substack{m_2\ge 1\\ n_2\in\mathbf{R}_{>0}}} a(m_1)b(n_1)c(m_2)d(n_2) n_2^{\ell-2} m_2^{k-2}\int_0^\infty e^{-2\pi(n_1 n_2 y + \frac{m_1 m_2}{y})}\frac{dy}{y^k}\\
&= \int_0^\infty \Bigg(\sum_{m_1,m_2\ge 1} a(m_1)c(m_2) m_2^{k-2} e^{-2\pi m_1 m_2/y}\Bigg)\Bigg(\sum_{n_1,n_2\in\mathbf{R}_{>0}} b(n_1)d(n_2) n_2^{\ell-2} e^{-2\pi n_1 n_2 y}\Bigg)\frac{dy}{y^k}.
\end{aligned}$$

This computation will be summarised with the formal transformation $ab\otimes cd \to ac\otimes bd$.

Now the term T_1 is a linear combination of four terms, involving substitutions $(m_i, n_i) \to (-m_i, -n_i)$ for $i = 1, 2$. As a shortcut, write $f^-(x) = f(-x)$ for a function $f\colon \mathbf{R}\to\mathbf{C}$. Then the computation of T_1 can be written formally

$$\begin{aligned}
&(ab + (-1)^k a^- b^-) \otimes (cd + (-1)^\ell c^- d^-)\\
&\qquad \to ac\otimes bd + (-1)^\ell a c^- \otimes b d^- + (-1)^k a^- c \otimes b^- d + (-1)^{k+\ell} a^- c^- \otimes b^- d^-.
\end{aligned}$$

This linear combination does not produce Eisenstein series: for example $\sum b(n_1)d(n_2) q^{n_1 n_2}$ has no modularity property, because of the lack of parity conditions in b and d. To get Eisenstein series, we have to take the imaginary part of T_1; this corresponds to considering the Beilinson regulator map with values in *real* Deligne–Beilinson cohomology. Noting that $\bar{a} = a^-$, $\bar{c} = c^-$, $\bar{b} = b$ and $\bar{d} = d$, we see that $T_1 - \overline{T_1}$ can be computed as

$$\begin{aligned}
&(ab + (-1)^k a^- b^-) \otimes (cd + (-1)^\ell c^- d^-) + (-1)^{k-1}(a^- b + (-1)^k ab^-) \otimes (c^- d + (-1)^\ell cd^-) \\
&\to ac \otimes bd + (-1)^\ell ac^- \otimes bd^- + (-1)^k a^- c \otimes b^- d + (-1)^{k+\ell} a^- c^- \otimes b^- d^- \\
&\quad + (-1)^{k-1} a^- c^- \otimes bd + (-1)^{k+\ell-1} a^- c \otimes bd^- - ac^- \otimes b^- d + (-1)^{\ell-1} ac \otimes b^- d^- \\
&= (ac + (-1)^{k-1} a^- c^-) \otimes (bd + (-1)^{\ell-1} b^- d^-) + (-1)^k (a^- c + (-1)^{k-1} ac^-) \otimes (b^- d + (-1)^{\ell-1} bd^-).
\end{aligned}$$

Up to the constant terms, we recognise the sum of two pairwise products of Eisenstein series of weights $k-1$ and $\ell-1$, respectively. Denoting by $f^0 = f - a_0(f)$ the rapidly decreasing part of f, we have

$$\text{(57)} \qquad \operatorname{Im}(T_1) = \frac{i^{-k-1}}{2} \int_0^\infty \Big(H^{(k-1),0}_{u_1,v_2}\big(\frac{i}{y}\big) G^{(\ell-1),0}_{v_1,-u_2}(iy) - H^{(k-1),0}_{u_1,-v_2}\big(\frac{i}{y}\big) G^{(\ell-1),0}_{v_1,u_2}(iy) \Big) \frac{dy}{y^k},$$

where for $\boldsymbol{x} = (x_1, x_2) \in (\mathbf{R}/\mathbf{Z})^2$, the Eisenstein series $H^{(k)}_{\boldsymbol{x}}$ is given by

$$H^{(k)}_{\boldsymbol{x}}(\tau) = a_0(H^{(k)}_{\boldsymbol{x}}) + \sum_{m,n \ge 1} \big(e(mx_1 + nx_2) + (-1)^k e(-mx_1 - nx_2) \big) n^{k-1} q^{mn},$$

with

$$a_0(H^{(1)}_{\boldsymbol{x}}) = \begin{cases} 0 & \text{if } \boldsymbol{x} = \mathbf{0}, \\ \frac{1}{2} \frac{1+e(x_2)}{1-e(x_2)} & \text{if } x_1 = 0 \text{ and } x_2 \neq 0, \\ \frac{1}{2} \frac{1+e(x_1)}{1-e(x_1)} & \text{if } x_1 \neq 0 \text{ and } x_2 = 0, \\ \frac{1}{2} \Big(\frac{1+e(x_1)}{1-e(x_1)} + \frac{1+e(x_2)}{1-e(x_2)} \Big) & \text{if } x_1 \neq 0 \text{ and } x_2 \neq 0, \end{cases}$$

$$(k \ge 2) \qquad a_0(H^{(k)}_{\boldsymbol{x}}) = (-1)^k \hat{\zeta}(-x_2, 1-k).$$

The Eisenstein series $G^{(k)}_{\boldsymbol{x}}$ and $H^{(k)}_{\boldsymbol{x}}$ are related as follows.

Lemma 60. *Let $k \ge 1$ and $\boldsymbol{x} = (x_1, x_2) \in (\mathbf{R}/\mathbf{Z})^2$, with $x_1 \neq 0$ in the case $k = 2$. Then $H^{(k)}_{\boldsymbol{x}}|_k \sigma = G^{(k)}_{\boldsymbol{x}}$. In particular, we have $H^{(k)}_{\boldsymbol{x}}(i/y) = (iy)^k G^{(k)}_{\boldsymbol{x}}(iy)$ for any $y > 0$.*

Proof. In the case $\boldsymbol{x} \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2$, this is [7, Lemme 3.10]. The general case follows since both sides are continuous in $\boldsymbol{x}$. □

We compute (57) by 'completing' the Eisenstein series $H^{(k-1)}$ and $G^{(\ell-1)}$, and separating the contribution from the constant terms, using also Lemma 60:

$$\begin{aligned}
\text{(58)} \quad &\int_0^\infty H^{(k-1),0}_{\boldsymbol{a}}\big(\frac{i}{y}\big) G^{(\ell-1),0}_{\boldsymbol{b}}(iy) y^s \frac{dy}{y} \\
&= \mathcal{M}\Big(H^{(k-1)}_{\boldsymbol{a}}\big(\frac{i}{y}\big) G^{(\ell-1)}_{\boldsymbol{b}}(iy), s \Big) - a_0(H^{(k-1)}_{\boldsymbol{a}}) \mathcal{M}(G^{(\ell-1)}_{\boldsymbol{b}}, s) - a_0(G^{(\ell-1)}_{\boldsymbol{b}}) \mathcal{M}\Big(H^{(k-1)}_{\boldsymbol{a}}\big(\frac{i}{y}\big), s \Big) \\
&= i^{k-1} \mathcal{M}(G^{(k-1)}_{\boldsymbol{a}} G^{(\ell-1)}_{\boldsymbol{b}}, s+k-1) \\
&\quad - a_0(H^{(k-1)}_{\boldsymbol{a}}) \mathcal{M}(G^{(\ell-1)}_{\boldsymbol{b}}, s) - i^{k-1} a_0(G^{(\ell-1)}_{\boldsymbol{b}}) \mathcal{M}(G^{(k-1)}_{\boldsymbol{a}}, s+k-1).
\end{aligned}$$

Putting (57) and (58) together, we get the following formula for the imaginary part of T_1:

$$\begin{aligned}
&\operatorname{Im}(T_1) = T_1' + T_2' + T_3', \\
&\quad T_1' = -\frac{1}{2} \mathcal{M}^*\big(G^{(k-1)}_{u_1,v_2} G^{(\ell-1)}_{v_1,-u_2} - G^{(k-1)}_{u_1,-v_2} G^{(\ell-1)}_{v_1,u_2}, 0 \big), \\
&\quad T_2' = \frac{i^{1-k}}{2} a_0(H^{(k-1)}_{u_1,v_2}) \mathcal{M}(G^{(\ell-1)}_{v_1,-u_2}, 1-k) - i^{1-k} a_0(H^{(k-1)}_{u_1,-v_2}) \mathcal{M}(G^{(\ell-1)}_{v_1,u_2}, 1-k), \\
&\quad T_3' = \frac{1}{2} a_0(G^{(\ell-1)}_{v_1,-u_2}) \mathcal{M}^*(G^{(k-1)}_{u_1,v_2}, 0) - \frac{1}{2} a_0(G^{(\ell-1)}_{v_1,u_2}) \mathcal{M}^*(G^{(k-1)}_{u_1,-v_2}, 0).
\end{aligned}$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are N-torsion (and $u_1, v_1 \neq 0$), the main term T_1' is the (completed) L-value of a modular form of weight $k+\ell-2$ and level $\Gamma(N)$ with rational Fourier coefficients.

8.3. **The constant terms.** We henceforth assume that $k=2$, which is enough for our purpose. Also, we put ourselves in the generic situation where the coordinates of $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-zero. In this case, the Eisenstein series appearing in T_1' have no constant term (see Definition 39), so that the Mellin transform in T_1' is holomorphic at $s=0$. Moreover $a_0(G^{(\ell-1)}_{v_1,-u_2}) = 0$ and thus $T_3' = 0$.

Let us compute $\operatorname{Im}(T_2)$. The Mellin transform of the Eichler integral $\widetilde{E}^{(\ell)}_{\boldsymbol{v}}$ is given by

$$\mathcal{M}(\widetilde{E}^{(\ell)}_{\boldsymbol{v}}, s) = \frac{2\pi}{s}\mathcal{M}(E^{(\ell)}_{\boldsymbol{v}}, s+1) = \frac{2\pi}{s}\Big(-\frac{a_0(E^{(\ell)}_{\boldsymbol{v}})}{s+1} + \mathcal{M}^*(E^{(\ell)}_{\boldsymbol{v}}, 0) + O_{s\to -1}(s+1)\Big),$$

from which we deduce

$$T_2 = -a_0(E^{(2)}_{u\sigma})\mathcal{M}^*(\widetilde{E}^{(\ell)}_{\boldsymbol{v}}, -1) = \pi B_2(\{u_2\})\big(\mathcal{M}^*(E^{(\ell)}_{\boldsymbol{v}}, 0) - a_0(E^{(\ell)}_{\boldsymbol{v}})\big).$$

Using Lemma 41 and equation (13), this leads to

$$\begin{aligned}\operatorname{Im}(T_2) &= -\frac{\pi i}{2}B_2(\{u_2\})\lim_{s\to 0}\Gamma(s)\big(-\zeta(v_1, s-\ell+1) + (-1)^\ell\zeta(-v_1, s-\ell+1)\big)\big(\hat{\zeta}(v_2, s) - \hat{\zeta}(-v_2, s)\big)\\ &= \frac{\pi i}{2}B_2(\{u_2\})\frac{1+e(v_2)}{1-e(v_2)}\lim_{s\to 0}\Gamma(s)\big(\zeta(v_1, s-\ell+1) + (-1)^{\ell+1}\zeta(-v_1, s-\ell+1)\big).\end{aligned}$$

For the term T_3, we rewrite it using (56):

$$T_3 = -2\pi a_0(E^{(\ell)}_{\boldsymbol{v}})\int_0^\infty E^{(2)}_{\boldsymbol{u}}(iy)y\,dy = -2\pi\frac{B_\ell(\{v_1\})}{\ell}\mathcal{M}^*(E^{(2)}_{\boldsymbol{u}}, 2).$$

Moreover,

$$\begin{aligned}\lim_{\substack{s\to 2\\ s\in\mathbf{R}}}\operatorname{Im}(\mathcal{M}(E^{(2)}_{\boldsymbol{u}}, s)) &= (2\pi)^{-2}\lim_{\substack{s\to 2\\ s\in\mathbf{R}}}\big(-\zeta(u_1, s-1)\operatorname{Im}(\hat{\zeta}(u_2, s)) - \zeta(-u_1, s-1)\operatorname{Im}(\hat{\zeta}(-u_2, s))\big)\\ &= (2\pi)^{-2}\operatorname{Im}(\hat{\zeta}(u_2, 2)) \times \pi i\frac{1+e(u_1)}{1-e(u_1)}.\end{aligned}$$

Therefore,

$$\operatorname{Im}(T_3) = -\frac{i}{2}\frac{B_\ell(\{v_1\})}{\ell}\cdot\operatorname{Im}(\hat{\zeta}(u_2, 2))\cdot\frac{1+e(u_1)}{1-e(u_1)}.$$

It remains to compute T_2'. We have $T_2' = A + B$ with

$$\begin{aligned}A &= \frac{i}{4}\frac{1+e(u_1)}{1-e(u_1)}(\mathcal{M}(G^{(\ell-1)}_{v_1,u_2}, -1) - \mathcal{M}(G^{(\ell-1)}_{v_1,-u_2}, -1)),\\ B &= -\frac{i}{4}\frac{1+e(v_2)}{1-e(v_2)}(\mathcal{M}(G^{(\ell-1)}_{v_1,u_2}, -1) + \mathcal{M}(G^{(\ell-1)}_{v_1,-u_2}, -1)).\end{aligned}$$

Let us compute A. Using Lemma 41, we obtain

$$\begin{aligned}&\mathcal{M}(G^{(\ell-1)}_{v_1,u_2}, -1) - \mathcal{M}(G^{(\ell-1)}_{v_1,-u_2}, -1)\\ &\quad= 2\pi\lim_{s\to -1}\Gamma(s)\big(\zeta(v_1, s-\ell+2) + (-1)^\ell\zeta(-v_1, s-\ell+2)\big)\big(\zeta(u_2, s) - \zeta(u_2, -s)\big)\\ &\quad= -\frac{4\pi}{\ell}B_\ell(\{v_1\})\lim_{s\to -1}\Gamma(s)(\zeta(u_2, s) - \zeta(-u_2, s)).\end{aligned}$$

Now using the Hurwitz formula [7, eq. (6)], we have

$$\zeta(u_2, s) - \zeta(-u_2, s) = (2\pi)^{s-1}\Gamma(1-s)(e^{-\pi i(1-s)/2} - e^{\pi i(1-s)/2})(\hat{\zeta}(u_2, 1-s) - \hat{\zeta}(-u_2, 1-s)),$$

which gives

$$\lim_{s\to -1}\Gamma(s)(\zeta(u_2, s) - \zeta(-u_2, s)) = -\frac{1}{2\pi}\operatorname{Im}(\hat{\zeta}(u_2, 2)).$$

Therefore,

$$A = \frac{i}{2}\frac{B_\ell(\{v_1\})}{\ell}\frac{1+e(u_1)}{1-e(u_1)}\operatorname{Im}(\hat{\zeta}(u_2,2)).$$

Similarly, the term B is equal to

$$B = -\frac{\pi i}{2}\frac{1+e(v_2)}{1-e(v_2)}B_2(\{u_2\})\lim_{s\to 0}\Gamma(s)\big(\zeta(v_1,s-\ell+1)+(-1)^{\ell+1}\zeta(-v_1,s-\ell+1)\big).$$

Collecting everything, we see that $\operatorname{Im}(T_2)+B=0$ and $\operatorname{Im}(T_3)+A=0$. Thus, $\operatorname{Im}(I^{(2,\ell)}_{\boldsymbol{u},\boldsymbol{v}}) = T_1'$, as summarised in the following theorem.

Theorem 61. *Let $\ell \geq 2$ be an integer, and $u=(u_1,u_2)$, $v=(v_1,v_2)$ in $(\mathbf{R}/\mathbf{Z})^2$, where all u_i and v_i are non-zero. Then*

$$\operatorname{Im}(I^{(2,\ell)}_{\boldsymbol{u},\boldsymbol{v}}) = -\frac{1}{2}\mathcal{M}\big(G^{(1)}_{u_1,v_2}G^{(\ell-1)}_{v_1,-u_2} - G^{(1)}_{u_1,-v_2}G^{(\ell-1)}_{v_1,u_2}, 0\big).$$

9. Getting to the L-value

In Section 7, we established that the a_2-derivative of the (interpolated) Goncharov regulator of $\tilde{\xi}(\boldsymbol{a},\boldsymbol{b})$ is

$$\frac{\partial}{\partial a_2}\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) = -4\pi^2\operatorname{Im}\Big(\int_0^\infty \big(E^{(2)}_{\boldsymbol{b}}(\tau)-E^{(2)}_{\boldsymbol{a}}(\tau)\big)\big(\widetilde{E}^{(3)}_{\boldsymbol{b}}(\tau)-\tfrac{1}{2}\widetilde{E}^{(3)}_{\boldsymbol{a}}(\tau)-\tfrac{1}{2}\widetilde{E}^{(3)}_{\boldsymbol{c}}(\tau)\big)\,d\tau\Big),$$

see formula (55). This holds in the domain where all the coordinates of $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in(\mathbf{R}/\mathbf{Z})^2$ are non-zero, with $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=\boldsymbol{0}$ as usual. Using Theorem 61, we have

(59)
$$\begin{aligned}\frac{\partial}{\partial a_2}\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) = 2\pi^2\mathcal{M}\Big(&(G^{(1)}_{b_1,b_2}G^{(2)}_{b_1,-b_2} - G^{(1)}_{b_1,-b_2}G^{(2)}_{b_1,b_2}) - \frac{1}{2}(G^{(1)}_{b_1,a_2}G^{(2)}_{a_1,-b_2} - G^{(1)}_{b_1,-a_2}G^{(2)}_{a_1,b_2})\\ &-\frac{1}{2}(G^{(1)}_{b_1,c_2}G^{(2)}_{c_1,-b_2} - G^{(1)}_{b_1,-c_2}G^{(2)}_{c_1,b_2}) - (G^{(1)}_{a_1,b_2}G^{(2)}_{b_1,-a_2} - G^{(1)}_{a_1,-b_2}G^{(2)}_{b_1,a_2})\\ &+\frac{1}{2}(G^{(1)}_{a_1,a_2}G^{(2)}_{a_1,-a_2} - G^{(1)}_{a_1,-a_2}G^{(2)}_{a_1,a_2}) + \frac{1}{2}(G^{(1)}_{a_1,c_2}G^{(2)}_{c_1,-a_2} - G^{(1)}_{a_1,-c_2}G^{(2)}_{c_1,a_2}), 0\Big).\end{aligned}$$

Let us write $f = f_1+\cdots+f_6$ for the modular form inside (59). We rewrite f using Theorems 58 and 59. Theorem 59 gives

$$\mathcal{M}(f_1,0) = \mathcal{M}(G^{(3)}_{0,b_2},0), \qquad \mathcal{M}(f_5,0) = \frac{1}{2}\mathcal{M}(G^{(3)}_{0,a_2},0).$$

Using Theorem 58 with $x_1=c_1$, $y_1=a_1$, $u_2=a_2$ and $v_2=-c_2$, we have

$$G^{(1)}_{-b_1,a_2}G^{(2)}_{a_1,b_2} + G^{(1)}_{a_1,-c_2}G^{(2)}_{c_1,a_2} - G^{(1)}_{-b_1,-c_2}G^{(2)}_{c_1,-b_2} = G^{(1)}_{a_1,b_2}G^{(2)}_{b_1,-a_2}; \tag{60}$$

and with $x_1=c_1$, $y_1=b_1$, $u_2=b_2$ and $v_2=-c_2$, we obtain

$$G^{(1)}_{b_1,-c_2}G^{(2)}_{c_1,b_2} - G^{(1)}_{-a_1,-c_2}G^{(2)}_{c_1,-a_2} - G^{(1)}_{b_1,a_2}G^{(2)}_{-a_1,b_2} = G^{(1)}_{a_1,-b_2}G^{(2)}_{b_1,a_2}. \tag{61}$$

Combining (60) and (61), we have

$$f_2+f_3+f_6 = -\frac{1}{2}\times(60) + \frac{1}{2}\times(61) = -\frac{1}{2}G^{(1)}_{a_1,b_2}G^{(2)}_{b_1,-a_2} + \frac{1}{2}G^{(1)}_{a_1,-b_2}G^{(2)}_{b_1,a_2} = \frac{1}{2}f_4.$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial a_2}\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) &= 2\pi^2\mathcal{M}\Big(G^{(3)}_{0,b_2} + \frac{1}{2}G^{(3)}_{0,a_2} + \frac{3}{2}f_4, 0\Big)\\ &= -3\pi^2\mathcal{M}(G^{(1)}_{a_1,b_2}G^{(2)}_{b_1,-a_2} - G^{(1)}_{a_1,-b_2}G^{(2)}_{b_1,a_2},0) + \pi^2\mathcal{M}(G^{(3)}_{0,a_2}+2G^{(3)}_{0,b_2},0).\end{aligned} \tag{62}$$

To find the a_2-antiderivative of the right-hand side of (62), we use Lemma 40. We have formally

$$\begin{aligned}\mathcal{M}(G^{(1)}_{a_1,-b_2}G^{(2)}_{b_1,a_2},0) &= \int_0^\infty G^{(1)}_{a_1,-b_2}(iy)G^{(2)}_{b_1,a_2}(iy)\frac{dy}{y}\\ &= -\frac{1}{2\pi}\int_0^\infty G^{(1)}_{a_1,-b_2}(iy)\frac{\partial}{\partial a_2}G^{(1)}_{b_1,a_2}(iy)\frac{dy}{y^2}\\ &= -\frac{1}{2\pi}\frac{\partial}{\partial a_2}\mathcal{M}(G^{(1)}_{a_1,-b_2}G^{(1)}_{b_1,a_2},-1).\end{aligned} \tag{63}$$

Lemma 62. *For $x \in \mathbf{R}/\mathbf{Z}$, $x \neq 0$, we have $\mathcal{M}(G^{(3)}_{0,x},0) = -2\zeta'(-2)B_1(\{x\})$.*

Proof. Using (20), we obtain

$$\mathcal{M}(G^{(3)}_{0,x},0) = \lim_{s\to 0}\Gamma(s)\zeta(s-2)(\zeta(x,s)-\zeta(-x,s)) = \zeta'(-2)(\zeta(x,0)-\zeta(-x,0)).$$

We conclude using the evaluation $\zeta(x,0) = -B_1(\{x\})$ [7, Section 2, p. 1123]. □

From (62), (63) and Lemma 62, we get

$$\frac{\partial}{\partial a_2}\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) = -\frac{3\pi}{2}\frac{\partial}{\partial a_2}\mathcal{M}(G^{(1)}_{a_1,b_2}G^{(1)}_{b_1,-a_2} + G^{(1)}_{a_1,-b_2}G^{(1)}_{b_1,a_2},-1) + \frac{\zeta(3)}{2}\big(B_1(a_2)+2B_1(b_2)\big). \tag{64}$$

This identity holds in the domain

$$D_0 = \{(\boldsymbol{a},\boldsymbol{b}) : 0 < a_1,a_2,b_1,b_2 < 1,\ a_1+b_1 \neq 1,\ a_2+b_2\neq 1\},$$

which has four connected components:

$$D_{++} = \{a_1+b_1>1,\ a_2+b_2>1\}, \qquad D_{+-} = \{a_1+b_1>1,\ a_2+b_2<1\},$$
$$D_{-+} = \{a_1+b_1<1,\ a_2+b_2>1\}, \qquad D_{--} = \{a_1+b_1<1,\ a_2+b_2<1\}.$$

We can integrate (64) on each of these domains, with possibly different integration constants. So for $\square \in \{++,+-,-+,--\}$ and $(\boldsymbol{a},\boldsymbol{b}) \in D_\square$, we have

$$\begin{aligned}\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) &= -\frac{3\pi}{2}\mathcal{M}(G^{(1)}_{a_1,b_2}G^{(1)}_{b_1,-a_2} + G^{(1)}_{a_1,-b_2}G^{(1)}_{b_1,a_2},-1)\\ &\quad + \frac{\zeta(3)}{4}\big(B_2(a_2)+B_2(b_2)+4B_1(a_2)B_1(b_2)\big) + C_\square(\boldsymbol{a},\boldsymbol{b}),\end{aligned} \tag{65}$$

where $C_\square(\boldsymbol{a},\boldsymbol{b})$ does not depend on a_2. For convenience, write

$$L(\boldsymbol{a},\boldsymbol{b}) = -\frac{3\pi}{2}\mathcal{M}(G^{(1)}_{a_1,b_2}G^{(1)}_{b_1,-a_2} + G^{(1)}_{a_1,-b_2}G^{(1)}_{b_1,a_2},-1).$$

To get further, note that the symmetry $(\boldsymbol{a},\boldsymbol{b}) \to (\boldsymbol{b},\boldsymbol{a})$ leaves stable the connected components $D_\square$. And we have

$$\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) = \mathcal{G}(\boldsymbol{b},\boldsymbol{a}) \qquad ((\boldsymbol{a},\boldsymbol{b}) \in D_\square),$$

which follows from the identity of cocycles $\tilde{\xi}(\boldsymbol{a},\boldsymbol{b}) = \tilde{\xi}(\boldsymbol{b},\boldsymbol{a})$, or from the expression of $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ in terms of triple modular values. Taking into account $L(\boldsymbol{a},\boldsymbol{b}) = L(\boldsymbol{b},\boldsymbol{a})$, we see from (65) that $C_\square(\boldsymbol{a},\boldsymbol{b})$ is symmetric in $\boldsymbol{a},\boldsymbol{b}$. Therefore $C_\square(\boldsymbol{a},\boldsymbol{b})$ does not depend on b_2 either, and we can write

$$C_\square(\boldsymbol{a},\boldsymbol{b}) = C'_\square(a_1,b_1).$$

(The function $C'_\square(\alpha,\beta)$ is defined either on the domain $\alpha+\beta>1$ or on the domain $\alpha+\beta<1$, depending on the first sign in $\square$.)

Now, let us use the matrix $\sigma = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ acting as

$$(\boldsymbol{a},\boldsymbol{b}) = (a_1,a_2,b_1,b_2) \to (\boldsymbol{a}\sigma,\boldsymbol{b}\sigma) = (a_2,-a_1,b_2,-b_1).$$

It permutes the connected components of the domain by $D_{++} \to D_{+-} \to D_{--} \to D_{-+} \to D_{++}$. With the regulator, we have

$$\begin{aligned}\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) &= \int_0^\infty r_3(2)(\tilde{\xi}(\boldsymbol{a},\boldsymbol{b})) = \int_\infty^0 r_3(2)(\tilde{\xi}(\boldsymbol{a},\boldsymbol{b}))|\sigma \\ &= -\int_0^\infty r_3(2)(\tilde{\xi}(\boldsymbol{a},\boldsymbol{b})|\sigma) = -\int_0^\infty r_3(2)(\tilde{\xi}(\boldsymbol{a}\sigma,\boldsymbol{b}\sigma)) = -\mathcal{G}(\boldsymbol{a}\sigma,\boldsymbol{b}\sigma).\end{aligned}$$

One also checks that $L(\boldsymbol{a},\boldsymbol{b}) = -L(\boldsymbol{a}\sigma,\boldsymbol{b}\sigma)$, using the identity of Eisenstein series $G^{(1)}_{x_1,x_2} = G^{(1)}_{x_2,x_1} = -G^{(1)}_{-x_1,-x_2}$. Therefore,

$$\begin{aligned}0 &= \mathcal{G}(\boldsymbol{a},\boldsymbol{b}) + \mathcal{G}(\boldsymbol{a}\sigma,\boldsymbol{b}\sigma) \\ &= \frac{\zeta(3)}{4}\big(B_2(a_2)+B_2(b_2)+4B_1(a_2)B_1(b_2)\big) + C'_\square(a_1,b_1) \\ &\quad + \frac{\zeta(3)}{4}\big(B_2(a_1)+B_2(b_1)+4B_1(a_1)B_1(b_1)\big) + C'_{\sigma(\square)}(a_2,b_2).\end{aligned}$$

This identity can be rewritten as

$$\begin{aligned}&\frac{\zeta(3)}{4}\big(B_2(a_1)+B_2(b_1)+4B_1(a_1)B_1(b_1)\big) + C'_\square(a_1,b_1) \\ &\qquad = -\frac{\zeta(3)}{4}\big(B_2(a_2)+B_2(b_2)+4B_1(a_2)B_1(b_2)\big) - C'_{\sigma(\square)}(a_2,b_2).\end{aligned}$$

The left-hand side depends only on a_1,b_1, while the right-hand side depends only on a_2,b_2. Therefore, they do not depend on $(\boldsymbol{a},\boldsymbol{b})$ in $D_\square$ and we can write

$$\begin{aligned}C'_\square(a_1,b_1) &= -\frac{\zeta(3)}{4}\big(B_2(a_1)+B_2(b_1)+4B_1(a_1)B_1(b_1)\big) + C''_\square, \\ C'_{\sigma(\square)}(a_2,b_2) &= -\frac{\zeta(3)}{4}\big(B_2(a_2)+B_2(b_2)+4B_1(a_2)B_1(b_2)\big) - C''_\square.\end{aligned}$$

Reporting into (65) we have, for $(\boldsymbol{a},\boldsymbol{b}) \in D_\square$,

$$\begin{aligned}\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) &= -\frac{3\pi}{2}\mathcal{M}(G^{(1)}_{a_1,b_2}G^{(1)}_{b_1,-a_2} + G^{(1)}_{a_1,-b_2}G^{(1)}_{b_1,a_2}, -1) \\ &\quad - \frac{\zeta(3)}{4}\big(B_2(a_1)+B_2(b_1)+4B_1(a_1)B_1(b_1) \\ &\qquad - B_2(a_2)-B_2(b_2)-4B_1(a_2)B_1(b_2)\big) + C''_\square.\end{aligned} \tag{66}$$

Substituting $(\boldsymbol{a},\boldsymbol{b}) \to (\boldsymbol{a}\sigma,\boldsymbol{b}\sigma)$ in this relation, we see that $C''_{\sigma(\square)} = -C''_\square$ for any component $\square \in \{++,+-,-+,--\}$.

Finally, let us take $\boldsymbol{a} = \boldsymbol{b}$ in (66). Since the cocycle $\tilde{\xi}(\boldsymbol{a},\boldsymbol{a})$ is zero, we have $\mathcal{G}(\boldsymbol{a},\boldsymbol{a}) = 0$. Specialising even further to $\boldsymbol{a} = \boldsymbol{b} = (\alpha,\alpha)$ with $\alpha \in (0,1)$, $\alpha \neq \frac{1}{2}$, the L-value part in (66) vanishes since $G^{(1)}_{\alpha,-\alpha} = 0$. Moreover, the $\zeta(3)$ part also vanishes. It follows that $C''_{++} = C''_{--} = 0$, hence $C''_\square = 0$ for every $\square$. We have thus shown the following.

Theorem 63. *For any $\boldsymbol{a},\boldsymbol{b} \in (\mathbf{R}/\mathbf{Z})^2$ such that the coordinates of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}+\boldsymbol{b}$ are non-zero, we have*

$$\begin{aligned}\mathcal{G}(\boldsymbol{a},\boldsymbol{b}) &= -\frac{3\pi}{2}\mathcal{M}(G^{(1)}_{a_1,b_2}G^{(1)}_{b_1,-a_2} + G^{(1)}_{a_1,-b_2}G^{(1)}_{b_1,a_2}, -1) \\ &\quad - \frac{\zeta(3)}{4}\big(B_2(a_1)+B_2(b_1)+4B_1(a_1)B_1(b_1) \\ &\qquad - B_2(a_2)-B_2(b_2)-4B_1(a_2)B_1(b_2)\big).\end{aligned} \tag{67}$$

Theorem 1 follows by specialising Theorem 63 to the case of N-torsion points. More precisely, using (18), we have the relation, for $\boldsymbol{x},\boldsymbol{y} \in (\mathbf{Z}/N\mathbf{Z})^2$,

$$\mathcal{M}(G^{(1)}_{\boldsymbol{x}/N}G^{(1)}_{\boldsymbol{y}/N}, -1) = \frac{1}{N}\mathcal{M}(G^{(1);N}_{\boldsymbol{x}}G^{(1);N}_{\boldsymbol{y}}, -1) = -\frac{2\pi}{N}L'(G^{(1);N}_{\boldsymbol{x}}G^{(1);N}_{\boldsymbol{y}}, -1). \tag{68}$$

10. Relation to the Beilinson regulator

In [8, Conjecture 9.3], the first author conjectured that the elements $\xi((0,a),(0,b))$, $a,b \in \mathbf{Z}/N\mathbf{Z}$, are proportional to the Beilinson elements in $K_4^{(3)}(Y_1(N))$. For $\boldsymbol{a},\boldsymbol{b} \in (\mathbf{Z}/N\mathbf{Z})^2$, let $\mathrm{Eis}^{0,0,1}(\boldsymbol{a},\boldsymbol{b})$ denote the associated Beilinson element in $K_4^{(3)}(Y(N))$ [23, Definition 2.3.6]. There is an explicit representative $\mathrm{Eis}^{0,0,1}_{\mathcal{D}}(\boldsymbol{a},\boldsymbol{b})$ of the Beilinson regulator of $\mathrm{Eis}^{0,0,1}(\boldsymbol{a},\boldsymbol{b})$ [23, Proposition 2.4.2]. This is a differential 1-form on $Y(N)(\mathbf{C})$, and Weijia Wang proved the following explicit formula [23, Example 6.1.5], for $\boldsymbol{a},\boldsymbol{b} \neq \mathbf{0}$:

$$\mathcal{B}(\boldsymbol{a},\boldsymbol{b}) := \int_0^\infty \mathrm{Eis}^{0,0,1}_{\mathcal{D}}(\boldsymbol{a},\boldsymbol{b}) = \frac{9\pi}{N^3}\mathcal{M}(G^{(1);N}_{a_2,-b_1}G^{(1);N}_{b_2,a_1} + G^{(1);N}_{a_2,b_1}G^{(1);N}_{b_2,-a_1},-1).$$

Using the relations $G^{(1);N}_{x_1,x_2} = G^{(1);N}_{x_2,x_1} = -G^{(1);N}_{-x_1,-x_2}$, as well as (68), we obtain

$$\mathcal{B}(\boldsymbol{a},\boldsymbol{b}) = -\frac{9\pi}{N^2}\mathcal{M}(G^{(1)}_{a_1,b_2}G^{(1)}_{b_1,-a_2} + G^{(1)}_{a_1,-b_2}G^{(1)}_{b_1,a_2},-1) \qquad (\boldsymbol{a},\boldsymbol{b} \neq \mathbf{0}),$$

where we identify $\mathbf{Z}/N\mathbf{Z}$ and $\frac{1}{N}\mathbf{Z}/\mathbf{Z}$. The modular form on the right-hand side matches with the one in (67), and from comparing the two expressions we deduce Theorem 2.

As explained in the introduction, Theorem 2 gives evidence for the conjectural coincidence of the motivic cohomology classes $\xi(\boldsymbol{a},\boldsymbol{b})$ and $\pm\frac{N^2}{3}\mathrm{Eis}^{0,0,1}(\boldsymbol{a},\boldsymbol{b})$. This was formulated in [8, Conjecture 9.3] for the modular curve $Y_1(N)$, taking indices of the form $(0,x)$ with $x \in \mathbf{Z}/N\mathbf{Z}$, however we expect it to hold also for $Y(N)$ with general indices in $(\mathbf{Z}/N\mathbf{Z})^2$. Note a different from $\pm N^2/3$ factor $N^2/6$ in Theorem 2: this is due to the fact that the Beilinson regulator, which is used to define $\mathrm{Eis}^{0,0,1}_{\mathcal{D}}(\boldsymbol{a},\boldsymbol{b})$, is expected to be ± 2 times the Goncharov regulator $r_3(2)$, via De Jeu's map. De Jeu has proved this compatibility for general curves under some assumptions [15, Theorem 5.4]; see the discussion in [8, Section 5.4].

11. Conclusion

One important application of Theorem 1 is to proving the longstanding conjecture of Boyd and Rodriguez Villegas on the Mahler measure [10] of the three-variable polynomial $P = (1+x)\times(1+y)+z$, namely $\mathrm{m}(P) = -2L'(E,-1)$, where E is the elliptic curve over $\mathbf{Q}$ defined by the affine equation $(1+x)(1+y)(1+\frac{1}{x})(1+\frac{1}{y}) = 1$. To do this, the starting point is the work of Lalín [19] expressing this Mahler measure as a Goncharov regulator on the elliptic curve E:

$$\mathrm{m}((1+x)(1+y)+z) = \frac{1}{4\pi^2}\int_{\gamma_E^+} r_3(2)(\xi_P), \tag{69}$$

where ξ_P is a degree 2 cohomology class in the weight 3 Goncharov complex of E, and γ_E^+ is a generator of $H_1(E(\mathbf{C}),\mathbf{Z})^+$, the subgroup of invariants under complex conjugation in the homology of E. What allows one to compute the regulator integral (69) is that E is actually isomorphic to the modular curve $X_1(15)$, and using this identification, the class ξ_P has the simple expression $\xi_P = 20\xi((0,4),(0,6)) - 20\xi((0,6),(0,7))$. The details of this are given by the first author in [9].

Though our Theorems 1 and 2 do not cover the boundary cases, where some coordinates of $\boldsymbol{a},\boldsymbol{b},\boldsymbol{a}+\boldsymbol{b} \in (\mathbf{Z}/N\mathbf{Z})^2$ are zero, they indicate some interesting behaviour when the parameters approach the boundary. The rational multiple of $\zeta(3)$ in Theorems 1 and 2 has discontinuities at the boundary due to the Bernoulli polynomial B_1, which may have to be replaced by the sawtooth wave or by regularised values as in Propositions 44 and 45. It is also not clear, to begin with, whether the Goncharov regulator $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ can be interpolated as a continuous function along the boundary. It would be interesting to gain a more conceptual understanding of these continuity issues; some numerical experiments may shed light on that.

In essence, the explicit relation between the regulator integrals $\mathcal{G}(\boldsymbol{a},\boldsymbol{b})$ and $\mathcal{B}(\boldsymbol{a},\boldsymbol{b})$ should be enough to prove [8, Conjecture 9.3] at the level of Deligne–Beilinson cohomology (as well as its more general version for the modular curve $Y(N)$). At the motivic level, however, the conjecture looks more difficult and seems to require new ideas; a Hodge theoretic interpretation of the computations in this article would be already very interesting.

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ÉNS Lyon, Unité de mathématiques pures et appliquées, 46 allée d'Italie, 69007 Lyon, France
Email address: `francois.brunault@ens-lyon.fr`
URL: `http://perso.ens-lyon.fr/francois.brunault`

IMAPP, Radboud University Nijmegen, PO Box 9010, 6500 GL Nijmegen, The Netherlands
Email address: `w.zudilin@math.ru.nl`
URL: `https://www.math.ru.nl/~wzudilin/`