NOTE ON FOURIER EXPANSIONS AT CUSPS

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Abstract. This was originally an appendix to our paper 'Fourier expansions at cusps' [1]. The purpose of this note is to give a proof of a Theorem of Shimura on the action of $\mathrm{Aut}(\mathbb{C})$ on modular forms for $\Gamma(N)$ from the perspective of algebraic modular forms. As the theorem is well-known, we do not intend to publish this note but want to keep it available as a preprint.

In this note we give a new proof of the following theorem which is originally due to Shimura, see [5, Theorem 8] and [6, Lemma 10.5]. It gives the interaction between the $\mathrm{SL}_2(\mathbb{Z})$-action and the $\mathrm{Aut}(\mathbb{C})$-action on spaces of modular forms on the group $\Gamma(N)$. These actions on a modular form $f(\tau) = \sum_n a_n e^{2\pi i n\tau/w}$ of weight $k \geq 1$ are defined as follows:

$$f|g(\tau) = \frac{1}{(C\tau+D)^k} f\Big(\frac{A\tau+B}{C\tau+D}\Big), \qquad f^\sigma(\tau) = \sum_n \sigma(a_n) e^{2\pi i n\tau/w}$$

for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$. For any integer $N \geq 1$ we denote $\zeta_N = e^{2\pi i/N}$.

Theorem 1. [5, 6] Let $f \in M_k(\Gamma(N))$ be a modular form of weight $k \geq 1$ on $\Gamma(N)$. Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $\sigma(\zeta_N) = \zeta_N^\lambda$ with $\lambda \in (\mathbb{Z}/N\mathbb{Z})^\times$. Then

$$(f|g)^\sigma = f^\sigma | g_\lambda,$$

where g_λ is any lift in $\mathrm{SL}_2(\mathbb{Z})$ of the matrix $\left(\begin{smallmatrix} A & \lambda B \\ \lambda^{-1} C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

This theorem immediately implies that if a modular form f of level N has Fourier coefficients in a field K_f, then the Fourier coefficients of $f|\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ for any $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ will lie in $K_f(\zeta_N)$. In [1] we obtain improved results if f is a modular form for $\Gamma_0(N)$ or $\Gamma_1(N)$ and in the case of newforms on $\Gamma_0(N)$, we determine the number field generated by the Fourier coefficients of $f|\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ explicitly.

We recall the theory of algebraic modular forms, in order to give a new proof of Theorem 1. For more details on this theory, see [3, Chap. II] and the references therein.

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Definition 2. Let R be an arbitrary commutative ring, and let $N \geq 1$ be an integer. A *test object* of level N over R is a triple $T = (E, \omega, \beta)$ where E/R is an elliptic curve, $\omega \in \Omega^1(E/R)$ is a nowhere vanishing invariant differential, and β is a *level N structure* on E/R, that is an isomorphism of R-group schemes

$$\beta : (\mu_N)_R \times (\mathbb{Z}/N\mathbb{Z})_R \xrightarrow{\cong} E[N]$$

satisfying $e_N(\beta(\zeta, 0), \beta(1, 1)) = \zeta$ for every $\zeta \in (\mu_N)_R$. Here $\mu_N = \operatorname{Spec} \mathbb{Z}[t]/(t^N - 1)$ is the scheme of N-th roots of unity, and e_N is the Weil pairing on $E[N]$ [1].

If $\phi : R \to R'$ is a ring morphism, we denote by $T_{R'} = (E_{R'}, \omega_{R'}, \beta_{R'})$ the base change of T to R' along ϕ.

The isomorphism classes of test objects over $\mathbb{C}$ are in bijection with the set of lattices L in $\mathbb{C}$ endowed with a symplectic basis of $\frac{1}{N}L/L$ [3, 2.4]. Another example is given by the Tate curve $\operatorname{Tate}(q) = \mathbb{G}_m/q^{\mathbb{Z}}$ [2, §8]. It is an elliptic curve over $\mathbb{Z}((q))$ endowed with the canonical differential $\omega_{\text{can}} = dx/x$ and the level N structure $\beta_{\text{can}}(\zeta, n) = \zeta q^{n/N} \bmod q^{\mathbb{Z}}$. The test object $(\operatorname{Tate}(q), \omega_{\text{can}}, \beta_{\text{can}})$ is defined over $\mathbb{Z}((q^{1/N}))$.

Definition 3. An *algebraic modular form* of weight $k \in \mathbb{Z}$ and level N over R is the data, for each R-algebra R', of a function

$$F = F_{R'} : \{\text{isomorphism classes of test objects of level } N \text{ over } R'\} \to R'$$

satisfying the following properties:

(1) $F(E, \lambda^{-1}\omega, \beta) = \lambda^k F(E, \omega, \beta)$ for every $\lambda \in (R')^\times$;
(2) F is compatible with base change: for every morphism of R-algebras $\psi : R' \to R''$ and for every test object T of level N over R', we have $F_{R''}(T_{R''}) = \psi(F_{R'}(T))$.

We denote by $M_k^{\text{alg}}(\Gamma(N); R)$ the R-module of algebraic modular forms of weight k and level N over R.

Evaluating at the Tate curve provides an injective R-linear map

$$M_k^{\text{alg}}(\Gamma(N); R) \hookrightarrow \mathbb{Z}((q^{1/N})) \otimes_{\mathbb{Z}} R$$

called the q-expansion map. The *q-expansion principle* states that if R' is a subring of R, then an algebraic modular form $F \in M_k^{\text{alg}}(\Gamma(N); R)$ belongs to $M_k^{\text{alg}}(\Gamma(N); R')$ if and only if the q-expansion of F has coefficients in R'.

Algebraic modular forms are related to classical modular forms as follows. To any algebraic modular form $F \in M_k^{\text{alg}}(\Gamma(N); \mathbb{C})$, we associate the function $F^{\text{an}} : \mathcal{H} \to \mathbb{C}$ defined by

$$F^{\text{an}}(\tau) = F\Big(\frac{\mathbb{C}}{2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z}}, dz, \beta_\tau\Big)$$

with $\beta_\tau(\zeta_N^m, n) := [2\pi i(m + n\tau)/N]$.

Proposition 4. The map $F \mapsto F^{\text{an}}$ induces an isomorphism between $M_k^{\text{alg}}(\Gamma(N); \mathbb{C})$ and the space $M_k^!(\Gamma(N))$ of weakly holomorphic modular forms on $\Gamma(N)$ (that is, holomorphic on $\mathcal{H}$ and meromorphic at the cusps). Moreover, the q-expansion of F coincides with that of F^{an}.

[1] Our definition of the Weil pairing is the reciprocal of Silverman's definition [7, III.8]. With our definition, we have $e_N(1/N, \tau/N) = e^{2\pi i/N}$ on the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with $\operatorname{Im}(\tau) > 0$.

We now interpret the action of $\mathrm{SL}_2(\mathbb{Z})$ on modular forms in algebraic terms. Let $F \in M_k^{\mathrm{alg}}(\Gamma(N);\mathbb{C})$ with $f = F^{\mathrm{an}}$, and let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. A simple computation shows that

$$(f|_k g)(\tau) = F\Big(\frac{\mathbb{C}}{2\pi i(\mathbb{Z}+\tau\mathbb{Z})}, dz, \beta'_\tau\Big) \tag{3}$$

where the level N structure β'_τ is given by

$$\beta'_\tau(\zeta_N^m, n) = \beta_\tau(\zeta_N^{md+nb}, mc+na). \tag{4}$$

Let $\psi : (\mathbb{Z}/N\mathbb{Z})^2 \to \mu_N(\mathbb{C}) \times \mathbb{Z}/N\mathbb{Z}$ be the isomorphism defined by $\psi(a,b) = (\zeta_N^b, a)$. Let us identify the level structure β_τ (resp. β'_τ) with the map $\alpha_\tau = \beta_\tau \circ \psi$ (resp. $\alpha'_\tau = \beta'_\tau \circ \psi$). Then (4) shows that

$$\alpha'_\tau(a,b) = \alpha_\tau((a,b)g). \tag{5}$$

What we have here is the right action of $\mathrm{SL}_2(\mathbb{Z})$ on the row space $(\mathbb{Z}/N\mathbb{Z})^2$, which induces a left action on the set of level N structures. As we will see, all this makes sense algebraically. For any $\mathbb{Z}[\zeta_N]$-algebra R, we denote by $\zeta_{N,R}$ the image of $\zeta_N = e^{2\pi i/N}$ under the structural morphism $\mathbb{Z}[\zeta_N] \to R$.

Lemma 5. If R is a $\mathbb{Z}[\zeta_N, 1/N]$-algebra, then there is an isomorphism of R-group schemes $(\mathbb{Z}/N\mathbb{Z})_R \xrightarrow{\cong} (\mu_N)_R$ sending 1 to $\zeta_{N,R}$.

Proof. Note that $(\mu_N)_R = \operatorname{Spec} R[t]/(t^N-1) = \operatorname{Spec} R[\mathbb{Z}/N\mathbb{Z}]$ and $(\mathbb{Z}/N\mathbb{Z})_R = \operatorname{Spec} R^{\mathbb{Z}/N\mathbb{Z}}$. If $R = \mathbb{C}$, then $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}] \cong \mathbb{C}^{\mathbb{Z}/N\mathbb{Z}}$ because all irreducible representations of $\mathbb{Z}/N\mathbb{Z}$ have dimension 1. This isomorphism $\mathcal{F}_\mathbb{C}$ is given by the Fourier transform, and both $\mathcal{F}_\mathbb{C}$ and $\mathcal{F}_\mathbb{C}^{-1}$ have coefficients in $\mathbb{Z}[\zeta_N, 1/N]$ with respect to the natural bases. It follows that in general $R[\mathbb{Z}/N\mathbb{Z}] \cong R^{\mathbb{Z}/N\mathbb{Z}}$ and this isomorphism sends [1] to $(\zeta_{N,R}^a)_{a\in\mathbb{Z}/N\mathbb{Z}}$. □

Let R be a $\mathbb{Z}[\zeta_N, 1/N]$-algebra. We have an isomorphism of R-group schemes

$$\psi_R : (\mathbb{Z}/N\mathbb{Z})^2_R \to (\mu_N)_R \times (\mathbb{Z}/N\mathbb{Z})_R$$

given by $\psi_R(a,b) = (\zeta_{N,R}^b, a)$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts from the right on the row space $(\mathbb{Z}/N\mathbb{Z})^2_R$ by R-automorphisms, and for $\alpha : (\mathbb{Z}/N\mathbb{Z})^2_R \xrightarrow{\cong} E[N]$ we define

$$(g\cdot\alpha)(a,b) = \alpha((a,b)g) \qquad ((a,b) \in (\mathbb{Z}/N\mathbb{Z})^2). \tag{6}$$

Using ψ_R, we transport this to a left action of $\mathrm{SL}_2(\mathbb{Z})$ on the set of level N structures of an elliptic curve over R. Given a test object $T = (E,\omega,\beta)$ over R, we define $g\cdot T := (E,\omega,g\cdot\beta)$. For any $F \in M_k^{\mathrm{alg}}(\Gamma(N);R)$, we define $F|g \in M_k^{\mathrm{alg}}(\Gamma(N);R)$ by the rule $(F|g)(T) = F(g\cdot T)$ for any test object T over any R-algebra R'. The computation (3) then shows that the right action of $\mathrm{SL}_2(\mathbb{Z})$ on $M_k^{\mathrm{alg}}(\Gamma(N);\mathbb{C})$ corresponds to the usual slash action on $M_k^!(\Gamma(N))$.

Remark 6. The action of $\mathrm{SL}_2(\mathbb{Z})$ on algebraic modular forms over $\mathbb{Z}[\zeta_N, 1/N]$-algebras has the following consequence: if a classical modular form $f \in M_k(\Gamma(N))$ has Fourier coefficients in some subring A of $\mathbb{C}$, then for any $g \in \mathrm{SL}_2(\mathbb{Z})$, the Fourier expansion of $f|g$ lies in $\mathbb{Z}[[q^{1/N}]] \otimes A[\zeta_N, 1/N]$.

We now interpret the action of $\mathrm{Aut}(\mathbb{C})$ in algebraic terms (see [4, p. 88]). Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. For any $\mathbb{C}$-algebra R, we define $R^\sigma := R \otimes_{\mathbb{C},\sigma^{-1}} \mathbb{C}$, which means that $(ax)\otimes 1 = x \otimes \sigma^{-1}(a)$ for all $a \in \mathbb{C}$, $x \in R$. We endow R^σ with the structure of a $\mathbb{C}$-algebra using the map

$a \in \mathbb{C} \mapsto 1 \otimes a \in R^\sigma$. We denote by $\phi_\sigma : R \to R^\sigma$ the map defined by $\phi_\sigma(x) = x \otimes 1$. The map ϕ_σ is a ring isomorphism, but one should be careful that ϕ_σ is not a morphism of $\mathbb{C}$-algebras, as it is only σ^{-1}-linear. For any test object T over R, we denote by T^σ its base change to R^σ using the ring morphism ϕ_σ.

Let $F \in M_k^{\mathrm{alg}}(\Gamma(N); \mathbb{C})$ be an algebraic modular form. For any $\mathbb{C}$-algebra R, we define

$$F_R^\sigma : \{\text{isomorphism classes of test objects of level } N \text{ over } R\} \to R$$
$$T \mapsto \phi_\sigma^{-1}\big(F_{R^\sigma}(T^\sigma)\big).$$

One may check that the collection of functions F_R^σ satisfies the conditions (1) and (2) above, hence defines an algebraic modular form $F^\sigma \in M_k^{\mathrm{alg}}(\Gamma(N); \mathbb{C})$. Moreover, since the Tate curve is defined over $\mathbb{Z}((q))$, one may check that the map $F \mapsto F^\sigma$ corresponds to the usual action of $\mathrm{Aut}(\mathbb{C})$ on the Fourier expansions of modular forms: for every $F \in M_k^{\mathrm{alg}}(\Gamma(N); \mathbb{C})$ and every $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have $(F^\sigma)^{\mathrm{an}} = (F^{\mathrm{an}})^\sigma$.

We finally come to the proof of Theorem 1.

Proof. Let $f \in M_k(\Gamma(N))$ with corresponding algebraic modular form $F \in M_k^{\mathrm{alg}}(\Gamma(N); \mathbb{C})$. Let $g \in \mathrm{SL}_2(\mathbb{Z})$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$. We take as test object $T = (\mathrm{Tate}(q), \omega_{\mathrm{can}}, \beta_{\mathrm{can}})$ over $R = \mathbb{Z}((q^{1/N})) \otimes \mathbb{C}$. Since a modular form is determined by its Fourier expansion, and unravelling the definitions of $F|g$ and F^σ, it suffices to check that the test objects $g \cdot T^\sigma$ and $(g_\lambda \cdot T)^\sigma$ over R^σ are isomorphic. Since $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ acts only on the level structures of the test objects, we have to show that

$$g \cdot \beta_{\mathrm{can}}^\sigma \cong (g_\lambda \cdot \beta_{\mathrm{can}})^\sigma. \tag{7}$$

For any scheme X over R, let X^σ denote its base change to R^σ along ϕ_σ. Since ϕ_σ is a ring isomorphism, the canonical projection map $X^\sigma \to X$ is an isomorphism of schemes, and we also denote by $\phi_\sigma : X \to X^\sigma$ the inverse map.

Put $E = \mathrm{Tate}(q)$ and $\beta = \beta_{\mathrm{can}}$. Let $\alpha = \beta \circ \psi_R : (\mathbb{Z}/N\mathbb{Z})_R^2 \xrightarrow{\cong} E[N]$. By functoriality, the level structure β^σ is given by the following commutative diagram

$$\begin{tikzcd}
(\mathbb{Z}/N\mathbb{Z})_R^2 \arrow[r, "\psi_R"] \arrow[d, dotted, "\gamma"'] \arrow[rr, bend left, "\alpha"] & (\mu_N)_R \times (\mathbb{Z}/N\mathbb{Z})_R \arrow[r, "\beta"] \arrow[d, "\cong"', "\phi_\sigma"] & E[N] \arrow[d, "\cong"', "\phi_\sigma"] \\
(\mathbb{Z}/N\mathbb{Z})_{R^\sigma}^2 \arrow[r, "\psi_{R^\sigma}"] \arrow[rr, bend right, "\alpha^\sigma"'] & (\mu_N)_{R^\sigma} \times (\mathbb{Z}/N\mathbb{Z})_{R^\sigma} \arrow[r, "\beta^\sigma"] & E^\sigma[N].
\end{tikzcd} \tag{8}$$

Let us compute the dotted arrow γ. Since ϕ_σ is σ^{-1}-linear, we have $\phi_\sigma(\zeta_{N,R}) = \zeta_{N,R^\sigma}^{\lambda^{-1}}$. It follows that

$$\phi_\sigma(\psi_R(a,b)) = \phi_\sigma(\zeta_{N,R}^b, a) = (\zeta_{N,R^\sigma}^{\lambda^{-1}b}, a) = \psi_{R^\sigma}(a, \lambda^{-1}b) \tag{9}$$

so that $\gamma(a,b) = (a, \lambda^{-1}b)$. We may thus express α^σ in terms of α by

$$\alpha^\sigma(a,b) = \phi_\sigma \circ \alpha \circ \gamma^{-1}(a,b) = \phi_\sigma \circ \alpha(a, \lambda b) = \phi_\sigma \circ \alpha\left((a,b)\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\right). \tag{10}$$

Let us make explicit both sides of (7). By (6) and (10), the left hand side is given by

$$(g \cdot \alpha^\sigma)(a,b) = \alpha^\sigma((a,b)g) = \phi_\sigma \circ \alpha\left((a,b)g \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\right). \tag{11}$$

Let us now turn to the right hand side of (7). By (6), we have $(g_\lambda \cdot \alpha)(a,b) = \alpha((a,b)g_\lambda)$. Applying the commutative diagram (8) with α replaced by $g_\lambda \cdot \alpha$, we get

$$(g_\lambda \cdot \alpha)^\sigma(a,b) = \phi_\sigma \circ (g_\lambda \cdot \alpha)\left((a,b)\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\right) = \phi_\sigma \circ \alpha\left((a,b)\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} g_\lambda\right). \tag{12}$$

Finally, we note that $g \left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right) g_\lambda$.

□

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