The Rogers-Zudilin method

To calculate the L-function of a product of Eisenstein series, Rogers-Zudilin introduced the following trick. Let $E_1, E_2 : \mathcal{H} \to \mathbf{C}$ be defined by

$$E_1(\tau) = \sum_{m_1,n_1\geq 1} a_1(m_1) q^{m_1 n_1} \qquad E_2(\tau) = \sum_{m_2,n_2\geq 1} a_2(m_2) q^{m_2 n_2} \qquad (q = e^{2\pi i \tau}).$$

$$\int_0^\infty E_1(iy)E_2(i/y)y^s \frac{dy}{y}$$

$$= \int_0^\infty \Big(\sum_{m_1,n_1\geq 1} a_1(m_1) e^{-2\pi m_1 n_1 y}\Big)\Big(\sum_{m_2,n_2\geq 1} a_2(m_2) e^{-2\pi \frac{m_2 n_2}{y}}\Big) y^s \frac{dy}{y}$$

$$= \sum_{m_1,n_1,m_2,n_2\geq 1} \int_0^\infty a_1(m_1)a_2(m_2) e^{-2\pi(m_1 n_1 y + \frac{m_2 n_2}{y})} y^s \frac{dy}{y}$$

$$\overset{y=m_2 y'/m_1}{=} \sum_{m_1,n_1,m_2,n_2\geq 1} \int_0^\infty a_1(m_1)a_2(m_2) e^{-2\pi(m_2 n_1 y' + \frac{m_1 n_2}{y'})} \left(\frac{m_2}{m_1}\right)^s y'^s \frac{dy'}{y'}$$

$$= \int_0^\infty \Big(\sum_{m_1,n_2\geq 1} a_1(m_1) m_1{}^{-s} e^{-2\pi \frac{m_1 n_2}{y'}}\Big)\Big(\sum_{m_2,n_1\geq 1} a_2(m_2) m_2{}^{s} e^{-2\pi m_2 n_1 y'}\Big) y'^s \frac{dy'}{y'}$$