Explicit $p$-adic regulators on $K_2$ of modular curves

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Construction of \( p \)-adic \( L \)-functions by interpolation

- \( \zeta_p(s), L_p(\chi, s) \) (Kubota-Leopoldt)
- \( L_p(E, s), L_p(f, s) \) (Manin, Vishik, Amice-Vélu)
- Deligne-Ribet, Katz, Hida, Panchishkin. . .

Another possibility is to define \( p \)-adic \( L \)-functions using compatible systems of global objects (e.g. Coleman series, Euler systems).

Perrin-Riou has given a very general conjectural definition of \( p \)-adic \( L \)-functions associated to motives, as well as conjectures on their special values.

→ What is the \( p \)-adic analogue of Beilinson conjectures on special values of \( L \)-functions?

In this talk, we will consider the special case of curves.
$X/\mathbb{Q}$ smooth projective curve of genus $g$

$T_X = H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)$ \quad $V_X = T_X \otimes \mathbb{Z}_p \mathbb{Q}_p$.

$V_X$ is a (global) $p$-adic representation, $\dim_{\mathbb{Q}_p} V_X = 2g$.

- $T_X(1) \cong T_p(J)$ with $J = \text{Jac}(X)$
- Perfect pairing $T_X \times T_X \to \mathbb{Z}_p(-1)$
- $T_X$ is unramified at any prime $\ell \neq p$ such that $J$ has good reduction at $\ell$.

Fontaine has shown that $V_X$ is de Rham at $p$.

$D_{\text{dR}}(V_X) := (V_X \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{G_{\mathbb{Q}_p}} \cong H^1_{\text{dR}}(X/\mathbb{Q}_p)$.

$$\text{Fil}^i D_{\text{dR}}(V_X) = \begin{cases} 
H^1_{\text{dR}}(X/\mathbb{Q}_p) & \text{if } i \leq 0 \\
\Omega^1(X/\mathbb{Q}_p) & \text{if } i = 1 \\
\{0\} & \text{if } i \geq 2.
\end{cases}$$
Theorem (Coleman-Iovita, Breuil)

\( V_X \) crystalline at \( p \) ⇔ \( J \) has good reduction at \( p \).

\( V_X \) semi-stable at \( p \) ⇔ \( J \) has semi-stable reduction at \( p \).

Let \( D_{\text{cris}}(V_X) := (V_X \otimes B_{\text{cris}})^{G_{\mathbb{Q}_p}} \to D_{\text{dR}}(V_X) \).

\( D_{\text{cris}}(V_X) \) is a filtered \( \phi \)-module.

If \( J \) has good reduction at \( p \), we have (Katz-Messing)

\[
\det(1 - T \phi|D_{\text{cris}}(V_X)) = P_{X,p}(T) \in \mathbb{Z}[T]
\]

where \( P_{X,p} \) is the classical Euler factor of \( L(X,s) \) at \( p \).
Iwasawa cohomology (local case)

\[ G_n = \text{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) \quad (n \in \mathbb{N} \cup \{\infty\}) \]
\[ \Lambda = \mathbb{Z}_p[[G_\infty]] := \lim_{\leftarrow n \geq 0} \mathbb{Z}_p[G_n]. \]
\[ H^i_{Iw}(\mathbb{Q}_p, T_X) := \lim_{\leftarrow n \geq 0} H^i(\mathbb{Q}_p(\zeta_{p^n}), T_X) = \text{fin. gen.} \ \Lambda\text{-module} \]
\[ H^i_{Iw}(\mathbb{Q}_p, V_X) := H^i_{Iw}(\mathbb{Q}_p, T_X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \]

- \[ H^i_{Iw} = 0 \text{ if } i \notin \{1, 2\} \]
- \[ H^2_{Iw}(\mathbb{Q}_p, T_X) \text{ is a torsion } \Lambda\text{-module} \]
- \[ \text{rk}_\Lambda H^1_{Iw}(\mathbb{Q}_p, T_X) = \text{rk}_{\mathbb{Z}_p}(T_X) = 2g \]
- Compatibility with Tate twists:

\[ H^i_{Iw}(\mathbb{Q}_p, T_X) \overset{\cong}{\rightarrow} H^i_{Iw}(\mathbb{Q}_p, T_X(k)) \quad (k \in \mathbb{Z}) \]
\[ (c_n)_{n \geq 0} \mapsto (c_n \otimes \zeta_{p^n}^k)_{n \geq 0} \]
Iwasawa cohomology (global cae)

\[ G_\infty \cong \text{Gal}(\mathbb{Q}(\zeta_p^\infty)/\mathbb{Q}) \]

\[ H^i_{Iw}(\mathbb{Z}[\frac{1}{p}], T_X) := \lim_{\leftarrow n \geq 0} H^i(\mathbb{Z}[\zeta_p^n, \frac{1}{p}], T_X) = \text{fin. gen. } \Lambda\text{-module} \]

\[ H^i_{Iw}(\mathbb{Z}[\frac{1}{p}], V_X) := H^i_{Iw}(\mathbb{Z}[\frac{1}{p}], T_X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \]

- Localization maps \( H^i_{Iw}(\mathbb{Z}[\frac{1}{p}], T_X) \rightarrow H^i_{Iw}(\mathbb{Q}_p, T_X) \)

- Weak Leopoldt conjecture : \( H^2_{Iw}(\mathbb{Z}[\frac{1}{p}], T_X) \) is a torsion \( \Lambda\)-module. This implies \( \text{rk}_\Lambda H^1_{Iw}(\mathbb{Z}[\frac{1}{p}], T_X) = \text{rk}_{\mathbb{Z}_p}(T_X^-) = g. \)

**Theorem (Kato)**

*If X is covered by a modular curve then WL is true for T_X.*

The proof uses Euler systems.
Definition

A *Euler system* for $T = T_X(2)$ is a collection of classes

$$z_m \in H^1(\mathbb{Z}[\zeta_m, \frac{1}{p}], T)$$

satisfying the corestriction conditions

$$\text{cores}(z_{m'}) = \prod_{\ell | m'} P_{X,\ell}(\sigma_{\ell}^{-1}) \cdot z_m \quad (m | m')$$

where $\sigma_{\ell} \in \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ is the arithmetic Frobenius.
Remarks

- The set of Euler systems is a $\Lambda$-module.
- If $z = (z_m)$ is a Euler system then $(z_{p^n})_{n \geq 1} \in H^1_{Iw}(\mathbb{Z}[\frac{1}{p}], T)$.

Definition

Let $Z$ be the $\Lambda$-submodule of $H^1_{Iw}(\mathbb{Z}[\frac{1}{p}], T)$ consisting of Euler systems.

Optimistic guess: $Z$ is $\Lambda$-cotorsion in $H^1_{Iw}(\mathbb{Z}[\frac{1}{p}], T)$.

Assuming this, and assuming $V_X$ is crystalline, we explain, following Perrin-Riou, how to construct the « full » $p$-adic $L$-function of $X$. 
\( \Lambda \subset \mathcal{H} = \text{large Iwasawa algebra} \subset \mathbb{Q}_p[[G_\infty]]. \)

Perrin-Riou has constructed a « logarithme élargi »

\[
\mathcal{L} : H^1_{Iw}(\mathbb{Q}_p, V_X(2)) \to \mathcal{H} \otimes_{\mathbb{Q}_p} D_{\text{cris}}(V_X)^\vee
\]

by interpolation of the Bloch-Kato exponential maps for \( V_X(k) \). \( \mathcal{L} \) is a map of \( \Lambda \)-modules and it is known that \( \mathcal{L} \otimes \text{Frac}(\mathcal{H}) \) is an isomorphism, hence \( \text{rk}_\Lambda \mathcal{L}(Z) = g \).

**Definition**

The module of \( p \)-adic \( L \)-functions of \( X \) is \( L_X = \det_\Lambda \mathcal{L}(Z) \).

**Remark**

\( L_X \) is a \( \Lambda \)-line in \( \mathcal{H} \otimes_{\mathbb{Q}_p} \Lambda^g \otimes_{\mathbb{Q}_p} D_{\text{cris}}(V_X)^\vee \).
This defines the $p$-adic $L$-function of $X$ up to a unit of $\Lambda$. We normalize it using an interpolation property.

Let $\omega_1, \ldots, \omega_g$ be a $\mathbb{Q}$-basis of $\Omega^1(X)$.

Let $\eta_1, \ldots, \eta_g \in D_{\text{cris}}(V_X) \otimes \mathbb{Q}_p$ such that

- $\phi(\eta_i) = \alpha_i \eta_i$ with $v_p(\alpha_i) < 1$ for all $1 \leq i \leq g$;
- $\det\langle \omega_i, \eta_j \rangle = 1$.

**Definition**

The full $p$-adic $L$-function of $X$ (with values in $\wedge^g D^\vee$) is

$$L_p(X, s) = \int_{\mathbb{Z}_p^\times} \langle x \rangle^{s-1} \mu_X(x) \quad (s \in \mathbb{Z}_p)$$

where $\mu_X$ is the unique generator of $L_X \otimes \mathbb{Q}_p$ such that
\[
\int_{\mathbb{Z}_p^\times} \chi \cdot \mu_X(\eta_1 \wedge \ldots \wedge \eta_g) = \frac{\tau(\chi)^g}{\alpha_1^n \cdots \alpha_g^n} \cdot \frac{L(J, \chi, 1)}{\Omega_j^{\chi(-1)}}
\]

for every primitive character \( \chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \mathbb{Q}^\times \).

\( \tau(\chi) \) = Gauss sum of \( \chi \)

\( \Omega_j^{\pm} \) = periods of \( J \) with respect to \( \omega_1 \wedge \ldots \wedge \omega_g \).

**Remarks**

- This definition is completely conjectural... In fact the analytic continuation of \( L(J, s) \) and algebraicity of \( L(J, 1)/\Omega_j^+ \) is not known in general as soon as \( g \geq 2 \).
- In the case \( X = E \) is an elliptic curve, one recovers the usual \( p \)-adic \( L \)-function by projecting to the \( \varphi \)-eigenspace.
Let $X/\mathbb{Q}_p$ be any smooth curve. Let $\text{ch}_X : K_2(X) \to H^2_{\text{ét}}(X, \mathbb{Z}_p(2))$ be the Chern class map.

**Lemma**

$H^2_{\text{ét}}(X, \mathbb{Z}_p(2)) \cong H^1(\mathbb{Q}_p, H^1_{\text{ét}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p(2)))$

So we get a regulator map

$$K_2(X) \to H^1(\mathbb{Q}_p, V_X(2))$$

Bloch and Kato conjectured that the integral subspace of $K_2(X)$ is mapped into $H^1_f(\mathbb{Q}_p, V_X(2))$ (proved in special cases by Nekovar and Niziol).
Lemma

The Bloch-Kato exponential map $D_{dR}(V_X(2)) \rightarrow H^1_f(Q_p, V_X(2))$ is an isomorphism.

In this way one gets a regulator map

$$\text{reg}_X : K_2(X)_{\mathbb{Z}} \rightarrow D_{dR}(V_X) \cong H^1_{dR}(X/\mathbb{Q}_p).$$

Remark

Other definitions of the $p$-adic regulator map on $K_2(X)$:

- Coleman-de Shalit (using Coleman integrals)
- Besser (syntomic regulator)

Besser has proved that all these constructions coincide (in the good reduction case).
Let $X/Q$ be a smooth projective curve of genus $g$, with good reduction at $p$.

**Conjecture (p-adic Beilinson for $X$ at $s = 0$)**

- $K_2(X) \otimes \mathbb{Q}$ has dimension $g$.
- The image of $\text{reg}_X \otimes \mathbb{Z}_p$ is a free $\mathbb{Z}_p$-module of rank $g$.
- Up to standard identifications, we have

$$\det \text{reg}_X(K_2(X) \otimes \mathbb{Z}) \sim L_p(X, \omega^{-1}, 0)$$

where $\sim$ indicates equality up to a nonzero rational factor and $\omega : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ is the Teichmüller character.
Remark
Using the syntomic regulator, one could also formulate a $p$-adic Beilinson conjecture at any non-critical integer (at least in the good reduction case).
For any $m \geq 0$, the value of $L_p(X, s)$ at $s = -m$ should be linked with the syntomic regulator on $K_{2m+2}^2(X)$.

Theorem (Coleman-de Shalit)
If $E/\mathbb{Q}$ is a CM elliptic curve with good ordinary reduction at $p$, then $L_{p,\alpha}(E, 0)$ can be expressed as the regulator of an explicit element of $K_2(E)$.

Aim of the talk: investigate the case of non CM elliptic curves using the deep results of Kato, Perrin-Riou, Colmez.
$E/\mathbb{Q}$ elliptic curve of conductor $N$, without CM.
$L(E, s) = \sum_{n \geq 1} a_n / n^s$ complex $L$-function of $E$.
$f = \sum_{n \geq 1} a_n q^n \in S_2(\Gamma_0(N))$
$\varphi : X_1(N) \to E$ modular parametrization
$p$ prime number not dividing $2N$.

We have $V_E \xrightarrow{\varphi^*} V_{X_1(N)} \hookrightarrow V_{Y_1(N)}$ which induces an isomorphism $V_E \cong V_{Y_1(N)}/\langle T_n - a_n; n \geq 1 \rangle$.

$\eta_\alpha \in D_{\text{cris}}(V_E) \otimes \overline{\mathbb{Q}}_p$ such that $\varphi(\eta_\alpha) = \alpha \eta_\alpha$ with $v_p(\alpha) < 1$ and $\langle \omega_f, \eta_\alpha \rangle = 1$.

On $Y(N)$ we have the Siegel modular units $g_{a,b}$ ($a, b \in \mathbb{Z}/N\mathbb{Z}$).

So we can form the cup product $\{g_{a,b}, g_{c,d}\} \in K_2(Y(N)) \otimes \mathbb{Q}$.
We consider the following regulator map

\[
\begin{align*}
K_2(Y(N)) \xrightarrow{\text{trace}} K_2(Y_1(N)) \xrightarrow{\text{reg}_{Y_1(N)}} D_{dR}(V_{Y_1(N)}) & \to D_{dR}(V_E).
\end{align*}
\]

**Theorem (B.)**

\[
\langle \text{reg}\{g_{1,0}, g_{0,1}\}, \eta_\alpha \rangle = \\
(\prod_{\ell|N} 1 - a_\ell) \frac{L(E, 1)\Omega_E}{\iota(f, f)} [(1 - p^{\alpha})(1 - 1/p^{\alpha})]^{-1} L_{p,\alpha}(E, \omega^{-1}, 0)
\]

A similar formula for higher weight modular forms, at all non-critical integers, was obtained by Gealy (unpublished).
\[
\langle \text{reg}\{g_1,0,g_0,1\}, \eta_\alpha \rangle = \\
(\prod_{\ell|N} 1 - a_\ell) \frac{L(E,1)\Omega_E^\ell}{i\langle f, f \rangle} \left[\left(1 - \frac{p}{\alpha}\right)\left(1 - \frac{1}{p\alpha}\right)\right]^{-1} L_{p,\alpha}(E,\omega^{-1},0)
\]

**Remarks**

- The formula is similar to the formula for the *complex* regulator of \(\{g_1,0,g_0,1\}\) (Beilinson, Kato).
- The presence of \[\left(1 - \frac{p}{\alpha}\right)\left(1 - \frac{1}{p\alpha}\right)\] is a well-known phenomenon (extra Euler factor at \(p\)).
- The formula is not optimal (the RHS can be zero).
\[
\langle \text{reg}\{g_{1,0}, g_{0,1}\}, \eta_\alpha \rangle = \\
(\prod_{\ell | N} 1 - a_\ell) \frac{L(E, 1) \Omega^-_E}{\langle f, f \rangle} [(1 - \frac{p}{\alpha})(1 - \frac{1}{p\alpha})]^{-1} L_{p, \alpha}(E, \omega^{-1}, 0)
\]

Remarks

- We can also compute \( \text{reg}\{g_{a,b}, g_{c,d}\} \) for any
  \[
  \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \quad (\rightarrow \text{modular symbol in the RHS}).
  \]
- The method doesn’t seem to work for non-invertible matrices
  (such matrices would be needed in order to prove the surjectivity of the regulator map).
- We can also treat the case where \( E \) has non-split multiplicative reduction at \( p \).
Ingredients of the proof :

1. Kato’s Euler system
2. Perrin-Riou’s « logarithme élargi »
3. Explicit reciprocity law

1. Idea : there is a commutative diagram

\[
\begin{array}{ccl}
K_2(Y(Np^{n+1})) & \longrightarrow & H^1(\mathbb{Q}(\zeta_{p^{n+1}}), V_{Y_1(N)}(2)) \\
\downarrow\text{trace} & & \downarrow\text{cores} \\
K_2(Y(Np^n)) & \longrightarrow & H^1(\mathbb{Q}(\zeta_{p^n}), V_{Y_1(N)}(2))
\end{array}
\]

The elements \( z_{Np^n} = \{g_{1,0}, g_{0,1}\} \in K_2(Y(Np^n)) \otimes \mathbb{Q} \) are compatible for the trace when \( n \geq 1 \).
By considering the image of \((z_{Np^n})_{n \geq 1}\) under the regulator map and projecting to the elliptic curve \(E\), one gets

\[ z_E(2) \in H_{Iw}^1(\mathbb{Q}, V_E(2)). \]

Kato’s Euler system is defined by \(z_{Kato}(2) = \lambda^{-1}z_E(2)\) for some explicit \(\lambda \in \Lambda\) (which explains the bad Euler factors \(1 - a_\ell\) in the final formula).

2. Perrin-Riou has constructed an exponential map

\[ \Omega_{V_E} : D_{\text{cris}}(V_E) \rightarrow (H_{Iw}^1(\mathbb{Q}_p, V_E)/V_E^{G_{\mathbb{Q}_p(\zeta_{p^\infty})}}) \otimes \mathcal{H} \]

interpolating the Bloch-Kato exponential maps for \(V_E(k), k \geq 1\).
Dualizing, we get $\mathcal{L} : H^1_{Iw}(\mathbb{Q}_p, V_E(2)) \to \mathcal{H} \otimes D_{\text{cris}}(V_E)^\vee$.

**Theorem (Kato)**

$\mathcal{L}(z_{\text{Kato}}(2))(\eta_\alpha) = L_{p,\alpha}(E)$

Evaluating at the trivial character of $G_\infty$, we get

$$L_{p,\alpha}(E, \omega^{-1}, 0) = \langle \pi_0(z_{\text{Kato}}(2)), \pi_0(\Omega_{V_E}(\eta_\alpha)) \rangle$$

$$= \langle \log \pi_0(z_{\text{Kato}}(2)), \exp^* \pi_0(\Omega_{V_E}(\eta_\alpha)) \rangle$$

Note that $\pi_0(z_E(2)) = \pi_0(\lambda) \pi_0(z_{\text{Kato}}(2))$ and that $\log \pi_0(z_E(2))$ is equal to $\text{reg}(z_N)$ up to a simple rational factor (involving taking the trace from $Y(Np)$ to $Y(N)$). Thus it remains to compute $\exp^* \pi_0(\Omega_{V_E}(\eta_\alpha))$. 

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3. We use the explicit reciprocity law (proved by Colmez, Kato-Kurihara-Tsuji, Benois). It states that $\Omega_{V_E}$, although defined in terms of the positive twists of $V_E$, is also linked with the negative twists $V_E(k), k \leq 0$. More precisely if $\exp^* : H^1(\mathbb{Q}_p, V_E) \to D_{dR}(V_E)$ denotes Bloch-Kato’s dual exponential map, we have

$$\exp^* \pi_0(\Omega_{V_E}(\eta_\alpha)) = (1 - \frac{1}{p\phi})(1 - \phi)^{-1}\eta_\alpha.$$ 

This accounts for the extra Euler factor at $p$. Combining all these ingredients, we get the formula.
Some questions

- Extensions to higher weight modular forms
- Is it possible to prove that the image of the $p$-adic regulator map has the correct rank? (assuming the $p$-adic $L$-values are nonzero)
- In the case of split multiplicative reduction, does there exist a link between the regulator and the $L$-invariant?
- Formulate $p$-adic Beilinson for more general varieties