BEILINSON-KATO ELEMENTS IN K_2 OF MODULAR CURVES

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Abstract. This article investigates explicit linear dependence relations in the K_2-group of modular curves. In particular, it is shown that the Beilinson-Kato elements in K_2 of the modular curve $Y(N)$ satisfy the Manin relations when N is not divisible by 3. Similar results are obtained for the modular curves $X_1(N)$ and $X_0(N)$ when N is prime. Finally we exhibit explicit generators of K_2, assuming the Beilinson conjecture.

Introduction

Let X be a smooth projective curve over $\mathbf{Q}$, and $L(h^1(X), s)$ be the associated L-function. A very special case of Beilinson's conjectures predicts that the special value $L(h^1(X), 2)$ can be expressed in terms of a suitable regulator map on the algebraic K-group $K_2(X)$ (see [8] for a nice overview and a precise statement of this conjecture). Beilinson proved a part of his conjecture in the case where X is a modular curve [18]. Beilinson's work was also partially anticipated by Bloch, who studied the particular case of CM elliptic curves [1].

Despite these profound results, the K-group itself remains very mysterious. There's quite an art to constructing special elements in this group and, as soon as the genus of X is not zero, it is not even known whether $K_2(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a finite dimensional $\mathbf{Q}$-vector space.

I showed in [5] how Beilinson's theorem can be made explicit in the case of the modular curve $X_1(N)$. This raised the question of determining linear dependence relations in the group $K_2(X_1(N))$ [5, §8].

The main point of this article is to make these relations explicit. Let $Y(N)$ be the open modular curve associated to the congruence subgroup $\Gamma(N)$. By taking cup-products of Siegel units, there is a natural map

$$\rho : M_2(\mathbf{Z}/N\mathbf{Z}) \to K_2(Y(N)) \otimes \mathbf{Q}. \tag{1}$$

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Under the hypothesis that N is not divisible by 3, I show that ρ satisfies the Manin relations (Theorem 1.4). This was also proved by Goncharov [9] using a different method, and his proof works for all N. Thus ρ can be seen as a Manin symbol (or modular symbol) with values in $K_2(Y(N)) \otimes \mathbf{Q}$. This result is similar to constructions of Borisov and Gunnells [2, 3] and Paşol [17] in the case of modular forms. In these works, the product of two Eisenstein series plays the role of the cup-product.

I then use this result to study the case of the modular curves $X_1(p)$ and $X_0(p)$, where p is prime (Theorems 4.2, 4.4 and 4.8). In particular, the Beilinson conjecture implies that the elements so constructed span the vector space $K_2(X_0(p))_{\mathbf{Z}} \otimes \mathbf{Q}$, and I determine all the relations between them.

Some questions would deserve further study. I do not know (even conjecturally) whether the image of ρ spans $K_2(Y(N)) \otimes \mathbf{Q}$ (see Remark 1.7). In view of the arithmetic applications of Kato's Euler system [10], it would be also of interest to describe the action of Hecke correspondences on these elements, in the spirit of Merel's results for modular symbols [15].

1. The Beilinson-Kato elements in K_2

Let us first state some standard facts on modular curves (see [20, 13, 7, 11] for more detailed accounts). Let $N \geq 3$ be an integer and $Y(N)$ be the modular curve classifying elliptic curves E with a level N structure, that is a basis (e_1, e_2) of $E[N]$ over $\mathbf{Z}/N\mathbf{Z}$. The curve $Y(N)$ is a smooth projective curve defined over $\mathbf{Q}$, whose affine ring $\mathcal{O}(Y(N))$ contains the cyclotomic field $\mathbf{Q}(\zeta_N)$ generated by $\zeta_N := e^{2i\pi/N}$. The curve $Y(N)$ is not geometrically connected. Indeed, there is an isomorphism $Y(N)(\mathbf{C}) \cong (\mathbf{Z}/N\mathbf{Z})^* \times (\Gamma(N)\backslash\mathcal{H})$, where $\mathcal{H}$ is the Poincaré upper half-plane and $\Gamma(N) \subset \mathrm{SL}_2(\mathbf{Z})$ is the congruence subgroup of matrices satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}.$$

For any $z \in \mathcal{H}$ and $\lambda \in \mathbf{Q}$, let us set $q = e^{2i\pi z}$ and $q^\lambda = e^{2i\pi\lambda z}$.

The curve $Y(N)$ has a smooth compactification $X(N)$ over $\mathbf{Q}$ which is obtained by adding on the cusps. The function field of $X(N)$ will be referred to by $\mathbf{Q}(X(N))$. It is naturally embedded into the function field of the compactification of $\Gamma(N)\backslash\mathcal{H}$. There is also a natural inclusion of $\mathbf{Q}(X(N))$ into the field of formal Laurent series $\mathbf{Q}(\zeta_N)((q^{1/N}))$, by looking at the q-expansion.

1.1. **Siegel units.** Let us give the definition of Siegel units (see [6, 10, 12] for further reference). The group of modular units of $X(N)$ will be denoted by $\mathcal{O}^*(Y(N))$. In order to avoid torsion problems, Siegel units will always be considered in the $\mathbf{Q}$-vector space $\mathcal{O}^*(Y(N)) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Let $B_2(X) = X^2 - X + \frac{1}{6}$ be the second Bernoulli polynomial.

Definition 1.1. For any $(\alpha, \beta) \in (\mathbf{Z}/N\mathbf{Z})^2 - \{(0,0)\}$ let us define

$$g_{\alpha,\beta}(z) = q^{\frac{1}{2}B_2(\widetilde{\alpha}/N)} \prod_{n\geq 0} \left(1 - q^n q^{\widetilde{\alpha}/N} \zeta_N^{\beta}\right) \prod_{n\geq 1} \left(1 - q^n q^{-\widetilde{\alpha}/N} \zeta_N^{-\beta}\right) \tag{2}$$

where $\widetilde{\alpha} \in \mathbf{Z}$ is the unique representative of α satisfying $0 \leq \widetilde{\alpha} < N$. By convention $g_{0,0} = 1$.

Thus $g_{\alpha,\beta}$ is a holomorphic function on $\mathcal{H}$. It is known that some power of $g_{\alpha,\beta}$ (in fact $g_{\alpha,\beta}^{12N}$) is modular with respect to $\Gamma(N)$, and lies in $\mathcal{O}^*(Y(N))$ [13, Chap 19 §2]. Therefore $g_{\alpha,\beta}$ is well-defined as an element of $\mathcal{O}^*(Y(N)) \otimes \mathbf{Q}$.

Let G be the group $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$. It acts from the left on $Y(N)$, by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (E, e_1, e_2) = (E, ae_1 + be_2, ce_1 + de_2) \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\right). \tag{3}$$

This induces on $\mathcal{O}^*(Y(N)) \otimes \mathbf{Q}$ a right action of G. It turns out that G acts on the set of Siegel units. More precisely, we have [10, Lemma 1.7]

$$g_{\alpha,\beta}|\gamma = g_{(\alpha,\beta)\cdot\gamma} \qquad (\gamma \in G). \tag{4}$$

Since $-1 \in G$ acts trivially on $Y(N)$, we get the relation $g_{-\alpha,-\beta} = g_{\alpha,\beta}$. Kubert and Lang proved that the Siegel units of level N generate $\mathcal{O}^*(Y(N))\otimes\mathbf{Q}$ [12].

1.2. **The construction of Beilinson and Kato.** Let us consider the Quillen K-group $K_2(Y(N))$, which enjoys a right action of G by functoriality. Beilinson constructed special elements in it using cup-products of modular units. This motivates the following definition.

Definition 1.2. Let ρ be the map

$$\begin{aligned} \rho : M_2(\mathbf{Z}/N\mathbf{Z}) &\to K_2(Y(N)) \otimes_{\mathbf{Z}} \mathbf{Q} \\ \begin{pmatrix} s & t \\ u & v \end{pmatrix} &\mapsto \{g_{s,t}, g_{u,v}\}. \end{aligned} \tag{5}$$

Remark 1.3. Colmez [6, 1.4.2] constructed an algebraic distribution on $M_2(\mathbf{Q} \otimes \widehat{\mathbf{Z}})$ with values in K_2, which generalizes Definition 1.2. I shall not use this more conceptual point of view in what follows.

Let ε (resp. σ, τ) be the image of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$) in G.

Theorem 1.4. *The elements $\rho(M)$ satisfy the following relations*

(6)
$$\rho(\varepsilon M) = \rho(M) \qquad \textit{and} \qquad \rho(M) + \rho(\sigma M) = 0 \qquad (M \in M_2(\mathbf{Z}/N\mathbf{Z})).$$

Let us suppose further that 3 *does not divide* N*. Then we have*

$$\rho(M) + \rho(\tau M) + \rho(\tau^2 M) = 0 \qquad (M \in M_2(\mathbf{Z}/N\mathbf{Z})). \tag{7}$$

Remark 1.5. The Manin relations (6) and (7) have also been established by Goncharov [9, Corollary 2.17], without any assumption on the level N, using a different method.

Remark 1.6. The Manin relations (6) and (7) are consistent with the formula of Kato [10, Thm 2.6] giving the regulator of $z_N = \rho(I)$. The element z_N plays a prominent role in the construction of Kato's Euler system [10, §5].

Remark 1.7. It would be interesting to know whether the elements $\rho(M)$ span the $\mathbf{Q}$-vector space $K_2(Y(N)) \otimes \mathbf{Q}$. A related question is to determine whether $K_2(Y(N))$ is generated by the symbols $\{u, v\}$ with $u, v \in \mathcal{O}^*(Y(N))$. Since $K_2(Y(N)) \otimes \mathbf{Q}$ is in general not known to be finite-dimensional, it is more reasonable to ask whether the Manin relations make up a complete set of relations between the elements $\rho(M)$. A natural way to tackle this problem would be to compute the Beilinson regulator of $\rho(M)$. However, the formula of Kato [10, Thm 2.6] seems to indicate that in general $\rho(G)$ cannot span $K_2(Y(N)) \otimes \mathbf{Q}$.

Proposition 1.8. *For any $M \in M_2(\mathbf{Z}/N\mathbf{Z})$ the relations (6) hold.*

Proof. Let $M = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$. We have

$$\begin{aligned} \rho(\varepsilon M) &= \{g_{-s,-t}, g_{u,v}\} = \{g_{s,t}, g_{u,v}\} = \rho(M) \\ \text{and } \rho(\sigma M) &= \{g_{-u,-v}, g_{s,t}\} = -\{g_{s,t}, g_{u,v}\} = -\rho(M), \end{aligned}$$

because of the relation $g_{-s,-t} = g_{s,t}$ and the antisymmetry of the Milnor symbol. □

The relation (7) can be seen as an analogue of the Manin 3-term relation for modular symbols. The proof of this relation lies deeper, and will be given in the next two sections.

2. Weierstrass units

For any $z \in \mathcal{H}$, we let $\wp(z,u)$ be the Weierstrass $\wp$-function associated to the lattice $\Lambda_z = \mathbf{Z}z + \mathbf{Z} \subset \mathbf{C}$. It is defined for $u \in \mathbf{C} - \Lambda_z$.

Definition 2.1. For any $a = (a_1, a_2) \in (\mathbf{Z}/N\mathbf{Z})^2 - \{(0,0)\}$, let us define

$$\wp_a(z) = \wp\Big(z, \frac{\widetilde{a_1} z + \widetilde{a_2}}{N}\Big) \qquad (z \in \mathcal{H}), \tag{8}$$

where $\widetilde{a_1}$ and $\widetilde{a_2}$ are any representatives of a_1 and a_2 in $\mathbf{Z}$.

We use these functions to construct the *Weierstrass units*. This classical construction is undertaken in [12, Chap 2 §6]. We give some details for the sake of completeness.

Theorem ([12])**.** *Let a, b, c, d be four nonzero elements of $(\mathbf{Z}/N\mathbf{Z})^2$ satisfying $a \neq \pm b$ and $c \neq \pm d$. The function*

$$\frac{\wp_a - \wp_b}{\wp_c - \wp_d} \tag{9}$$

defines an element of $\mathcal{O}^*(Y(N))$.

Proof. The function $\wp_a$ is holomorphic on $\mathcal{H}$ and defines a modular form of weight 2 for the group $\Gamma(N)$. For any $z \in \mathcal{H}$, we have $\wp_a(z) = \wp_b(z)$ if and only if $a = \pm b$. Thus $(\wp_a - \wp_b)/(\wp_c - \wp_d)$ is well-defined and does not vanish on $\mathcal{H}$. The fact that it belongs to $\mathbf{Q}(X(N))$ is a consequence of results of Shimura ([19, §4], [20, Chap 6]). It essentially amounts to express $(\wp_a - \wp_b)/(\wp_c - \wp_d)$ in terms of the x-coordinates of N-torsion points of the universal elliptic curve over $Y(N)$. The fact that (9) is a modular unit is proved in [12, Chap 2 Thm 6.1]. □

Now we express the Weierstrass units in terms of Siegel units. Once again this is done in [12, Chap 2 §6].

Proposition 2.2. *Let a, b, c, d be four nonzero elements of $(\mathbf{Z}/N\mathbf{Z})^2$ satisfying $a \neq \pm b$ and $c \neq \pm d$. Then the following identity holds in $\mathcal{O}^*(Y(N)) \otimes \mathbf{Q}$*

$$\frac{\wp_a - \wp_b}{\wp_c - \wp_d} = \frac{g_{a+b} g_{a-b}}{g_a^2 g_b^2} \cdot \frac{g_c^2 g_d^2}{g_{c+d} g_{c-d}}. \tag{10}$$

Proof. We start with the following classical formula from the theory of elliptic functions [13, Chap 18 Thm 2]

$$\wp(z,u) - \wp(z,v) = -\frac{\sigma(z,u+v)\sigma(z,u-v)}{\sigma^2(z,u)\sigma^2(z,v)} \qquad (z \in \mathcal{H}), \tag{11}$$

where σ refers to the Weierstrass sigma function. For any $(a_1, a_2) \in \mathbf{Z}^2$, let us define in the same way as (8)

$$\sigma_{a_1,a_2}(z) = \sigma(z, \frac{a_1 z + a_2}{N}) \qquad (z \in \mathcal{H}).$$

We write abusively $a = (a_1, a_2)$ and $b = (b_1, b_2)$ for representatives of a and b in $\mathbf{Z}^2$. The formula (11) can then be rewritten as

$$\wp_a - \wp_b = -\frac{\sigma_{a+b}\sigma_{a-b}}{\sigma_a^2 \sigma_b^2}.$$

Using the expression of σ as an infinite q-product [13, Chap 18 Thm 4], we get the following formula (compare with [12, p. 29 and 51])

$$\wp_a - \wp_b = (2i\pi)^2 q^{b_1/N} \zeta_N^{b_2} \prod_{n \geq 1} (1 - q^n)^4 \cdot \frac{\gamma(q, a+b)\gamma(q, a-b)}{\gamma^2(q, a)\gamma^2(q, b)}$$

where γ is defined by

$$\gamma(q, a_1, a_2) = \prod_{n \geq 0} (1 - q^n q^{a_1/N} \zeta_N^{a_2}) \cdot \prod_{n \geq 1} (1 - q^n q^{-a_1/N} \zeta_N^{-a_2}).$$

Using the obvious notation for c and d, this gives

$$\frac{\wp_a - \wp_b}{\wp_c - \wp_d} = q^{(b_1 - d_1)/N} \zeta_N^{b_2 - d_2} \frac{\gamma(q, a+b)\gamma(q, a-b)}{\gamma^2(q, a)\gamma^2(q, b)} \cdot \frac{\gamma^2(q, c)\gamma^2(q, d)}{\gamma(q, c+d)\gamma(q, c-d)}.$$

Using the expression (2) for Siegel units, we get the equation

$$\frac{\wp_a - \wp_b}{\wp_c - \wp_d} = \zeta_N^{b_2 - d_2} \frac{g_{a+b} g_{a-b}}{g_a^2 g_b^2} \cdot \frac{g_c^2 g_d^2}{g_{c+d} g_{c-d}}.$$

It is a priori a relation between q-products, but raising it to an appropriate power yields an equality in $\mathbf{Q}(\zeta_N)((q^{1/N}))$ and thus in $\mathcal{O}^*(Y(N))$. Therefore the formula (10) is valid in $\mathcal{O}^*(Y(N)) \otimes \mathbf{Q}$. □

3. The three-term relation

Weierstrass units (9) satisfy additive relations. These have already been used by Kubert and Lang to get diophantine results on modular curves [12, Chap 8]. In fact the whole proof of (7) is based on the following simple identity

$$\frac{\wp_a - \wp_b}{\wp_a - \wp_c} + \frac{\wp_b - \wp_c}{\wp_a - \wp_c} = 1. \tag{12}$$

The relation (12) also has applications to the S-unit equation and is connected to the arithmetic of Fermat curves (see the nice introduction of [12, Chap 8] for precise statements and references).

Since the canonical bilinear map $\mathcal{O}^*(Y(N)) \times \mathcal{O}^*(Y(N)) \to K_2(Y(N))$ enjoys Steinberg relations [16, 9.8], the identity (12) implies the following relation in $K_2(Y(N))$

$$\left\{ \frac{\wp_a - \wp_b}{\wp_a - \wp_c}, \frac{\wp_b - \wp_c}{\wp_a - \wp_c} \right\} = 0. \tag{13}$$

Using the expression of Weierstrass units in terms of Siegel units gives linear dependence relations between the elements $\rho(M)$ in $K_2(Y(N)) \otimes \mathbf{Q}$. The main task will be to show that the 3-term relation is a consequence of these relations.

Let a, b, c be three nonzero elements of $(\mathbf{Z}/N\mathbf{Z})^2$ such that $a \neq \pm b$, $b \neq \pm c$ and $c \neq \pm a$. Using (10) and (13) we have the following identity in $K_2(Y(N)) \otimes \mathbf{Q}$

$$\left\{ \frac{g_{a+b}g_{a-b}}{g_a^2 g_b^2} \cdot \frac{g_a^2 g_c^2}{g_{a+c}g_{a-c}}, \frac{g_{b+c}g_{b-c}}{g_b^2 g_c^2} \cdot \frac{g_a^2 g_c^2}{g_{a+c}g_{a-c}} \right\} = 0.$$

Expanding this and using the relation $g_{-a} = g_a$, we get the more symmetric identity

$$\begin{aligned} \{g_{a+b}g_{a-b}g_c^2, g_{b+c}g_{b-c}g_a^2\} + \{g_{b+c}g_{b-c}g_a^2, g_{c+a}g_{c-a}g_b^2\} \\ + \{g_{c+a}g_{c-a}g_b^2, g_{a+b}g_{a-b}g_c^2\} = 0. \end{aligned} \tag{14}$$

We remark that when $a = 0$ the relation (14) still makes sense and holds. Similarly it holds in the cases $b = 0$, $c = 0$, $a = \pm b$, $b = \pm c$ or $c = \pm a$. Thus (14) is true for any values of $a, b, c \in (\mathbf{Z}/N\mathbf{Z})^2$.

We now wish to write (14) as a linear combination of 3-term relations. Let us define $\psi(M) = \rho(M) + \rho(\tau M) + \rho(\tau^2 M)$ for any $M \in M_2(\mathbf{Z}/N\mathbf{Z})$. Let $M = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$. An elementary computation yields

$$\psi(M) = \{g_{s,t}, g_{u,v}\} + \{g_{u,v}, g_{s-u,t-v}\} + \{g_{s-u,t-v}, g_{s,t}\}. \tag{15}$$

For any two elements a and b of $(\mathbf{Z}/N\mathbf{Z})^2$, let us write $\begin{pmatrix} a \\ b \end{pmatrix}$ for the 2 by 2 matrix with row vectors a and b. Then (15) can be rewritten as

$$\psi \begin{pmatrix} a \\ b \end{pmatrix} = \rho \begin{pmatrix} a \\ b \end{pmatrix} + \rho \begin{pmatrix} b \\ a-b \end{pmatrix} + \rho \begin{pmatrix} a-b \\ a \end{pmatrix}. \tag{16}$$

We also have

$$\psi \begin{pmatrix} a \\ -b \end{pmatrix} = \rho \begin{pmatrix} a \\ b \end{pmatrix} + \rho \begin{pmatrix} b \\ a+b \end{pmatrix} + \rho \begin{pmatrix} a+b \\ a \end{pmatrix}. \tag{17}$$

Lemma 3.1. *For any $a, b, c \in (\mathbf{Z}/N\mathbf{Z})^2$, the left hand side of the relation (14) can be written as*

$$
\begin{aligned}
&2\psi\begin{pmatrix} a \\ b \end{pmatrix} + 2\psi\begin{pmatrix} a \\ -b \end{pmatrix} + 2\psi\begin{pmatrix} b \\ c \end{pmatrix} + 2\psi\begin{pmatrix} b \\ -c \end{pmatrix} + 2\psi\begin{pmatrix} c \\ a \end{pmatrix} + 2\psi\begin{pmatrix} c \\ -a \end{pmatrix} \\
&\quad + \psi\begin{pmatrix} b+a \\ b+c \end{pmatrix} + \psi\begin{pmatrix} b+a \\ b-c \end{pmatrix} + \psi\begin{pmatrix} b-a \\ b+c \end{pmatrix} + \psi\begin{pmatrix} b-a \\ b-c \end{pmatrix}.
\end{aligned} \tag{18}
$$

Proof. By expanding (14) completely, we obtain

$$
\begin{aligned}
&\{g_{a+b}, g_{b+c}\} + \{g_{b+c}, g_{c-a}\} + \{g_{c-a}, g_{a+b}\} \\
&+ \{g_{a+b}, g_{b-c}\} + \{g_{b-c}, g_{c+a}\} + \{g_{c+a}, g_{a+b}\} \\
&+ 2\{g_{a+b}, g_a\} + 4\{g_a, g_b\} + 2\{g_b, g_{a+b}\} \\
&+ \{g_{a-b}, g_{b+c}\} + \{g_{b+c}, g_{c+a}\} + \{g_{c+a}, g_{a-b}\} \\
&+ \{g_{a-b}, g_{b-c}\} + \{g_{b-c}, g_{c-a}\} + \{g_{c-a}, g_{a-b}\} \\
&+ 2\{g_{a-b}, g_a\} + 2\{g_b, g_{a-b}\} \\
&+ 2\{g_c, g_{b+c}\} + 2\{g_{b+c}, g_b\} + 4\{g_b, g_c\} \\
&+ 2\{g_c, g_{b-c}\} + 2\{g_{b-c}, g_b\} \\
&+ 4\{g_c, g_a\} \\
&+ 2\{g_a, g_{c+a}\} + 2\{g_{c+a}, g_c\} \\
&+ 2\{g_a, g_{c-a}\} + 2\{g_{c-a}, g_c\} = 0.
\end{aligned} \tag{19}
$$

In most lines of (19) we recognize an expression of type (16) or (17), but there are incomplete terms. We can arrange the picture by splitting the terms with a coefficient 4 and moving them to the right places. This gives exactly (18). □

We now make use of the relation (14) with a particular choice of a, b and c. Let us assume that $c = a + b$. This gives (for any choice of a and b)

$$
\begin{aligned}
&2\psi\begin{pmatrix} a \\ b \end{pmatrix} + 2\psi\begin{pmatrix} a \\ -b \end{pmatrix} + 2\psi\begin{pmatrix} b \\ a+b \end{pmatrix} + 2\psi\begin{pmatrix} b \\ -a-b \end{pmatrix} \\
&\quad + 2\psi\begin{pmatrix} a+b \\ a \end{pmatrix} + 2\psi\begin{pmatrix} a+b \\ -a \end{pmatrix} + \psi\begin{pmatrix} a+b \\ a+2b \end{pmatrix} \\
&\quad + \psi\begin{pmatrix} a+b \\ -a \end{pmatrix} + \psi\begin{pmatrix} -a+b \\ a+2b \end{pmatrix} + \psi\begin{pmatrix} -a+b \\ -a \end{pmatrix} = 0.
\end{aligned} \tag{20}
$$

Using the notation $M = \begin{pmatrix} a \\ b \end{pmatrix}$ and letting T (resp. T') be the image of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$) in G, we can rewrite (20) as

$$2\psi(M)+2\psi(-\varepsilon M)+2\psi(-\tau\varepsilon M)+2\psi(-\tau T^2 M)+2\psi(\tau^2\varepsilon M) \\ +3\psi(\tau^2 T'^2 M)+\psi(-\tau^2 T^2 M)+\psi(\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix} M)+\psi(\tau^2 M)=0.$$

Since $\psi(M)=\psi(-M)=\psi(\tau M)$ for any M, this simplifies to

$$3\psi(M)+6\psi(\varepsilon M)+3\psi(T^2 M)+3\psi(T'^2 M)+\psi(\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix} M)=0. \tag{21}$$

Let us consider the formal linear combination of matrices in $\mathbf{Z}[M_2(\frac{\mathbf{Z}}{N\mathbf{Z}})]$

$$D(M)=3[M]+6[\varepsilon M]+3[T^2 M]+3[T'^2 M]+[\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix} M].$$

By assumption, we have $\det\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix}=-3\in(\frac{\mathbf{Z}}{N\mathbf{Z}})^*$.

Lemma 3.2. *The elements $D(M)$ span $\mathbf{Q}[M_2(\frac{\mathbf{Z}}{N\mathbf{Z}})]$ when M runs through $M_2(\frac{\mathbf{Z}}{N\mathbf{Z}})$.*

Proof. We remark that $D(M)$ is congruent mod 3 to the single matrix $\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix} M$. Therefore the determinant of the vectors $D(M)$ in the canonical basis of $\mathbf{Z}[M_2(\frac{\mathbf{Z}}{N\mathbf{Z}})]$ is not zero mod 3, and thus a nonzero integer. $\square$

Using (21) and Lemma 3.2 gives $\psi(M)=0$ for any $M\in M_2(\mathbf{Z}/N\mathbf{Z})$, which concludes the proof of Theorem 1.4.

4. Varying the modular curve

In this section I study special elements in the groups $K_2(X_1(N))\otimes\mathbf{Q}$ and $K_2(X_0(N))\otimes\mathbf{Q}$, in the case of prime level. In particular, I make explicit the link between the Beilinson-Kato elements and the elements which come up during my PhD thesis [4].

Let us first recall the definition of particular modular units on $X_1(N)$ [5, (95)]. Let $Y_1(N)$ be the modular curve over $\mathbf{Q}$ classifying elliptic curves E with a point P of order N, and let $X_1(N)$ be the smooth compactification of $Y_1(N)$. The set of cusps of $X_1(N)(\mathbf{C})$ is identified with $\Gamma_1(N)\backslash\mathbf{P}^1(\mathbf{Q})$, and with this convention the cusp $[0]$ is defined over $\mathbf{Q}$. Let $W_N: X_1(N)\to X_1(N)$ be the Atkin-Lehner involution, which is defined over $\mathbf{Q}(\zeta_N)$. For any $\lambda\in(\mathbf{Z}/N\mathbf{Z})^*$, the Diamond operator $\langle\lambda\rangle$ associated to λ is defined by $(E,P)\mapsto(E,\lambda P)$. On the complex points of $X_1(N)$ we have $\langle\lambda\rangle[z]=[m_\lambda z]$ where $m_\lambda\in\mathrm{SL}_2(\mathbf{Z})$ is any matrix congruent to $\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}$ mod N.

Definition 4.1. For any $\lambda \in (\mathbf{Z}/N\mathbf{Z})^*$, let $u_\lambda \in \mathcal{O}^*(Y_1(N)) \otimes \mathbf{Q}$ be the unique modular unit satisfying

$$\operatorname{div}(u_\lambda) = \langle \lambda \rangle [0] - [0] \qquad \text{and} \qquad u_\lambda \circ W_N \text{ is normalized.} \tag{22}$$

Note that we use the cusp $[0]$ instead of $[\infty]$. It essentially amounts to the same thing, because the two definitions are related by W_N. In [5, Prop 6.1] I show that the element $\{u_\lambda, u_\mu\}$ belongs to $K_2(X_1(N)) \otimes \mathbf{Q}$ for any choice of $\lambda, \mu \in (\mathbf{Z}/N\mathbf{Z})^*$.

From now on, let us suppose that $N = p$ is an odd prime. In [5, §8] I remark that the Beilinson conjecture should imply some linear dependence relations between the elements $\{u_\lambda, u_\mu\}$. It turns out that these relations can be worked out explicitly and even rigorously proved, as follows.

Let $\overline{B_2} : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ be the 1-periodic function obtained from B_2 by defining $\overline{B_2}(\overline{t}) = B_2(t)$ for any $0 \leq t \leq 1$. For any $u, v \in (\mathbf{Z}/p\mathbf{Z})^*$, let us define

$$\gamma(u,v) = \sum_{\lambda,\mu \in (\mathbf{Z}/p\mathbf{Z})^*} \overline{B_2}(\frac{\lambda u}{p}) \overline{B_2}(\frac{\mu v}{p}) \{u_\lambda, u_\mu\} \in K_2(X_1(p)) \otimes \mathbf{Q}. \tag{23}$$

By convention, we put $\gamma(u,v) = 0$ when $u = 0$ or $v = 0$.

Theorem 4.2. *The elements $\gamma(u,v)$ $(u, v \in \mathbf{Z}/p\mathbf{Z})$ satisfy the following relations*

$$\gamma(u,v) = \gamma(\pm u, v) = \gamma(u, \pm v), \tag{24}$$

$$\gamma(u,v) + \gamma(v,-u) = 0, \tag{25}$$

$$\gamma(u,v) + \gamma(v,-u-v) + \gamma(-u-v,u) = 0. \tag{26}$$

Proof. Since $\overline{B_2}$ is an even function and $u_{-\lambda} = u_\lambda$, we have the relations $\gamma(\pm u, v) = \gamma(u, \pm v) = \gamma(u,v)$. The antisymmetry of the Milnor symbol yields $\gamma(v,u) = -\gamma(u,v)$, which proves (25).

In order to prove the three-term relation (26), we jump to $X(p)$. We have a finite morphism $\pi : Y(p) \to Y_1(p)$ which is defined over $\mathbf{Q}$, given by $(E, e_1, e_2) \mapsto (E, e_2)$.

Let $\mathcal{M}(p)$ be the field of meromorphic functions on the compactification of $\Gamma(p)\backslash\mathcal{H}$. It is a Galois extension of $\mathbf{C}(j)$ with Galois group $\mathrm{SL}_2(\mathbf{Z}/p\mathbf{Z})/\pm 1$. We say that a function $f \in \mathbf{C}((q^{1/n}))^*$ (for some $n \geq 1$) is *normalized* when the leading coefficient of its q-expansion is one. This definition extends naturally to $\mathbf{C}((q^{1/n}))^* \otimes \mathbf{Q}$. Two functions $f, g \in \mathcal{M}(p)^*$ coincide if and only if their divisors are equal and f/g is normalized. Since we have an inclusion $\mathcal{O}^*(Y(p)) \subset \mathcal{M}(p)^*$, we will apply this principle to check equality between modular units in $\mathcal{O}^*(Y(p)) \otimes \mathbf{Q}$.

The set of cusps of $\Gamma(p)\backslash\mathcal{H}$ is identified with $\Gamma(p)\backslash\mathbf{P}^1(\mathbf{Q})$, and the restriction of π to the cusps is the natural projection $\Gamma(p)\backslash\mathbf{P}^1(\mathbf{Q}) \to \Gamma_1(p)\backslash\mathbf{P}^1(\mathbf{Q})$. The inverse image of a cusp $[x]$ by π is given by

$$\pi^*[x] = \sum_{k=0}^{p-1}[x+k] \qquad (x \in \mathbf{P}^1(\mathbf{Q})).$$

The set of cusps $\Gamma(p)\backslash\mathbf{P}^1(\mathbf{Q})$ can be identified with the set of nonzero column vectors of $(\mathbf{Z}/p\mathbf{Z})^2$ quotiented by ± 1, the bijection being induced by $[a/c] \in \mathbf{P}^1(\mathbf{Q}) \mapsto \begin{bmatrix}\overline{a}\\ \overline{c}\end{bmatrix}$ for any two relatively prime integers a and c. We now consider $\pi^* u_\lambda \in \mathcal{O}^*(Y(p)) \otimes \mathbf{Q} \subset \mathcal{M}(p)^* \otimes \mathbf{Q}$. Its divisor is given by

$$\operatorname{div} \pi^* u_\lambda = \pi^* \operatorname{div} u_\lambda = \sum_{k=0}^{p-1}[\langle\lambda\rangle 0 + k] - [k] = \sum_{k \in \mathbf{Z}/p\mathbf{Z}} \begin{bmatrix}k\\ \lambda\end{bmatrix} - \begin{bmatrix}k\\ 1\end{bmatrix}. \tag{27}$$

On the other hand, the order of the Siegel unit $g_{\alpha,\beta}$ at the cusp $[\infty]$ can be deduced from the q-product (2). Since $q^{1/p}$ is a uniformizing parameter at $[\infty]$, we have

$$\operatorname{ord}_{[\infty]} g_{\alpha,\beta} = \frac{p}{2}\overline{B_2}(\frac{\alpha}{p}) \qquad (\alpha,\beta) \neq (0,0).$$

Using the transformation formula (4), we deduce the order of $g_{\alpha,\beta}$ at any cusp :

$$\operatorname{ord}_{[\frac{a}{c}]} g_{\alpha,\beta} = \frac{p}{2}\overline{B_2}(\frac{\alpha\overline{a}+\beta\overline{c}}{p}) \qquad (\alpha,\beta) \neq (0,0).$$

A straightforward computation gives

$$\operatorname{div} g_{0,\beta} = \frac{p}{4} \sum_{\substack{\lambda \in (\mathbf{Z}/p\mathbf{Z})^* \\ k \in \mathbf{Z}/p\mathbf{Z}}} \overline{B_2}(\frac{\beta\lambda}{p}) \begin{bmatrix}k\\ \lambda\end{bmatrix} + \frac{p}{24} \sum_{k \in (\mathbf{Z}/p\mathbf{Z})^*} \begin{bmatrix}k\\ 0\end{bmatrix} \qquad (\beta \neq 0). \tag{28}$$

From (27) and (28), it follows that the divisor

$$\operatorname{div} g_{0,\beta} - \frac{p}{4} \sum_{\lambda \in (\mathbf{Z}/p\mathbf{Z})^*} \overline{B_2}(\frac{\beta\lambda}{p}) \operatorname{div} \pi^* u_\lambda$$

does not depend on $\beta \in (\mathbf{Z}/p\mathbf{Z})^*$. Moreover, we have

$$g_{0,\beta}(-\frac{1}{pz}) = g_{\beta,0}(pz) \quad \text{in } \mathbf{C}^* \otimes \mathbf{Q} \qquad (z \in \mathcal{H}).$$

and $g_{\beta,0}(pz)$ is a normalized function. Since each $u_\lambda \circ W_p$ is normalized, we can write

$$g_{0,\beta} = h \cdot \prod_{\lambda \in (\mathbf{Z}/p\mathbf{Z})^*} \pi^* u_\lambda \otimes \left(\frac{p}{4} \overline{B_2}\left(\frac{\beta\lambda}{p}\right)\right)$$

where $h \in \mathcal{O}^*(Y(p)) \otimes \mathbf{Q}$ is well-defined and independent of β. We then have

$$\left\{\frac{g_{0,u}}{h}, \frac{g_{0,v}}{h}\right\} = \frac{p^2}{16} \pi^* \gamma(u,v) \qquad (u, v \in (\mathbf{Z}/p\mathbf{Z})^*). \tag{29}$$

We are now ready to prove (26). Since the map $\pi^* : K_2(Y_1(p)) \otimes \mathbf{Q} \to K_2(Y(p)) \otimes \mathbf{Q}$ is injective, it suffices to work in the latter vector space. The cases $u = 0$, $v = 0$ and $u + v = 0$ are easily treated. In the general case, we write

$$\left\{\frac{g_{0,u}}{h}, \frac{g_{0,v}}{h}\right\} = \{g_{0,u}, g_{0,v}\} + \left\{h, \frac{g_{0,u}}{g_{0,v}}\right\}. \tag{30}$$

Thanks to Theorem 1.4, we already know that $(u,v) \mapsto \{g_{0,u}, g_{0,v}\}$ satisfies the three-term relation. Since $\{h, \frac{g_{0,u}}{g_{0,v}}\}$ is a "boundary element", we get the desired result. $\square$

Remark 4.3. In general, the relations (24), (25) and (26) between the elements $\gamma(u,v)$ do not make up a complete set of relations. It can be seen by working out the case $p = 5$ explicitly. In that case $X_1(p)$ is isomorphic to $\mathbf{P}^1$ over $\mathbf{Q}$ and $K_2(X_1(p)) \otimes \mathbf{Q}$ is known to be 0. In the general case however, if we average under the action of Diamond operators (see below), we can produce special elements in $K_2(X_0(p)) \otimes \mathbf{Q}$ together with a full set of relations.

A theorem of Schappacher and Scholl [18, 1.1.2 (iii)] implies that $\gamma(u,v)$ belongs to the integral subsapce $K_2(X_1(p))_{\mathbf{Z}} \otimes \mathbf{Q}$, and we can ask about the span of the elements $\gamma(u,v)$. Let

$$r_p : K_2(X_1(p))_{\mathbf{Z}} \otimes \mathbf{Q} \to \mathrm{Hom}_{\mathbf{Q}}(\Omega^1(X_1(p)), \mathbf{R}) \tag{31}$$

be the Beilinson regulator map, as defined in [5, §1].

Theorem 4.4. *The Beilinson conjecture for $L(h^1(X_1(p)), 2)$ implies that $K_2(X_1(p))_{\mathbf{Z}} \otimes \mathbf{Q}$ is generated by the elements $\gamma(u,v)$, with $u, v \in (\mathbf{Z}/p\mathbf{Z})^*$.*

Proof. Beilinson's conjecture predicts that r_p is injective and that its image is a $\mathbf{Q}$-structure of the target vector space. We already know that Beilinson's conjecture implies that $K_2(X_1(p))_{\mathbf{Z}} \otimes \mathbf{Q}$ is generated by the elements $\{u_\lambda, u_\mu\}$ [5, §8]. It is sufficient to show that each $\{u_\lambda, u_\mu\}$ is a $\mathbf{Q}$-linear combination of the elements $\gamma(u,v)$. Let us consider

$$\theta = \sum_{\lambda\in(\mathbf{Z}/p\mathbf{Z})^*} \overline{B_2}(\frac{\lambda}{p})[\lambda] \in \mathbf{Q}[(\mathbf{Z}/p\mathbf{Z})^*/\pm 1].$$

For every even Dirichlet character $\chi : (\mathbf{Z}/p\mathbf{Z})^* \to \mathbf{C}^*$, we have

$$\chi(\theta) = \sum_{\lambda\in(\mathbf{Z}/p\mathbf{Z})^*} \overline{B_2}(\frac{\lambda}{p})\chi(\lambda) = \begin{cases} \frac{1-p}{6p} & (\chi = 1) \\ \frac{\tau(\chi)}{\pi^2} L(\chi,2) & (\chi \neq 1) \end{cases} \tag{32}$$

where $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)e^{2ia\pi/p}$ is the Gauß sum of χ. But for any character χ, we have $L(\chi,2) \neq 0$, so that θ is invertible in the group algebra $\mathbf{Q}[(\mathbf{Z}/p\mathbf{Z})^*/\pm 1]$. □

We finally investigate the group $K_2(X_0(p)) \otimes \mathbf{Q}$. The natural morphism $X_1(p) \to X_0(p)$ identifies $K_2(X_0(p))\otimes\mathbf{Q}$ with the fixed part of $K_2(X_1(p))\otimes \mathbf{Q}$ under the Diamond operators.

Definition 4.5. For any $x \in (\mathbf{Z}/p\mathbf{Z})^*$, let

$$\gamma_0(x) = \sum_{u\in(\mathbf{Z}/p\mathbf{Z})^*} \gamma(u,ux). \tag{33}$$

Besides, we define $\gamma_0(0) = \gamma_0(\infty) = 0$.

Lemma 4.6. *For any $x \in (\mathbf{Z}/p\mathbf{Z})^*$, we have $\gamma_0(x) \in K_2(X_0(p)) \otimes \mathbf{Q}$.*

Proof. It suffices to prove that $\pi^*\gamma_0(x)$ is invariant under any matrix $t = \begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}/p\mathbf{Z})$. Because of (4), we have $g_{0,\beta}|t = g_{0,d\beta}$. Using (29), we remark that

$$\begin{aligned}\frac{p^2}{16}\pi^*\gamma_0(x) &= \sum_{u\in(\mathbf{Z}/p\mathbf{Z})^*} \{\frac{g_{0,u}}{h}, \frac{g_{0,ux}}{h}\} \\ &= \sum_{u\in(\mathbf{Z}/p\mathbf{Z})^*} \{g_{0,u}, g_{0,ux}\}\end{aligned}$$

which is clearly invariant under t. □

Remark 4.7. The element $\gamma_0(x) \in K_2(X_0(p))\otimes\mathbf{Q}$ is defined only implicitly. By this I mean that the actual definition uses Milnor symbols with functions on $X_1(p)$, and not on $X_0(p)$, which only contains two cusps. It is posible to rewrite $\gamma_0(x)$ as follows

$$\begin{aligned}\gamma_0(x) &= \sum_{u\in(\mathbf{Z}/p\mathbf{Z})^*}\ \sum_{\lambda,\mu\in(\mathbf{Z}/p\mathbf{Z})^*} \overline{B_2}(\frac{\lambda u}{p})\overline{B_2}(\frac{\mu u x}{p})\{u_\lambda,u_\mu\} \\ &= \sum_{\nu\in(\mathbf{Z}/p\mathbf{Z})^*} \Big(\sum_{u\in(\mathbf{Z}/p\mathbf{Z})^*} \overline{B_2}(\frac{u}{p})\overline{B_2}(\frac{u\nu x}{p})\Big)\Big(\sum_{\lambda\in(\mathbf{Z}/p\mathbf{Z})^*} \{u_\lambda, u_{\lambda\nu}\}\Big).\end{aligned} \tag{34}$$

In (34), each sum over λ already lies in $K_2(X_0(p)) \otimes \mathbf{Q}$. Moreover, we recognize the sum over u to be a Dedekind sum.

For any $x \in \mathbf{P}^1(\mathbf{Z}/p\mathbf{Z})$, let $\xi(x) \in H_1(X_0(p)(\mathbf{C}), \text{cusps}, \mathbf{Z})$ be the modular symbol $\{g_x 0, g_x \infty\}$ where $g_x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ is any matrix satisfying $x = c/d \mod p$. Let $\xi^{\pm}(x) = \frac{1}{2}\big(\xi(x) + \xi(-x)\big)$. For any cusp form $f \in S_2(\Gamma_1(p))$, we define $\xi_f(x) = \int_{\xi(x)} \omega_f$ and $\xi_f^{\pm}(x) = \int_{\xi^{\pm}(x)} \omega_f$ where $\omega_f = 2i\pi f(z)dz$.

Theorem 4.8. (1) *For any newform $f \in S_2(\Gamma_0(p))$, we have*

$$\langle r_p(\gamma_0(x)), f \rangle = \frac{8(p-1)}{p\pi} L(f,2) \xi_f^{+}(x) \qquad (x \in (\mathbf{Z}/p\mathbf{Z})^*). \tag{35}$$

(2) *For any $x \in \mathbf{P}^1(\mathbf{Z}/p\mathbf{Z})$, the following relations hold*

$$\begin{aligned} \gamma_0(x) &= \gamma_0(-x) \\ \gamma_0(x) + \gamma_0(-1/x) &= 0 \\ \gamma_0(x) + \gamma_0(-\frac{1}{x-1}) + \gamma_0(1-\frac{1}{x}) &= 0. \end{aligned} \tag{36}$$

(3) *The equations (36) make up a complete set of relations for the elements $\gamma_0(x)$.*

Proof. The point (1) will be a consequence of the explicit computation of Beilinson's regulator for the modular curve $X_1(p)$ [5, Thm 1.1]. Let X be the set of even non-trivial characters of $(\mathbf{Z}/p\mathbf{Z})^*$. For any $\chi \in X$, we define a modular unit $u_\chi \in \mathcal{O}^*(Y_1(p)) \otimes \mathbf{C}$ by

$$u_\chi = \prod_{\lambda \in (\mathbf{Z}/p\mathbf{Z})^*} u_\lambda \otimes \big(-\frac{L(\chi,2)\overline{\chi}(\lambda)}{2\pi^2}\big). \tag{37}$$

Now let us compute the following element in $K_2(X_1(p)) \otimes \mathbf{C}$

$$\gamma_x = \sum_{\chi \in X} \chi(x) \{u_\chi, u_{\overline{\chi}}\} \qquad (x \in (\mathbf{Z}/p\mathbf{Z})^*).$$

Using (37) gives

$$\gamma_x = \frac{1}{4\pi^4} \sum_{\lambda,\mu \in (\mathbf{Z}/p\mathbf{Z})^*} \Big(\sum_{\chi \in X} \chi(\frac{x\mu}{\lambda}) L(\chi,2) L(\overline{\chi},2)\Big) \{u_\lambda, u_\mu\}. \tag{38}$$

The inner sum can be computed using the formula (32), which gives

$$(39) \qquad \frac{\pi^4(p-1)}{2p} \sum_{\substack{\alpha,\beta\in(\mathbf{Z}/p\mathbf{Z})^* \\ \alpha x\mu=\pm\beta\lambda}} \overline{B_2}(\frac{\alpha}{p})\overline{B_2}(\frac{\beta}{p}) - \frac{\pi^4}{p} \sum_{\alpha,\beta\in(\mathbf{Z}/p\mathbf{Z})^*} \overline{B_2}(\frac{\alpha}{p})\overline{B_2}(\frac{\beta}{p}).$$

The second term of (39) contributes to zero in (38) by antisymmetry of the Milnor symbol. Finally, we get

$$\gamma_x = \frac{p-1}{4p} \sum_{\substack{\alpha,\beta,\lambda,\mu\in(\mathbf{Z}/p\mathbf{Z})^* \\ \alpha x\mu=\beta\lambda}} \overline{B_2}(\frac{\alpha}{p})\overline{B_2}(\frac{\beta}{p})\{u_\lambda, u_\mu\} = \frac{p-1}{4p}\gamma_0(x).$$

In order to use [5, Thm 1.1], we have to take care of the Atkin-Lehner involution W_p. Let $w(f)$ be the W_p-eigenvalue of f. We let temporarily $\widetilde{u}_\chi$ (resp. $\widetilde{u}_\lambda$) be the modular unit defined in [5, (5)] (resp. in [5, (95)]). We have $u_\lambda|W_p = \widetilde{u}_{\lambda^{-1}}$ and for any $\chi \in X$

$$\begin{aligned}
\{u_\chi, u_{\overline{\chi}}\}|W_p &= \frac{L(\chi,2)L(\overline{\chi},2)}{4\pi^4} \sum_{\lambda,\mu\in(\mathbf{Z}/p\mathbf{Z})^*} \overline{\chi}(\lambda)\chi(\mu)\{u_\lambda, u_\mu\}|W_p \\
&= \frac{L(\chi,2)L(\overline{\chi},2)}{4\pi^4} \sum_{\lambda,\mu\in(\mathbf{Z}/p\mathbf{Z})^*} \chi(\lambda/\mu)\{\widetilde{u}_\lambda, \widetilde{u}_\mu\} \\
&= \{\widetilde{u}_{\overline{\chi}}, \widetilde{u}_\chi\}
\end{aligned}$$

because of [5, Prop 5.4]. Let $f \in S_2(\Gamma_0(p))$ be a newform and $w(f)$ be the W_p-eigenvalue of f. Using [5, Thm 1.1], we have

$$\begin{aligned}
\langle r_p(\{u_\chi, u_{\overline{\chi}}\}), f\rangle &= \langle r_p(\{u_\chi, u_{\overline{\chi}}\}|W_p), W_p f\rangle \\
&= w(f)\langle r_p(\{\widetilde{u}_\chi, \widetilde{u}_{\overline{\chi}}\}), f\rangle \\
&= \frac{2(p-1)w(f)}{p\pi\tau(\chi)} L(f,2)L(f,\chi,1).
\end{aligned}$$

A classical computation [14] yields

$$L(f,\chi,1) = -\frac{w(f)\tau(\chi)}{p} \sum_{a\in(\mathbf{Z}/p\mathbf{Z})^*} \chi(a)\xi_f^+(a) \qquad (\chi \in X).$$

By taking the sum over characters χ, we obtain

$$\langle r_p(\gamma_x), f\rangle = \frac{2(p-1)^2}{p^2\pi} L(f,2)\xi_f^+(x).$$

This proves (35).

The relations (36) are an easy consequence of Theorem 4.2 and the definition (33) of $\gamma_0(x)$. Note that they are consistent with the regulator formula (35).

Finally, for the point (3), let $\widetilde{\gamma_0}$ be the map

$$\begin{aligned} \widetilde{\gamma_0} : \mathbf{Q}[(\mathbf{Z}/p\mathbf{Z})^*] &\to K_2(X_0(p)) \otimes \mathbf{Q} \\ [x] &\mapsto \gamma_0(x). \end{aligned}$$

Let R be the kernel of $\widetilde{\gamma_0}$. We wish to show that R is generated by the relations (36). For this we use the theory of Manin symbols. For any $x \in (\mathbf{Z}/p\mathbf{Z})^*$, the cycle $\xi(x)$ has trivial boundary. Thus we have a map

$$\xi^+ : \mathbf{Q}[(\mathbf{Z}/p\mathbf{Z})^*] \to H_1^+(X_0(p)(\mathbf{C}), \mathbf{Q}).$$

Manin's theorem implies that the kernel of ξ^+ is generated by the relations (36), so that $\ker \xi^+ \subset R$. In order to prove the reverse inclusion, it suffices to consider the dimensions. Let $g(X_0(p))$ be the genus of $X_0(p)$. From (35) we know that the image of $\widetilde{\gamma_0}$ has dimension at least $g(X_0(p))$. Manin's theorem implies that the dimension of the image of ξ^+ is precisely $g(X_0(p))$ (the element $\xi(0) = \{0, \infty\} = -\xi(\infty)$ has non-trivial boundary). We conclude that $\dim R \leq \dim \ker \xi^+$, so that R is generated by (36). □

References

[1] S. J. Bloch. *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves*, volume 11 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2000.

[2] L. A. Borisov and P. E. Gunnells. Toric modular forms and nonvanishing of L-functions. *J. Reine Angew. Math.*, 539:149–165, 2001.

[3] L. A. Borisov and P. E. Gunnells. Toric modular forms of higher weight. *J. Reine Angew. Math.*, 560:43–64, 2003.

[4] F. Brunault. *Étude de la valeur en $s = 2$ de la fonction L d'une courbe elliptique.* PhD thesis, Université Paris 7, décembre 2005. arXiv:math/0602186v1.

[5] F. Brunault. Valeur en 2 de fonctions L de formes modulaires de poids 2 : théorème de Beilinson explicite. *Bull. Soc. Math. France*, 135(2):215–246, 2007.

[6] P. Colmez. La conjecture de Birch et Swinnerton-Dyer p-adique. *Astérisque*, 294:251–319, 2004.

[7] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 143–316. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973.

[8] T. Dokchitser, R. de Jeu, and D. Zagier. Numerical verification of Beilinson's conjecture for K_2 of hyperelliptic curves. *Comp. Math.*, 142(2):339–373, 2006.

[9] A. B. Goncharov. Euler complexes and geometry of modular varieties. *Geom. Funct. Anal.*, 17(6):1872–1914, 2008. Special volume dedicated to J. Bernstein.

[10] K. Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, 295:117–290, 2004. Cohomologies p-adiques et applications arithmétiques (III).

[11] N. M. Katz and B. Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[12] D. S. Kubert and S. Lang. *Modular units*, volume 244 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York, 1981.

[13] S. Lang. *Elliptic functions*, volume 112 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1987. With an appendix by J. Tate.

[14] Y. I. Manin. Parabolic points and zeta functions of modular curves. *Math. USSR Izvestija*, 6(1):19–64, 1972.

[15] L. Merel. Universal Fourier expansions of modular forms. In *On Artin's conjecture for odd 2-dimensional representations*, volume 1585 of *Lecture Notes in Math.*, pages 59–94. Springer, Berlin, 1994.

[16] J. Milnor. *Introduction to algebraic K-theory*. Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.

[17] V. Paşol. A Modular Symbol with Values in Cusp Forms. arXiv:math/0611704v1, November 2006.

[18] N. Schappacher and A. J. Scholl. Beilinson's theorem on modular curves. In *Beilinson's conjectures on special values of L-functions*, volume 4 of *Perspect. Math.*, pages 273–304. Academic Press, Boston, MA, 1988.

[19] G. Shimura. Correspondances modulaires et les fonctions ζ de courbes algébriques. *J. Math. Soc. Japan*, 10:1–28, 1958.

[20] G. Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kano Memorial Lectures, 1.

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