# Mahler measures and multiple Eisenstein values

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## Definition (Mahler, 1962)

For $P \in \mathbf{C}[x_1,\ldots,x_n] \smallsetminus \{0\}$, define

$$m(P) = \frac{1}{(2\pi i)^n} \int_{T^n} \log|P(x_1,\ldots,x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

where $T^n$: $|x_1| = \ldots = |x_n| = 1$ is the $n$-torus.

- The integral converges absolutely.
- If $P$ has coefficients in $\overline{\mathbf{Q}}$ then $m(P)$ should be a period in the sense of Kontsevich–Zagier.
- $m(P)$ measures the "size" of a polynomial in $\mathbf{Z}[x_1,\ldots,x_n]$.
- Lehmer's problem (1933): For $P \in \mathbf{Z}[x]$ monic irreducible, not cyclotomic, can $m(P) > 0$ be arbitrarily small?

## Theorem (Jensen, 1899)

For $P \in \mathbf{C}[x] \smallsetminus \{0\}$, $P = a_d \prod_{i=1}^{d} (x - \alpha_i)$, we have

$$m(P) = \log|a_d| + \sum_{\substack{k=1 \\ |\alpha_k| \geq 1}}^{d} \log|\alpha_k|.$$

- Jensen's formula is still useful for multivariate polynomials: it reduces an $n$-dimension integral to an $(n-1)$-dimensional one.
- Example: using Jensen's formula with respect to $y$, we have

$$m(1+x+y) = \frac{1}{2\pi i} \int_{\substack{|x|=1 \\ |1+x| \geq 1}} \log|1+x| \frac{dx}{x} = \frac{1}{2\pi} \int_{-2\pi/3}^{2\pi/3} \log|1+e^{i\theta}| d\theta$$

- How to evaluate further?

## Timeline of identities

**Smyth (1981)**: $m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_3, 2)$

Here $L(\chi_3, s) = \sum_{n=1}^{\infty} \chi_3(n)/n^s$ is the Dirichlet $L$-function for

$$\chi_3(n) = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 3 \\ -1 & \text{if } n \equiv 2 \bmod 3 \\ 0 & \text{if } n \equiv 0 \bmod 3 \end{cases}$$

The proof uses the series expansion

$$\log\left|1+e^{i\theta}\right| = -\mathrm{Re} \sum_{n=1}^{\infty} \frac{(-1)^n e^{in\theta}}{n}.$$

and then integration from $\theta = -2\pi/3$ to $2\pi/3$.

## Timeline of identities

**Smyth (1981)**: $m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$

**Boyd and Deninger (1997)**:

$$m\Big(x + \frac{1}{x} + y + \frac{1}{y} + 1\Big) \stackrel{?}{=} \frac{15}{4\pi^2} L(E,2) = L'(E,0)$$

where $L(E,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is the $L$-function of the elliptic curve

$$E : x + \frac{1}{x} + y + \frac{1}{y} + 1 = 0.$$

- Discovered using numerical experiments + theoretical insights.
- Proved by Rogers and Zudilin (2011).

**Boyd (1998)**: Families of conjectural identities, such as

$$m\Big(x + \frac{1}{x} + y + \frac{1}{y} + k\Big) \stackrel{?}{=} c_k L'(E_k, 0) \qquad (k \in \mathbf{Z},\ k \neq 0, \pm 4)$$

for some rational number $c_k \in \mathbf{Q}^\times$.

- Generalises to other families $m(P(x,y)+k)$ where the Newton polygon of $P(x,y)$ has $(0,0)$ as the only interior point.
- Only finitely many such identities are proved.
- Related to the algebraic $K$-group $K_2(E_k)$ and the Bloch-Beilinson regulator map $K_2(E_k) \to \mathbf{R}$.

**Conjecture (Boyd and Rodriguez Villegas, 2003)**:

$$m((1+x)(1+y)+z) \stackrel{?}{=} \frac{15^2}{4\pi^4} L(E,3) = -2L'(E,-1)$$

where $E$ is an elliptic curve of conductor 15.

- There are several other $L(E,3)$ identities, but they do not seem to come in families.
- Why does an elliptic curve appear here?
- Because

$$E : \begin{cases} (1+x)(1+y)+z=0 \\ (1+\frac{1}{x})(1+\frac{1}{y})+\frac{1}{z}=0. \end{cases}$$

- Note that $\{(1+x)(1+y)+z=0\} \cap T^3 \subset E$.

Regulator maps

closer — Beilinson's conjectures

Mahler measures — — — — — — — — — — — — — $L$-functions

In this talk, we will consider $L$-functions of *modular forms*.
If $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$ is a modular form on a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$, its $L$-function is defined by

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- Integral representation: $(2\pi)^{-s}\Gamma(s)L(f,s) = \int_0^{\infty} (f(iy) - a_0) y^s \frac{dy}{y}$.
- Meromorphic continuation to $\mathbf{C}$ and functional equation.

**Theorem (B. 2023)**
*We have* $m((1+x)(1+y)+z) = -2L'(E,-1)$.

- Now related to the $K$-group $K_4(E)$.
- Uses joint work with Zudilin on $K_4$ regulators.
- Key tool: Multiple modular values

$$\int_0^\infty f_1(iy_1)y_1^{s_1-1}dy_1 \int_{y_1}^\infty f_2(iy_2)y_2^{s_2-1} \ldots \int_{y_{n-1}}^\infty f_n(iy_n)y_n^{s_n-1}dy_n$$

where $f_1, \ldots, f_n$ are modular forms.

Let $P = (1+x)(1+y)+z$.

**Step 1: Deninger's method**

Use Jensen's formula with respect to $z$.

$$\rightsquigarrow \quad m(P) = \frac{1}{(2\pi i)^2} \int_\Gamma \eta(x,y,z)$$

where:

- $\eta$ is an explicit closed 2-form on $V_P = \{P(x,y,z) = 0\}$.
- $\Gamma = \{(x,y,z) \in V_P : |x| = |y| = 1, |z| \geq 1\}$.

## Step 2: Stokes's theorem

In our case, the form $\eta$ happens to be *exact*. Writing $\eta = d\rho$, we have by Stokes's theorem

$$m(P) = \frac{1}{(2\pi i)^2} \int_\Gamma d\rho = \frac{1}{(2\pi i)^2} \int_\gamma \rho$$

with

$$\gamma = \partial\Gamma = \{(x,y,z) \in V_P : |x| = |y| = |z| = 1\}.$$

- $\gamma = V_P \cap T^3$ is contained in $E$.
- $\rho$ is a *closed* 1-form on $E$.
- So we now have a 1-dimensional integral on $E$.

For any two functions $f, g$ on $E$, define

$$\rho(f,g) = -D(f)\mathrm{darg}(g) + \frac{1}{3}\log|g|\Big(\log|1-f|\mathrm{dlog}|f| - \log|f|\mathrm{dlog}|1-f|\Big)$$

where $D\colon \mathbf{P}^1(\mathbf{C}) \to \mathbf{R}$ is the Bloch-Wigner dilogarithm

$$D(z) = \mathrm{Im}\Big(\sum_{n=1}^{\infty} \frac{z^n}{n^2}\Big) + \log|z| \arg(1-z).$$

## Theorem (Lalín, 2015)

- $\rho = \rho(-y,x) - \rho(-x,y)$.
- $\gamma$ *is a generator of* $H_1(E(\mathbf{C}),\mathbf{Z})^+$.

## Step 3: Translate in the modular world

The elliptic curve $E$ is isomorphic to the modular curve $X_1(15)$.

$$X_1(N) = \Gamma_1(N)\backslash\mathcal{H} \cup \{\mathrm{cusps}\}$$

where

$$\Gamma_1(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) : a, d \equiv 1 \bmod N,\ c \equiv 0 \bmod N\right\}.$$

Nice feature: the functions $x$ and $y$ on $E$ correspond to *modular units* on $X_1(15)$, that is, all their zeros and poles are at the cusps.

Key fact: if $u$ is a modular unit, then $\mathrm{dlog}(u) = E_2(z)dz$ where $E_2$ is an Eisenstein series of weight 2.

We want to understand

$$\rho(u,v) = -D(u)\,\mathrm{darg}(v) + \frac{1}{3}\log|v|\Big(\log|1-u|\,\mathrm{dlog}|u| - \log|u|\,\mathrm{dlog}|1-u|\Big).$$

when $u$ and $v$ are modular units on $X_1(N)$.

- $\mathrm{dlog}(u)$ and $\mathrm{dlog}(v)$ are Eisenstein series, so the log terms of the formula are well-understood.
- The challenging piece is $D(u)$. We use

$$d(D(u)) = \log|u|\,\mathrm{darg}(1-u) - \log|1-u|\,\mathrm{darg}(u)$$

- If $u$ and $1-u$ are modular units, then $D(u)$ is an *iterated integral* of Eisenstein series.

## Definition

For $k \geq 1$ and $\boldsymbol{x} = (x_1, x_2) \in (\mathbf{Z}/N\mathbf{Z})^2$, define the Eisenstein series

$$E_{\boldsymbol{x}}^{(k)}(\tau) = \sum_{m,n \in \mathbf{Z}} \frac{\exp\left(\frac{2\pi i}{N}(mx_2 - nx_1)\right)}{(m\tau + n)^k} \in M_k(\Gamma(N))$$

For $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in (\mathbf{Z}/N\mathbf{Z})^2$, define the *multiple Eisenstein values* (Manin, Brown)

$$\Lambda(\boldsymbol{x}, \boldsymbol{y}) := \int_0^{i\infty} E_{\boldsymbol{x}}^{(2)}(\tau_1) d\tau_1 \int_{\tau_1}^{i\infty} E_{\boldsymbol{y}}^{(2)}(\tau_2) d\tau_2$$

$$\Lambda(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) := \int_0^{i\infty} E_{\boldsymbol{x}}^{(2)}(\tau_1) d\tau_1 \int_{\tau_1}^{i\infty} E_{\boldsymbol{y}}^{(2)}(\tau_2) d\tau_2 \int_{\tau_2}^{i\infty} E_{\boldsymbol{z}}^{(2)}(\tau_3) d\tau_3.$$

$\rightsquigarrow$ The Mahler measure of $P$ can be written as an explicit linear combination of multiple Eisenstein values.

## Theorem (B.–Zudilin, 2023)

Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in (\mathbf{Z}/N\mathbf{Z})^2$ such that $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} = 0$. If all the coordinates of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are non-zero, then

$$\begin{aligned}\mathrm{Re}\Big(&\Lambda(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}) - \Lambda(\boldsymbol{z},\boldsymbol{y},\boldsymbol{y}) + \Lambda(\boldsymbol{y},\boldsymbol{x},\boldsymbol{x}) - \Lambda(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}) + \Lambda(\boldsymbol{z},\boldsymbol{y},\boldsymbol{x}) + \Lambda(\boldsymbol{z},\boldsymbol{x},\boldsymbol{y}) \\ &- (\Lambda(\boldsymbol{y}) - \Lambda(\boldsymbol{x}))(\Lambda(\boldsymbol{x},\boldsymbol{y}) + \Lambda(\boldsymbol{y},\boldsymbol{z}) + \Lambda(\boldsymbol{z},\boldsymbol{x}))\Big) = L'(F_{\boldsymbol{x},\boldsymbol{y}}, -1) + c_{\boldsymbol{x},\boldsymbol{y}}\zeta(3)\end{aligned}$$

for some explicit $F_{\boldsymbol{x},\boldsymbol{y}} \in M_2(\Gamma(N))$, and $c_{\boldsymbol{x},\boldsymbol{y}} \in \mathbf{Q}$.

Proving this formula requires two ingredients:

- Interpolate the multiple Eisenstein values to continuous parameters, viewing $(\mathbf{Z}/N\mathbf{Z})^2$ inside $(\mathbf{R}/\mathbf{Z})^2$ using $(x_1, x_2) \mapsto \left(\frac{x_1}{N}, \frac{x_2}{N}\right)$.
- Differentiate with respect to these parameters to reduce the length of the iterated integrals.

## Key lemma

For $\boldsymbol{x} = (x_1, x_2) \in (\mathbf{R}/\mathbf{Z})^2$, we have

$$\frac{d}{dx_2} E_{\boldsymbol{x}}^{(2)}(\tau) = \frac{d}{d\tau} E_{\boldsymbol{x}}^{(1)}(\tau).$$

So for example

$$\begin{aligned}
\frac{d}{dy_2}\Lambda(\boldsymbol{x}, \boldsymbol{y}) &= \int_0^{i\infty} E_{\boldsymbol{x}}^{(2)}(\tau_1) d\tau_1 \int_{\tau_1}^{i\infty} \frac{d}{dy_2} E_{\boldsymbol{y}}^{(2)}(\tau_2) d\tau_2 \\
&= \int_0^{i\infty} E_{\boldsymbol{x}}^{(2)}(\tau_1) d\tau_1 \int_{\tau_1}^{i\infty} \frac{d}{d\tau_2} E_{\boldsymbol{y}}^{(1)}(\tau_2) d\tau_2 \\
&= \int_0^{i\infty} E_{\boldsymbol{x}}^{(2)}(\tau_1) \Big( E_{\boldsymbol{y}}^{(1)}(i\infty) - E_{\boldsymbol{y}}^{(1)}(\tau_1) \Big) d\tau_1.
\end{aligned}$$

This reduces a double integral to a single integral.

To prove the formula

$$\mathrm{Re}\Big(\Lambda(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y})-\Lambda(\boldsymbol{z},\boldsymbol{y},\boldsymbol{y})+\Lambda(\boldsymbol{y},\boldsymbol{x},\boldsymbol{x})-\Lambda(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x})+\Lambda(\boldsymbol{z},\boldsymbol{y},\boldsymbol{x})+\Lambda(\boldsymbol{z},\boldsymbol{x},\boldsymbol{y}) \\ -(\Lambda(\boldsymbol{y})-\Lambda(\boldsymbol{x}))(\Lambda(\boldsymbol{x},\boldsymbol{y})+\Lambda(\boldsymbol{y},\boldsymbol{z})+\Lambda(\boldsymbol{z},\boldsymbol{x}))\Big)=L'(F_{\boldsymbol{x},\boldsymbol{y}},-1)+c_{\boldsymbol{x},\boldsymbol{y}}\zeta(3)$$

we differentiate the LHS with respect to $x_2$.
We get a sum of double integrals of the form

$$\int_0^{i\infty} E_{\boldsymbol{a}}^{(2)}(\tau_1)d\tau_1 \int_{\tau_1}^{i\infty} E_{\boldsymbol{b}}^{(2)}(\tau_2)E_{\boldsymbol{c}}^{(1)}(\tau_2)d\tau_2.$$

Miracle: The (complicated) linear combination of products $E_{\boldsymbol{b}}^{(2)}E_{\boldsymbol{c}}^{(1)}$ is actually an Eisenstein series of weight 3!
This means that we have a *double Eisenstein value*.

The double Eisenstein values can be computed using the *Rogers-Zudilin method*. We get

$$\frac{d}{dx_2}(\mathrm{LHS}) = \text{sum of } L\text{-values } L'(G_{\boldsymbol{a}}^{(1)} G_{\boldsymbol{b}}^{(2)}, 0)$$

for some (other) Eisenstein series $G^{(1)}$ and $G^{(2)}$.

This can be integrated to

$$\mathrm{LHS} = \text{sum of } L\text{-values } L'(G_{\boldsymbol{a}}^{(1)} G_{\boldsymbol{b}}^{(1)}, -1)$$

We arrive at our $L$-value $L'(F_{\boldsymbol{x},\boldsymbol{y}}, -1)$.

### Remark

We have no good understanding of the $\zeta(3)$ term in the formula.

The proof of the theorem also builds on:

- The Siegel modular units $g_{\boldsymbol{x}}$ for $\boldsymbol{x} \in (\mathbf{Z}/N\mathbf{Z})^2$ on the modular curve $Y(N) = \Gamma(N)\backslash\mathcal{H}$
- Milnor symbols $\{g_{\boldsymbol{x}}, g_{\boldsymbol{y}}\}$ in $K_2(Y(N)) \otimes \mathbf{Q}$
- Three-term relations: if $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} = 0$ then

$$\{g_{\boldsymbol{x}}, g_{\boldsymbol{y}}\} + \{g_{\boldsymbol{y}}, g_{\boldsymbol{z}}\} + \{g_{\boldsymbol{z}}, g_{\boldsymbol{x}}\} = 0.$$

- We can actually find a "triangulation"

$$g_{\boldsymbol{x}} \wedge g_{\boldsymbol{y}} + g_{\boldsymbol{y}} \wedge g_{\boldsymbol{z}} + g_{\boldsymbol{z}} \wedge g_{\boldsymbol{x}} = \sum_i m_i \cdot u_i \wedge (1 - u_i)$$

  where $u_i$ and $1 - u_i$ are modular units, and $m_i \in \mathbf{Q}$.
- This triangulation leads to an element of $K_4(Y(N)) \otimes \mathbf{Q}$.

This should extend in higher weight: for $k \geq 0$ and $\boldsymbol{x} \in (\mathbf{Z}/N\mathbf{Z})^2$, there is the *Eisenstein symbol*

$$\mathrm{Eis}^k(\boldsymbol{x}) \in K_{k+1}(E(N)^k) \otimes \mathbf{Q}$$

where $E(N)^k$ is the $k$-fold fibre product of the universal elliptic curve $E(N)$ over the modular curve $Y(N)$.

## Definition

For $k, \ell \geq 0$ and $\boldsymbol{x}, \boldsymbol{y} \in (\mathbf{Z}/N\mathbf{Z})^2$, define

$$X^k Y^\ell(\boldsymbol{x}, \boldsymbol{y}) = p_1^* \mathrm{Eis}^k(\boldsymbol{x}) \cup p_2^* \mathrm{Eis}^\ell(\boldsymbol{y}) \in K_{k+\ell+2}(E(N)^{k+\ell}) \otimes \mathbf{Q},$$

where $p_1 : E^{k+\ell} \to E^k$ and $p_2 : E^{k+\ell} \to E^\ell$ are the projections.

## Conjecture

Let $k, \ell \geq 0$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in (\mathbf{Z}/N\mathbf{Z})^2$ with $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} = 0$. Then

$$X^k Y^\ell (\boldsymbol{x}, \boldsymbol{y}) + X^\ell (-X - Y)^k (\boldsymbol{y}, \boldsymbol{z}) + Y^k (-X - Y)^\ell (\boldsymbol{z}, \boldsymbol{x}) = 0.$$

- One should be able to prove this in Deligne cohomology.
- Induction on the weight, using differentiation with respect to the parameters of the Eisenstein symbols.
- Open question: what is the triangulation?
- In this range, Deligne cohomology is just de Rham cohomology, so this amounts to say that a particular differential form is exact. Can we make explicit a primitive?

## Beyond the reach of current technology

### Conjecture (Rodriguez Villegas, 2003)

$$m(1+x_1+x_2+x_3+x_4) = -L'(f,-1)$$
$$m(1+x_1+x_2+x_3+x_4+x_5) = -8L'(g,-1)$$

*for modular forms $f \in S_3(\Gamma_1(15))$ and $g \in S_4(\Gamma_0(6))$.*

### Conjecture (B.–Pengo, 2023)

$$m(xyt+xzt+yzt+xy+xz-yz-yt+zt-y+z-t+1) = \frac{1}{6}L'(E,-2)$$

*where $E = 32a2$ is an elliptic curve of conductor 32.*

## How we found the polynomial

Take $P(x, y, z, t)$ of the form

$$P = a(x, y) + b(x, y)z + c(c, y)t + d(x, y)zt.$$

Eliminating $t$ in $P(x, y, z, t) = P(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{t}) = 0$ gives

$$W_P : A(x, y)z^2 + B(x, y)z + C(x, y) = 0.$$

Want: $\Delta = B^2 - 4AC$ is a square $\delta(x, y)^2$ in $\mathbf{Q}(x, y)$.
Then $W_P = W_1 \cup W_2$ with

$$W_1 \cap W_2 : \delta(x, y) = 0.$$

We look for $a, b, c, d$ such that $W_1 \cap W_2$ is an elliptic curve.

## Numerical computation of $m(P)$

**Rodriguez Villegas**: $2m(P) = \log k - \int_0^{1/k} \phi_P(u)du$ where $k$ is the constant coefficient of $P(x,y,z,t)P(\frac{1}{x},\frac{1}{y},\frac{1}{z},\frac{1}{t})$ and

$$\phi_P(u) = \frac{1}{(2\pi i)^n} \int_{T^n} \frac{Q}{1-uQ} \cdot \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dt}{t}$$

with $Q = P(x,y,z,t)P(\frac{1}{x},\frac{1}{y},\frac{1}{z},\frac{1}{t}) - k$.

**Pengo–Ringeling:** Using creative telescoping, one can find a polynomial ODE satisfied by $\phi_P$. This takes a long time, but then $m(P)$ can be computed quickly with high precision.