ON THE MAHLER MEASURE ASSOCIATED TO $X_1(13)$

by

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Abstract. — We show that the Mahler measure of a defining equation of the modular curve $X_1(13)$ is equal to the derivative at $s=0$ of the L-function of a cusp form of weight 2 and level 13 with integral Fourier coefficients. This generalizes a result on the Mahler measure of $X_1(11)$. The proof combines Deninger's method, an explicit version of Beilinson's theorem together with an idea of Merel to express the regulator integral as a linear combination of periods. Finally, we present further examples related to the modular curves of level 16, 18 and 25.

The Mahler measure of a polynomial $P \in \mathbf{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is defined by

$$m(P) = \frac{1}{(2\pi i)^n} \int_{|z_1|=1} \cdots \int_{|z_n|=1} \log |P(z_1, \dots, z_n)| \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}.$$

In a fascinating paper, Boyd [**5**] developed a body of conjectures relating Mahler measures of 2-variable polynomials and special values of L-functions of elliptic curves. Deninger [**11**] provided a bridge between the world of Mahler measures and certain K-theoretic regulators, and showed the relevance of Beilinson's conjectures to prove relations between Mahler measures and special values of L-functions. In the case of curves, such identities have been proven rigorously only in rare instances, mainly in the case of genus 0 and 1 ([**7**], [**20**], [**21**]...). There has been some recent work, however, in the case of genus 2 (see [**2**] and the references therein).

The aim of this paper is to achieve such a relation in the case of a curve of genus 2. We work with the modular curve $X_1(13)$. Thanks to [**14**, p. 56], a defining equation of $X_1(13)$ is

$$P = y^2 x(x-1) + y(-x^3 + x^2 + 2x - 1) - x^2 + x.$$

We prove the following theorem.

Theorem 1. — *We have the identity $m(P) = 2L'(f,0)$, where f is the cusp form of weight 2 and level 13 whose Fourier expansion begins with*

$$f = 2q - 3q^2 - 2q^3 + q^4 + 6q^6 - q^9 - 3q^{10} - 4q^{12} - 5q^{13} + O(q^{15}).$$

Note that the cusp form f is not a newform; rather, it is the trace of the unique (up to Galois conjugacy) newform of weight 2 on the group $\Gamma_1(13)$.

Theorem 1 generalizes a result on the Mahler measure associated to the modular curve $X_1(11)$, which has genus 1 [**7**]. As in this previous work, the proof builds on Deninger's method [**11**] and an explicit version of Beilinson's theorem for the modular curve $X_1(N)$ [**8**]. As a difference with [**7**], we also prove a variant of a theorem of Merel in order to express the Petersson product arising in Beilinson's regulator in terms of 1-dimensional periods.

Another approach to the main theorem would be to use the Rogers-Zudilin method [**25**], but we have not carried out the details of this computation.

In the last section, we present further examples of relations between Mahler measures and L-values in the case of the modular curves $X_1(16)$, $X_1(18)$ and $X_1(25)$ (whose respective genera are 2, 2 and 12). We obtain various numerical identities relating Mahler measures and L-values of cusp forms or Dirichlet characters. As we explain in the beginning of Section 6, it would be interesting to generalize this phenomenon for modular units of arbitrary level. It would be also interesting to understand when the identities obtained involve cusp forms, are of Dirichlet type, or are of mixed type.

This article grew out of results in my PhD thesis (see especially [**6**, §3.8 and Remarque 112]). I would like to thank Odile Lecacheux for helpful exchanges having led to the discovery of these identities. I would also like to thank Wadim Zudilin for useful comments.

1. Deninger's method

In this section we express the Mahler measure of P in terms of the integral of a differential 1-form on the modular curve $X_1(13)$, following Deninger's method [**11**].

We view P as a polynomial in h:

$$P(H,h) = -H + (-H^2 + 2H + 1)h + (H^2 + H - 1)h^2 - Hh^3.$$

Note that the constant term of P is given by $P^*(H) = -H$.

Let $Z \subset \mathbf{G}_m^2$ be the curve defined by the equation $P = 0$. Then Z identifies with an affine open subscheme of $X_1(13)$ by [**14**, p. 56]. In particular Z is smooth.

Looking at the resultant of the polynomials $P(H,h)$ and $H^2h^3P(\frac{1}{H},\frac{1}{h})$ with respect to h, it can be checked that P doesn't vanish on the torus $T^2 = \{(H,h) \in \mathbf{C} : |H| = |h| = 1\}$. Moreover, we check numerically that for each $H \in T$, there exists a unique $h(H) \in \mathbf{C}$ such that $P(H,h(H)) = 0$ and $0 < |h(H)| < 1$. The map $H \in T \mapsto h(H)$ defines a closed cycle γ_P in $H_1(Z(\mathbf{C}),\mathbf{Z})$. We call γ_P the *Deninger cycle* associated to P. We give γ_P the canonical orientation coming from T.

Since P^* doesn't vanish on T, the polynomial P satisfies the assumptions [**11**, 3.2], so that the discussion in *loc. cit.* applies. Consider the differential form $\eta = \log|h|\frac{dH}{H}$ on $Z(\mathbf{C})$. Using Jensen's formula, and noting that $m(P^*) = 0$, we have [**11**, (23)]

$$m(P) = -\frac{1}{2\pi i}\int_{\gamma_P} \eta.$$

Now we may express this as an integral of a closed differential form. By [**11**, Prop. 3.3], we get

$$m(P) = -\frac{1}{2\pi i}\int_{\gamma_P} \log|H|\cdot(\partial-\overline{\partial})\log|h| - \log|h|\cdot(\partial-\overline{\partial})\log|H|.$$

We now introduce a standard notation.

Definition 2. — For any two meromorphic functions u, v on a Riemann surface, define

$$\eta(u,v) := \log|u|\,\mathrm{darg}(v) - \log|v|\,\mathrm{darg}(u).$$

The 1-form $\eta(u,v)$ is well-defined outside the set of zeros and poles of u and v. It is closed, so we may integrate it over cycles. Moreover, we have $\mathrm{darg}(u) = -i(\partial-\overline{\partial})\log|u|$. Thus we have proved the following proposition.

Proposition 3. — *We have* $m(P) = \frac{1}{2\pi}\int_{\gamma_P} \eta(h,H)$.

Lemma 4. — *Let c denote complex conjugation on $Z(\mathbf{C})$. We have $c_*\gamma_P = -\gamma_P$.*

Proof. — For every $H \in T$, we have $h(\overline{H}) = \overline{h(H)}$. It follows that $c_*\gamma_P = -\gamma_P$. □

2. Determining Deninger's cycle

In this section, we determine γ_P explicitly in terms of modular symbols.

The space $S_2(\Gamma_1(13))$ of cusp forms of weight 2 and level 13 has dimension 2 over $\mathbf{C}$. Let $\varepsilon : (\mathbf{Z}/13\mathbf{Z})^\times \to \mathbf{C}^\times$ be the unique Dirichlet character satisfying $\varepsilon(2) = \zeta_6 := e^{\frac{2\pi i}{6}}$. It is even and has order 6. A basis of $S_2(\Gamma_1(13))$ is given by $(f_\varepsilon, f_{\overline{\varepsilon}})$, where f_ε (resp. $f_{\overline{\varepsilon}}$) is a newform having character ε (resp. $\overline{\varepsilon}$). The Fourier coefficients of f_ε and $f_{\overline{\varepsilon}}$ belong to the field $\mathbf{Q}(\zeta_6)$ and are complex conjugate to each other. We define $f = f_\varepsilon + f_{\overline{\varepsilon}}$.

We denote by $\langle d\rangle$ the diamond automorphism of $X_1(13)$ associated to $d \in (\mathbf{Z}/13\mathbf{Z})^\times/\pm 1$.

Let $\hat{\mathcal{H}} = H_1(X_1(13)(\mathbf{C}), \{\text{cusps}\}, \mathbf{Z})$ be the homology group of $X_1(13)(\mathbf{C})$ relative to the cusps. Let E_{13} be the set of non-zero vectors of $(\mathbf{Z}/13\mathbf{Z})^2$. For any $x \in E_{13}$, we let $\xi(x) = \{g_x 0, g_x\infty\}$, where $g_x \in \mathrm{SL}_2(\mathbf{Z})$ is any matrix whose bottom line is congruent to x modulo 13. Using Manin's algorithm [**17**] and its implementation in Magma [**3**], we find that a $\mathbf{Z}$-basis of $\mathcal{H} = H_1(X_1(13)(\mathbf{C}), \mathbf{Z})$ is given by

$$\begin{aligned}
\gamma_1 &= \xi(1,-5) - \xi(2,5) - \xi(1,-2) = \left\{\frac{1}{5}, \frac{2}{5}\right\}\\
\gamma_2 &= \langle 2\rangle_*\gamma_1 = \xi(2,3) - \xi(4,-3) - \xi(2,-4)\\
\gamma_3 &= \xi(1,-3) - \xi(1,3) = \left\{\frac{1}{3}, -\frac{1}{3}\right\}\\
\gamma_4 &= \langle 2\rangle_*\gamma_3 = \xi(2,-6) - \xi(2,6).
\end{aligned}$$

Consider the pairing

$$\begin{aligned}
\langle \cdot, \cdot\rangle : \hat{\mathcal{H}} \times S_2(\Gamma_1(13)) &\to \mathbf{C}\\
(\gamma, f) &\mapsto \int_\gamma 2\pi i f(z)dz.
\end{aligned}$$

Definition 5. — Let $\mathcal{H}^- := \{\gamma \in \mathcal{H} : c_*\gamma = -\gamma\}$. We define the map

$$\begin{aligned}
\iota : \mathcal{H}^- &\to \mathbf{C}\\
\gamma &\mapsto \langle \gamma, f_\varepsilon\rangle.
\end{aligned}$$

Lemma 6. — *The map ι is injective.*

Proof. — If $\iota(\gamma) = 0$ then $\langle \gamma, f_{\overline{\varepsilon}}\rangle = \overline{\langle c_*\gamma, f_\varepsilon\rangle} = -\overline{\langle \gamma, f_\varepsilon\rangle} = 0$. Thus γ is orthogonal to $S_2(\Gamma_1(13))$, which implies $\gamma = 0$. □

Lemma 7. — *The image of ι is the hexagonal lattice generated by $\iota(\gamma_3)$ and $\iota(\gamma_4) = \zeta_6\iota(\gamma_3)$.*

Proof. — The action of complex conjugation on $\mathcal{H}$ is given by

$$\begin{aligned}
c_*(\gamma_1) &= \gamma_1 + \gamma_4\\
c_*(\gamma_2) &= \gamma_2 - \gamma_3 + \gamma_4\\
c_*(\gamma_3) &= -\gamma_3\\
c_*(\gamma_4) &= -\gamma_4.
\end{aligned}$$

From these formulas, it is clear that a $\mathbf{Z}$-basis of $\mathcal{H}^-$ is given by (γ_3, γ_4). By Lemma 6, we have $\iota(\gamma_3) \neq 0$. Then

$$\iota(\gamma_4) = \langle \langle 2\rangle_*\gamma_3, f_\varepsilon\rangle = \langle \gamma_3, f_\varepsilon|\langle 2\rangle\rangle = \varepsilon(2)\iota(\gamma_3) = \zeta_6\iota(\gamma_3).$$

□

We have $\gamma_3 = \{\frac{1}{3}, -\frac{1}{3}\} = \{\frac{1}{3}, g_1(\frac{1}{3})\}$ with $g_1 = \begin{pmatrix} 14 & -5 \\ -39 & 14 \end{pmatrix} \in \Gamma_1(13)$. Let us choose $z_0 = \frac{14+i}{39}$. Then $g_1(z_0) = \frac{-14+i}{39}$. We have

$$\langle \gamma_3, f_\varepsilon \rangle = \int_{z_0}^{g_1 z_0} 2\pi i f_\varepsilon(z) dz = \sum_{n=1}^{\infty} \frac{a_n(f_\varepsilon)}{n} \left(e^{\frac{-28\pi i n}{39}} - e^{\frac{28\pi i n}{39}} \right) e^{-\frac{2\pi n}{39}}.$$

Using Magma, we get numerically

$$\langle \gamma_3, f_\varepsilon \rangle \sim 1.06759 - 2.60094 i.$$

Proposition 8. — *Let $\gamma_P \in \mathcal{H}^-$ be Deninger's cycle. We have $\gamma_P = \gamma_3$.*

Proof. — A $\mathbf{Q}$-basis of $\Omega^1(X_1(13))$ is given by $(\omega, h\omega)$ where

$$\omega = \frac{(h^2-h)H - h^3 + h^2 + 2h - 1}{h^4 - 2h^3 + 3h^2 - 2h + 1} dH.$$

Using Magma, we compute the Fourier expansion of ω and $h\omega$ at infinity, and deduce

$$2\pi i f_\varepsilon(z) dz = \alpha\omega + \beta h\omega \tag{1}$$

with

$$\alpha \sim 0.71163 + 0.70256 i \qquad \beta \sim 0.25262 - 0.96757 i.$$

Note that α and β are algebraic numbers, but we won't need an explicit formula for them. With Pari/GP [**22**], we find

$$\int_{\gamma_P} \omega \sim -3.21731 i \qquad \int_{\gamma_P} h\omega \sim -1.23275 i. \tag{2}$$

From (1) and (2), it follows that

$$\langle \gamma_P, f_\varepsilon \rangle \sim 1.06759 - 2.60094 i \sim \langle \gamma_3, f_\varepsilon \rangle.$$

Since the image of ι is a lattice by Lemma 7, we may ascertain that $\gamma_P = \gamma_3$. □

We will also need to make explicit the action of the Atkin-Lehner involution W_{13} on γ_P.

Proposition 9. — *We have $W_{13}\gamma_P = \gamma_4 - \gamma_3$.*

Proof. — By [**1**, Thm 2.1], we have $W_{13} f_\varepsilon = w \cdot f_{\overline{\varepsilon}}$ with

$$w = \frac{3\zeta_6 - 4}{13} \tau(\varepsilon) \sim -0.96425 + 0.26501 i. \tag{3}$$

We deduce

$$\iota(W_{13}\gamma_P) = \langle \gamma_P, W_{13} f_\varepsilon \rangle = w \langle \gamma_P, f_{\overline{\varepsilon}} \rangle = w \overline{\langle c_* \gamma_P, f_\varepsilon \rangle} = -w \overline{\langle \gamma_P, f_\varepsilon \rangle} \sim 1.71869 + 2.22503 i.$$

Moreover, we have

$$\iota(\gamma_4) = \zeta_6 \iota(\gamma_3) \sim 2.78628 - 0.37591 i \sim \iota(W_{13}\gamma_P) + \iota(\gamma_3).$$

Using Lemma 7 again, we conclude that $W_{13}\gamma_P = \gamma_4 - \gamma_3$. □

3. Beilinson's theorem

We now recall the explicit version of Beilinson's theorem on the modular curve $X_1(N)$ [**8**]. Let $\mathbf{C}(X_1(N))$ be the function field of $X_1(N)$. The *regulator map* on $X_1(N)$ is defined by

$$\begin{aligned} r_N : K_2(\mathbf{C}(X_1(N))) &\to \operatorname{Hom}_{\mathbf{C}}(S_2(\Gamma_1(N)), \mathbf{C}) \\ \{u,v\} &\mapsto \left(f \mapsto \int_{X_1(N)(\mathbf{C})} \eta(u,v) \wedge \omega_f \right) \end{aligned}$$

where $\omega_f \coloneqq 2\pi i f(z)dz$. After tensoring with $\mathbf{C}$, we get a linear map

$$r_N : K_2(\mathbf{C}(X_1(N))) \otimes \mathbf{C} \to \operatorname{Hom}_{\mathbf{C}}(S_2(\Gamma_1(N)), \mathbf{C}).$$

For any even non-trivial Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, there exists a modular unit $u_\chi \in \mathcal{O}^\times(Y_1(N)(\mathbf{C})) \otimes \mathbf{C}$ satisfying

$$\log|u_\chi(z)| = \frac{1}{\pi} \lim_{\substack{s\to 1 \\ \operatorname{Re}(s)>1}} \left(\sum_{(m,n)\in\mathbf{Z}^2}{}' \frac{\chi(n)\cdot \operatorname{Im}(z)^s}{|Nmz+n|^{2s}} \right) \qquad (z \in \mathfrak{H}),$$

where $\sum'$ denotes that we omit the term $(m,n) = (0,0)$ (see [**8**, Prop 5.3]).

Remark 10. — We are working with the model of $X_1(N)$ in which the ∞-cusp is not defined over $\mathbf{Q}$, but rather over $\mathbf{Q}(\zeta_N)$. Therefore, the modular unit u_χ is not defined over $\mathbf{Q}$ but rather over $\mathbf{Q}(\zeta_N)$.

Theorem 11. — [**8**, Thm 1.1] *Let $f \in S_2(\Gamma_1(N), \psi)$ be a newform of weight 2, level N and character ψ. For any even primitive Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, with $\chi \neq \overline{\psi}$, we have*

$$L(f,2)L(f,\chi,1) = \frac{N\pi\tau(\chi)}{2\phi(N)} \langle r_N(\{u_{\overline{\chi}}, u_{\psi\chi}\}), f\rangle \tag{4}$$

where $L(f,\chi,s) \coloneqq \sum_{n=1} a_n(f)\chi(n)n^{-s}$ denotes the L-function of f twisted by χ, $\tau(\chi) \coloneqq \sum_{a\in(\mathbf{Z}/N\mathbf{Z})^\times} \chi(a)e^{\frac{2\pi i a}{N}}$ denotes the Gauss sum of χ, and $\phi(N)$ denotes Euler's function.

We will also need the following lemma.

Lemma 12. — *Let c denote complex conjugation on $Y_1(N)(\mathbf{C})$. For any even non-trivial Dirichlet characters $\chi, \chi' : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, we have $c^*\eta(u_\chi, u_{\chi'}) = -\eta(u_\chi, u_{\chi'})$.*

Proof. — Recall that c is given by $c(z) = -\overline{z}$ on $\mathfrak{H}$. We have $c^* \log|u_\chi| = \log|u_\chi|$, and c^* exchanges the holomorphic and anti-holomorphic parts of $\operatorname{dlog}|u_\chi|$. Since $\operatorname{darg}(u_\chi) = -i(\partial - \overline{\partial})\log|u_\chi|$, we get $c^* \operatorname{darg}(u_\chi) = -\operatorname{darg}(u_\chi)$, and thus $c^*\eta(u_\chi, u_{\chi'}) = -\eta(u_\chi, u_{\chi'})$. □

Remark 13. — By [**8**, Prop. 5.4 and Prop. 6.1], we have $\{u_\chi, u_{\chi'}\} \in K_2(X_1(N)(\mathbf{C})) \otimes \mathbf{C}$. This implies that for $\gamma \in H_1(Y_1(N)(\mathbf{C}), \mathbf{Z})$, the integral $\int_\gamma \eta(u_\chi, u_{\chi'})$ depends only on the image of γ in $H_1(X_1(N)(\mathbf{C}), \mathbf{Z})$ (see for example the discussion in [**12**, §3]). Therefore, we have a well-defined map

$$\int \eta(u_\chi, u_{\chi'}) : H_1(X_1(N)(\mathbf{C}), \mathbf{Z}) \to \mathbf{C}.$$

It can be extended by linearity to $H_1(X_1(N)(\mathbf{C}), \mathbf{C})$.

Remark 14. — Since $c^*\eta(u_\chi, u_{\chi'}) = -\eta(u_\chi, u_{\chi'})$ by Lemma 12, we have $\int_\gamma \eta(u_\chi, u_{\chi'}) = \int_{\gamma^-} \eta(u_\chi, u_{\chi'})$ with $\gamma^- = \frac{1}{2}(\gamma - c_*\gamma)$.

4. Merel's formula

In this section, we express the regulator integral appearing in the right hand side of (4) as a linear combination of periods. In order to do this, we use an idea of Merel to express the integral over $X_1(N)(\mathbf{C})$ as a linear combination of products of 1-dimensional integrals.

Let $N \geq 1$ be an integer. Let E_N be the set of vectors $(u,v) \in (\mathbf{Z}/N\mathbf{Z})^2$ such that $(u,v,N) = 1$. For any $f \in S_2(\Gamma_1(N))$ and any $x \in E_N$, we define the *Manin symbol*

$$\xi_f(x) = -\frac{1}{2\pi}\langle \xi(x), f\rangle = -i\int_{g_x 0}^{g_x\infty} f(z)dz,$$

where $g_x \in \mathrm{SL}_2(\mathbf{Z})$ is any matrix whose bottom row is congruent to x modulo N.

Let $\rho = e^{\frac{\pi i}{3}}$ and $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.

The following theorem is a variant of a theorem of Merel which expresses the Petersson scalar product of two cusp forms f and g of weight 2 as a linear combination of products of Manin symbols of f and g [**18**, Théorème 2].

Theorem 15. — *Let $f \in S_2(\Gamma_1(N))$ be a cusp form of weight 2 and level N, and let $u,v \in \mathcal{O}^\times(Y_1(N)(\mathbf{C}))$ be two modular units. We have*

$$\int_{X_1(N)(\mathbf{C})} \eta(u,v)\wedge\omega_f = \frac{\pi}{2}\sum_{x\in E_N}\left(\int_{g_x\rho}^{g_x\rho^2}\eta(u,v)\right)\xi_f(x). \tag{5}$$

Proof. — Let $\mathcal{F}$ be the standard fundamental domain of $\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}$:

$$\mathcal{F} = \{z \in \mathfrak{H} : |\mathrm{Re}(z)| \leq \frac{1}{2}, |z| \geq 1\}.$$

Its boundary $\partial\mathcal{F}$ is the hyperbolic triangle with vertices ρ^2, ρ, ∞. Define

$$F_x(z) = \int_\infty^z \omega_f|g_x \qquad (x \in E_N, z \in \mathfrak{H}).$$

We have

$$\int_{X_1(N)(\mathbf{C})} \eta(u,v)\wedge\omega_f = \sum_{x\in E_N/\pm 1}\int_{\mathcal{F}}(\eta(u,v)\wedge\omega_f)|g_x.$$

Since $\eta(u,v)$ is closed, we have $(\eta(u,v)\wedge\omega_f)|g_x = -d(F_x\cdot(\eta(u,v)|g_x))$ and Stokes' formula gives

$$\begin{aligned}\int_{X_1(N)(\mathbf{C})} \eta(u,v)\wedge\omega_f &= -\sum_{x\in E_N/\pm 1}\int_{\partial\mathcal{F}} F_x\cdot(\eta(u,v)|g_x)\\ &= -\sum_{x\in E_N/\pm 1}\left(\int_{\rho^2}^{\rho}+\int_\rho^\infty+\int_\infty^{\rho^2}\right)F_x\cdot(\eta(u,v)|g_x). \qquad (6)\end{aligned}$$

The matrix T fixes ∞ and maps ρ^2 to ρ. We have

$$F_x(Tz) = \int_\infty^{Tz}\omega_f|g_x = \int_\infty^z \omega_f|g_xT = F_{xT}(z).$$

It follows that

$$\begin{aligned}\sum_{x\in E_N/\pm 1}\int_\rho^\infty F_x\cdot(\eta(u,v)|g_x) &= \sum_{x\in E_N/\pm 1}\int_{\rho^2}^\infty F_x|T\cdot(\eta(u,v)|g_xT)\\ &= \sum_{x\in E_N/\pm 1}\int_{\rho^2}^\infty F_{xT}\cdot(\eta(u,v)|g_{xT})\\ &= \sum_{x\in E_N/\pm 1}\int_{\rho^2}^\infty F_x\cdot(\eta(u,v)|g_x).\end{aligned}$$

Hence (6) simplifies to

$$\int_{X_1(N)(\mathbf{C})} \eta(u,v) \wedge \omega_f = \sum_{x \in E_N/\pm 1} \int_{\rho}^{\rho^2} F_x \cdot (\eta(u,v)|g_x).$$

Similarly, let us use the matrix σ, which exchanges ρ and ρ^2, as well as 0 and ∞. Since $F_x(\sigma z) = F_{x\sigma}(z) + 2\pi\xi_f(x)$, we get

$$\int_{\rho}^{\rho^2} F_x \cdot (\eta(u,v)|g_x) = \int_{\rho^2}^{\rho} F_{x\sigma} \cdot (\eta(u,v)|g_{x\sigma}) + 2\pi\xi_f(x) \int_{\rho^2}^{\rho} \eta(u,v)|g_x.$$

Summing over x and using the fact that $\xi_f(x\sigma) = -\xi_f(x)$, we get

$$\begin{aligned}\int_{X_1(N)(\mathbf{C})} \eta(u,v) \wedge \omega_f &= \frac{1}{2} \sum_{x \in E_N/\pm 1} 2\pi\xi_f(x) \int_{\rho^2}^{\rho} \eta(u,v)|g_{x\sigma} \\ &= \pi \sum_{x \in E_N/\pm 1} \xi_f(x) \int_{\rho}^{\rho^2} \eta(u,v)|g_x.\end{aligned}$$

□

Remark 16. — It can be shown that if $\{u,v\}$ defines an element in $K_2(X_1(N)(\mathbf{C})) \otimes \mathbf{Q}$, then the cycle $\sum_{x \in E_N} \left(\int_{g_x\rho}^{g_x\rho^2} \eta(u,v) \right) \xi(x)$ is *closed*. This follows from the fact that if γ_P denotes a small loop around a cusp P of $X_1(N)(\mathbf{C})$, then $\int_{\gamma_P} \eta(u,v) = 2\pi \log|\partial_P(u,v)|$, where $\partial_P(u,v)$ denotes the tame symbol of $\{u,v\}$ at P (see for example [**23**, §4, Lemma]).

Definition 17. — Let $f \in S_2(\Gamma_1(N))$ be a cusp form of weight 2 and level N. Consider the following relative cycle on $Y_1(N)(\mathbf{C})$:

$$\gamma_f := \sum_{x \in E_N} \xi_f(x)\{g_x\rho, g_x\rho^2\}.$$

Furthermore, let us define $\gamma_f^- := \frac{1}{2}(\gamma_f - c_*\gamma_f)$.

Combining Theorem 11, Theorem 15 and Remark 14, we get the following result.

Theorem 18. — *Let $f \in S_2(\Gamma_1(N), \psi)$ be a newform of weight 2, level N and character ψ. For any even primitive Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, with $\chi \neq \overline{\psi}$, we have*

$$L(f,2)L(f,\chi,1) = \frac{N\pi^2\tau(\chi)}{4\phi(N)} \int_{\gamma_f} \eta(u_{\overline{\chi}}, u_{\psi\chi}) = \frac{N\pi^2\tau(\chi)}{4\phi(N)} \int_{\gamma_f^-} \eta(u_{\overline{\chi}}, u_{\psi\chi}). \tag{7}$$

We will also need an explicit expression of γ_f in terms of Manin symbols. For any $f \in S_2(\Gamma_1(N))$ and any $x = (u,v) \in E_N$, let us define $x^c = (-u,v)$ and

$$\xi_f^+(x) = \frac{1}{2}(\xi_f(x) + \xi_f(x^c)) = \frac{1}{2}(\xi_f(x) + \overline{\xi_{f^*}(x)}),$$

where f^* denotes the cusp form with complex conjugate Fourier coefficients.

Proposition 19. — *Let $f \in S_2(\Gamma_1(N))$ be a cusp form of weight 2 and level N. The cycle γ_f is closed, and its image in $H_1(X_1(N)(\mathbf{C}), \mathbf{Z})$ can be expressed as follows:*

$$\gamma_f = -\frac{1}{3} \sum_{x \in E_N} (\xi_f(x) + 2\xi_f(x\tau))\,\xi(x). \tag{8}$$

Moreover, we have

$$\gamma_f^- = -\frac{1}{3} \sum_{x \in E_N} \left(\xi_f^+(x) + 2\xi_f^+(x\tau)\right) \xi(x). \tag{9}$$

Proof. — Let us compute the boundary of γ_f. Since $\sigma(\rho) = \rho^2$ and $\xi_f(x\sigma) = -\xi_f(x)$, we have

$$\begin{aligned}\partial\gamma_f &= \sum_{x\in E_N} \xi_f(x)([g_x\rho^2] - [g_x\rho])\\ &= \sum_{x\in E_N} \xi_f(x)([g_{x\sigma}\rho] - [g_x\rho])\\ &= -2\sum_{x\in E_N} \xi_f(x)[g_x\rho].\end{aligned}$$

Since $\tau(\rho) = \rho$ and because of Manin's relation $\xi_f(x) + \xi_f(x\tau) + \xi_f(x\tau^2) = 0$, we get

$$\begin{aligned}\partial\gamma_f &= -\frac{2}{3}\sum_{x\in E_N} \xi_f(x)([g_x\rho] + [g_{x\tau}\rho] + [g_{x\tau^2}\rho])\\ &= -\frac{2}{3}\sum_{x\in E_N} (\xi_f(x) + \xi_f(x\tau) + \xi_f(x\tau^2))[g_x\rho] = 0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\gamma_f &= \sum_{x\in E_N} \xi_f(x)(\{g_x\rho, g_x\infty\} + \{g_x\infty, g_x\rho^2\})\\ &= \sum_{x\in E_N} \xi_f(x)(\{g_x\rho, g_x\infty\} - \sum_{x\in E_N} \xi_f(x)\{g_x0, g_x\rho\})\\ &= 2\sum_{x\in E_N} \xi_f(x)\{g_x\rho, g_x\infty\} - \sum_{x\in E_N} \xi_f(x)\xi(x).\end{aligned}$$

Using the matrix τ, we get

$$\begin{aligned}\gamma_f &= \frac{2}{3}\sum_{x\in E_N} \big(\xi_f(x)\{g_x\rho, g_x\infty\} + \xi_f(x\tau)\{g_{x\tau}\rho, g_{x\tau}\infty\} + \xi_f(x\tau^2)\{g_{x\tau^2}\rho, g_{x\tau^2}\infty\}\big) - \sum_{x\in E_N} \xi_f(x)\xi(x)\\ &= \frac{2}{3}\sum_{x\in E_N} \big(\xi_f(x)\{g_x\rho, g_x\infty\} + \xi_f(x\tau)\{g_x\rho, g_x0\} + \xi_f(x\tau^2)\{g_x\rho, g_x1\}\big) - \sum_{x\in E_N} \xi_f(x)\xi(x)\\ &= \frac{2}{3}\sum_{x\in E_N} \big(\xi_f(x\tau)\{g_x\infty, g_x0\} + \xi_f(x\tau^2)\{g_x\infty, g_x1\}\big) - \sum_{x\in E_N} \xi_f(x)\xi(x)\\ &= \frac{2}{3}\sum_{x\in E_N} \big(-\xi_f(x\tau)\xi(x) + \xi_f(x\tau^2)\{g_{x\tau^2}0, g_{x\tau^2}\infty\}\big) - \sum_{x\in E_N} \xi_f(x)\xi(x)\\ &= \frac{2}{3}\sum_{x\in E_N} (-\xi_f(x\tau)\xi(x) + \xi_f(x)\xi(x)) - \sum_{x\in E_N} \xi_f(x)\xi(x)\\ &= \frac{1}{3}\sum_{x\in E_N} (\xi_f(x) - 2\xi_f(x\tau))\xi(x).\end{aligned}$$

This gives (8). The action of complex conjugation on γ_f is given by

$$\begin{aligned}c_*\gamma_f &= \sum_{x\in E_N} \xi_f(x)\{c(g_x\rho), c(g_x\rho^2)\}\\ &= \sum_{x\in E_N} \xi_f(x)\{g_{x^c}\rho^2, g_{x^c}\rho\}\\ &= -\sum_{x\in E_N} \xi_f(x^c)\{g_x\rho, g_x\rho^2\}.\end{aligned}$$

It follows that

$$\gamma_f^- = \sum_{x\in E_N} \xi_f^+(x)\{g_x\rho, g_x\rho^2\}.$$

Since the quantities $\xi_f^+(x)$ satisfy the Manin relations, the same proof as above gives (9). □

5. Proof of the main theorem

Let us return to the case $N = 13$. Using Theorem 18 with $f = f_\varepsilon$, $\psi = \varepsilon$ and $\chi = \varepsilon^3$, we get

$$L(f_\varepsilon, 2)L(f_\varepsilon, \varepsilon^3, 1) = \frac{13\pi^2\tau(\varepsilon^3)}{48} \int_{\gamma^-_{f_\varepsilon}} \eta(u_{\varepsilon^3}, u_{\overline{\varepsilon}^2}). \tag{10}$$

We are going to make explicit each term in this formula. Note that $\tau(\varepsilon^3) = \sqrt{13}$.

Definition 20. — For any Dirichlet character $\psi : (\mathbf{Z}/13\mathbf{Z})^\times \to \mathbf{C}^\times$, let us denote $\mathcal{H}(\psi)$ (resp. $\hat{\mathcal{H}}(\psi)$) the ψ-isotypical component of $\mathcal{H}\otimes\mathbf{C}$ (resp. $\hat{\mathcal{H}}\otimes\mathbf{C}$) with respect to the action of diamond operators $\langle d\rangle_*$, $d \in (\mathbf{Z}/13\mathbf{Z})^\times$. For any $\gamma \in \hat{\mathcal{H}} \otimes \mathbf{C}$, let γ^ψ denote its ψ-isotypical component. Moreover, let us define $\hat{\mathcal{H}}^\pm(\psi) = (\hat{\mathcal{H}}^\pm \otimes \mathbf{C}) \cap \hat{\mathcal{H}}(\psi)$ and $\mathcal{H}^\pm(\psi) = (\mathcal{H}^\pm \otimes \mathbf{C}) \cap \mathcal{H}(\psi)$.

Lemma 21. — *Let $\psi = \varepsilon$ or $\overline{\varepsilon}$. Then $\mathcal{H}^\pm(\psi)$ has dimension* 1, *and a generator is given by*

$$\gamma^+_\psi := \sum_{a\in(\mathbf{Z}/13\mathbf{Z})^\times} \varepsilon^3(a)\xi(1,a)^\psi$$
$$\gamma^-_\psi := \xi(1,-3)^\psi - \xi(1,3)^\psi.$$

Moreover, we have $W_{13}\gamma^+_\psi = \psi(2)\gamma^+_\psi$.

Proof. — The pairing $\langle\cdot,\cdot\rangle$ induces a perfect pairing

$$\mathcal{H}^\pm(\psi) \times S_2(\Gamma_1(13), \psi) \to \mathbf{C}.$$

Since $S_2(\Gamma_1(13),\psi)$ is 1-dimensional, we get $\dim_{\mathbf{C}} \mathcal{H}^\pm(\psi) = 1$. From the definition, it is clear that $\gamma^+_\psi \in \hat{\mathcal{H}}^+(\psi)$ and $\gamma^-_\psi \in \hat{\mathcal{H}}^-(\psi)$. Moreover, since $\gamma^-_\psi = \gamma_3^\psi$, we have $\gamma^-_\psi \in \mathcal{H}^-(\psi)$.

Let us compute the boundary of γ^+_ψ. For any $u, v \in (\mathbf{Z}/13\mathbf{Z})^\times$, we have $\partial\xi(u,v) = P_u - P_v$ with $P_d := \langle d\rangle(0)$. Moreover, for any $x \in E_{13}$, we have

$$\xi(x)^\psi = \frac{1}{12}\sum_{d\in(\mathbf{Z}/13\mathbf{Z})^\times} \overline{\psi}(d)\langle d\rangle_*\xi(x) = \frac{1}{12}\sum_{d\in(\mathbf{Z}/13\mathbf{Z})^\times} \overline{\psi}(d)\xi(dx).$$

It follows that

$$\begin{aligned}
\partial\gamma^+_\psi &= \sum_{a\in(\mathbf{Z}/13\mathbf{Z})^\times} \varepsilon^3(a)\partial(\xi(1,a)^\psi)\\
&= \frac{1}{12}\sum_{a\in(\mathbf{Z}/13\mathbf{Z})^\times} \varepsilon^3(a) \sum_{d\in(\mathbf{Z}/13\mathbf{Z})^\times} \overline{\psi}(d)\partial\xi(d,da)\\
&= \frac{1}{12}\sum_{a\in(\mathbf{Z}/13\mathbf{Z})^\times} \varepsilon^3(a) \sum_{d\in(\mathbf{Z}/13\mathbf{Z})^\times} \overline{\psi}(d)(P_d - P_{da})\\
&= \frac{1}{12}\sum_{d\in(\mathbf{Z}/13\mathbf{Z})^\times} \left(\sum_{a\in(\mathbf{Z}/13\mathbf{Z})^\times} \varepsilon^3(a) - \varepsilon^3\psi(a)\right)\overline{\psi}(d)\cdot P_d = 0.
\end{aligned}$$

Hence $\gamma^+_\psi \in \mathcal{H}^+(\psi)$. By [**19**, Lemme 5], the elements $\xi(1,0)^\psi, \xi(1,2)^\psi, \xi(1,3)^\psi, \xi(1,-3)^\psi$ form a basis of $\hat{\mathcal{H}}(\psi)$, and we can express γ^+_ψ in terms of this basis. This gives

$$\gamma^+_\psi = (2 - 4\psi(2))\xi(1,2)^\psi + \xi(1,3)^\psi + \xi(1,-3)^\psi. \tag{11}$$

In particular γ^+_ψ and γ^-_ψ are nonzero, and thus they generate $\mathcal{H}^\pm(\psi)$.

It remains to compute the action of W_{13} on γ^+_ψ. In view of (11), it is enough to determine the action of W_{13} on $\xi(1,2)$ and $\xi(1,3)$. We have

$$\begin{aligned}
W_{13}\xi(1,2) &= \left\{\frac{2}{13}, \infty\right\} = \left\{\frac{2}{13}, \frac{1}{6}\right\} + \left\{\frac{1}{6}, 0\right\} + \{0, \infty\}\\
&= -\xi(0,-6) + \xi(1,-6) + \xi(0,1).
\end{aligned}$$

Hence, using [**19**, Lemme 5] again, we get

$$\begin{aligned}W_{13}(\xi(1,2)^{\psi}) &= -\xi(0,-6)^{\overline{\psi}} + \xi(1,-6)^{\overline{\psi}} + \xi(0,1)^{\overline{\psi}}\\ &= (\overline{\psi}(6)-1)\xi(1,0)^{\overline{\psi}} - \overline{\psi}(6)\xi(1,2)^{\overline{\psi}}.\end{aligned}$$

Similarly, we find

$$\begin{aligned}W_{13}(\xi(1,3)^{\psi}) &= (\overline{\psi}(4)-1)\xi(1,0)^{\overline{\psi}} - \overline{\psi}(4)\xi(1,-3)^{\overline{\psi}}\\ W_{13}(\xi(1,-3)^{\psi}) &= (\overline{\psi}(4)-1)\xi(1,0)^{\overline{\psi}} - \overline{\psi}(4)\xi(1,3)^{\overline{\psi}}.\end{aligned}$$

Since we know that $W_{13}\gamma_{\psi}^{+}$ is a multiple of $\gamma_{\overline{\psi}}^{+}$, we deduce $W_{13}\gamma_{\psi}^{+} = -\overline{\psi}(4)\gamma_{\overline{\psi}}^{+} = \psi(2)\gamma_{\overline{\psi}}^{+}$. □

Proposition 22. — *We have* $L(f_{\varepsilon},\varepsilon^3,1) = \frac{\overline{\varepsilon}(2)}{\sqrt{13}}\langle \gamma_{\varepsilon}^{+}, f_{\varepsilon}\rangle$.

Proof. — By [**17**, Thm 4.2.b)], we have

$$L(f_{\varepsilon},\varepsilon^3,1) = \frac{1}{\sqrt{13}}\sum_{a\in(\mathbf{Z}/13\mathbf{Z})^\times}\varepsilon^3(a)\int_{a/13}^{\infty}\omega_{f_{\varepsilon}}.$$

Let us compute the cycle $\theta = \sum_{a\in(\mathbf{Z}/13\mathbf{Z})^\times}\varepsilon^3(a)\{\frac{a}{13},\infty\}$ in terms of Manin symbols. We have

$$W_{13}(\theta^{\varepsilon}) = (W_{13}\theta)^{\overline{\varepsilon}} = \sum_{a\in(\mathbf{Z}/13\mathbf{Z})^\times}\varepsilon^3(a)\left\{-\frac{1}{a},0\right\}^{\overline{\varepsilon}} = \sum_{a\in(\mathbf{Z}/13\mathbf{Z})^\times}\varepsilon^3(a)\xi(1,a)^{\overline{\varepsilon}} = \gamma_{\overline{\varepsilon}}^{+}.$$

By Lemma 21, it follows that

$$\langle\theta,f_{\varepsilon}\rangle = \langle\theta^{\varepsilon},f_{\varepsilon}\rangle = \langle W_{13}(\gamma_{\overline{\varepsilon}}^{+}),f_{\varepsilon}\rangle = \overline{\varepsilon}(2)\langle\gamma_{\varepsilon}^{+},f_{\varepsilon}\rangle.$$

□

Proposition 23. — *We have* $\gamma_{f_{\varepsilon}}^{-} = \frac{1-2\zeta_6}{\pi}\langle\gamma_{\varepsilon}^{+},f_{\varepsilon}\rangle\cdot\gamma_{\overline{\varepsilon}}^{-}$.

Proof. — By Proposition 19, we have

$$\gamma_{f_{\varepsilon}}^{-} = -\frac{1}{3}\sum_{x\in E_{13}}(\xi_{f_{\varepsilon}}^{+}(x)+2\xi_{f_{\varepsilon}}^{+}(x\tau))\xi(x).$$

This sum involves 168 terms, but we may reduce it to 14 terms by considering the action of diamond operators. Let $\mathcal{E}$ be the set of 2-tuples $(0,1)$ and $(1,v)$, $v\in\mathbf{Z}/13\mathbf{Z}$. We have

$$\begin{aligned}\gamma_{f_{\varepsilon}}^{-} &= -\frac{1}{3}\sum_{x\in\mathcal{E}}\sum_{d\in(\mathbf{Z}/13\mathbf{Z})^\times}(\xi_{f_{\varepsilon}}^{+}(dx)+2\xi_{f_{\varepsilon}}^{+}(dx\tau))\xi(dx)\\ &= -\frac{1}{3}\sum_{x\in\mathcal{E}}\sum_{d\in(\mathbf{Z}/13\mathbf{Z})^\times}(\xi_{f_{\varepsilon}}^{+}(x)+2\xi_{f_{\varepsilon}}^{+}(x\tau))\cdot\varepsilon(d)\langle d\rangle_*\xi(x)\\ &= -4\sum_{x\in\mathcal{E}}(\xi_{f_{\varepsilon}}^{+}(x)+2\xi_{f_{\varepsilon}}^{+}(x\tau))\xi(x)^{\overline{\varepsilon}}.\end{aligned}$$

A simple computation shows that the terms $x=(0,1)$ and $x=(1,0)$ cancel each other. Hence

$$\gamma_{f_{\varepsilon}}^{-} = -4\sum_{v\in(\mathbf{Z}/13\mathbf{Z})^*}\left(\xi_{f_{\varepsilon}}^{+}(1,v)+2\varepsilon(v)\xi_{f_{\varepsilon}}^{+}(1,1+\frac{1}{v})\right)\cdot\xi(1,v)^{\overline{\varepsilon}}.$$

Using [**19**, Lemme 5], we may express $\xi_{f_{\varepsilon}}^{+}(1,v)$, $v\neq 0$ in terms of $\xi_{f_{\varepsilon}}^{+}(1,2)$ and $\xi_{f_{\varepsilon}}^{+}(1,3)$. We find $\xi_{f_{\varepsilon}}^{+}(1,-v)=\xi_{f_{\varepsilon}}^{+}(1,v)$ and

$$\begin{aligned}&\xi_{f_{\varepsilon}}^{+}(1,1)=0 && \xi_{f_{\varepsilon}}^{+}(1,4)=(1-\zeta_6)\xi_{f_{\varepsilon}}^{+}(1,3)\\ &\xi_{f_{\varepsilon}}^{+}(1,5)=(\zeta_6-1)\big(\xi_{f_{\varepsilon}}^{+}(1,2)-\xi_{f_{\varepsilon}}^{+}(1,3)\big) && \xi_{f_{\varepsilon}}^{+}(1,6)=(\zeta_6-1)\xi_{f_{\varepsilon}}^{+}(1,2).\end{aligned}$$

Moreover, also by [**19**, Lemme 5], the cycles $\xi(1,v)^{\overline{\varepsilon}}$, $v\neq 0$, are linear combinations of $\xi(1,2)^{\overline{\varepsilon}}$, $\xi(1,3)^{\overline{\varepsilon}}$ and $\xi(1,-3)^{\overline{\varepsilon}}$. Thus the same is true for $\gamma_{f_{\varepsilon}}^{-}$. But we know that $\gamma_{f_{\varepsilon}}^{-}$ is a multiple of

$\gamma^-_{\overline{\varepsilon}} = \xi(1,3)^{\overline{\varepsilon}} - \xi(1,-3)^{\overline{\varepsilon}}$. It is thus enough to compute the coefficient in front of $\xi(1,3)^{\overline{\varepsilon}}$, which leads to the identity

$$\gamma^-_{f_\varepsilon} = \big(12\xi^+_{f_\varepsilon}(1,2) + (8\zeta_6 - 4)\xi^+_{f_\varepsilon}(1,3)\big) \cdot \gamma^-_{\overline{\varepsilon}}.$$

Using (11) with $\psi = \varepsilon$, we get the proposition. □

Consider the modular units $x = W_{13}(h)$ and $y = W_{13}(H)$.

Proposition 24. — *We have* $\int_{\gamma^-_{\overline{\varepsilon}}} \eta(x,y) = \frac{13^2\sqrt{13}}{48}(1+\zeta_6)\tau(\varepsilon^2) \int_{\gamma^-_{\overline{\varepsilon}}} \eta(u_{\varepsilon^3}, u_{\overline{\varepsilon}^2})$.

Proof. — Since h and H are supported in the cusps above $0 \in X_0(13)(\mathbf{Q})$, it follows that x and y are supported in the cusps above $\infty \in X_0(13)(\mathbf{Q})$, namely the cusps $\langle d\rangle\infty$, $d \in (\mathbf{Z}/13\mathbf{Z})^\times/\pm 1$. The method of proof is simple : we decompose the divisors of x and y as linear combinations of Dirichlet characters.

Let us write $\begin{pmatrix} n_1 & n_2 & \cdots & n_6 \end{pmatrix}$ for the divisor $\sum_{d=1}^6 n_d \cdot \langle d\rangle\infty$. By [**14**, p. 56], we have

$$\operatorname{div}(x) = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 & -1 \end{pmatrix}$$
$$\operatorname{div}(y) = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

The divisors of u_{ε^3} and $u_{\overline{\varepsilon}^2}$ are given by [**8**, Prop 5.4]. We have

$$\operatorname{div}(u_{\varepsilon^3}) = -\frac{L(\varepsilon^3,2)}{\pi^2} \cdot \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & -1 \end{pmatrix} = -\frac{4\sqrt{13}}{13^2}\operatorname{div}(y).$$

Since the divisor of x is invariant under the diamond operator $\langle 5\rangle$, it is a linear combination of $\operatorname{div}(u_{\varepsilon^2})$ and $\operatorname{div}(u_{\overline{\varepsilon}^2})$. We find explicitly

$$\begin{aligned}\operatorname{div}(x) &= \frac{1-2\zeta_6}{3}\Big(\frac{\operatorname{div}(u_{\varepsilon^2})}{L(\varepsilon^2,2)/\pi^2} - \frac{\operatorname{div}(u_{\overline{\varepsilon}^2})}{L(\overline{\varepsilon}^2,2)/\pi^2}\Big)\\ &= \frac{13}{12}\big((2-\zeta_6)\tau(\overline{\varepsilon}^2)\operatorname{div}(u_{\varepsilon^2}) + (1+\zeta_6)\tau(\varepsilon^2)\operatorname{div}(u_{\overline{\varepsilon}^2})\big).\end{aligned}$$

Here we have used the classical formula [**9**, (1.80) and (3.87)]

$$\frac{L(\chi,2)}{\pi^2} = \frac{\tau(\chi)}{N}\sum_{a=0}^{N-1}\overline{\chi}(a)B_2\Big(\frac{a}{N}\Big)$$

where χ is an even non-trivial Dirichlet character modulo N, and $B_2(x) = x^2 - x + \frac{1}{6}$ is the second Bernoulli polynomial.

Considering u_{ε^3} and $u_{\overline{\varepsilon}^2}$ as elements of $\mathcal{O}^*(Y_1(13)(\mathbf{C})) \otimes \mathbf{C}$ and following the notations of [**8**, (65)], we have $\widehat{u_{\varepsilon^3}}(\infty) = \widehat{u_{\overline{\varepsilon}^2}}(\infty) = 1$ by [**8**, Prop. 5.3]. Moreover, looking at the behaviour of x and y at ∞, we find $x(\infty) = 1$ and $\widehat{y}(\infty) = -1$. Hence $x \otimes 1$ can be expressed as a linear combination of u_{ε^2} and $u_{\overline{\varepsilon}^2}$ in $\mathcal{O}^*(Y_1(13)(\mathbf{C})) \otimes \mathbf{C}$, while $y \otimes 1$ is proportional to u_{ε^3}. Thus

$$\eta(x,y) = -\frac{13^2}{4\sqrt{13}} \cdot \frac{13}{12}\big((2-\zeta_6)\tau(\overline{\varepsilon}^2)\eta(u_{\varepsilon^2}, u_{\varepsilon^3}) + (1+\zeta_6)\tau(\varepsilon^2)\eta(u_{\overline{\varepsilon}^2}, u_{\varepsilon^3})\big).$$

Since the differential form $\eta(u_{\varepsilon^2}, u_{\varepsilon^3})$ has character ε, we have $\int_{\gamma^-_{\overline{\varepsilon}}} \eta(u_{\varepsilon^2}, u_{\varepsilon^3}) = 0$, and the proposition follows. □

Proof of Theorem 1. — Combining (10) with Propositions 22, 23, 24, we get

$$L(f_\varepsilon, 2) = \frac{\pi}{\sqrt{13}} \cdot \frac{1-\zeta_6}{\tau(\varepsilon^2)}\int_{\gamma^-_{\overline{\varepsilon}}} \eta(x,y). \tag{12}$$

Formula (12) simplifies if we use the functional equation of $L(f_\varepsilon, s)$. Recall that $W_{13}(f_\varepsilon) = wf_{\overline{\varepsilon}}$. Let $\Lambda(f,s) := 13^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$. Then the functional equation of $L(f_\varepsilon,s)$ reads

$$\Lambda(f_\varepsilon, s) = -w\Lambda(f_{\overline{\varepsilon}}, 2-s).$$

Using (3), we deduce that

$$L(f_\varepsilon,2) = \frac{4\pi^2}{13^2}(4-3\zeta_6)\tau(\varepsilon)L'(f_{\overline{\varepsilon}},0).$$

Replacing in (12) and using $\tau(\varepsilon^2)\tau(\varepsilon) = (4\zeta_6 - 3)\sqrt{13}$, we get

$$\int_{\gamma_{\overline{\varepsilon}}^-} \eta(x,y) = 4\pi(\zeta_6 - 1)L'(f_{\overline{\varepsilon}},0). \tag{13}$$

Taking complex conjugation, and since $\overline{\eta(x,y)} = \eta(x,y)$, we obtain

$$\int_{\gamma_\varepsilon^-} \eta(x,y) = -4\pi\zeta_6 L'(f_\varepsilon,0). \tag{14}$$

We have a direct sum decomposition $\mathcal{H}^- \otimes \mathbf{C} = \mathcal{H}^-(\varepsilon) \oplus \mathcal{H}^-(\overline{\varepsilon})$. Write $\gamma_3 = \gamma_3^\varepsilon + \gamma_3^{\overline{\varepsilon}}$. Then $\gamma_4 = \langle 2\rangle_* \gamma_3 = \varepsilon(2)\gamma_3^\varepsilon + \overline{\varepsilon}(2)\gamma_3^{\overline{\varepsilon}}$. By Proposition 9, we deduce

$$W_{13}\gamma_P = \gamma_4 - \gamma_3 = (\zeta_6 - 1)\gamma_3^\varepsilon + (\overline{\zeta_6} - 1)\gamma_3^{\overline{\varepsilon}} = (\zeta_6 - 1)\gamma_\varepsilon^- + (\overline{\zeta_6} - 1)\gamma_{\overline{\varepsilon}}^-.$$

By (13) and (14), we then have

$$\begin{aligned}\int_{W_{13}\gamma_P} \eta(x,y) &= (\zeta_6 - 1)\int_{\gamma_\varepsilon^-} \eta(x,y) + (\overline{\zeta_6} - 1)\int_{\gamma_{\overline{\varepsilon}}^-} \eta(x,y)\\ &= 4\pi(L'(f_\varepsilon,0) + L'(f_{\overline{\varepsilon}},0)).\end{aligned}$$

By Proposition 3, we conclude that

$$m(P) = \frac{1}{2\pi}\int_{\gamma_P} \eta(h,H) = \frac{1}{2\pi}\int_{W_{13}\gamma_P} \eta(x,y) = 2L'(f,0).$$

□

Remark 25. — There may have been a quicker way to proceed. Starting from Theorem 11 in the particular case $N = 13$, probably all we need is a symplectic basis of $H_1(X_1(13)(\mathbf{C}),\mathbf{Z})$ with respect to the intersection pairing (see the formula [**4**, A.2.5]). But this is less canonical than Theorem 15.

Remark 26. — Another approach to Theorem 1 would be to use the Rogers-Zudilin method, more precisely to use the main formula of [**25**]. We have not worked out the details of this computation.

Question 27. — Let $g = f|\langle 2\rangle = \zeta_6 f_\varepsilon + \overline{\zeta_6} f_{\overline{\varepsilon}}$. Then (f,g) is a basis of the space $S_2(\Gamma_1(13),\mathbf{Q})$ of cusp forms with rational Fourier coefficients. Is there a polynomial $Q \in \mathbf{Z}[x,y]$ such that $m(Q)$ is proportional to $L'(g,0)$?

6. Examples in higher level

We note that the functions H and h used in the proof of Theorem 1 are modular units on $X_1(13)$ and that P is their minimal polynomial. There is a similar story for the modular curve $X_1(11)$ [**7**, Cor 3.3] and we may try to generalize this phenomenon.

Let $N \geq 1$ be an integer, and let u and v be two modular units on $X_1(N)$. Let $P \in \mathbf{C}[x,y]$ be an irreducible polynomial such that $P(u,v) = 0$. Then the map $z \mapsto (u(z),v(z))$ is a modular parametrization of the curve $C_P : P(x,y) = 0$ and we have a natural map $Y_1(N) \to C_P$. Assuming P satisfies Deninger's conditions, we may express $m(P)$ in terms of the integral of $\eta(u,v)$ over a (non necessarily closed) cycle γ_P.

The most favourable case is when the curve C_P intersects the torus $T^2 = \{|x| = |y| = 1\}$ only at cusps. In this case γ_P is a modular symbol and we may use [**25**] to compute $\int_{\gamma_P} \eta(u,v)$ in terms of special values of L-functions.

In this section, we work out this idea for some examples of increasing complexity. We work with the modular units provided by [**24**]. These modular units are supported on the cusps above $\infty \in X_0(N)$, so that [**8**, Prop 6.1] implies that P is automatically tempered.

In all examples below, we found that γ_P can be written as the sum of a closed path γ_0 and a path γ_1 joining cusps. The integral of $\eta(u,v)$ over γ_1 can be computed using [**25**, Thm 1]. The integral of $\eta(u,v)$ over γ_0 can be dealt with using either [**25**, Thm 1] or the explicit version of Beilinson's theorem – we have not carried out the details of the computation. So in order to establish the identities below rigorously, it only remains to express γ_0 in terms of modular symbols and to compute $\int_{\gamma_0} \eta(u,v)$ using the tools explained above.

It would be interesting to understand when the identities obtained involve cusp forms (like (15)), are of Dirichlet type (like (16)), or of mixed type (like (17)). In the general case, it would be also interesting to find conditions on the modular units u and v so that the boundary of γ_P consists of cusps or other interesting points.

6.1. $N = 16$. — The modular curve $X_1(16)$ has genus 2 and has been studied in [**16**]. Let u and v be the following modular units:

$$u = q \prod_{\substack{n \geq 1 \\ n \equiv \pm 1, \pm 5 (16)}} (1-q^n) / \prod_{\substack{n \geq 1 \\ n \equiv \pm 3, \pm 7 (16)}} (1-q^n)$$

$$v = q \prod_{\substack{n \geq 1 \\ n \equiv \pm 14 (16)}} (1-q^n) / \prod_{\substack{n \geq 1 \\ n \equiv \pm 10 (16)}} (1-q^n).$$

Their minimal polynomial is given by

$$P_{16} = y - x - xy - xy^2 + x^2y + xy^3.$$

This polynomial vanishes on the torus at the points $(x,y) = (1,1)$, $(1,\pm i)$, $(-1,-1)$, but the Deninger cycle $\gamma_{P_{16}}$ is *closed*. So we may expect that $m(P_{16})$ is equal to $L'(f,0)$ for some cusp form f of level 16 with rational coefficients. Indeed, we find numerically

$$m(P_{16}) \stackrel{?}{=} L'(f,0) \tag{15}$$

where f is the trace of the unique newform of weight 2 and level 16, having coefficients in $\mathbf{Z}[i]$.

6.2. $N = 18$. — The modular curve $X_1(18)$ has genus 2 and has been studied in [**13**]. It has 3 cusps above ∞, so we may form essentially two modular units supported on these cusps. Let u and v be the following modular units:

$$u = q^3 \prod_{\substack{n \geq 1 \\ n \equiv \pm 1, \pm 2 (18)}} (1-q^n) / \prod_{\substack{n \geq 1 \\ n \equiv \pm 7, \pm 8 (18)}} (1-q^n)$$

$$v = q^2 \prod_{\substack{n \geq 1 \\ n \equiv \pm 1, \pm 4 (18)}} (1-q^n) / \prod_{\substack{n \geq 1 \\ n \equiv \pm 5, \pm 8 (18)}} (1-q^n).$$

Their minimal polynomial is given by

$$P_{18} = -x^2 + y^3 + xy^2 - x^2y + x^2y^2 - x^3y^2.$$

Note that $P_{18}(\frac{1}{x}, -\frac{1}{y})$ gives Lecacheux's equation [**13**, (2)]. The polynomial P_{18} vanishes on the torus at the points $(x,y) = (1,\pm 1)$, $(-1,\pm 1)$, (ζ_6^2, ζ_6) and $(\overline{\zeta_6}^2, \overline{\zeta_6})$ with $\zeta_6 = e^{2\pi i/6}$. The points (ζ_6^2, ζ_6) and $(\overline{\zeta_6}^2, \overline{\zeta_6})$ correspond respectively to the cusps $\frac{1}{6}$ and $-\frac{1}{6}$, and the Deninger cycle $\gamma_{P_{18}}$ is given by $\gamma_0 + \{-\frac{1}{6}, \frac{1}{6}\}$, where γ_0 is a closed cycle. Using [**25**, Thm 1], we find

$$\int_{-1/6}^{1/6} \eta(u,v) = \frac{1}{4\pi} L(F,2)$$

where F is a modular form of weight 2 and level (at most) 18^2. Actually F has level 18 and [**25**, Thm 1] simplifies if we use the functional equation $L(F,2) = -\frac{2\pi^2}{9}L'(W_{18}F,0)$. In fact [**25**, Lemma 2] guarantees that $W_{18}F$ will be a modular form with *integral* Fourier coefficients. In this case, we find

$$W_{18}F = -36E_2^{\psi}$$

where $E_2^{\psi} = \sum_{n=1}^{\infty}(\sum_{d|n} d)\psi(n)q^n$ is an Eisenstein series of level 9, and $\psi : (\mathbf{Z}/3\mathbf{Z})^\times \to \{\pm 1\}$ is the unique Dirichlet character of conductor 3. Since $L(E_2^{\psi},s) = L(\psi,s)L(\psi,s-1)$, we may expect that $m(P_{18})$ involves L-values of Dirichlet characters. Indeed, we find numerically

$$m(P_{18}) \stackrel{?}{=} 2L'(\psi,-1). \tag{16}$$

6.3. $N = 25$. — The modular curve $X_1(25)$ has genus 12 and the quotient $X = X_1(25)/\langle 7\rangle$ has genus 4. The curve X and its modular units have been studied by Lecacheux [**15**] and Darmon [**10**]. Consider the following modular units:

$$u = q \prod_{\substack{n\geq 1\\ n\equiv \pm 3,\pm 4 (25)}} (1-q^n) / \prod_{\substack{n\geq 1\\ n\equiv \pm 2,\pm 11 (25)}} (1-q^n)$$
$$v = q^{-1} \prod_{\substack{n\geq 1\\ n\equiv \pm 9,\pm 12 (25)}} (1-q^n) / \prod_{\substack{n\geq 1\\ n\equiv \pm 6,\pm 8 (25)}} (1-q^n).$$

Their minimal polynomial is given by

$$P_{25} = y^2x^4 + (y^3+y^2)x^3 + (3y^3-y^2-2y)x^2 + (y^4-4y^2+y-1)x - y^3.$$

This polynomial vanishes on the torus at the points $(x,y) = (\zeta,-\zeta)$ for each primitive 5-th root of unity ζ. These points are cusps: letting $\zeta_5 = e^{2\pi i/5}$, we have

$$u(1/5) = \zeta_5^2 = -v(1/5) \qquad u(-1/5) = \zeta_5^{-2} = -v(-1/5)$$
$$u(2/5) = \zeta_5 = -v(2/5) \qquad u(-2/5) = \zeta_5^{-1} = -v(-2/5).$$

The Deninger cycle associated to P_{25} is given by $\gamma_{P_{25}} = \gamma_0 + \gamma_1$ where γ_0 is a closed cycle and $\gamma_1 = \{\frac{1}{5}, -\frac{1}{5}\} + \{-\frac{2}{5}, \frac{2}{5}\}$. Using [**25**, Thm 1], we get

$$\int_{\gamma_1} \eta(u,v) = \frac{1}{4\pi}L(F,2)$$

where F is a modular form of weight 2 and level 25. This time F is a linear combination of newforms and Eisenstein series. Let $\varepsilon : (\mathbf{Z}/25\mathbf{Z})^\times \to \mathbf{C}^\times$ be the unique Dirichlet character such that $\varepsilon(2) = \zeta_5$. A basis of eigenforms of $\Omega^1(X)\otimes \mathbf{C}$ is given by newforms $(f_a)_{a\in(\mathbf{Z}/5\mathbf{Z})^\times}$ having Fourier coefficients in $\mathbf{Q}(\zeta_5)$ and forming a single Galois orbit. The newform f_a has character ε^a and for any $\sigma \in \mathrm{Gal}(\mathbf{Q}(\zeta_5)/\mathbf{Q})$, we have $\sigma(f_a) = f_{\chi(\sigma)a}$ where χ is the cyclotomic character. Moreover, let $\psi : (\mathbf{Z}/5\mathbf{Z})^\times \to \mathbf{C}^\times$ be the Dirichlet character defined by $\psi(2) = i$. Then $W_{25}F$ has integral coefficients and is given by

$$W_{25}F = -10\,\mathrm{Tr}_{\mathbf{Q}(\zeta_5)/\mathbf{Q}}(\lambda f_1) - 25(1+i)E_2^{\psi,\overline{\psi}} - 25(1-i)E_2^{\overline{\psi},\psi}$$

where $\lambda = 2\zeta_5 + \zeta_5^{-1} + 2\zeta_5^{-2}$ and $E_2^{\psi,\overline{\psi}}$ is the Eisenstein series defined by

$$E_2^{\psi,\overline{\psi}} = \sum_{m,n=1}^{\infty} m\overline{\psi}(m)\psi(n)q^{mn}.$$

We may therefore expect $m(P_{25})$ being a linear combination of $L'(\psi,-1)$, $L'(\overline{\psi},-1)$ and $L'(f,0)$, where f is a cusp form with rational Fourier coefficients. Indeed, we find numerically

$$m(P_{25}) \stackrel{?}{=} L'(f,0) + \frac{1+2i}{5}L'(\overline{\psi},-1) + \frac{1-2i}{5}L'(\psi,-1) \tag{17}$$

where

$$f = \frac{1}{5}\operatorname{Tr}\big((2+\zeta_5+2\zeta_5^{-2})f_1\big) = q + q^2 - q^3 - q^4 - 3q^5 - 2q^9 + 3q^{10} + 4q^{11} + O(q^{12}).$$

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