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Zagier's conjectures on special values of L-functions (**)

I'll use the following notations through the text. For any set X, the free abelian group with basis X will be denoted by $\boldsymbol{Z}[X]$. I'll also use the symbol $\sim_{\boldsymbol{Q}^*}$ to express the fact that two numbers equal up to a multiplicative factor in $\boldsymbol{Q}^*$.

1 - Number fields

Let F be a number field and $\mathcal{O}_F$ be the ring of integers of F. Let $\sigma_1, \sigma_2, \ldots, \sigma_{r_1+2r_2}$ be the embeddings of F into $\boldsymbol{C}$ (with $r_1+2r_2=[F:\boldsymbol{Q}]$), such that $\sigma_1, \sigma_2, \ldots, \sigma_{r_1}$ are real and $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$ are nonreal and pairwise inequivalent under complex conjugation. Let Δ_F be the discriminant of F.

Let $\zeta_F(s)$ be the Dedekind zeta function of F, that is

$$\zeta_F(s) = \sum_{I \subset \mathcal{O}_F} (\mathrm{N}I)^{-s} \quad (\Re(s) > 1), \tag{1}$$

where the sum is taken over all nonzero ideals I of $\mathcal{O}_F$, and $\mathrm{NI} = |\mathcal{O}_F/\mathrm{I}|$ is the norm of the ideal I. We wish to investigate the special values of $\zeta_F(s)$ at $s=m$, where m is an integer $\geqslant 2$. We start with the case $m=2$.

Definition 1 (The classical polylogarithm function). *For any integer $k \geqslant 1$, the k-th polylogarithm function* Li_k *is defined by*

$$\mathrm{Li}_k(z) = \sum_{n \geqslant 1} \frac{z^n}{n^k} \qquad (z \in \boldsymbol{C},\ |z| < 1). \tag{2}$$

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The function Li_k can be extended to a multivalued function on $\boldsymbol{C}\backslash\{0, 1\}$. This means that Li_k can be defined (at least) on the universal covering of $\boldsymbol{C}\backslash\{0, 1\}$. We have for example

$$\mathrm{Li}_2(z) = \mathrm{Li}_2\left(\frac{1}{2}\right) - \int_\gamma \frac{\log(1-z)}{z}\,dz \qquad (z \in \boldsymbol{C}\backslash\{0, 1\}), \tag{3}$$

which depends on a continuous path γ from $\frac{1}{2}$ to z in $\boldsymbol{C}\backslash\{0, 1\}$. We now define the Bloch-Wigner function, a single-valued version of Li_2.

Definition 2 (The Bloch-Wigner function). *Let D be the function defined by*

$$\begin{aligned} D : \boldsymbol{P}^1(\boldsymbol{C}) &\to \boldsymbol{R} \\ z &\mapsto \begin{cases} \Im(\mathrm{Li}_2(z)) + \log|z| \arg(1-z) & \text{if } z \notin \{0, 1, \infty\} \\ 0 & \text{if } z \in \{0, 1, \infty\}, \end{cases} \end{aligned} \tag{4}$$

where $\mathrm{Li}_2(z)$ (resp. $\arg(1-z)$*) is defined using* (2) *(resp. using the principal branch of the logarithm) for $|z| < 1$.*

It can be checked that D is indeed single-valued. It is easy to see that D is continuous on $\boldsymbol{P}^1(\boldsymbol{C})$ and real-analytic on $\boldsymbol{C}\backslash\{0, 1\}$. The function D has many interesting properties, among which

$$D(\bar{z}) = D\left(\frac{1}{z}\right) = D(1-z) = -D(z) \qquad (z \in \boldsymbol{P}^1(\boldsymbol{C})). \tag{5}$$

By linearity, the function D extends to a homomorphism $D : \boldsymbol{Z}[\boldsymbol{P}^1(\boldsymbol{C})] \to \boldsymbol{R}$, where $\boldsymbol{Z}[\boldsymbol{P}^1(\boldsymbol{C})]$ is the group of divisors on $\boldsymbol{P}^1(\boldsymbol{C})$.

Now comes an algebraic construction. For any field K, we define following Bloch and Suslin an abelian group $\mathcal{B}(K)$ called the Bloch group of K. In order to do this we consider the free abelian group $\boldsymbol{Z}[K^*]$ on K^* and we put $[0] = [\infty] = 0$ in $\boldsymbol{Z}[K^*]$. Let β be the map

$$\begin{aligned} \beta : \boldsymbol{Z}[K^*] &\to (K^* \wedge_{\boldsymbol{Z}} K^*) \otimes_{\boldsymbol{Z}} \boldsymbol{Z}\left[\frac{1}{2}\right] \\ [x] &\mapsto \begin{cases} x \wedge (1-x) & \text{if } x \neq 1 \\ 0 & \text{if } x = 1\,. \end{cases} \end{aligned} \tag{6}$$

Define subgroups $\mathcal{A}(K)$ and $\mathcal{C}(K)$ of $\boldsymbol{Z}[K^*]$ by

$$\mathcal{A}(K) = \ker \beta \,, \tag{7}$$

$$\mathcal{C}(K) = \left\langle [x] + [y] + \left[\frac{1-x}{1-xy} \right] + [1-xy] + \left[\frac{1-y}{1-xy} \right],\ (x, y) \in K^2 \setminus \{(1, 1)\} \right\rangle. \tag{8}$$

We can check that $\mathcal{C}(K) \subset \mathcal{A}(K)$.

Definition 3 (The Bloch group). *For any field K, let $\mathcal{B}(K)$ be the abelian group defined by*

$$\mathcal{B}(K) = \frac{\mathcal{A}(K)}{\mathcal{C}(K)} \,. \tag{9}$$

The following proposition can be seen as a functional equation for the Bloch-Wigner function D. It is a consequence of the classical 5-term functional equation of Li_2, which is known since the nineteenth century [O], p. 10.

Proposition 4. *In the case $K = \boldsymbol{C}$, we have $D(\mathcal{C}(\boldsymbol{C})) = 0$.*

Proposition 4 can be proved by differentiation. Now we return to the special case of a number field F. We define a dilogarithmic map on the Bloch group $\mathcal{B}(F)$ as follows.

Definition 5 (The map D_F). *Let D_F be the homomorphism defined by*

$$\begin{aligned} D_F \colon \mathcal{B}(F) &\to \boldsymbol{R}^{r_2} \\ [x] &\mapsto (D(\sigma_{r_1+i}(x)))_{1 \le i \le r_2} . \end{aligned} \tag{10}$$

The map D_F is well-defined thanks to Proposition 4. Note that we use only the nonreal embeddings of F (the relation $D(\bar{z}) = -D(z)$ implies that D vanishes on $\boldsymbol{R}$).

The fundamental theorem is the following.

Theorem 6. *Let F be a number field. The kernel of D_F is the finite subgroup $\mathcal{B}(F)_{\mathrm{tors}}$ of torsion elements in $\mathcal{B}(F)$. The image $D_F(\mathcal{B}(F))$ is a full lattice of $\boldsymbol{R}^{r_2}$ whose covolume equals up to a nonzero rational factor*

$$\frac{\zeta_F(2)\,|\Delta_F|^{1/2}}{\pi^{2(r_1+r_2)}} \,. \tag{11}$$

Corollary 7 (Zagier's conjecture for $\zeta_F(2)$). *For any number field F, there exists a family of divisors $x_1, x_2, \dots, x_{r_2} \in \boldsymbol{Z}[F^*]$ such that*

$$\zeta_F(2) \sim_{\boldsymbol{Q}^*} \frac{\pi^{2(r_1+r_2)}}{|\Delta_F|^{1/2}} \det\big(D(\sigma_{r_1+i}(x_j))\big)_{\substack{1 \leqslant i \leqslant r_2 \\ 1 \leqslant j \leqslant r_2}}. \tag{12}$$

Note that Theorem 6 also implies that for any family of divisors $x_1, x_2, \dots, x_{r_2} \in \mathcal{A}(F)$, we have

$$\frac{\pi^{2(r_1+r_2)}}{|\Delta_F|^{1/2}} \det\big(D(\sigma_{r_1+i}(x_j))\big)_{\substack{1 \leqslant i \leqslant r_2 \\ 1 \leqslant j \leqslant r_2}} \in \boldsymbol{Q}\cdot\zeta_F(2). \tag{13}$$

It is not clear who proved Theorem 6. The main ingredient of the proof is the construction of a homomorphism $\phi_F \colon \mathcal{B}(F) \to K_3(F)$ fitting the following diagram

$$\begin{tikzcd} K_3(F) \arrow[r, "r_{\mathrm{Borel}}"] & \boldsymbol{R}^{r_2}, \\ \mathcal{B}(F) \arrow[u, "\phi_F"] \arrow[ur, "D_F"'] & \end{tikzcd} \tag{14}$$

where $K_3(F)$ is Quillen's K-group and r_{Borel} denotes Borel's regulator.

Bloch first constructed a map $\phi_F \colon \mathcal{B}(F) \to K_3(F)$ (see [Bl2], Section 7.2). Then Suslin proved that ϕ_F has finite kernel and cokernel (see [Su1], [Su2]). It is not clear to me who proved the commutativity of the diagram (14) (but see the ideas in [Bl2], Section 5.1, Section 7.4). According to Borel's theorem, the kernel of r_{Borel} is the finite group $K_3(F)_{\mathrm{tors}}$, and the image of r_{Borel} is a full lattice of $\boldsymbol{R}^{r_2}$ whose covolume is proportional to (11). Putting everything together yields Theorem 6. For another proof of the theorem, we refer to [Go3], Theorem 2.1, p. 250.

Zagier [Z] gave a beautiful conjectural generalization of Theorem 6 to the case of $\zeta_F(m)$ where m is any integer $\geqslant 2$. In analogy with the case $m = 2$, Zagier constructed subgroups

$$\mathcal{C}_m(F) \subset \mathcal{A}_m(F) \subset \boldsymbol{Z}[F^*]. \tag{15}$$

He defined the m-th Bloch group of F as

$$\mathcal{B}_m(F) = \frac{\mathcal{A}_m(F)}{\mathcal{C}_m(F)}, \tag{16}$$

together with a polylogarithmic map $P_{m,F}$:

$$P_{m,F}\colon \mathcal{B}_m(F) \to \begin{cases} \boldsymbol{R}^{r_2} & \text{if } m \text{ is even}, \\ \boldsymbol{R}^{r_1+r_2} & \text{if } m \text{ is odd}. \end{cases} \tag{17}$$

His conjecture can be stated as follows.

Conjecture 8 (Zagier's conjecture for $\zeta_F(m)$). *Let F be a number field and $m \geqslant 2$ be an even (resp. odd) integer. The kernel of $P_{m,F}$ is the subgroup $\mathcal{B}_m(F)_{\mathrm{tors}}$ of torsion elements in $\mathcal{B}_m(F)$. The image $P_{m,F}(\mathcal{B}_m(F))$ is a full lattice of $\boldsymbol{R}^{r_2}$ (resp. $\boldsymbol{R}^{r_1+r_2}$) whose covolume equals up to a nonzero rational factor*

$$\frac{\zeta_F(m)\,|\Delta_F|^{1/2}}{\pi^{m(r_1+r_2)}}\left(\text{resp. } \frac{\zeta_F(m)\,|\Delta_F|^{1/2}}{\pi^{mr_2}}\right). \tag{18}$$

Beĭlinson and Deligne (unpublished) and de Jeu [dJ] constructed a homomorphism $\phi_{m,F}\colon \mathcal{B}_m(F) \to K_{2m-1}(F) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ fitting the following commutative diagram

$$\begin{tikzcd} K_{2m-1}(F) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \arrow[r, "r_{\mathrm{Borel}}"] & \boldsymbol{R}^r, \\ \mathcal{B}_m(F) \arrow[u, "\phi_{m,F}"] \arrow[ur, "P_{m,F}"'] & \end{tikzcd} \tag{19}$$

where $K_{2m-1}(F)$ is Quillen's K-group, r_{Borel} is Borel's regulator, and $r = r_2$ or $r_1 + r_2$ depending on whether m is even or odd. De Jeu also proved [dJ, Remark 5.4] that there exists an integer $N_m \geqslant 1$ depending only on m such that $N_m \phi_{m,F}(\mathcal{B}_m(F))$ is contained in the image of $K_{2m-1}(F)$ in $K_{2m-1}(F) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. In particular $\phi_{m,F}(\mathcal{B}_m(F))$ is a finitely generated abelian group. Together with Borel's theorem [Bo], this implies that the first (resp. second) part of Conjecture 8 reduces to the injectivity (resp. surjectivity) of

$$\phi_{m,F} \otimes_{\boldsymbol{Z}} \boldsymbol{Q} : \mathcal{B}_m(F) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to K_{2m-1}(F) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}\,. \tag{20}$$

The injectivity of $\phi_{m,F} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ seems to be a difficult problem. It amounts to finding *all* functional equations of the polylogarithm function $P_{m,F}$. Examples of functional equations are known only in the cases $2 \leqslant m \leqslant 7$. We refer to [O], [Wo], [Ga1], [Ga2] for more details.

Now what is known about the surjectivity of $\phi_{m,F} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$? In the case $m = 3$, Goncharov has proved [Go3] that $\phi_{3,F} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ is surjective for any number field F (see also [Go1], [Go2]). It should be also pointed out that for an explicitely given number field F and a given integer $m \geqslant 2$, the surjectivity of $\phi_{m,F} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ can be proved «by hand». It suffices indeed to find elements $x_1, x_2, \dots, x_r \in \mathcal{A}_m(F)$ whose images $P_{m,F}(x_1), P_{m,F}(x_2), \dots, P_{m,F}(x_r) \in \boldsymbol{R}^r$ are linearly independant over $\boldsymbol{R}$. This amounts to showing that some determinant is nonzero and therefore can be ascertained by a finite computation. From this we can deduce a formula for $\zeta_F(m)$ up to a rational factor. However, *proving* that this rational factor is the one suggested by the computer is hard. In fact, it relies on knowing explicitly the rational factor occurring in Borel's theorem (or at least, on bounding its denominator). The rational factors are predicted by the conjectures of Lichtenbaum and Bloch-Kato.

2 - Elliptic curves

Let E be an elliptic curve which is defined over $\boldsymbol{Q}$. We now recall the definition of the Hasse-Weil zeta function $L(E, s)$ which is associated to E.

For any prime number p, choose a Weierstraß equation for E which is minimal at p [Si], Chapter 7, § 1. Let E_p be its reduction mod p. Put $a_p = p + 1 - |E_p(\boldsymbol{F}_p)|$. By definition, E has good reduction at p if and only if E_p is an elliptic curve over $\boldsymbol{F}_p$. In this case we define

$$L_p(E, s) = \frac{1}{1 - a_p p^{-s} + p^{1-2s}} \qquad \left(s \in \boldsymbol{C},\ \Re(s) > \frac{1}{2}\right). \tag{21}$$

In the other case, we have $a_p \in \{-1, 0, 1\}$ and we define

$$L_p(E, s) = \frac{1}{1 - a_p p^{-s}} \qquad (s \in \boldsymbol{C},\ \Re(s) > 0). \tag{22}$$

Definition 9 (The Hasse-Weil zeta function of E). *Let $L(E, s)$ be the function defined by*

$$L(E, s) = \prod_{p \text{ prime}} L_p(E, s) \qquad \left(s \in \boldsymbol{C},\ \Re(s) > \frac{3}{2}\right). \tag{23}$$

The infinite product defining $L(E, s)$ converges for $\Re(s) > \frac{3}{2}$ and defines there an holomorphic function of s. Now it is known that $L(E, s)$ has an holomorphic continuation to the whole complex plane.

The elliptic analogue of the Bloch-Wigner function was discovered by Bloch

[Bl2]. It is called the elliptic dilogarithm. In order to define it, we choose an isomorphism

$$\eta : E(\boldsymbol{C}) \xrightarrow{\cong} \frac{\boldsymbol{C}}{\boldsymbol{Z} + \tau \boldsymbol{Z}}, \tag{24}$$

with $\tau \in \boldsymbol{C}$ satisfying $\Im(\tau) > 0$. Composing η with the map $z \mapsto \exp(2\pi i z)$ yields an isomorphism $E(\boldsymbol{C}) \cong \boldsymbol{C}^*/q^{\boldsymbol{Z}}$, where q is defined by $q = \exp(2\pi i \tau)$.

Definition 10 (The elliptic dilogarithm). *Let D_E be the function defined by*

$$\begin{aligned} D_E \colon E(\boldsymbol{C}) \cong \frac{\boldsymbol{C}^*}{q^{\boldsymbol{Z}}} &\to \boldsymbol{R} \\ [x] &\mapsto \sum_{n \in \boldsymbol{Z}} D(xq^n). \end{aligned} \tag{25}$$

Remark 11. *The definition works for any elliptic curve over $\boldsymbol{C}$ but depends on the choice of τ and η. Since E is defined over $\boldsymbol{R}$, we can make the function D_E well-defined up to sign by choosing*

$$\tau = \frac{\int_{\gamma_2} \omega}{\int_{\gamma_1} \omega} \quad \text{and} \quad \eta : P \mapsto \left[\frac{\int_0^P \omega}{\int_{\gamma_1} \omega} \right], \tag{26}$$

where (γ_1, γ_2) is any oriented basis of $H_1(E(\boldsymbol{C}), \boldsymbol{Z})$ such that γ_1 generates $H_1^+(E(\boldsymbol{C}), \boldsymbol{Z})$, and ω is any nonzero holomorphic 1-form on $E(\boldsymbol{C})$. The sign depends only on an orientation of $E(\boldsymbol{R})$.

The series (25) defining D_E converges absolutely and gives rise to a function which is continuous on $E(\boldsymbol{C})$ and real-analytic on $E(\boldsymbol{C}) \setminus \{0\}$. The function D_E also satisfies

$$D_E(\overline{P}) = D_E(P) \qquad D_E(-P) = -D_E(P) \qquad (P \in E(\boldsymbol{C})) \tag{27}$$

$$D_E(nP) = n \sum_{Q \in E[n]} D_E(P + Q) \qquad (P \in E(\boldsymbol{C}),\ n \geqslant 1) \tag{28}$$

where $E[n]$ denotes the subgroup of n-torsion points of $E(\boldsymbol{C})$. By linearity, the function D_E induces a homomorphism $D_E \colon \boldsymbol{Z}[E(\overline{\boldsymbol{Q}})]^{\mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})} \to \boldsymbol{R}$ (this does not depend on the embedding $\overline{\boldsymbol{Q}} \hookrightarrow \boldsymbol{C}$).

The main theorem reads as follows.

Theorem 12 (Goncharov, Levin [GL]). *Let E be a modular elliptic curve defined over $\boldsymbol{Q}$. There exists a divisor* $l = \sum_i n_i [P_i] \in \boldsymbol{Z}[E(\overline{\boldsymbol{Q}})]^{\mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})}$ *such that the following conditions are fulfilled*

1. *We have* $L(E, 2) \sim_{\boldsymbol{Q}^*} \pi D_E(l)$;
2. *We have* $\sum_i n_i P_i^3 = 0$ *in* $\mathrm{Sym}^3_{\boldsymbol{Z}} E(\overline{\boldsymbol{Q}})$;
3. *For every non-trivial absolute value v of $\overline{\boldsymbol{Q}}$, we have*

$$\sum_i n_i h_v(P_i) \otimes P_i = 0 \quad \text{in } \boldsymbol{R} \otimes_{\boldsymbol{Z}} E(\overline{\boldsymbol{Q}}) \tag{29}$$

where $h_v \colon E(\overline{\boldsymbol{Q}}) \to \boldsymbol{R}$ denotes the canonical local height at v [Si], *Appendix C*, § 18;

4. *Last but not least, the divisor l satisfies an integrality condition at every prime p where E has split multiplicative reduction* [Wi], *condition* (iii) *of Examples* 1.11. (a), p. 376.

We refer to [ZG], pp. 605-606 for a sketch of the proof of Theorem 12. At the bottom of [ZG], p. 606, $\boldsymbol{Q}[E_{\mathrm{tors}}]$ should be replaced by $\boldsymbol{Q}[E]$ (see the example of the curve $E = 37A$ discussed in [SS]). We note that Beĭlinson's theorem on modular curves [SS] already implies that for every modular elliptic curve E over $\boldsymbol{Q}$, there is a divisor $l \in \boldsymbol{Z}[E(\overline{\boldsymbol{Q}})]^{\mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})}$ satisfying condition 1.

Now we let

$$\mathcal{A}(E) = \{ l \in \boldsymbol{Z}[E(\overline{\boldsymbol{Q}})]^{\mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})} \text{satisfying conditions } 2., 3. \text{ and } 4. \}. \tag{30}$$

Since the conditions 2., 3. and 4. are linear in l, $\mathcal{A}(E)$ is a subgroup of $\boldsymbol{Z}[E(\overline{\boldsymbol{Q}})]^{\mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})}$. It is the analogue of the group $\mathcal{A}(K)$ defined by (7). It is possible to define a subgroup $\mathcal{C}(E) \subset \mathcal{A}(E)$ using the functional equations of D_E (see [ZG], p. 603). In accordance with the first section, we define the Bloch group of E by $\mathcal{B}(E) = \dfrac{\mathcal{A}(E)}{\mathcal{C}(E)}$.

Using the minimal proper regular model of E over $\boldsymbol{Z}$, it is possible to define a subgroup $K_2(E)_{\boldsymbol{Z}}$ of Quillen's K-group $K_2(E)$. Beĭlinson's regulator can be seen as a map $r_E \colon K_2(E)_{\boldsymbol{Z}} \to \boldsymbol{R}$. Theorem 12 and results of Wildeshaus [Wi] and Rolshausen and Schappacher [RS] imply the following theorem.

Theorem 13. *There is a homomorphism of abelian groups* $\phi_E \colon \mathcal{B}(E) \to K_2(E)_{\boldsymbol{Z}} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ *such that*

- *The following diagram commutes.*

$$\begin{tikzcd} K_2(E)_{\boldsymbol{Z}} \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \arrow[r, "\pi \cdot r_E"] & \boldsymbol{R} \\ \mathcal{B}(E) \arrow[u, "\phi_E"] \arrow[ur, "D_E"'] & \end{tikzcd} \tag{31}$$

- *The map $\phi_E \otimes_{\boldsymbol{Z}} \boldsymbol{Q} : \mathcal{B}(E) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to K_2(E)_{\boldsymbol{Z}} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ is an isomorphism.*

Conjecture 14 (Zagier's conjecture for $L(E, 2)$). *Let E be an elliptic curve over $\boldsymbol{Q}$. The kernel of D_E: $\mathcal{B}(E) \to \boldsymbol{R}$ is the subgroup $\mathcal{B}(E)_{\text{tors}}$ of torsion elements in $\mathcal{B}(E)$. The image $D_E(\mathcal{B}(E))$ is the lattice in $\boldsymbol{R}$ which is generated, up to a non-zero rational factor, by $\dfrac{L(E, 2)}{\pi}$.*

Theorem 13 reduces the proof of Conjecture 14 to the proof of the following statements.

- The analogue of Borel's theorem holds for r_E. In other words, $r_E \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ is injective (note that the surjectivity of $r_E \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ follows from Beĭlinson's theorem [SS]).
- There is an integer $N \geq 1$ such that $N\phi_E(\mathcal{B}(E))$ is contained in the image of $K_2(E)_{\boldsymbol{Z}}$ in $K_2(E)_{\boldsymbol{Z}} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ (this should not be difficult to prove).

Remark 15. *In order to show the analogue of Borel's theorem for r_E: $K_2(E)_{\boldsymbol{Z}} \to \boldsymbol{R}$, it suffices to show that the abelian group $K_2(E)_{\boldsymbol{Z}}$ is of finite type and has rank 1. This problem seems to be very difficult: it is not even known that $K_2(E)_{\boldsymbol{Z}} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ is finite-dimensional over $\boldsymbol{Q}$.*

3 - Towards a generalization

We wish to generalize the statement of Zagier's conjecture to the case of curves of higher genus.

Let X be a smooth projective curve of genus $g \geq 1$, which is defined over $\boldsymbol{Q}$ and geometrically irreducible. It is known that the homology group $H_1(X(\boldsymbol{C}), \boldsymbol{Z})$ is free abelian of rank $2g$. Since X is defined over $\boldsymbol{R}$, the complex conjugation c acts on $X(\boldsymbol{C})$ and therefore on $H_1(X(\boldsymbol{C}), \boldsymbol{Z})$. The subgroup

$$H_1^+(X(\boldsymbol{C}), \boldsymbol{Z}) = \{\gamma \in H_1(X(\boldsymbol{C}), \boldsymbol{Z});\ c_* \gamma = \gamma\} \tag{32}$$

is free abelian of rank g. Beĭlinson's regulator can be seen as a map

(33) $$r_X\colon K_2(X)_{\boldsymbol{Z}} \to H_1^+(X(\boldsymbol{C}), \boldsymbol{R}).$$

We'll denote by $L(H^1(X), s)$ the L-function associated to the jacobian variety of X (see [R, pp. 73-74] for the definition of the L-function associated to an abelian variety over $\boldsymbol{Q}$).

Conjecture 16 (Beĭlinson's conjecture for $L(H^1(X), 2)$). *The image $r_X(K_2(X)_{\boldsymbol{Z}})$ of Beĭlinson's regulator is a full lattice of $H_1^+(X(\boldsymbol{C}), \boldsymbol{R})$ whose covolume with respect to $H_1^+(X(\boldsymbol{C}), \boldsymbol{Z})$ equals up to a nonzero rational factor*

(34) $$\frac{L(H^1(X), 2)}{\pi^{2g}}.$$

The theory we outlined in the first two sections makes the following question natural. Is it possible to construct an abelian group $\mathcal{B}(X)$ (preferably the most explicit) and a dilogarithmic map

(35) $$D_X\colon \mathcal{B}(X) \to H_1^+(X(\boldsymbol{C}), \boldsymbol{R})$$

such that there is a commutative diagram

(36) $$\begin{tikzcd} K_2(X)_{\boldsymbol{Z}} \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \arrow[r, "r_X"] & H_1^+(X(\boldsymbol{C}), \boldsymbol{R}) \\ \mathcal{B}(X) \arrow[u, "\phi_X"] \arrow[ur, "D_X"'] & \end{tikzcd}$$

with the property that

(37) $$\phi_X \otimes_{\boldsymbol{Z}} \boldsymbol{Q} : \mathcal{B}(X) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to K_2(X)_{\boldsymbol{Z}} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$$

is an isomorphism?

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Abstract

A conjecture of Zagier links special values of Dedekind zeta functions to special values of polylogarithms. In this article we give a short account of recent results towards this conjecture. We also describe its analogue for the special value $L(E, 2)$, where E is an elliptic curve over $\mathbf{Q}$. Finally we discuss the possibility of replacing E by a smooth projective curve over $\mathbf{Q}$.

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