REGULATORS OF SIEGEL UNITS AND APPLICATIONS

by

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Abstract. — We present a formula for the regulator of two arbitrary Siegel units in terms of L-values of pairwise products of Eisenstein series of weight one. We give applications to Boyd's conjectures and Zagier's conjectures for elliptic curves of conductors 14, 21, 35, 48 and 54.

1. Introduction

In a recent work [**21**], W. Zudilin proved a formula for the regulator of two modular units. The aim of this article is to generalize this result to arbitrary Siegel units and give applications to elliptic curves.

For any two holomorphic functions f and g on a Riemann surface, define the real 1-form

$$\eta(f,g) = \log|f| \,\mathrm{d}\arg g - \log|g| \,\mathrm{d}\arg f.$$

Note that if f and g are two holomorphic functions which do not vanish on the upper half-plane $\mathcal{H}$, then $\eta(f,g)$ is a well-defined 1-form on $\mathcal{H}$. We prove the following theorem.

Theorem 1. — *Let $N \geq 1$ be an integer. Let $u = (a,b)$ and $v = (c,d)$ be two nonzero vectors in $(\mathbf{Z}/N\mathbf{Z})^2$, and let g_u and g_v be the Siegel units associated to u and v (see Section 2 for the definition). We have*

$$\int_0^{i\infty} \eta(g_u, g_v) = \pi \Lambda^*(e_{a,d}e_{b,-c} + e_{a,-d}e_{b,c}, 0) \tag{1}$$

where $e_{a,b}$ is the Eisenstein series of weight 1 and level N^2 defined by

$$e_{a,b}(\tau) = \alpha_0(a,b) + \sum_{\substack{m,n\geq 1\\ m\equiv a,\, n\equiv b(N)}} q^{mn} - \sum_{\substack{m,n\geq 1\\ m\equiv -a,\, n\equiv -b(N)}} q^{mn} \qquad (q = e^{2\pi i\tau}) \tag{2}$$

with

$$\alpha_0(a,b) = \begin{cases} 0 & \text{if } a=b=0 \\ \frac{1}{2} - \{\frac{b}{N}\} & \text{if } a=0 \text{ and } b\neq 0 \\ \frac{1}{2} - \{\frac{a}{N}\} & \text{if } a\neq 0 \text{ and } b=0 \\ 0 & \text{if } a\neq 0 \text{ and } b\neq 0. \end{cases}$$

Here $\Lambda^*(f,0)$ denotes the regularized value of the completed L-function $\Lambda(f,s)$ at $s=0$ (see Section 3 for the definition).

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Remark 2. — Let $Y(N)$ be the affine modular curve of level N over $\mathbf{Q}$, and let $X(N)$ be the smooth compactification of $Y(N)$. The homology group $H = H_1(X(N)(\mathbf{C}), \{\text{cusps}\}, \mathbf{Z})$ is generated by the modular symbols $\xi(\gamma) = \{\gamma 0, \gamma\infty\}$ with $\gamma \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$. If c is an element of H, we may write $c = \sum_i \lambda_i \xi(\gamma_i)$, and it follows that

$$\int_c \eta(g_u, g_v) = \sum_i \lambda_i \int_0^\infty \eta(g_{u\gamma_i}, g_{v\gamma_i}).$$

Moreover, the Siegel units generate (up to constants) the group $\mathcal{O}(Y(N))^\times \otimes \mathbf{Q}$. Therefore Theorem 1 (together with Lemma 5) gives a formula for all possible regulator integrals $\int_c \eta(u,v)$ with $c \in H$ and $u, v \in \mathcal{O}(Y(N))^\times$.

Remark 3. — Theorem 1 is a generalization of [**21**, Thm 1]. More precisely, let $\tilde{g}_a$, $a \in \mathbf{Z}/N\mathbf{Z}$ denote the modular units arising in [**21**]. Then for every $c \in \mathbf{Z}/N\mathbf{Z}$, we have

$$\tilde{g}_a(c/N + it) = g_{a,ac}(iNt) \qquad (t > 0).$$

We recover [**21**, Thm 1] by taking $u = (a, ac)$ and $v = (b, bc)$ in Theorem 1. Note that in this case $f_{a,b,c} = e_{a,bc}e_{ac,-b} + e_{a,-bc}e_{ac,b}$ belongs to $\mathbf{Q}[[q^N]]$, and $f_{a,b,c}(\tau/N)$ is a modular form on $\Gamma_1(N)$. More generally, if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}/N\mathbf{Z})$ is any matrix such that $\det(M) = 0$, then $F_M(\tau/N) = (e_{a,d}e_{b,-c} + e_{a,-d}e_{b,c})(\tau/N)$ is a modular form of weight 2 on $\Gamma_1(N)$. It would be interesting to study further the properties of these modular forms and to understand their possible relations with the toric modular forms introduced by L. Borisov and P. Gunnells [**1**].

The proof of Theorem 1 follows the strategy of [**21**]. We express the logarithms of Siegel units as a double infinite sum (Lemma 16) and deduce an expression for the regulator as a quadruple sum. We then perform the same analytical change of variables from [**16**], leading to the Mellin transform of a product of Eisenstein series. The key lemma to do this (Lemma 8) suggests that similar results should hold in higher weight.

We point out that the simple shape of the Eisenstein series $e_{a,b}$ makes Theorem 1 particularly amenable to explicit computations. We give some applications of Theorem 1 in Section 6, for elliptic curves which are parametrized by modular units [**6**].

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2. Siegel units

We recall some basic definitions and results about Siegel units, for which we refer the reader to [**9**, §1] and [**10**].

Let $B_2 = X^2 - X + \frac{1}{6}$ be the second Bernoulli polynomial. For $x \in \mathbf{R}$, we define $B(x) = B_2(\{x\}) = \{x\}^2 - \{x\} + \frac{1}{6}$, where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

Let $\mathcal{H}$ be the upper half-plane. Let $N \geq 1$ be an integer and $\zeta_N = e^{2\pi i/N}$. For any $(a,b) \in (\mathbf{Z}/N\mathbf{Z})^2$, $(a,b) \neq (0,0)$, the Siegel unit $g_{a,b}$ on $\mathcal{H}$ is defined by

$$g_{a,b}(\tau) = q^{B(a/N)/2} \prod_{n \geq 0} (1 - q^n q^{\tilde{a}/N} \zeta_N^b) \prod_{n \geq 1} (1 - q^n q^{-\tilde{a}/N} \zeta_N^{-b}) \qquad (q = e^{2\pi i \tau}) \tag{3}$$

where $\tilde{a}$ is the representative of a satisfying $0 \leq \tilde{a} < N$. Here $q^\alpha = e^{2\pi i \alpha\tau}$ for $\alpha \in \mathbf{Q}$. It is known that the function $g_{a,b}^{12N}$ is modular for the group

$$\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) : \gamma \equiv I_2 \pmod{N}\}.$$

In fact $g_{a,b}$ defines an element of $\mathcal{O}(Y(N))^\times \otimes \mathbf{Q}$, where $Y(N)$ denotes the affine modular curve of level N over $\mathbf{Q}$. Recall that the group $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ acts on $Y(N)$ by $\mathbf{Q}$-automorphisms. For any $\gamma \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$, we have the identity in $\mathcal{O}(Y(N))^\times \otimes \mathbf{Q}$

$$g_{a,b}|\gamma = g_{(a,b)\gamma}. \tag{4}$$

Lemma 4. — *Let $(a,b) \in (\mathbf{Z}/N\mathbf{Z})^2$, $(a,b) \neq (0,0)$. We have*

$$g_{a,b}(-1/\tau) = e^{-2\pi i(\{\frac{a}{N}\}-\frac12)(\{\frac{b}{N}\}-\frac12)} g_{b,-a}(\tau) \qquad (\tau \in \mathcal{H}). \tag{5}$$

Proof. — By taking the matrix $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in (4), we see that $g_{a,b}(-1/\tau) = w_{a,b} g_{b,-a}(\tau)$ for some root of unity $w_{a,b}$. The formula for $w_{a,b}$ follows from [**10**, Chap. 2, §1, K1, K4]. □

Lemma 5. — *For any $a,b \in \mathbf{Z}/N\mathbf{Z}$, we have*

$$\int_0^\infty \mathrm{d}\arg g_{a,b} = \begin{cases} 0 & \text{if } a=0 \text{ or } b=0 \\ 2\pi(\{\frac{a}{N}\}-\frac12)(\{\frac{b}{N}\}-\frac12) & \text{if } a\neq 0 \text{ and } b \neq 0. \end{cases} \tag{6}$$

Proof. — If $a=0$ or $b=0$ then $g_{a,b}$ has constant argument on the imaginary axis $\tau = it$, $t>0$, hence $\int_0^\infty \mathrm{d}\arg g_{a,b} = 0$.

If $a \neq 0$ and $b \neq 0$, it is easily seen that $\arg g_{a,b}(it) \xrightarrow{t\to\infty} 0$. Moreover, by Lemma 4, we have $\arg g_{a,b}(it) \xrightarrow{t\to 0} -2\pi(\{\frac{a}{N}\}-\frac12)(\{\frac{b}{N}\}-\frac12) \pmod{2\pi}$. This proves (6) up to a multiple of 2π. In order to establish the exact equality, let us introduce the Klein forms [**10**, Chap. 2, §1, p. 27]:

$$\mathfrak{k}_{\alpha,\beta}(\tau) = e^{-\frac12 \eta(\alpha\tau+\beta,\tau)(\alpha\tau+\beta)}\sigma(\alpha\tau+\beta,\tau) \qquad (\alpha,\beta \in \mathbf{R}; \tau \in \mathcal{H})$$

where η and σ denote the Weierstrass functions. The link with Siegel units is given by

$$g_{a,b}(\tau) = w\mathfrak{k}_{a/N,b/N}(\tau)\Delta(\tau)^{1/12} \qquad (1 \le a,b \le N-1)$$

where w is a root of unity [**10**, p. 29]. Since Δ is positive on the imaginary axis, it follows that

$$\int_0^\infty \mathrm{d}\arg g_{a,b} = \int_0^\infty \mathrm{d}\arg \mathfrak{k}_{a/N,b/N}.$$

Using the q-product formula for the σ function [**11**, Chap. 18, §2] and the Legendre relation $\eta_2\omega_1 - \eta_1\omega_2 = 2\pi i$, we find

$$\mathfrak{k}_{\alpha,\beta}(it) = \frac{1}{2\pi i} e^{-\pi\alpha^2 t} e^{\pi i \alpha\beta}(e^{\pi i\beta}e^{-\pi\alpha t} - e^{-\pi i \beta}e^{\pi\alpha t}) \prod_{n\ge 1} \frac{(1-e^{-2\pi(n+\alpha)t}e^{2\pi i\beta})(1-e^{-2\pi(n-\alpha)t}e^{-2\pi i\beta})}{(1-e^{-2\pi n t})^2}. \tag{7}$$

Assume $0<\alpha,\beta<1$. Then by (7), we have $\arg \mathfrak{k}_{\alpha,\beta}(it) \xrightarrow{t\to\infty} \pi(\alpha\beta - \beta + \frac12)$. Moreover, the Klein forms are homogeneous of weight -1 [**10**, p. 27, K1], which implies

$$\mathfrak{k}_{\alpha,\beta}(-1/\tau) = \frac{1}{\tau}\mathfrak{k}_{\beta,-\alpha}(\tau).$$

From this we get $\arg \mathfrak{k}_{\alpha,\beta}(it) \xrightarrow{t\to 0} \pi(-\alpha\beta+\alpha) \pmod{2\pi}$ and

$$\int_0^\infty \mathrm{d}\arg \mathfrak{k}_{\alpha,\beta} \equiv 2\pi(\alpha - \frac12)(\beta - \frac12) \pmod{2\pi}.$$

Moreover, using the fact that $\int_0^\infty \mathrm{d}\arg \mathfrak{k}_{\alpha,\beta} = \int_i^\infty \mathrm{d}\arg \mathfrak{k}_{\alpha,\beta} - \int_i^\infty \mathrm{d}\arg \mathfrak{k}_{\beta,-\alpha}$ and taking the imaginary part of the logarithm of (7), we may express $\int_0^\infty \mathrm{d}\arg \mathfrak{k}_{\alpha,\beta}$ as an infinite sum, which shows that it is a continuous function of $(\alpha,\beta) \in (0,1)^2$. But for $\beta = \frac12$, the Klein form $\mathfrak{k}_{\alpha,\frac12}(it)$ has constant argument. This implies that $\int_0^\infty \mathrm{d}\arg \mathfrak{k}_{\alpha,\beta} = 2\pi(\alpha-\frac12)(\beta-\frac12)$ for any $0<\alpha,\beta<1$. □

3. L-functions of modular forms

In this section we recall basic results on the functional equation satisfied by L-functions of modular forms.

Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ be a modular form of weight $k \geq 1$ on the group $\Gamma_1(N)$. The L-function of f is defined by $L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}$, $\mathfrak{R}(s) > k$. Define the completed L-function

$$\Lambda(f,s) := N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s) = N^{s/2}\int_0^{\infty} (f(iy) - a_0)y^s \frac{\mathrm{d}y}{y}.$$

Recall that the Atkin-Lehner involution W_N on $M_k(\Gamma_1(N))$ is defined by

$$(W_N f)(\tau) = i^k N^{-k/2}\tau^{-k} f(-1/(N\tau)).$$

Note that in the case $k = 2$ this W_N is the opposite of the usual involution acting on differential 1-forms. The following theorem is classical (see [**14**, Thm 4.3.5]).

Theorem 6. — *Let $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_1(N))$. The function $\Lambda(f,s)$ can be analytically continued to the whole s-plane, and satisfies the functional equation $\Lambda(f,s) = \Lambda(W_N f, k-s)$. Moreover, write $W_N f = \sum_{n=0}^{\infty} b_n q^n$. Then the function*

$$\Lambda(f,s) + \frac{a_0}{s} + \frac{b_0}{k-s}$$

is holomorphic on the whole s-plane.

Definition 7. — The notations being as in Theorem 6, we define the regularized values of $\Lambda(f,s)$ at $s = 0$ and $s = k$ by

$$\Lambda^*(f,0) := \lim_{s\to 0} \Lambda(f,s) + \frac{a_0}{s} \tag{8}$$

$$\Lambda^*(f,k) := \lim_{s\to k} \Lambda(f,s) + \frac{b_0}{k-s}. \tag{9}$$

Note that the functional equation translates into the equalities of regularized values

$$\Lambda^*(f,0) = \Lambda^*(W_N f, k) \qquad \Lambda^*(f,k) = \Lambda^*(W_N f, 0). \tag{10}$$

We will need the following lemma.

Lemma 8. — *Let $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_1(N))$ and $g = \sum_{n=0}^{\infty} b_n q^n \in M_\ell(\Gamma_1(N))$ with $k, \ell \geq 1$. Let $h = W_N(g)$. Write $f^* = f - a_0$ and $g^* = g - b_0$. Then for any $s \in \mathbf{C}$, we have*

$$N^{s/2}\int_0^{\infty} f^*(iy) g^*\big(\frac{i}{Ny}\big) y^s \frac{\mathrm{d}y}{y} = \Lambda(fh, s+\ell) - a_0\Lambda(h, s+\ell) - b_0\Lambda(f,s). \tag{11}$$

Proof. — Note that the integral in (11) is absolutely convergent because $f^*(\tau)$ and $g^*(\tau)$ have exponential decay when $\mathfrak{I}(\tau)$ tends to $+\infty$. Moreover, it is easy to check, using Theorem 6, that the right hand side of (11) is holomorphic on the whole s-plane. Therefore it suffices to establish (11) when $\mathfrak{R}(s) > k$. Since $W_N g = h$, we have

$$\begin{aligned} N^{s/2}\int_0^{\infty} f^*(iy) g^*\big(\frac{i}{Ny}\big) y^s \frac{\mathrm{d}y}{y} &= N^{s/2}\int_0^{\infty} f^*(iy)\big(g\big(\frac{i}{Ny}\big) - b_0\big) y^s \frac{\mathrm{d}y}{y} \\ &= N^{s/2}\int_0^{\infty} f^*(iy)(N^{\ell/2}y^\ell h(iy) - b_0) y^s \frac{\mathrm{d}y}{y}. \end{aligned}$$

Now, we remark that $f^* h = fh - a_0 h = (fh)^* - a_0 h^*$. Thus

$$\begin{aligned} N^{s/2}\int_0^{\infty} f^*(iy) g^*\big(\frac{i}{Ny}\big) y^s \frac{\mathrm{d}y}{y} &= N^{s/2}\int_0^{\infty} \big(N^{\ell/2}y^\ell((fh)^*(iy) - a_0 h^*(iy)) - b_0 f^*(iy)\big) y^s \frac{\mathrm{d}y}{y} \\ &= \Lambda(fh, s+\ell) - a_0\Lambda(h, s+\ell) - b_0\Lambda(f,s). \end{aligned}$$

□

Specializing Lemma 8 to the (regularized) value at $s=k$, we get the following formula.

Lemma 9. — *Let $f=\sum_{n=0}^\infty a_n q^n \in M_k(\Gamma_1(N))$ and $g=\sum_{n=0}^\infty b_n q^n \in M_\ell(\Gamma_1(N))$ with $k,\ell \geq 1$. Let $h=W_N(g)$. Write $f^*=f-a_0$ and $g^*=g-b_0$. Then we have*

$$N^{k/2}\int_0^\infty f^*(iy)g^*\Big(\frac{i}{Ny}\Big)y^k\frac{\mathrm{d}y}{y} = \Lambda^*(fh,k+\ell)-a_0\Lambda(h,k+\ell)-b_0\Lambda^*(f,k). \tag{12}$$

4. Eisenstein series of weight 1

In this section we define some Eisenstein series of weight 1. These are the same as those arising in [**21**].

Definition 10. — For any $a,b\in \mathbf{Z}/N\mathbf{Z}$, we let

$$e_{a,b} = \alpha_0(a,b) + \sum_{\substack{m,n\geq 1\\ m\equiv a,\ n\equiv b(N)}} q^{mn} - \sum_{\substack{m,n\geq 1\\ m\equiv -a,\ n\equiv -b(N)}} q^{mn} \tag{13}$$

where

$$\alpha_0(a,b) = \begin{cases} 0 & \text{if } a=b=0\\ \frac12 - \{\frac{b}{N}\} & \text{if } a=0 \text{ and } b\neq 0\\ \frac12 - \{\frac{a}{N}\} & \text{if } a\neq 0 \text{ and } b= 0\\ 0 & \text{if } a\neq 0 \text{ and } b\neq 0.\end{cases}$$

Lemma 11. — *The function $e_{a,b}(\tau/N)$ is an Eisenstein series of weight 1 on the group $\Gamma(N)$, and the function $e_{a,b}$ is an Eisenstein series of weight 1 on $\Gamma_1(N^2)$.*

Proof. — In [**17**, Chap. VII, §2.3], for any $(a,b)\in(\mathbf{Z}/N\mathbf{Z})^2$ the following Eisenstein series are introduced

$$G_{1,(a,b)}(\tau) = -\frac{2\pi i}{N}\Big(\gamma_0(a,b) + \sum_{\substack{m,n\geq 1\\ n\equiv a(N)}} \zeta_N^{bm}q^{mn/N} - \sum_{\substack{m,n\geq 1\\ n\equiv -a(N)}} \zeta_N^{-bm}q^{mn/N}\Big)$$

where

$$\gamma_0(a,b) = \begin{cases} 0 & \text{if } a=b=0\\ \frac12 \frac{1+\zeta_N^b}{1-\zeta_N^b} & \text{if } a=0 \text{ and } b\neq 0\\ \frac12 - \{\frac{a}{N}\} & \text{if } a\neq 0.\end{cases}$$

The function $G_{1,(a,b)}$ is an Eisenstein series of weight 1 on the group $\Gamma(N)$. We have

$$\begin{aligned} e_{a,b}\Big(\frac{\tau}{N}\Big) &= \alpha_0(a,b) + \sum_{\substack{m,n\geq 1\\ m\equiv a,\ n\equiv b(N)}} q^{mn/N} - \sum_{\substack{m,n\geq 1\\ m\equiv -a,\ n\equiv -b(N)}} q^{mn/N}\\ &= \alpha_0(a,b) + \frac1N\sum_{c=0}^{N-1}\zeta_N^{ca}\Bigg(\sum_{\substack{m,n\geq 1\\ n\equiv b(N)}}\zeta_N^{-cm}q^{mn/N} - \sum_{\substack{m,n\geq 1\\ n\equiv -b(N)}}\zeta_N^{cm}q^{mn/N}\Bigg)\\ &= \alpha_0(a,b) - \frac1N\sum_{c=0}^{N-1}\zeta_N^{ca}\gamma_0(b,-c) - \frac{1}{2\pi i}\sum_{c=0}^{N-1}\zeta_N^{ca}G_{1,(b,-c)}.\end{aligned}$$

If $b\neq 0$ then

$$\frac1N\sum_{c=0}^{N-1}\zeta_N^{ca}\gamma_0(b,-c) = \frac1N\sum_{c=0}^{N-1}\zeta_N^{ca}\Big(\frac12 - \{\frac{b}{N}\}\Big) = \alpha_0(a,b),$$

hence $e_{a,b}(\tau/N)$ is an Eisenstein series of weight 1 on $\Gamma(N)$. If $a \neq 0$ then the same is true because $e_{a,b} = e_{b,a}$. Finally if $a = b = 0$ then

$$\alpha_0(a,b) - \frac{1}{N}\sum_{c=0}^{N-1} \zeta_N^{ca}\gamma_0(b,-c) = -\frac{1}{N}\sum_{c=0}^{N-1}\gamma_0(0,c) = 0$$

because $\gamma_0(0,-c) = -\gamma_0(0,c)$.

The second assertion follows from the fact that $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}\Gamma_1(N^2)\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} \subset \Gamma(N)$. □

Definition 12. — For any $a, b \in \mathbf{Z}/N\mathbf{Z}$, we let

$$f_{a,b} = \beta_0(a,b) + \sum_{m,n\geq 1} (\zeta_N^{am+bn} - \zeta_N^{-am-bn})q^{mn} \tag{14}$$

where

$$\beta_0(a,b) = \begin{cases} 0 & \text{if } a = b = 0 \\ \frac{1}{2}\frac{1+\zeta_N^b}{1-\zeta_N^b} & \text{if } a = 0 \text{ and } b \neq 0 \\ \frac{1}{2}\frac{1+\zeta_N^a}{1-\zeta_N^a} & \text{if } a \neq 0 \text{ and } b = 0 \\ \frac{1}{2}\Big(\frac{1+\zeta_N^a}{1-\zeta_N^a} + \frac{1+\zeta_N^b}{1-\zeta_N^b}\Big) & \text{if } a \neq 0 \text{ and } b \neq 0. \end{cases}$$

As the next lemma shows, the functions $f_{a,b}$ are also Eisenstein series; they relate to $e_{a,b}$ by the Atkin-Lehner involution of level N^2.

Lemma 13. — *We have the relation*

$$e_{a,b}\Big(-\frac{1}{N\tau}\Big) = -\frac{\tau}{N} f_{a,b}\Big(\frac{\tau}{N}\Big) \qquad (\tau \in \mathcal{H}). \tag{15}$$

The function $f_{a,b}(\tau/N)$ is an Eisenstein series of weight 1 on $\Gamma(N)$, and the function $f_{a,b}$ is an Eisenstein series of weight 1 on $\Gamma_1(N^2)$. Moreover, we have $W_{N^2}(e_{a,b}) = -\frac{i}{N} f_{a,b}$.

Proof. — The relation (15) follows from [**21**, Lemma 2] (the proof there works for arbitrary $a, b \in \mathbf{Z}/N\mathbf{Z}$). We deduce that $f_{a,b}(\tau/N)$ is a multiple of the function obtained from $e_{a,b}(\tau/N)$ by applying the slash operator $|\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in weight 1. Hence $f_{a,b}(\tau/N)$ is an Eisenstein series of weight 1 on $\Gamma(N)$. The last assertion follows from replacing τ by $N\tau$ in (15). □

We will need the following formula for the completed L-function of $f_{a,b}$.

Lemma 14. — *For any $a, b \in \mathbf{Z}/N\mathbf{Z}$, we have*

$$\Lambda(f_{a,b} + f_{-a,b}, s) = N^s\Gamma(s)(2\pi)^{-s}\Big(\sum_{m\geq 1}\frac{\zeta_N^{am} + \zeta_N^{-am}}{m^s}\Big)\Big(\sum_{n\geq 1}\frac{\zeta_N^{bn} - \zeta_N^{-bn}}{n^s}\Big). \tag{16}$$

Proof. — See the proof of [**21**, Lemma 3]. □

In the special cases $s = 1$ and $s = 2$, this gives the following formulas. Note that formula (18) is none other than [**21**, Lemma 3].

Lemma 15. — *We have*

$$\Lambda^*(f_{a,b} + f_{-a,b}, 1) = \begin{cases} 0 & \text{if } b = 0 \\ 2iN\gamma \cdot (\frac{1}{2} - \{\frac{b}{N}\}) & \text{if } a = 0 \text{ and } b \neq 0 \\ -2iN\log|1-\zeta_N^a| \cdot (\frac{1}{2} - \{\frac{b}{N}\}) & \text{if } a \neq 0 \text{ and } b \neq 0 \end{cases} \tag{17}$$

$$\Lambda(f_{a,b} + f_{-a,b}, 2) = iN^2 B\Big(\frac{a}{N}\Big)\,\mathrm{Cl}_2\Big(\frac{2\pi b}{N}\Big) \tag{18}$$

where γ is Euler's constant and

$$\mathrm{Cl}_2(x) = \sum_{m=1}^{\infty} \frac{\sin(mx)}{m^2} \qquad (x \in \mathbf{R})$$

denotes the Clausen dilogarithmic function.

Proof. — If $a = 0$ then $\sum_{n=1}^{\infty} \zeta_N^{an} n^{-s} = \zeta(s) = \frac{1}{s-1} + \gamma + O_{s\to 1}(s-1)$. If $a \neq 0$ then $\sum_{n=1}^{\infty} \zeta_N^{an}/n = -\log(1-\zeta_N^a)$ where we use the principal value of the logarithm. Formula (17) follows, noting that $-\log \frac{1-\zeta_N^b}{1-\zeta_N^{-b}} = 2\pi i(\frac{1}{2} - \{\frac{b}{N}\})$. Formula (18) is [**21**, Lemma 3]. □

5. The computation

Lemma 16. — *For any $(a,b) \in (\mathbf{Z}/N\mathbf{Z})^2$, $(a,b) \neq (0,0)$, we have*

$$\log g_{a,b}(it) = -\pi B(a/N)t + C_{a,b} - \sum_{m\geq 1} \sum_{\substack{n\geq 1 \\ n\equiv a(N)}} \frac{\zeta_N^{bm}}{m} e^{-\frac{2\pi mnt}{N}} - \sum_{m\geq 1} \sum_{\substack{n\geq 1 \\ n\equiv -a(N)}} \frac{\zeta_N^{-bm}}{m} e^{-\frac{2\pi mnt}{N}} \tag{19}$$

$$= -\frac{\pi B(b/N)}{t} + C_{b,-a} + i\theta_{a,b} - \sum_{m\geq 1} \sum_{\substack{n\geq 1 \\ n\equiv b(N)}} \frac{\zeta_N^{-am}}{m} e^{-\frac{2\pi mn}{Nt}} - \sum_{m\geq 1} \sum_{\substack{n\geq 1 \\ n\equiv -b(N)}} \frac{\zeta_N^{am}}{m} e^{-\frac{2\pi mn}{Nt}} \tag{20}$$

where $\theta_{a,b} = 2\pi(\{\frac{a}{N}\} - \frac{1}{2})(\{\frac{b}{N}\} - \frac{1}{2})$ and

$$C_{a,b} = \begin{cases} \log(1-\zeta_N^b) & \textit{if } a = 0, \\ 0 & \textit{if } a \neq 0. \end{cases} \tag{21}$$

Proof. — By the definition of Siegel units, we have

$$\log g_{a,b} = \pi i B(a/N)\tau + \sum_{n\geq 0} \log(1 - q^n q^{\tilde{a}/N} \zeta_N^b) + \sum_{n\geq 1} \log(1 - q^n q^{-\tilde{a}/N} \zeta_N^{-b})$$

Using the identity $\log(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}$ and substituting $\tau = it$, we get (19). Applying Lemma 4 with $\tau = i/t$, we have $g_{a,b}(it) = e^{i\theta_{a,b}} g_{b,-a}(i/t)$, whence (20). □

We will need the following lemma from [**21**].

Lemma 17. — *(See* [**21**, Lemma 4]*.) For any $a, b \in \mathbf{Z}/N\mathbf{Z}$, we have*

$$\begin{aligned} I(a,b) &:= \int_0^{\infty} \frac{1}{it} \mathrm{d} \sum_{m=1}^{\infty} \frac{\zeta_N^{am} - \zeta_N^{-am}}{m} \Bigg(\sum_{\substack{n\geq 1 \\ n\equiv b(N)}} - \sum_{\substack{n\geq 1 \\ n\equiv -b(N)}} \Bigg) \exp\Big(-\frac{2\pi mn}{Nt}\Big) \\ &= \begin{cases} 0 & \textit{if } a = 0 \textit{ or } b = 0 \\ -i\,\mathrm{Cl}_2(\frac{2\pi a}{N}) \frac{1+\zeta_N^b}{1-\zeta_N^b} & \textit{if } a \neq 0 \textit{ and } b \neq 0. \end{cases} \end{aligned} \tag{22}$$

Proof of Theorem 1. — By Lemma 16, we get

$$\log|g_u(it)| = -\frac{\pi B(b/N)}{t} + \Re(C_{b,-a}) - \frac{1}{2} \sum_{m\geq 1} \frac{\zeta_N^{am} + \zeta_N^{-am}}{m} \Bigg(\sum_{\substack{n\geq 1 \\ n\equiv b(N)}} + \sum_{\substack{n\geq 1 \\ n\equiv -b(N)}} \Bigg) e^{-\frac{2\pi mn}{Nt}} \tag{23}$$

and

$$\mathrm{d}\arg g_u(it) = -\frac{1}{2i}\mathrm{d}\sum_{m\geq 1}\frac{\zeta_N^{bm}-\zeta_N^{-bm}}{m}\Bigg(\sum_{\substack{n\geq 1\\ n\equiv a(N)}} - \sum_{\substack{n\geq 1\\ n\equiv -a(N)}}\Bigg)e^{-\frac{2\pi mnt}{N}} \tag{24}$$

$$= \frac{1}{2i}\mathrm{d}\sum_{m\geq 1}\frac{\zeta_N^{am}-\zeta_N^{-am}}{m}\Bigg(\sum_{\substack{n\geq 1\\ n\equiv b(N)}} - \sum_{\substack{n\geq 1\\ n\equiv -b(N)}}\Bigg)e^{-\frac{2\pi mn}{Nt}}. \tag{25}$$

Let $u=(a,b)$, $v=(c,d)\in(\mathbf{Z}/N\mathbf{Z})^2$, $u,v\neq(0,0)$. We have

$$\begin{aligned}
\eta(g_u,g_v) &= \Big(-\frac{\pi B(b/N)}{t}+\Re(C_{b,-a})\Big)\cdot\frac{1}{2i}\mathrm{d}\sum_{m\geq 1}\frac{\zeta_N^{cm}-\zeta_N^{-cm}}{m}\Bigg(\sum_{\substack{n\geq 1\\ n\equiv d(N)}} - \sum_{\substack{n\geq 1\\ n\equiv -d(N)}}\Bigg)e^{-\frac{2\pi mn}{Nt}}\\
&\quad -\frac{1}{2}\sum_{m_1\geq 1}\frac{\zeta_N^{am_1}+\zeta_N^{-am_1}}{m_1}\Bigg(\sum_{\substack{n_1\geq 1\\ n_1\equiv b(N)}} + \sum_{\substack{n_1\geq 1\\ n_1\equiv -b(N)}}\Bigg)e^{-\frac{2\pi m_1n_1}{Nt}}\\
&\quad \times -\frac{1}{2i}\mathrm{d}\sum_{m_2\geq 1}\frac{\zeta_N^{dm_2}-\zeta_N^{-dm_2}}{m_2}\Bigg(\sum_{\substack{n_2\geq 1\\ n_2\equiv c(N)}} - \sum_{\substack{n_2\geq 1\\ n_2\equiv -c(N)}}\Bigg)e^{-\frac{2\pi m_2n_2t}{N}}\\
&\quad -\Big(-\frac{\pi B(d/N)}{t}+\Re(C_{d,-c})\Big)\cdot\frac{1}{2i}\mathrm{d}\sum_{m\geq 1}\frac{\zeta_N^{am}-\zeta_N^{-am}}{m}\Bigg(\sum_{\substack{n\geq 1\\ n\equiv b(N)}} - \sum_{\substack{n\geq 1\\ n\equiv -b(N)}}\Bigg)e^{-\frac{2\pi mn}{Nt}}\\
&\quad +\frac{1}{2}\sum_{m_1\geq 1}\frac{\zeta_N^{cm_1}+\zeta_N^{-cm_1}}{m_1}\Bigg(\sum_{\substack{n_1\geq 1\\ n_1\equiv d(N)}} + \sum_{\substack{n_1\geq 1\\ n_1\equiv -d(N)}}\Bigg)e^{-\frac{2\pi m_1n_1}{Nt}}\\
&\quad \times -\frac{1}{2i}\mathrm{d}\sum_{m_2\geq 1}\frac{\zeta_N^{bm_2}-\zeta_N^{-bm_2}}{m_2}\Bigg(\sum_{\substack{n_2\geq 1\\ n_2\equiv a(N)}} - \sum_{\substack{n_2\geq 1\\ n_2\equiv -a(N)}}\Bigg)e^{-\frac{2\pi m_2n_2t}{N}}.
\end{aligned} \tag{26}$$

The terms involving double sums can be integrated using Lemmas 5 and 17. This gives

$$\begin{aligned}
\int_0^\infty \eta(g_u,g_v) &= -\frac{\pi}{2}B\Big(\frac{b}{N}\Big)I(c,d)+\frac{\pi}{2}B\Big(\frac{d}{N}\Big)I(a,b)\\
&\quad +\Re(C_{b,-a})\int_0^\infty \mathrm{d}\arg g_v - \Re(C_{d,-c})\int_0^\infty \mathrm{d}\arg g_u + I
\end{aligned} \tag{27}$$

with

$$\begin{aligned}
I = \frac{\pi i}{2N}\sum_{m_1,m_2\geq 1}&\Bigg((\zeta_N^{am_1}+\zeta_N^{-am_1})(\zeta_N^{dm_2}-\zeta_N^{-dm_2})\Bigg(\sum_{\substack{n_1\geq 1\\ n_1\equiv b(N)}} + \sum_{\substack{n_1\geq 1\\ n_1\equiv -b(N)}}\Bigg)\Bigg(\sum_{\substack{n_2\geq 1\\ n_2\equiv c(N)}} - \sum_{\substack{n_2\geq 1\\ n_2\equiv -c(N)}}\Bigg)\\
&-(\zeta_N^{cm_1}+\zeta_N^{-cm_1})(\zeta_N^{bm_2}-\zeta_N^{-bm_2})\Bigg(\sum_{\substack{n_1\geq 1\\ n_1\equiv d(N)}} + \sum_{\substack{n_1\geq 1\\ n_1\equiv -d(N)}}\Bigg)\Bigg(\sum_{\substack{n_2\geq 1\\ n_2\equiv a(N)}} - \sum_{\substack{n_2\geq 1\\ n_2\equiv -a(N)}}\Bigg)\Bigg)\cdot\\
&\cdot\frac{n_2}{m_1}\int_0^\infty \exp\left(-2\pi\left(\frac{m_1n_1}{Nt}+\frac{m_2n_2t}{N}\right)\right)\mathrm{d}t.
\end{aligned} \tag{28}$$

Making the change of variables $t'=\frac{n_2}{m_1}t$, we have

$$\frac{n_2}{m_1}\int_0^\infty \exp\left(-2\pi\left(\frac{m_1n_1}{Nt}+\frac{m_2n_2t}{N}\right)\right)\mathrm{d}t = \int_0^\infty \exp\left(-2\pi\left(\frac{n_1n_2}{Nt'}+\frac{m_1m_2t'}{N}\right)\right)\mathrm{d}t'. \tag{29}$$

Replacing in (28) and interchanging integral and summation, we get

(30)
$$\begin{aligned} I = \frac{\pi i}{2N} \int_0^\infty & \sum_{m_1,m_2 \geq 1} (\zeta_N^{am_1} + \zeta_N^{-am_1})(\zeta_N^{dm_2} - \zeta_N^{-dm_2}) e^{-\frac{2\pi m_1 m_2 t'}{N}} \cdot \\ & \cdot \Bigg(\sum_{\substack{n_1 \geq 1 \\ n_1 \equiv b(N)}} + \sum_{\substack{n_1 \geq 1 \\ n_1 \equiv -b(N)}} \Bigg) \Bigg(\sum_{\substack{n_2 \geq 1 \\ n_2 \equiv c(N)}} - \sum_{\substack{n_2 \geq 1 \\ n_2 \equiv -c(N)}} \Bigg) e^{-\frac{2\pi n_1 n_2}{N t'}} \\ & - \sum_{m_1,m_2 \geq 1} (\zeta_N^{cm_1} + \zeta_N^{-cm_1})(\zeta_N^{bm_2} - \zeta_N^{-bm_2}) e^{-\frac{2\pi m_1 m_2 t'}{N}} \cdot \\ & \cdot \Bigg(\sum_{\substack{n_1 \geq 1 \\ n_1 \equiv d(N)}} + \sum_{\substack{n_1 \geq 1 \\ n_1 \equiv -d(N)}} \Bigg) \Bigg(\sum_{\substack{n_2 \geq 1 \\ n_2 \equiv a(N)}} - \sum_{\substack{n_2 \geq 1 \\ n_2 \equiv -a(N)}} \Bigg) e^{-\frac{2\pi n_1 n_2}{N t'}} \mathrm{d}t'. \end{aligned}$$

Making the change of variables $y = t'/N$, we obtain

(31)
$$\begin{aligned} I = \frac{\pi i}{2} \int_0^\infty & (f^*_{a,d} + f^*_{-a,d})(iy) \cdot (e^*_{b,c} + e^*_{-b,c})\Big(\frac{i}{N^2 y}\Big) \\ & - (f^*_{c,b} + f^*_{-c,b})(iy) \cdot (e^*_{d,a} + e^*_{-d,a})\Big(\frac{i}{N^2 y}\Big) \mathrm{d}y. \end{aligned}$$

We compute this integral using Lemma 9 with $k = \ell = 1$, taking into account Lemma 13: for any $a, b, c, d \in \mathbf{Z}/N\mathbf{Z}$, we have

(32)
$$\int_0^\infty f^*_{a,b}(iy) e^*_{c,d}\Big(\frac{i}{N^2 y}\Big) \mathrm{d}y = -\frac{i}{N^2}\Big(\Lambda^*(f_{a,b} f_{c,d}, 2) - \beta_0(a,b)\Lambda(f_{c,d}, 2)\Big) - \frac{\alpha_0(c,d)}{N} \Lambda^*(f_{a,b}, 1).$$

Replacing in (31), we get $I = I_1 + I_2 + I_3$ with

(33)
$$I_1 = \frac{\pi}{2N^2} \Lambda^*\big((f_{a,d} + f_{-a,d})(f_{b,c} + f_{-b,c}) - (f_{c,b} + f_{-c,b})(f_{d,a} + f_{-d,a}), 2\big)$$

(34)
$$I_2 = -\frac{\pi}{2N^2}\Big((\beta_0(a,d) + \beta_0(-a,d))\Lambda(f_{b,c} + f_{-b,c}, 2) - (\beta_0(c,b) + \beta_0(-c,b))\Lambda(f_{d,a} + f_{-d,a}, 2)\Big)$$

(35)
$$I_3 = -\frac{\pi i}{2N}\Big((\alpha_0(b,c) + \alpha_0(-b,c))\Lambda^*(f_{a,d} + f_{-a,d}, 1) - (\alpha_0(d,a) + \alpha_0(-d,a))\Lambda^*(f_{c,b} + f_{-c,b}, 1)\Big)$$

Using the fact that $f_{a,b} = f_{b,a} = -f_{-a,-b}$, I_1 simplifies to

(36)
$$I_1 = \frac{\pi}{N^2} \Lambda^*(f_{a,d} f_{-b,c} - f_{a,-d} f_{b,c}, 2).$$

The terms involving $\Lambda(f_{a,b}, 2)$ can be evaluated with (18); they simplify with the terms involving $I(a,b)$ in (27):

(37)
$$I_2 = \frac{\pi}{2} B\Big(\frac{b}{N}\Big) I(c,d) - \frac{\pi}{2} B\Big(\frac{d}{N}\Big) I(a,b).$$

The terms involving $\Lambda^*(f_{a,b}, 1)$ can be evaluated with (17). Note that $\alpha_0(b,c) + \alpha_0(-b,c)$ is nonzero only in the case $b = 0$ and $c \neq 0$. Since we assumed $u \neq 0$, this implies $a \neq 0$ and the case of Lemma 15 involving Euler's constant does not happen. Anyway I_3 simplifies with the terms involving $\int_0^\infty \mathrm{d} \arg g_u$ in (27):

(38)
$$I_3 = -\Re(C_{b,-a}) \int_0^\infty \mathrm{d} \arg g_v + \Re(C_{d,-c}) \int_0^\infty \mathrm{d} \arg g_u.$$

Putting everything together, we get

$$\int_0^\infty \eta(g_u, g_v) = I_1 = \frac{\pi}{N^2}\Lambda^*(f_{a,d}f_{-b,c} - f_{a,-d}f_{b,c}, 2). \tag{39}$$

Theorem 1 now follows from (10), taking into account the fact that

$$W_{N^2}(f_{a,b}f_{c,d}) = W_{N^2}(f_{a,b})W_{N^2}(f_{c,d}) = -N^2 e_{a,b}e_{c,d}.$$

□

Remark 18. — It would be interesting to find a definition of $g_{0,0}$ so that Theorem 1 holds for any vectors $u, v \in (\mathbf{Z}/N\mathbf{Z})^2$.

6. Applications

In this section we investigate the applications of Theorem 1 to elliptic curves. Our strategy can be explained as follows. In [**6**], we determined a list of elliptic curves defined over $\mathbf{Q}$ which can be parametrized by modular units. Let E be such an elliptic curve, with modular parametrization $\varphi : X_1(N) \to E$. Let x, y be functions on E such that $u := \varphi^*(x)$ and $v := \varphi^*(y)$ are modular units. Assume that $\{x, y\} \in K_2(E) \otimes \mathbf{Q}$. Then the minimal polynomial P of (x, y) is tempered and in favorable cases, the Mahler measure of P can be expressed in terms of a regulator integral $\int_\gamma \eta(x, y)$ where γ is a (non necessarily closed) path on E. Using the techniques of [**6**], we compute the images of the various cusps under φ and deduce the divisors of u and v. Since the divisors of Siegel units are easily computed using (3) and (4), we get an expression of u and v in terms of Siegel units, and may apply Theorem 1.

We will need the following expression for the regulator integral in terms of Bloch's elliptic dilogarithm. Let $E/\mathbf{Q}$ be an elliptic curve, and let $D_E : E(\mathbf{C}) \to \mathbf{R}$ be the elliptic dilogarithm associated to a chosen orientation of $E(\mathbf{R})$. Extend D_E by linearity to a function $\mathbf{Z}[E(\mathbf{C})] \to \mathbf{R}$. Let γ_E^+ be the generator of $H_1(E(\mathbf{C}), \mathbf{Z})^+$ corresponding to the chosen orientation.

Proposition 19. — *Let $x \in K_2(E) \otimes \mathbf{Q}$. Choose rational functions f_i, g_i on E such that $x = \sum_i \{f_i, g_i\}$, and define $\eta(x) = \sum_i \eta(f_i, g_i)$. Then for every $\gamma \in H_1(E(\mathbf{C}), \mathbf{Z})$, we have*

$$\int_\gamma \eta(x) = -(\gamma_E^+ \bullet \gamma) D_E(\beta)$$

where $\bullet$ denotes the intersection product on $H_1(E(\mathbf{C}), \mathbf{Z})$, and β is the divisor given by

$$\beta = \sum_i \sum_{p,q \in E(\mathbf{C})} \operatorname{ord}_p(f_i) \operatorname{ord}_q(g_i)(p - q).$$

Proof. — Since $x \in K_2(E) \otimes \mathbf{Q}$, the integral of $\eta(x)$ over a closed path γ avoiding the zeros and poles of f_i, g_i depends only on the class of γ in $H_1(E(\mathbf{C}), \mathbf{Z})$. Let δ be an element of $H_1(E(\mathbf{C}), \mathbf{Z})$ such that $\gamma_E^+ \bullet \delta = 1$. Let c denote the complex conjugation on $E(\mathbf{C})$. Since $c^*\eta(x) = -\eta(x)$, we have $\int_{\gamma_E^+} \eta(x) = 0$ and it suffices to prove the formula for $\gamma = \delta$. Choose an isomorphism $E(\mathbf{C}) \cong \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ which is compatible with complex conjugation. We have

$$\overline{\int_{E(\mathbf{C})} \eta(x) \wedge \mathrm{d}z} = \int_{E(\mathbf{C})} \eta(x) \wedge \mathrm{d}\overline{z} = \int_{E(\mathbf{C})} c^*(-\eta(x) \wedge \mathrm{d}z) = \int_{E(\mathbf{C})} \eta(x) \wedge \mathrm{d}z$$

so that $\int_{E(\mathbf{C})} \eta(x) \wedge \mathrm{d}z \in \mathbf{R}$. By [**5**, Prop. 6], we get

$$\int_{E(\mathbf{C})} \eta(x) \wedge \mathrm{d}z = D_E(\beta).$$

Since (γ_E^+, δ) is a symplectic basis of $H_1(E(\mathbf{C}), \mathbf{Z})$, we have [**3**, A.2.5]

$$\int_{E(\mathbf{C})} \eta(x) \wedge \mathrm{d}z = \int_{\gamma_E^+} \eta(x) \cdot \int_\delta \mathrm{d}z - \int_{\gamma_E^+} \mathrm{d}z \cdot \int_\delta \eta(x) = -\int_\delta \eta(x).$$

□

The following proposition is a slight generalization of a technique introduced by A. Mellit [**13**] to prove identities involving elliptic dilogarithms. Let $E/\mathbf{Q}$ be an elliptic curve, which we view as a smooth cubic in $\mathbf{P}^2$.

Definition 20. — For any lines ℓ and m in $\mathbf{P}^2$, let $\beta_E(\ell,m)$ be the divisor of degree 9 on $E(\mathbf{C})$ defined by $\beta_E(\ell,m) = \sum_{x\in\ell\cap E}\sum_{y\in m\cap E}(x-y)$.

Proposition 21. — *Let ℓ_1,ℓ_2,ℓ_3 be three incident lines in $\mathbf{P}^2$. Then*

$$D_E(\beta_E(\ell_1,\ell_2)) + D_E(\beta_E(\ell_2,\ell_3)) + D_E(\beta_E(\ell_3,\ell_1)) = 0. \tag{40}$$

Proof. — Let f_1,f_2,f_3 be equations of ℓ_1,ℓ_2,ℓ_3 such that $f_1+f_2=f_3$. Using the Steinberg relation $\{\frac{f_1}{f_3},\frac{f_2}{f_3}\}=0$, we deduce $\{f_1,f_2\}+\{f_2,f_3\}+\{f_3,f_1\}=0$ in $K_2(\mathbf{C}(E))\otimes\mathbf{Q}$. Applying the regulator map and taking the real part [**5**, Prop. 6], we deduce

$$D_E(\beta(f_1,f_2)) + D_E(\beta(f_2,f_3)) + D_E(\beta(f_3,f_1)) = 0$$

where $\beta(f_i,f_{i+1})$ is defined as in Proposition 19. We have $\mathrm{div}(f_i) = (\ell_i\cap E) - 3(0)$ so that

$$\beta(f_i,f_{i+1}) = \beta_E(\ell_i,\ell_{i+1}) - 3(\ell_i\cap E) - 3\iota^*(\ell_{i+1}\cap E) + 9(0)$$

where ι denotes the map $p\mapsto -p$ on $E(\mathbf{C})$. Since D_E is odd, the proposition follows. □

Remark 22. — If the incidence point of ℓ_1,ℓ_2,ℓ_3 lies on E, then the relation (40) is trivial in the sense that it is a consequence of the fact that D_E is odd.

We will also need the following lemma to relate elliptic dilogarithms on isogenous curves.

Lemma 23. — *Let $\varphi : E\to E'$ be an isogeny between elliptic curves defined over $\mathbf{Q}$. Choose orientations of $E(\mathbf{R})$ and $E'(\mathbf{R})$ which are compatible under φ, and let d_φ be the topological degree of the map $E(\mathbf{R})^0\to E'(\mathbf{R})^0$, where $(\cdot)^0$ denotes the connected component of the origin. Then for any point $P'\in E'(\mathbf{C})$, we have*

$$D_{E'}(P') = d_\varphi\cdot\sum_{\varphi(P)=P'} D_E(P). \tag{41}$$

Proof. — Choose isomorphisms $E(\mathbf{C})\cong\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$ and $E'(\mathbf{C})\cong\mathbf{C}/(\mathbf{Z}+\tau'\mathbf{Z})$ which are compatible with complex conjugation. Then $E(\mathbf{R})^0=\mathbf{R}/\mathbf{Z}$ and $E'(\mathbf{R})^0=\mathbf{R}/\mathbf{Z}$ so that φ is given by $[z]\mapsto[d_\varphi z]$. We have isomorphisms $E(\mathbf{C})\cong\mathbf{C}^\times/q^{\mathbf{Z}}$ and $E'(\mathbf{C})\cong\mathbf{C}^\times/(q')^{\mathbf{Z}}$ with $q=e^{2\pi i\tau}$ and $q'=e^{2\pi i\tau'}$. Let $\pi:\mathbf{C}^\times\to E(\mathbf{C})$ and $\pi':\mathbf{C}^\times\to E'(\mathbf{C})$ be the canonical maps. Let P' be a point of $E'(\mathbf{C})$. By definition $D_{E'}(P')=\sum_{\pi'(x')=P'}D(x')$ where D is the Bloch-Wigner function, and similarly $D_E(P)=\sum_{\pi(x)=P}D(x)$. Now φ is induced by the map $x\mapsto x^{d_\varphi}$, so that (41) follows from the usual functional equation $D(x^r)=r\sum_{u^r=1}D(ux)$ for any $r\geq 1$ [**15**, (21)]. □

Note that in the particular case φ is the multiplication-by-n map on E, Lemma 23 gives the usual functional equation

$$D_E(nP) = n\sum_{Q\in E[n]} D_E(P+Q).$$

6.1. Conductors 14, 35 and 54. — We prove the following cases of Boyd's conjectures [**4**, Table 5, $k=-1,-2,-3$]. Note that the case of conductor 14 was proved by A. Mellit [**13**].

Theorem 24. — *Let P_k be the polynomial $P_k(x,y)=y^2+kxy+y-x^3$, and let E_k be the elliptic curve defined by the equation $P_k(x,y)=0$. We have the identities*

$$m(P_{-1}) = 2L'(E_{-1},0) \tag{42}$$

$$m(P_{-2}) = L'(E_{-2},0) \tag{43}$$

$$m(P_{-3}) = L'(E_{-3},0). \tag{44}$$

By the discussion in [**4**, p. 62], the polynomial P_k does not vanish on the torus for $k\in\mathbf{R}$, $k<-1$. For these values of k we thus have

$$m(P_k) = \frac{1}{2\pi}\int_{\gamma_k} \eta(x,y)$$

where γ_k is the closed path on $E_k(\mathbf{C})$ defined by

$$\gamma_k = \{(x,y)\in E_k(\mathbf{C}) : |x|=1, |y|\le 1\}.$$

The point $A=(0,0)$ on E_k has order 3 and the divisors of x and y are given by

$$\operatorname{div}(x) = (A)+(-A)-2(0) \qquad \operatorname{div}(y) = 3(A)-3(0).$$

The tame symbols of $\{x,y\}$ at 0, A, $-A$ are respectively equal to $1,-1,-1$, so that $\{x,y\}$ defines an element of $K_2(E_k)\otimes\mathbf{Q}$. Moreover γ_k is a generator of $H_1(E_k(\mathbf{C}),\mathbf{Z})^-$ which satisfies $\gamma_{E_k}^+ \bullet \gamma_k = -2$, so that Proposition 19 gives

$$m(P_k) = \frac{1}{\pi} D_{E_k}(\beta(x,y)) = \frac{9}{\pi} D_{E_k}(A) \qquad (k<-1). \tag{45}$$

Note that by continuity (45) also holds for $k=-1$.

Now assume $k\in\{-1,-2,-3\}$. The elliptic curves E_{-1}, E_{-2}, E_{-3} are respectively isomorphic to $14a4$, $35a3$ and $54a3$. By [**6**], these curves are parametrized by modular units. Since the functions x and y are supported in the rational torsion subgroup, their pull-back $u=\varphi^* x$ and $v=\varphi^* y$ are modular units, and we may express them in terms of Siegel units. For brevity, we put $g_b = g_{0,b}$ in what follows. We also let f_{-k} be the newform associated to E_{-k}, and we define $\omega_{f_{-k}} = 2\pi i f_{-k}(\tau)\mathrm{d}\tau$.

In the case $k=-1$, $N=14$, we find explicitly

$$u = \frac{g_5 g_6}{g_1 g_2} \qquad v = -\frac{g_3 g_5 g_6^2}{g_1^2 g_2 g_4}.$$

We now wish to express the Deninger path γ_{-1} in terms of modular symbols. Using Magma [**2**] and Pari/GP [**18**], we compute $\int_{2/7}^{-2/7}\omega_{f_{-1}} = -\Omega_{E_{-1}}^-$, where $\Omega_{E_{-1}}^- \in i\mathbf{R}_{>0}$ is the imaginary period of E_{-1}. The Magma and Pari/GP codes to evaluate numerically both sides of this identity are as follows:

```
// Magma code to evaluate the left hand side
E:=EllipticCurve("14a4");
M:=ModularSymbols(E);
phi:=PeriodMapping(M,1000);
phi(M!<1,[Cusps()|2/7,-2/7]>);

\\ Pari/GP code to evaluate the right hand side
e=ellinit("14a4");
-(e.omega[1]-2*e.omega[2])
```

Thus we get

$$\gamma_{-1} = \varphi_* \left\{\frac{2}{7}, -\frac{2}{7}\right\} = \varphi_* \left(-\xi\begin{pmatrix}2 & 1\\ 7 & 4\end{pmatrix} - \xi\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix} + \xi\begin{pmatrix}1 & 0\\ -4 & 1\end{pmatrix} + \xi\begin{pmatrix}-2 & 1\\ 7 & -4\end{pmatrix}\right).$$

Using Theorem 1, we obtain

$$\int_{\gamma_{-1}} \eta(x,y) = \int_{2/7}^{-2/7} \eta(u,v) = \pi L'(4f_{-1}, 0).$$

This proves (42).

In the case $k = -2$, $N = 35$, we find explicitly

$$u = \frac{g_2 g_9 g_{12} g_{15} g_{16}}{g_3 g_4 g_{10} g_{11} g_{17}} \qquad v = -\frac{g_2^2 g_5 g_9^2 g_{12}^2 g_{15} g_{16}^2}{g_1 g_3 g_4 g_6 g_8 g_{10}^2 g_{11} g_{13} g_{17}}.$$

Moreover the Deninger path is the following sum of modular symbols

$$\gamma_{-2} = \varphi_* \left\{\frac{1}{5}, -\frac{1}{5}\right\} = \varphi_* \left(\xi\begin{pmatrix}1 & 0\\ -5 & 1\end{pmatrix} - \xi\begin{pmatrix}1 & 0\\ 5 & 1\end{pmatrix}\right).$$

Using Theorem 1, we obtain

$$\int_{\gamma_{-2}} \eta(x,y) = \int_{1/5}^{-1/5} \eta(u,v) = \pi L'(2f_{-2}, 0).$$

This proves (43).

In the case $k = -3$, $N = 54$, we find explicitly

$$u = \frac{g_2 g_4 g_5^2 g_{13}^2 g_{14} g_{16} g_{20} g_{21} g_{22} g_{23}^2 g_{24}}{g_1 g_7 g_8^2 g_{10}^2 g_{11} g_{12} g_{15} g_{17} g_{19} g_{25} g_{26}^2} \qquad v = -\frac{g_2^3 g_3 g_5^3 g_{13}^3 g_{16}^3 g_{20}^3 g_{21} g_{23}^3 g_{24}^2}{g_1^3 g_6 g_8^3 g_{10}^3 g_{12} g_{15}^2 g_{17}^3 g_{19}^3 g_{26}^3}.$$

Moreover the Deninger path is the following sum of modular symbols

$$\gamma_{-3} = \varphi_* \left\{-\frac{1}{8}, \frac{1}{8}\right\} = \varphi_* \left(\xi\begin{pmatrix}1 & 0\\ 8 & 1\end{pmatrix} - \xi\begin{pmatrix}1 & 0\\ -8 & 1\end{pmatrix}\right).$$

Using Theorem 1, we obtain

$$\int_{\gamma_{-3}} \eta(x,y) = \int_{-1/8}^{1/8} \eta(u,v) = \pi L'(2f_{-3}, 0).$$

This proves (43).

Using (45), we also deduce Zagier's conjectures for these elliptic curves.

Theorem 25. — *We have the identities*

$$L(E_{-1},2) = \frac{9\pi}{7} D_{E_{-1}}(A) \qquad L(E_{-2},2) = \frac{36\pi}{35} D_{E_{-2}}(A) \qquad L(E_{-3},2) = \frac{2\pi}{3} D_{E_{-3}}(A). \tag{46}$$

6.2. Conductor 21. — The modular curve $X_0(21)$ has genus 1 and is isomorphic to the elliptic curve $E_0 = 21a1$ with minimal equation $y^2 + xy = x^3 - 4x - 1$. The Mordell-Weil group $E_0(\mathbf{Q})$ is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and is generated by the points $P = (5,8)$ and $Q = (-2,1)$, with respective orders 4 and 2. The modular curve $X_0(21)$ has 4 cusps: 0, 1/3, 1/7, ∞ and we may choose the isomorphism $\varphi_0 : X_0(21) \xrightarrow{\cong} E_0$ so that $\varphi_0(0) = 0$, $\varphi_0(1/3) = (-1,-1) = P + Q$, $\varphi_0(1/7) = Q$ and $\varphi_0(\infty) = P$. Let f_P and f_Q be functions on E with divisors

$$(f_P) = 4(P) - 4(0) \qquad (f_Q) = 2(Q) - 2(0).$$

These modular units can be expressed in terms of the Dedekind η function [**12**, §3.2]:

$$f_P \sim_{\mathbf{Q}^\times} \frac{\eta(3\tau)\eta(21\tau)^5}{\eta(\tau)^5\eta(7\tau)} \qquad f_Q \sim_{\mathbf{Q}^\times} \frac{\eta(3\tau)\eta(7\tau)^3}{\eta(\tau)^3\eta(21\tau)}.$$

They can in turn be expressed in terms of Siegel units using the formula

$$\frac{\eta(d\tau)}{\eta(\tau)} = C_d \prod_{k=1}^{(d-1)/2} g_{0,kN/d}(\tau) \qquad (C_d \in \mathbf{C}^\times).$$

Thus we can take

$$f_P = \frac{g_{0,7}(\prod_{b=1}^{10} g_{0,b})^5}{g_{0,3}g_{0,6}g_{0,9}} \qquad f_Q = \frac{g_{0,7}(g_{0,3}g_{0,6}g_{0,9})^3}{\prod_{b=1}^{10} g_{0,b}}.$$

The homology group $H_1(E_0(\mathbf{C}),\mathbf{Z})^-$ is generated by the modular symbol $\gamma = \{-\frac{1}{3},\frac{1}{3}\} = \xi\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix} - \xi\begin{pmatrix}1 & 0\\ -3 & 1\end{pmatrix}$. Using Theorem 1 and a computer algebra system, we find

$$\int_\gamma \eta(f_P,f_Q) = \pi\Lambda^*(F,0)$$

where F is the modular form of weight 2 and level 21 given by

$$F = 68q + 220q^2 + 68q^3 + 508q^4 + 440q^5 + 220q^6 + 508q^7 + 1068q^8 + 68q^9 + \cdots$$

The space $M_2(\Gamma_0(21))$ has dimension 4 and is generated by f_0, $E_{2,3}$, $E_{2,7}$ and $E_{2,21}$ where f_0 is the newform associated to E_0 and $E_{2,d}(\tau) = E_2(\tau) - dE_2(d\tau)$. We find explicitly

$$F = -4f_0 + 72E_{2,3} + \frac{72}{7}E_{2,7} - \frac{72}{7}E_{2,21}$$

We have $L(E_{2,d},s) = (1-d^{1-s})\zeta(s)\zeta(s-1)$ and a little computation gives

$$L(F,s) = -4L(E_0,s) + \frac{72}{7\cdot 21^s}(7\cdot 21^s - 21\cdot 7^s - 7\cdot 3^s + 21)\zeta(s)\zeta(s-1).$$

Thus $L(F,0)=0$ and using $\zeta(0) = -1/2$ and $\zeta(-1) = -1/12$, we find

$$\Lambda^*(F,0) = \Lambda(F,0) = L'(F,0) = -4L'(E_0,0) - 6\log 7.$$

The extraneous term $6\log 7$ stems from the fact that the Milnor symbol $\{f_P,f_Q\}$ does not extend to $K_2(E_0)\otimes\mathbf{Q}$. Indeed, the tame symbols are given by

$$\partial_0\{f_P,f_Q\} = 1 \qquad \partial_P\{f_P,f_Q\} = f_Q(P)^{-4} = \zeta_7^{-4}7^{-4} \qquad \partial_Q\{f_P,f_Q\} = \zeta_7^{-4}7^4.$$

Since f_P and f_Q are supported in torsion points, there is a standard trick (due to Bloch) to alter the symbol $\{f_P,f_Q\}$ to make an element of $K_2(E_0)\otimes\mathbf{Q}$. We will see that the corresponding regulator integral is proportional to $L'(E_0,0)$ alone. We put $x := \{f_P,f_Q\} + \{7, f_P/f_Q^2\}$, which belongs to $K_2(E_0)\otimes\mathbf{Q}$, and we define

$$\eta(x) := \eta(f_P,f_Q) + \eta(7,f_P/f_Q^2) = \eta(f_P,f_Q) + \log 7\cdot \mathrm{d}\arg(f_P/f_Q^2).$$

We can compute the integral of $\mathrm{d}\arg(f_P/f_Q^2)$ using Lemma 5, which results in

$$\int_\gamma \eta(x) = -4\pi L'(E_0,0).$$

On the other hand, we have $\int_\gamma \omega_{f_0} \approx 1.91099i$ which shows that $\gamma_{E_0}^+ \bullet \gamma > 0$. Since $E_0(\mathbf{R})$ has two connected components, this implies $\gamma_{E_0}^+\bullet\gamma = 1$ and Proposition 19 gives

$$\int_\gamma \eta(x) = -D_{E_0}(\beta).$$

We have $\beta = 8(P+Q) - 8(P) - 8(Q) + 8(0)$. Since D_{E_0} is odd, this gives

$$\int_\gamma \eta(x) = -8\big(D_{E_0}(P+Q) - D_{E_0}(P)\big).$$

Taking into account the functional equation $L'(E_0,0) = \frac{21}{4\pi^2}L(E_0,2)$, we have thus shown Zagier's conjecture for E_0.

Theorem 26. — *We have the identity* $L(E_0,2) = \frac{8\pi}{21}\big(D_{E_0}(P+Q) - D_{E_0}(P)\big)$.

We will now deduce Boyd's conjecture [**4**, Table 1, $k=3$] for the elliptic curve E_1 of conductor 21 given by the equation $P(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + 3 = 0$.

Theorem 27. — *We have the identity* $m(x+\frac{1}{x}+y+\frac{1}{y}+3) = 2L'(E_1,0)$.

The change of variables

$$X = x(x+y+3)+1 \qquad Y = x(x+1)(x+y+3)+1$$

puts E_1 in the Weierstrass form $Y^2+XY = X^3+X$. This is the elliptic curve labelled $21a4$ in Cremona's tables [**7**]. The Mordell-Weil group $E_1(\mathbf{Q})$ is isomorphic to $\mathbf{Z}/4\mathbf{Z}$ and is generated by $P_1 = (1,1)$.

The polynomial P satisfies Deninger's conditions [**8**, 3.2], so we have

$$m(P) = \frac{1}{2\pi}\int_{\gamma_P} \eta(x,y)$$

where γ_P is the path defined by $\gamma_P = \{(x,y) \in E_1(\mathbf{C}) : |x| = 1, |y| \leq 1\}$. The path γ_P joins the point $\bar{A} = (\bar{\zeta}_3,-1)$ to $A = (\zeta_3,-1)$. Note that these points have last coordinate -1, so the discussion in [**8**, p. 272] applies and γ_P defines an element of $H_1(E_1(\mathbf{C}),\mathbf{Q})$. After some computation, we find that $\gamma_P = \frac{1}{2}\gamma_1$ where γ_1 is a generator of $H_1(E_1(\mathbf{C}),\mathbf{Z})^-$ such that $\gamma_{E_1}^+ \bullet \gamma_1 = 2$ (note that $E_1(\mathbf{R})$ is connected). Using Proposition 19, it follows that

$$\int_{\gamma_P} \eta(x,y) = \frac{1}{2}\int_{\gamma_1}\eta(x,y) = -D_{E_1}(\beta)$$

where $\beta = \operatorname{div}(x) * \operatorname{div}(y)^-$ is the convolution of the divisors of x and y. We have

$$\operatorname{div}(x) = (P_1)+(2P_1)-(-P_1)-(0) \qquad \operatorname{div}(y) = (P_1)-(2P_1)-(-P_1)+(0)$$

so that $\beta = 4(P_1) - 4(-P_1)$. This gives

$$\int_{\gamma_P}\eta(x,y) = -8D_{E_1}(P_1).$$

We are now going to relate elliptic dilogarithms on E_1 and E_0 using Proposition 21 and Lemma 23. The curve E_1 is the $X_1(21)$-optimal elliptic curve in the isogeny class of E_0. We have a 2-isogeny $\lambda : E_1 \to E_0$ whose kernel is generated by $2P_1 = (0,0)$. Using Vélu's formulas [**19**], we find that an equation of λ is

$$\lambda(X,Y) = \Big(\frac{X^2+1}{X}, -\frac{1}{X} + \frac{X^2-1}{X^2}Y\Big).$$

The preimages of $P+Q$ under λ are the points $A = (\zeta_3, -1-\zeta_3)$ and $\bar{A} = (\bar{\zeta}_3, -1-\bar{\zeta}_3)$, while the preimages of P are given by $B = (\frac{5+\sqrt{21}}{2}, 4+\sqrt{21})$ and $B' = (\frac{5-\sqrt{21}}{2}, 4-\sqrt{21})$. Note that $2A = -P_1$ and $2B = P_1$ so that A and B have order 8 and we have the relations $\bar{A} = A + 2P_1 = 5A$ and $B' = 5B$. Moreover $C = A+B$ is the 2-torsion point given by $C = (\frac{-1+3i\sqrt{7}}{8}, \frac{1-3i\sqrt{7}}{16})$. Using Theorem 26 and Lemma 23, we have

$$L'(E_0,0) = \frac{4}{\pi}\big(D_{E_1}(A) + D_{E_1}(\bar{A}) - D_{E_1}(B) - D_{E_1}(B')\big)$$

so that Theorem 27 reduces to show

$$D_{E_1}(P_1) = -2\big(2D_{E_1}(A) - D_{E_1}(B) - D_{E_1}(B')\big).$$

We look for lines ℓ in $\mathbf{P}^2$ such that $\ell \cap E_1$ is contained in the subgroup generated by A and B. Using a computer search, we find that the tangents to E at A and $-A$ and the line

$\ell : Y + \frac{1}{2}X = 0$ passing through the 2-torsion points of E are incident. By Proposition 21, we deduce the relation

$$\begin{aligned}4D_{E_1}(2A) + 4D_{E_1}(3A) + D_{E_1}(4A) + 2D_{E_1}(-2A) + 4D_{E_1}(-A) \\ + 2D_{E_1}(2A+C) + 4D_{E_1}(3A+C) + 2D_{E_1}(-2A+C) + 4D_{E_1}(-A+C) = 0.\end{aligned}$$

Since D_{E_1} is odd and $D_{E_1}(3A) = -D_{E_1}(\bar{A}) = -D_{E_1}(A)$, this simplifies to

$$2D_{E_1}(2A) - 8D_{E_1}(A) + 4D_{E_1}(B) + 4D_{E_1}(B') = 0$$

which is the desired equality.

6.3. Conductor 48. — We prove the following case of Boyd's conjectures [**4**, Table 1, $k = 12$].

Theorem 28. — *We have the identity* $m(x + \frac{1}{x} + y + \frac{1}{y} + 12) = 2L'(E, 0)$, *where* E *is the elliptic curve defined by* $x + \frac{1}{x} + y + \frac{1}{y} + 12 = 0$.

The curve $x + \frac{1}{x} + y + \frac{1}{y} + 12 = 0$ is isomorphic to the elliptic curve $E = 48a5$. We have a commutative diagram

(47)
$$\begin{tikzcd} X_1(48) \arrow[r, "\pi"] \arrow[d, "\varphi_1"] & X_0(48) \arrow[d, "\varphi_0"] & \\ E_1 \arrow[r, "\lambda_0"] & E_0 \arrow[r, "\lambda"] & E. \end{tikzcd}$$

Here $E_1 = 48a4$ is the $X_1(48)$-optimal elliptic curve and $E_0 = 48a1$ is the strong Weil curve in the isogeny class of E. They are given by the equations

(48)
$$E_1 : y^2 = x^3 + x^2 + x \qquad E_0 : y^2 = x^3 + x^2 - 4x - 4.$$

The isogeny λ_0 has degree 2 and its kernel is generated by $P_1 = (0, 0)$. Using Vélu's formulas, we find an explicit equation for λ_0:

(49)
$$\lambda_0(x, y) = \left(x + \frac{1}{x}, (1 - \frac{1}{x^2})y\right).$$

The modular parametrization φ_0 has degree 2 and we have

$$\begin{gathered}\varphi_0(0) = \varphi_0(1/2) = 0 \qquad \varphi_0(1/3) = \varphi_0(1/6) = (-1, 0)\\ \varphi_0(1/8) = \varphi_0(1/16) = (-2, 0) \qquad \varphi_0(1/24) = \varphi_0(1/48) = (2, 0)\\ \varphi_0(1/4) = (0, 2i) \qquad \varphi_0(-1/4) = (0, -2i)\\ \varphi_0(1/12) = (-4, -6i) \qquad \varphi_0(-1/12) = (-4, 6i).\end{gathered}$$

Moreover the ramification indices of φ_0 at the cusps $\frac{1}{4}, -\frac{1}{4}, \frac{1}{12}, -\frac{1}{12}$ are equal to 2. Let S_0 be the set of points P of $E_0(\mathbf{C})$ such that $\varphi_0^{-1}(P)$ is contained in the set of cusps of $X_0(48)$, and similarly let S_1 be the set of points P of $E_1(\mathbf{C})$ such that $\varphi_1^{-1}(P)$ is contained in the set of cusps of $X_1(48)$. By the previous computation, we have

(50)
$$S_0 = E_0[2] \cup \{(0, \pm 2i), (-4, \pm 6i)\}.$$

The curve E_0 doesn't admit a parametrization by modular units, but the curve E_1 does. Indeed, consider the point $A = (i, i) \in E_1(\mathbf{C})$. It has order 8 and satisfies $\bar{A} = 3A$ and $4A = P_1$. Moreover $\lambda_0(A) = (0, 2i)$. Because of the commutative diagram (47), we know that S_1 contains $\lambda_0^{-1}(S_0)$; in particular S_1 contains the subgroup generated by A. Therefore the following functions on E_1 are modular units

(51)
$$(f) = 2(P_1) - 2(0) \qquad (g) = 2(A) + 2(\bar{A}) - 4(0).$$

We may take $f = x$ and $g = x^2 - 2y + 2x + 1$. It is plain that f and g parametrize E_1. Moreover the tame symbols of $\{f,g\}$ at $0, P_1, A, \bar{A}$ are equal to $1,1,-1,-1$ so that $\{f,g\}$ belongs to $K_2(E_1) \otimes \mathbf{Q}$. The expression of f and g in terms of Siegel units is

$$\varphi_1^* f = \frac{g_2 g_{20} g_{22}}{g_4 g_{10} g_{14}} \qquad \varphi_1^* g = \frac{g_1^2 g_2 g_{10} g_{11}^2 g_{12}^4 g_{13}^2 g_{14} g_{22} g_{23}^2}{g_4^3 g_5^2 g_6^2 g_7^2 g_{17}^2 g_{18}^2 g_{19}^2 g_{20}}. \tag{52}$$

A generator γ_1 of $H_1(E_1(\mathbf{C}),\mathbf{Z})^-$ is given by

$$\gamma_1 = (\varphi_1)_* \left\{-\frac{1}{7}, \frac{1}{7}\right\} = (\varphi_1)_* \left(\xi\begin{pmatrix} 1 & 0 \\ 7 & 1 \end{pmatrix} - \xi\begin{pmatrix} 1 & 0 \\ -7 & 1 \end{pmatrix}\right).$$

Using Theorem 1, we find

$$\int_{\gamma_1} \eta(f,g) = \int_{-1/7}^{1/7} \eta(\varphi_1^* f, \varphi_1^* g) = \pi L'(F_1, 0) \tag{53}$$

where F_1 is the modular form of weight 2 and level 48 given by

$$F_1 = 4q^2 + 8q^3 - 4q^6 - 8q^{10} - 32q^{11} - 16q^{15} + 4q^{18} + 32q^{19} + \ldots$$

This time F_1 is not a multiple of the newform f_{E_1} associated to E_1. We look for another modular symbol. Another generator γ_2 of $H_1(E_1(\mathbf{C}),\mathbf{Z})^-$ is given by

$$\gamma_2 = (\varphi_1)_* \left\{-\frac{2}{11}, \frac{2}{11}\right\} = (\varphi_1)_* \left(\xi\begin{pmatrix} 2 & 1 \\ 11 & 6 \end{pmatrix} + \xi\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} - \xi\begin{pmatrix} -2 & 1 \\ 11 & -6 \end{pmatrix} - \xi\begin{pmatrix} 1 & 0 \\ -6 & 1 \end{pmatrix}\right).$$

Using Theorem 1, we find

$$\int_{\gamma_2} \eta(f,g) = \int_{-2/11}^{2/11} \eta(\varphi_1^* f, \varphi_1^* g) = \pi L'(F_2, 0) \tag{54}$$

where F_2 is the modular form of weight 2 and level 48 given by

$$F_2 = -4q + 8q^2 + 12q^3 + 8q^5 - 8q^6 - 4q^9 - 16q^{10} - 48q^{11} + 8q^{13} - 24q^{15} - 8q^{17} + 8q^{18} + 48q^{19} + \ldots$$

A computation reveals that $2F_1 - F_2 = 4f_{E_1}$. Summing (53) and (54), we get

$$\int_{2\gamma_1 - \gamma_2} \eta(f,g) = 4\pi L'(E_1, 0). \tag{55}$$

Since $\gamma_{E_1}^+ \bullet \gamma_1 = \gamma_{E_1}^+ \bullet \gamma_2 = 2$, Proposition 19 gives

$$\int_{2\gamma_1 - \gamma_2} \eta(f,g) = -2D_{E_1}(\beta(f,g)) = -32 D_{E_1}(A). \tag{56}$$

Combining (55) and (56), we have thus shown Zagier's conjecture for E_1.

Theorem 29. — *We have the identities $L'(E_1,0) = -\frac{8}{\pi} D_{E_1}(A)$ and $L(E_1,2) = -\frac{2\pi}{3} D_{E_1}(A)$.*

Let us now turn to the elliptic curve E. Let P_k be the polynomial $P_k(x,y) = x + 1/x + y + 1/y + k$. For $k \notin \{0, \pm 4\}$, let C_k be the elliptic curve defined by $P_k(x,y) = 0$. The change of variables

$$X = 4x(x+y+k) \qquad Y = 8x^2(x+y+k)$$

puts C_k in Weierstrass form $Y^2 + 2kXY + 8kY = X^3 + 4X^2$. The point $Q = (0,0)$ on C_k has order 4. We show that the Mahler measure of P_k can be expressed in terms of the elliptic dilogarithm.

Proposition 30. — *Let k be a real number such that $|k| > 4$. We have*

$$m(P_k) = \begin{cases} -\frac{4}{\pi} D_{C_k}(Q) & \text{if } k > 0, \\ \frac{4}{\pi} D_{C_k}(Q) & \text{if } k < 0. \end{cases}$$

Proof. — Since $|k| > 4$, the polynomial P_k doesn't vanish on the torus, so that

$$m(P_k) = \frac{1}{2\pi}\int_{\gamma_k} \eta(x,y)$$

where γ_k is the closed path on $C_k(\mathbf{C})$ defined by

$$\gamma_k = \{(x,y) \in C_k(\mathbf{C}) : |x| = 1, |y| \leq 1\}.$$

It turns out that γ_k is a generator of $H_1(C_k(\mathbf{C}),\mathbf{Z})^-$ which satisfies $\gamma_{C_k}^+ \bullet \gamma_k = \operatorname{sgn}(k)$. The divisors of x and y are given by

$$\operatorname{div}(x) = (Q) + (2Q) - (-Q) - (0) \qquad \operatorname{div}(y) = (Q) - (2Q) - (-Q) + (0).$$

Since P_k is tempered, we have $\{x,y\} \in K_2(C_k) \otimes \mathbf{Q}$, and Proposition 19 gives

$$\int_{\gamma_k} \eta(x,y) = -\operatorname{sgn}(k) D_{C_k}(\beta(x,y)) = -8\operatorname{sgn}(k) D_{C_k}(Q).$$

□

Remark 31. — The fact that $m(P_k)$ can be expressed as an Eisenstein-Kronecker series was also proved by F. Rodriguez-Villegas [**20**].

We are now going to relate elliptic dilogarithms on $E = C_{12}$ and E_1. Let $\lambda' : E_1 \to E$ be the isogeny $\lambda \circ \lambda_0$ from (47). It is cyclic of degree 8 and its kernel is generated by the point $B = (-2-\sqrt{3}, 3i + 2i\sqrt{3})$. A preimage of Q under λ' is given by

$$C = \left(\frac{1}{2}(\alpha^3 + \alpha^2 + \alpha - 1), \frac{1}{2}(\alpha^3 + \alpha^2 - \alpha - 3)\right)$$

with $\alpha = \sqrt[4]{-3}$. The point C has order 4 and we have $A = B + 2C$. By Lemma 23, we have

$$D_E(Q) = 2 \sum_{k \in \mathbf{Z}/8\mathbf{Z}} D_{E_1}(C + kB). \tag{57}$$

Combining Theorem 29, Proposition 30 and (57), Theorem 28 reduces to show

$$\sum_{k \in \mathbf{Z}/8\mathbf{Z}} D_{E_1}(C + kB) = 2D_{E_1}(A). \tag{58}$$

Let T be the subgroup generated by B and C. It is isomorphic to $\mathbf{Z}/8\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$. There are 187 lines ℓ of $\mathbf{P}^2$ such that $\ell \cap E_1$ is contained in T. A computer search reveals that among them, there are 691 unordered triples of lines meeting at a point outside E_1. These incident lines yield a subgroup $\mathcal{R}$ of $\mathbf{Z}[T]$ of rank 18 such that $D_{E_1}(\mathcal{R}) = 0$. Let $\mathcal{R}_{\mathrm{triv}}$ be the subgroup of $\mathbf{Z}[T]$ generated by the following elements

$$[P] - [\bar{P}], \qquad [P] + [-P], \qquad [2P] - 2\sum_{Q \in E_1[2]} [P + Q] \qquad (P, Q \in T). \tag{59}$$

The group $\mathcal{R}_{\mathrm{triv}}$ has rank 26 and by Lemma 23, we have $D_{E_1}(\mathcal{R}_{\mathrm{triv}}) = 0$. Moreover $\mathcal{R} + \mathcal{R}_{\mathrm{triv}}$ has rank 27 and a generator of $(\mathcal{R} + \mathcal{R}_{\mathrm{triv}})/\mathcal{R}_{\mathrm{triv}}$ is given (for example) by the divisor

$$\beta = \beta_{E_1}(\ell_1, \ell_2) + \beta_{E_1}(\ell_2, \ell_3) + \beta_{E_1}(\ell_3, \ell_1)$$

where ℓ_1, ℓ_2, ℓ_3 are the lines defined by

$$\begin{aligned}
\ell_1 \cap E_1 &= (B) + (-B) + (0) \\
\ell_2 \cap E_1 &= (B + 2C) + (B - C) + (-2B - C) \\
\ell_3 \cap E_1 &= (4B + C) + (-3B + 2C) + (-B + C).
\end{aligned}$$

Computing explicitly, this gives

$$\beta = 2\left(\sum_{k\in\mathbf{Z}/8\mathbf{Z}} (2C+kB)+(3C+kB)\right) - 2(-A) - 2(-\bar{A}) + (4B) - (2C) - (4B+2C).$$

Using the functional equations (59) of D_{E_1}, we obtain (58).

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