# Non-Equilibrium Statistical Mechanics of the 2D Stochastic Navier-Stokes Equations and Geostrophic Turbulence

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$5^{th}$ Warsaw School of Statistical Physics.

# Collaborators

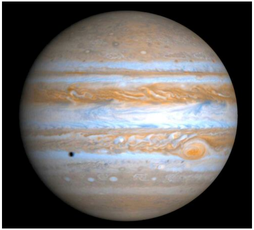
- Equilibrium statistical mechanics of ocean jets and vortices: A. Venaille (PHD student, post doc in Princeton, now in ENS-Lyon)
- Equilibrium statistical mechanics of the Great Red Spot of Jupiter: J. Sommeria (LEGI-Coriolis, Grenoble)
- Random changes of flow topology in the 2D Navier-Stokes equations: E. Simonnet (INLN-Nice) (ANR Statocean)
- Asymptotic stability and inviscid damping of the 2D-Euler equations: H. Morita (Tokyo university) (ANR Statflow)

## Collaborators

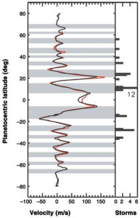
- Invariant measures of the 2D Euler and Vlasov equations: M. Corvellec (PHD student, INLN Nice, CNLS Los Alamos and ENS-Lyon)
- Instantons and large deviations for the 2D Navier-Stokes equations: J. Laurie (Post-doc ANR Statocean), O. Zaboronski (Warwick Univ.)
- Large deviations for systems with connected attractors: H. Touchette (Queen Mary Univ, London)
- Stochastic Averaging and Jet Formation in Geostrophic Turbulence: C. Nardini and T. Tangarife (ENS-Lyon)
- Phase transitions in rotating tank experiments: J. Sommeria (LEGI-Grenoble) and M. Mathur (Post-doc ANR Statocean, now in India)

# Earth and Jupiter's Zonal Jets

## We look for a theoretical description of zonal jets



Jupiter atmosphere



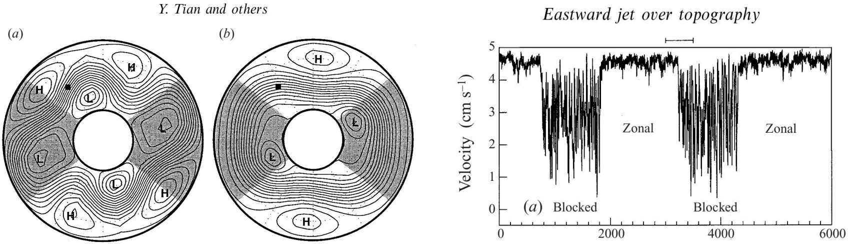

Jupiter Zonal wind (Voyager and Cassini, from Porco et al 2003)

How to theoretically predict such a velocity profile?

# Phase Transitions in Rotating Tank Experiments

## The rotation as an ordering field (Quasi Geostrophic dynamics)

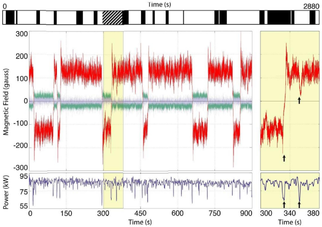
### Transitions between blocked and zonal states



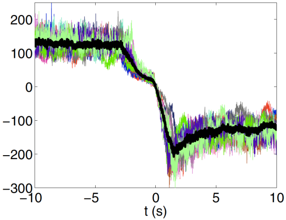

*Y. Tian and col, J. Fluid. Mech. (2001)* (groups of H. Swinney and M. Ghil)

# Random Transitions in Turbulence Problems

## Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)




Magnetic field timeseries




Zoom on reversal paths

(VKS experiment)

In turbulent flows, transitions from one attractor to another often occur through a predictable path.

# The 2D Navier-Stokes Equations

- Navier Stokes equations for an incompressible flow:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} = -\frac{\nabla P}{\rho} + \nu\Delta\mathbf{v} \text{ with } \nabla.\mathbf{v} = 0$$

- In a two-dimensional space: $\mathbf{v} = v_x(x,y,t)\mathbf{e}_x + v_y(x,y,t)\mathbf{e}_y$, the vorticity is a scalar

$$\omega = \nabla \times \mathbf{v} = \omega(x,y,t)\mathbf{e}_z.$$

- One easily checks that

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega.$$

- This equation is much simpler than its three dimensional counterpart.

# The 2D Stochastic-Navier-Stokes (SNS) Equations

- The simplest model for two dimensional turbulence
- Navier Stokes equations with random forces

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{\sigma} f_s$$

where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, $f_s$ is a random force, $\alpha$ is the Rayleigh friction coefficient.

- An academic model with experimental realizations (Sommeria and Tabeling experiments, rotating tanks, magnetic flows, and so on). Analogies with geophysical flows (Quasi Geostrophic and Shallow Water layer models)

# Equilibrium: the 2D Euler Equations

- 2D Euler equations:

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = 0$$

Vorticity $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$. Stream function $\psi$: $\mathbf{v} = \mathbf{e}_z \times \nabla \psi$, $\omega = \Delta \psi$

- Conservative dynamics - Hamiltonian (non canonical) and time reversible.

# Soap film experiments

*H. Kellay*

# Experiments in Thin Stratified Layers

J. Paret and P. Tabeling 3127

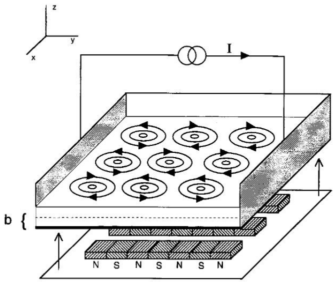



FIG. 1. The experimental set-up.

*P. Tabeling*

# Electron Plasma Experiments

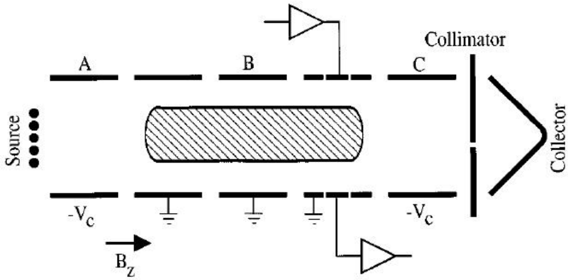



FIG. 1. Schematic of the cylindrical confinement geometry.

*C. F. Driscoll*

# Electron Plasma Experiments

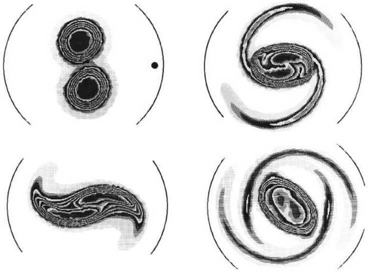


FIG. 8. $n(r,\theta,t)$ density (vorticity) plots of two symmetric vortices unstable to merger, at times 0, 16, 41 and 76 $\mu$s. The density between solid contours is $2.9\times 10^5$ cm$^{-3}$. Here, the vortices have radii $r_{v2}=r_{v1}=0.25$ and radial positions $r_2=r_1=0.30$.

*C. F. Driscoll*

## The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s,$$

  where $\omega=(\nabla\wedge\mathbf{v}).\mathbf{e}_z$ is the vorticity, $q=\omega+\beta_d y$ is the Potential Vorticity (PV), $f_s$ is a random force, $\alpha$ is the Rayleigh friction coefficient.
- Quasi-Gestrophic models: the basic models for midlatitude large scale turbulence.

# The Main Issues for Physicists

- What makes the dynamics of geophysical flows so peculiar?
- Why do the large scales of geophysical flows self-organize?
- Can we predict the statistics of the large scales of geophysical flows?
- Can we predict phase transitions for geophysical turbulent flows and their statistics?

# Statistical Mechanics for 2D and Geophysical Flows

- Statistical hydrodynamics ? Very complex problems.
- Example: Intermittency in 3D turbulence ; phenomenological approach, simplified models (Kraichnan).
- It may be much simpler for 2D or geophysical flows: conservative systems.
- Statistical equilibrium: A very old idea, some famous contributions Onsager (1949), Joyce and Montgomery (1970), Caglioti Marchioro Pulvirenti Lions (1990), Robert (1990), Miller (1990), Robert et Sommeria (1991), Eyink and Spohn (1994), Kiessling and Lebowitz (1994), Bodineau and Guionnet (1999), Boucher, Ellis and Turkington (1999)

# Non-Equilibrium Stat. Mech.

1. Stochastic averaging technics (kinetic theory in a stochastic framework).
2. Large deviations for transition probabilities (for rare events) through path integrals.
3. Tools from field theory in statistical physics.

# Global Outline

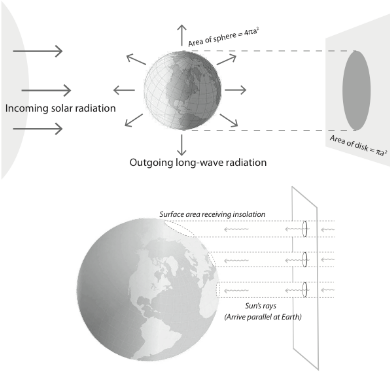
I) Introduction to geophysical fluid dynamics and the quasi-geostrophic model
II) Equilibrium Statistical mechanics of geostrophic turbulence
III) Non-equilibrium phase transitions, path integrals and instanton theory
IV) Kinetic theory (stochastic averaging) of zonal jet dynamics

# Outline for Today

1. Basics of mid-latitude atmosphere dynamics.
   - What is the energy source?
   - Weather, pressure and wind.
   - Quasi-geostrophic dynamics of atmosphere jets.
2. The 2D Euler and quasi-geostrophic equations
   - The inertial equations are Hamiltonian
   - Multiple invariants, steady states and attractors
   - Liouville theorem
3. Equilibrium statistical mechanics of 2D and geostrophic turbulence
   - Microcanonical measure
   - Large deviations, entropy, and the mean field variational problem
   - Applications of equilibrium statistical mechanics

# Outline

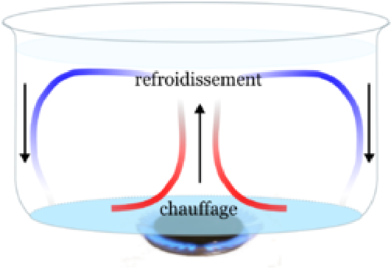
# The Sun is the main Source of Energy on Earth




Why and how insolation changes with latitude.

- The source of the motion is the temperature difference between equator and poles, due to differences in insolation.

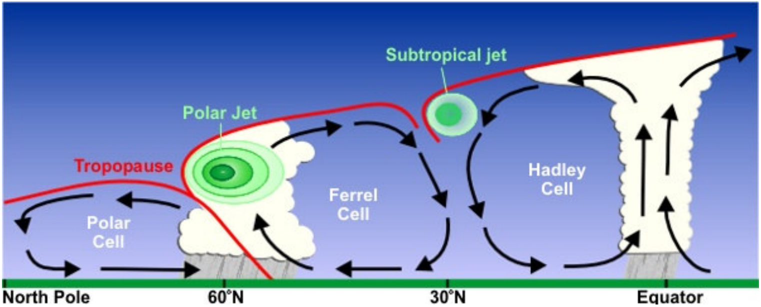
# Convection: Temperature Differences Create Motion in a Fluid




Convection in a pan heated from below

Rayleigh–Bénard convection

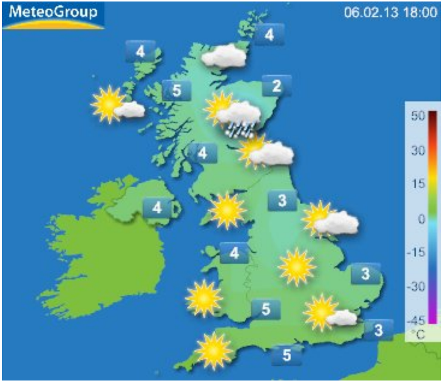
# Atmosphere Convection




Atmosphere convection cells

# Outline

# Weather Maps Dynamics

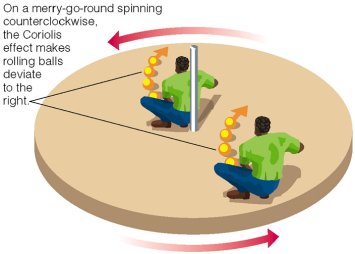



- What is the relation between weather maps and the convection picture?

# Pressure Maps Explain Weather


- Weather is correlated to pressure maps. Lows (low pressure) are cyclones. High (high pressure) are anticyclones.
- Can we understand the dynamics of pressure maps?

# Coriolis Force






- Coriolis force is a force felt by a body moving with respect to a rotating reference frame.
- Coriolis force (depends on velocity) is different from centrifugal force (does not depend on velocity).

# The Navier-Stokes Eq. in a Rotating Ref. Frame

- Second law of Newton:

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}.\nabla\mathbf{u}=-\frac{\nabla P}{\rho}-2\Omega\wedge\mathbf{u}+\nu\Delta\mathbf{u}+\mathbf{g}.$$

- Rossby number compares the order of magnitude of the Coriolis force and advection terms

$$\varepsilon=\frac{\mathscr{O}\left(\mathbf{u}.\nabla\mathbf{u}\right)}{\mathscr{O}\left(2\Omega\wedge\mathbf{u}\right)}=\frac{U}{2\Omega L}\quad\varepsilon=\frac{10\,\mathrm{m.s}^{-1}}{2\times2\pi/10^{5}\,\mathrm{s}^{-1}\times10^{6}\,\mathrm{m}}\simeq0,1$$

# The Navier-Stokes Eq. in a Rotating Ref. Frame

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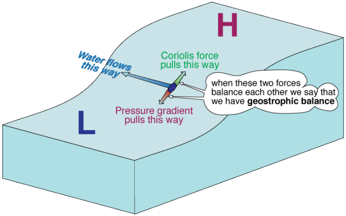
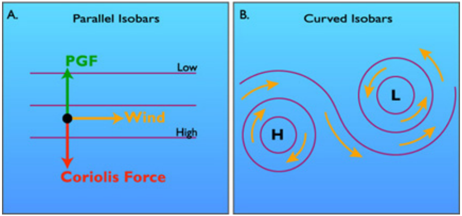
$$\varepsilon=\frac{\mathscr{O}(\mathbf{u}.\nabla\mathbf{u})}{\mathscr{O}(2\Omega\wedge\mathbf{u})}=\frac{U}{2\Omega L}\quad \varepsilon=\frac{10\,\mathrm{m.s}^{-1}}{2\times 2\pi/10^5\,\mathrm{s}^{-1}\times 10^6\,\mathrm{m}}\simeq 0,1$$

- In the atmosphere, horizontal dynamics is dominated by the geostrophic equilibrium (between pressure gradient and the Coriolis force).

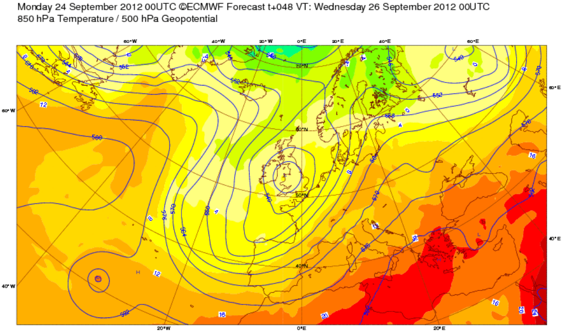
# Winds Flow along Constant Pressure Lines




- In the north hemisphere, winds flow clockwise around anticyclone (high pressure) and flow anti-clockwise around cyclones (high pressure).

# The Synoptic Scale

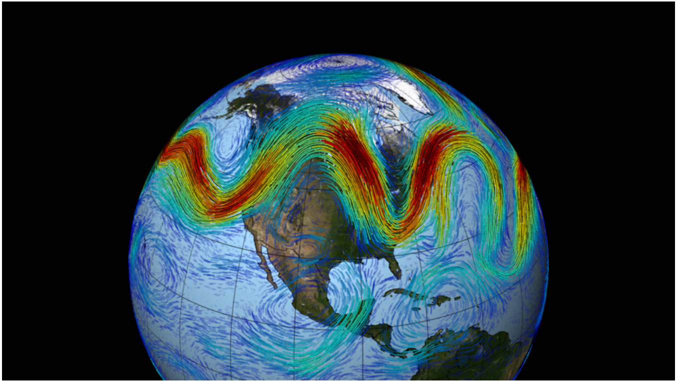



Temperature and pressure maps over Europe (26/09/2012)

- The good scale to understand atmosphere dynamics is synoptic scale (thousands of kilometers).

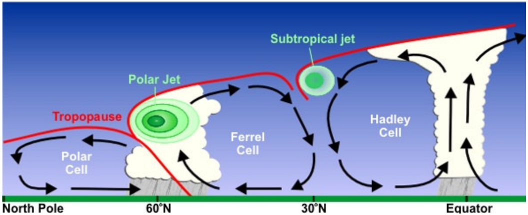
# Outline

# Atmosphere Jet Dynamics

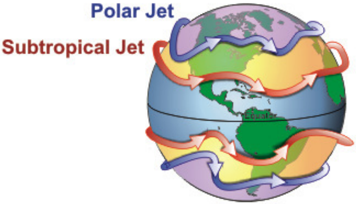


Upper atmosphere velocity

# Atmosphere Convection and Jet Streams




Atmosphere convection cells




Schematic jet stream configuration

# The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence. Obtained in the limit $\varepsilon \to 0$.
- Geostrophic balance: $\mathbf{v} = -\nabla\psi \wedge \mathbf{e}_z$ with $\psi$ proportional to the pressure. The Coriolis parameter is $2\Omega\cos y \simeq 2\Omega\cos y_0 + \beta y$.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu_d \Delta\omega - \lambda\omega + \sqrt{2\varepsilon} f_s,$$

  where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, $q = \omega + \beta y$ is the Potential Vorticity (PV), $f_s$ is a random Gaussian field with correlation $\langle f_S(\mathbf{x},t) f_S(\mathbf{x}',t')\rangle = C(\mathbf{x}-\mathbf{x}')\delta(t-t')$, $\varepsilon$ is the average energy input rate, $\lambda$ is the Rayleigh friction coefficient.
- Quasi-Geostrophic models: the basic models for midlatitude large scale dynamics.

# Outline

# The 2D Euler Equations

- 2D Euler equations:

$$\frac{\partial \omega}{\partial t}+\mathbf{v}\left[\omega\right].\nabla\omega=0,$$

Vorticity $\omega=\left(\nabla\wedge\mathbf{v}\right).\mathbf{e}_z$. Stream function $\psi$: $\mathbf{v}=\mathbf{e}_z\times\nabla\psi$, $\omega=\Delta\psi$.

- Conservative dynamics - Hamiltonian (non canonical) and time reversible.
- Invariants:

$$\text{Energy: } E\left[\omega\right]=\frac{1}{2}\int_{\mathscr{D}}\mathbf{v}^2\,d\mathbf{r}=-\frac{1}{2}\int_{\mathscr{D}}\omega\psi\,d\mathbf{r},$$

$$\text{Casimir's functionals: } \mathscr{C}_s[\omega]=\int_{\mathscr{D}}s(\omega)\,d\mathbf{r},$$

$$\text{Vorticity distribution: } D\left(\sigma\right)=\frac{dA}{d\sigma}\text{ with } A\left(\sigma\right)=\int_D\chi_{\{\omega(\mathbf{x})\leq\sigma\}}\,d\mathbf{r}.$$

# The (Inertial) Quasi-Geostrophic Eq.

- Quasi-Geostrophic equation with no forces and dissipation (inertial)
$$\frac{\partial q}{\partial t} + \mathbf{v}\left[q-h\right].\nabla q = 0,$$
with vorticity $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$, stream function $\psi$: $\mathbf{v} = \mathbf{e}_z \times \nabla \psi$, and PV $q = \Delta\psi + h(y)$.
- Conservative dynamics - Hamiltonian (non canonical) and time reversible.
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$$\text{Energy:}\quad E\left[q\right] = \frac{1}{2}\int_{\mathscr{D}} \mathbf{v}^2\,\mathrm{d}\mathbf{r} = -\frac{1}{2}\int_{\mathscr{D}} (q-h)\,\psi\,\mathrm{d}\mathbf{r},$$

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# Outline

# Consequences of Multiple Invariants of 2D flows I)

## Multiple steady solutions – Multiple stable steady solutions

- Any non degenerate minimum of a conserved quantity is a stable steady solution (think to mechanics and energy).
- Energy-Casimir functionals (Arnold 1966).
- Multiple invariants imply degeneracy of steady solutions to the 2D Euler Eq.: $\mathbf{v} = \mathbf{e}_z \times \nabla \psi$

$$\omega = \Delta \psi = f(\psi) \Rightarrow \mathbf{v}.\nabla \omega = (\nabla \psi \times \nabla \omega).\mathbf{e}_z = 0.$$

- To any solution of $\Delta \psi = f(\psi)$ corresponds an equilibrium solution.

# PV-Psi Relation for Jupiter's Great Red Spot

## From Dowling and Ingersoll empirical analysis (1989)

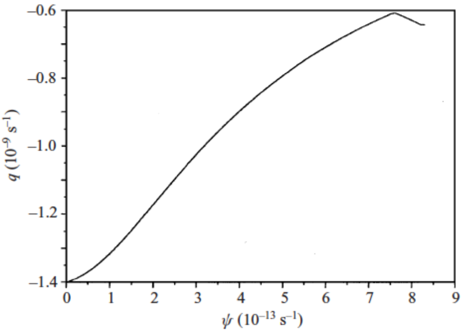
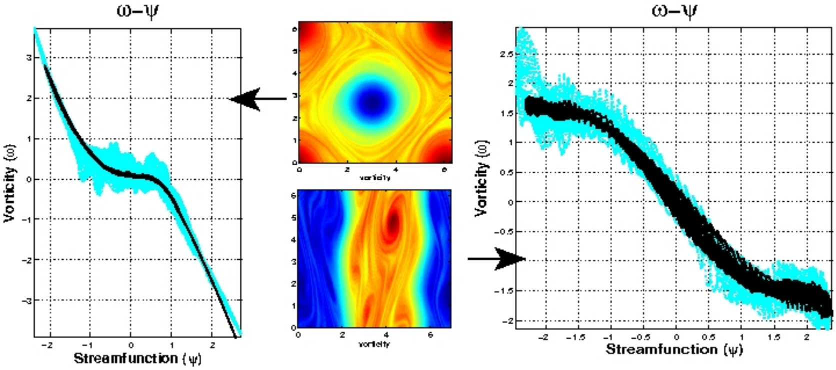



FIGURE 12. Potential vorticity $q^*_{SW}$ versus stream function $\Psi^*_{SW}$ from the determination of $q^*_{SW}(B^*_e)$ by Dowling & Ingersoll (1989) (their table 1) in the Great Red Spot (for $R = 2200$ km $q: 10^{-9}\ \mathrm{s}^{-1}$ and $\Psi: 10^{13}\ \mathrm{m\,s}^{-1}$). These SW potential vorticity and stream function are proportional to the QG ones in the QG limit. We observe that this function is in reasonable agreement with the tanh-like relation of the two-PV-level Gibbs states.

# Vorticity-Streamfunction Relation




Conclusion: we are close to steady states of the Euler Eq.

# Consequences of Multiple Invariants of 2D flows I)

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- Energy-Casimir functionals (Arnold 1966).
- Multiple invariants imply degeneracy of steady solutions to the 2D Euler Eq.:

$$\omega = \Delta \psi = f(\psi) \Rightarrow \mathbf{v}.\nabla\omega = (\nabla\psi \times \nabla\omega).\mathbf{e}_z = 0.$$

- Many steady solutions of 2D Euler equations are attractors. Degeneracy: what does select $f$?
- $f$ can be predicted using classical equilibrium (or non-equilibrium) statistical mechanics, or kinetic theory.

# Outline

# Invariant Measure and Divergence of the Vector Field

$$\dot{x}_i = F_i(x).$$

- Measure $\mu$ with density $f$: for any observable $A$, the average over $\mu$ is

$$\mu(A) \equiv \int \underbrace{\prod_{i=1}^{N} dx_i f(x)}_{d\mu} A(x_j).$$

- The measure $\mu$ is called invariant for the dynamics if

$$\frac{d}{dt}(\mu(A)) = 0 \quad \forall A \text{ or } \int \prod_{i=1}^{N} \mathrm{d}x_i \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left[f(x)F_j(x)\right] A(x) = 0 \;\; \forall A$$

- Then $\mu$ is invariant if the divergence of the vector field is zero

$$\nabla_f V \equiv \frac{1}{f(x)} \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left[f(x)F_j(x)\right] = 0$$

# Microcanonical Measures for Hamiltonian Systems

- Dynamics for canonical Hamiltonian systems

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}. \end{cases}$$

- Liouville theorem:

$$\sum_{i=1}^{N}\left(\frac{\partial \dot{q}_i}{\partial q_i}+\frac{\partial \dot{p}_i}{\partial p_i}\right)=0 \text{ then } \mathrm{d}\mu=\prod_{i=1}^{N}\mathrm{d}p_i\mathrm{d}q_i.$$

is dynamically invariant.

- The microcanonical measure is the natural invariant measure

$$\mathrm{d}\mu_m\left(I_1^0,\ldots,I_n^0\right)=\frac{1}{\Omega\left(I_1,\ldots,I_n\right)}\prod_{i=1}^{N}\mathrm{d}p_i\mathrm{d}q_i\prod_{k=1}^{n}\delta\left(I_k(q,p)-I_k^0\right).$$

# Liouville Theorem for the 2D Euler and Quasi-Geostrophic Eq.

$$\frac{\partial q}{\partial t} = \mathscr{F}\left[q\right](\mathbf{r})$$

- $\mathscr{F}$ verifies a Liouville theorem if

$$\nabla.\mathscr{F} \equiv \int_{\mathscr{D}} \frac{\delta \mathscr{F}}{\delta q(\mathbf{r})} \mathrm{d}\mathbf{r} = 0 \quad \left( \text{Generalization of } \nabla.\mathscr{F} \equiv \sum_{i=1}^{N} \frac{\partial \mathscr{F}}{\partial q_i} = 0 \right).$$

- Quasi-Geostrophic equations

$$\frac{\partial q}{\partial t} = -\mathbf{v}\left[q-h\right].\nabla q = -\nabla.\left[\mathbf{v}(q-h)q\right].$$

- Proof of a formal Liouville theorem:

$$\int_{\mathscr{D}} \frac{\delta}{\delta q(\mathbf{r})} \left\{\nabla.\left[\mathbf{v}(q-h)q\right]\right\} \mathrm{d}\mathbf{r} = \int_{\mathscr{D}} \nabla.\left\{ \frac{\delta}{\delta q(\mathbf{r})} \left[\mathbf{v}(q-h)q\right] \right\} \mathrm{d}\mathbf{r} = 0.$$

# Outline

# The (Inertial) Quasi-Geostrophic Eq.

- Quasi-Geostrophic equation with no forces and dissipation (inertial)
$$\frac{\partial q}{\partial t} + \mathbf{v}\left[q-h\right].\nabla q = 0,$$
with vorticity $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$, stream function $\psi$: $\mathbf{v} = \mathbf{e}_z \times \nabla \psi$, and PV $q = \Delta\psi + h(y)$.
- Conservative dynamics - Hamiltonian (non canonical) and time reversible.
- Invariants:

$$\text{Energy: } \mathscr{E}[q] = \frac{1}{2}\int_{\mathscr{D}} \mathbf{v}^2\, d\mathbf{r} == -\frac{1}{2}\int_{\mathscr{D}} (q-h)\,\psi\, d\mathbf{r},$$

$$\text{Casimir's functionals: } \mathscr{C}_s[q] = \int_{\mathscr{D}} s(q)\, d\mathbf{r},$$

$$\text{Vorticity distribution: } D[q](\sigma) = \frac{dA}{d\sigma} \text{ with } A[q](\sigma) = \int_D \chi_{\{q(\mathbf{r})\leq\sigma\}}\, d\mathbf{r}.$$

# Microcanonical Measures for the 2D Euler or Quasi-Geostrophic Eq.

- Microcanonical invariant measure

$$\mathrm{d}\mu_m\left(I_1^0,\ldots,I_n^0\right)=\frac{1}{\Omega\left(I_1,\ldots,I_n\right)}\prod_{i=1}^{N}\mathrm{d}p_i\mathrm{d}q_i\prod_{k=1}^{n}\delta\left(I_k(q,p)-I_k^0\right).$$

- Quasi-Geostrophic microcanonical measure:

$$\mathrm{d}\mu_m\left[E,D\right]=\mathscr{D}\left[q\right]\delta\left(\mathscr{E}\left[q\right]-E\right)\int_{-\infty}^{+\infty}\delta\left(D\left[q\right](\sigma)-D_0(\sigma)\right)\mathrm{d}\sigma.$$

- Is this meaningful? A mathematician: the Lebesgue measure does not exist for functional spaces! A physicist: don't you know about Rayleigh-Jeans paradox?

Y. Pomeau, Statistical approach (to 2D turbulence), in: P. Tabeling, O. Cardoso (Eds.), Turbulence: A Tentative Dictionary, Plenum Press, New York, 1995, pp. 117–123.

# Discretized Microstates for the 2D Euler Eq.

## The case with K levels of vorticity (for pedagogical purpose)

- We assume $D(\sigma) = \sum_{k=1}^{K} A_k \delta(\sigma - \sigma_k)$, with $\sum_{k=1}^{K} A_k = |\mathscr{D}|$. $(\omega(\mathbf{r}) \in \{\sigma_1, ..., \sigma_K\})$.
- We consider a $NxN$ lattice. At each lattice site $(i,j)$, $\omega$ takes one of the values $\sigma_k$ ($\omega_{ij} = \sigma_k$, It is a Potts model).

$$X_N = \left\{ \omega = (\omega_{ij})_{1 \le i,j \le N} \,|\, \forall i,j\, \omega_{ij} \in \{\sigma_k\}_{1 \le k \le K} \right\}\ ,\ \sharp(X_N) = K^{N^2}.$$

- The total number of sites such that $\omega_{ij} = \sigma_k$ is called $N_k[\omega]$.

$$\Gamma_N(\mathbf{A}) = \left\{ \omega \in X_N \,|\ \forall \mathrm{k}\ \ N_k[\omega] = \frac{N^2 A_k}{|\mathscr{D}|} \right\}.$$

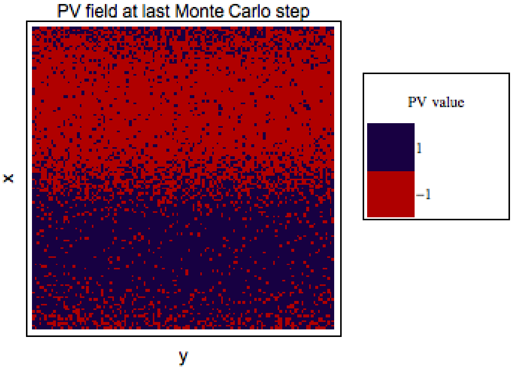
$$\Omega_N(\mathbf{A}) = \sharp[\Gamma_N(\mathbf{A})] = \frac{N^2!}{\prod_{k=1}^{K} \left( \frac{NA_k}{|\mathscr{D}|} \right)!}.$$

- $\omega \in X_N$ is a microstate. $X_N$ is the set of microstates (configuration space). $\Gamma_N(\mathbf{A})$ is a macrostate.

# Discretized Microstates for the 2D Euler Eq.

## The case with 2 levels of vorticity (for pedagogical purpose)

- We discuss the case $D(\sigma)=\frac{1}{2}\delta(\sigma+1)+\frac{1}{2}\delta(\sigma-1)$, $(\omega(\mathbf{r})\in\{-1,1\}$ with $\pm$ 1 values occupying equal areas, Ising model).




Vorticity on a $N$x$N$ lattice.

# Discretized Microcanonical Measures for the 2D Euler Eq.

## The case with K levels of vorticity (for pedagogical purpose)

- Vorticity points on a lattice of size $NxN$

$$\Gamma_N(\mathbf{A}) = \left\{ \omega \in X_N \mid \ \forall k \ \ N_k[\omega] = \frac{N^2 A_k}{|\mathscr{D}|} \right\}.$$

$$\Gamma_N(E,\mathbf{A},\Delta E) = \{\omega \in \Gamma_N(\mathbf{A}) \mid E \leq \mathscr{E}[\omega] \leq E + \Delta E\}, \ \Omega_N(E,\mathbf{A},\Delta E) = \sharp\{\Gamma_N(E,\mathbf{A},\Delta E)$$

- Finite dimensional approximate measures : equiprobability of all microstates with given energy

$$< \mu_N(E,\mathbf{A},\Delta E), \mathscr{A}[\omega] > = \frac{1}{\Omega_N(E,\mathbf{A},\Delta E)} \sum_{\omega \in \Gamma_N(E,\mathbf{A},\Delta E)} \mathscr{A}[\omega].$$

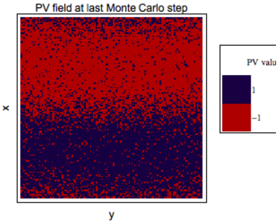
- Microcanonical measures for the 2D Euler equations:

$$\mu(E,\mathbf{A}) = \lim_{N\to\infty} \mu_N(E,\mathbf{A},\Delta E) \text{ and } S(E,\mathbf{A}) = \lim_{N\to\infty} \frac{1}{N^2} \ln(\Omega_N(E,\mathbf{A},\Delta E)).$$

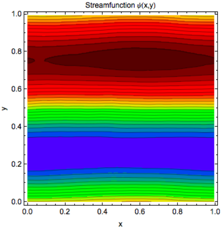
# A Typical Vorticity Field for the Microcanonical Measure

## The case with 2 levels of vorticity: $\omega \in \{-1,1\}$, $E = 0.6E_{max}$, $N$x$N = 128$x$128$

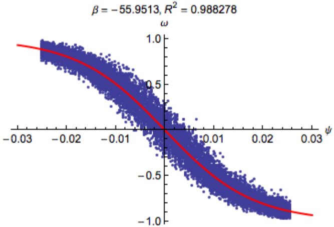
- Creutz's algorithm: a generalization of Metropolis-Hasting's algorithm that samples microcanonical measures.




Vorticity field




Stream function




Coarse-grained-vorticity vs stream function

# Long Range Interactions and Mean Field Behavior

## Microcanonical Measures for the 2D Euler Eq.

- Vorticity points on a lattice of size $NxN$

$$\Gamma_N(\mathbf{A}) = \left\{ \omega \in X_N \mid \forall \mathrm{k} \;\; N_k[\omega] = \frac{N^2 A_k}{|\mathscr{D}|} \right\}.$$

$$\Gamma_N(E,\mathbf{A},\Delta E) = \{\omega \in \Gamma_N(\mathbf{A}) \mid E \leq \mathscr{E}[\omega_N] \leq E + \Delta E\}.$$

- Energy: interaction through a potential with long range interaction

$$\mathscr{E}[\omega_N] = \frac{1}{2N^2} \sum_{i,j=1}^{N} G_{ij}\omega_i\omega_j, \text{ with } G_{ij} \simeq \frac{1}{4\pi}\log(\mathbf{r}_i - \mathbf{r}_j).$$

- G is the Laplacian Green functions in 2D.
- Mean field approximation will be valid in the limit $N \to \infty$.
- The 2D-Euler has a mean-field behavior. We can prove this using large deviation theory (a generalization of Sanov's theorem).

# Macrostates Through Coarse-Graining and Entropy

- Coarse-graining: we divide the $N \times N$ lattice into $(N/n) \times (N/n)$ boxes ($n^2$ sites per box).
- These boxes are centered on sites $(In, Jn)$. $(I, J)$ label the boxes $(0 \leq I, J \leq N/n - 1)$.
- $p_{IJ}^{\pm}$ is the frequency to find the value $\pm 1$ in the box $(I, J)$ $(p_{IJ}^{+} + p_{IJ}^{-} = 1)$

$$p_{IJ}^{\pm}[\omega]) = \frac{1}{n^2} \sum_{(i,j)\in(I,J)} \delta_d(\omega_{ij} - (\pm 1)).$$

- A macrostate $P = \left\{p_{IJ}^{\pm}\right\}_{0\leq I,J\leq N/n-1}$, is the set of all microstates $\left\{\omega \in X_N \middle| \text{ for all } I, J,\, p_{IJ}^{\pm}\left[\omega^N\right] = p_{IJ}^{\pm}\right\}$.
- Macrostate entropy = logarithm of the cardinal of the macrostate

$$S_N[p^N] = \frac{1}{N^2} \log \sharp \left(P^N\right).$$

# Outline

# Sum of Independent Random Variables

- We consider $n$ independent and identically distributed random variables $\{x_k\}_{1\leq k\leq n}$ with PDF $f_0(x)$.
- What can be said about the probability of the sum of these $n$ variables:

$$S_n = \frac{1}{n}\sum_{k=1}^{n} x_k .$$

- Law of large numbers:

$$S_n \underset{n\to\infty}{\to} m = \int f_0(x)\,\mathrm{d}x .$$

- Central limit theorem: the distribution of

$$\sqrt{n}S_n = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} x_k .$$

is asymptotically Gaussian, with average $m$ and variance

$$\sigma = \int f_0(x)\,x^2\mathrm{d}x .$$

# Large Deviations and Cramer's Theorem

- We are looking at the PDF of $S_n$:

$$S_n = \frac{1}{n}\sum_{k=1}^{n} x_k \ , \ P_n(S) \equiv \mathbb{E}\left[\delta(S_n - S)\right].$$

- Cramer's theorem:

$$P_n(S) \underset{n\to\infty}{\sim} Ce^{-nI(S)} \text{ with } I(S) = \sup_{\lambda}\left\{\lambda S - \log \int e^{\lambda x} f_0(x) \ dx\right\} \ .$$

- $I$: large deviation rate function. Large deviation principle:

$$\log P_n(S) \underset{n\to\infty}{\sim} -nI(S).$$

- When the central limit theorem is valid, expanding $S = m + 1/\sqrt{n}\delta S$ we have

$$P_n(\delta S) \underset{n\to\infty}{\sim} Ce^{-\sqrt{n}I''(S)\frac{(\delta S)^2}{2}} \text{ thus } \underset{S}{\mathrm{Min}} I(S) = I(m) = 0 \text{ and } I''(S) = \frac{1}{\sigma}.$$

# Sanov's Theorem: Large Deviations for the PDF

- We now look at the empirical distribution function (the PDF)

$$f_n(x) = \frac{1}{n}\sum_{k=1}^{n} \delta(x - x_k).$$

- We consider the probability distribution functional of $f_n$:

$$P_n[f] \equiv \mathbb{E}[\delta(f - f_n)].$$

- Sanov's theorem is a statement about the asymptotic behavior of $P_n[f]$. It states that

$$\ln P_n[f] \underset{n\to\infty}{\sim} -n \int f(x) \ln\left(\frac{f(x)}{f_0(x)}\right) \mathrm{d}x,$$

if $\int f \, \mathrm{d}x = 1$ and $-\infty$ otherwise.

# Sanov's Theorem for Variables with Discrete Values

- When the variable $x$ takes only $K$ possible values $\{\sigma_k\}_{1\leq k\leq K}$ with probability $\{\pi_k\}_{1\leq k\leq K}$

$$f_0(x) = \sum_{k=1}^{K} \pi_k \delta(x-\sigma_k),$$

with $\sum_{k=1}^{K} \pi_k = 1$, then $f_n(z) = \sum_{k=1}^{K} \rho_{k,n} \delta(z-\sigma_k)$.

- PDF of $(\rho_{1,n},...,\rho_{K,n})$ :

$$P_n(p_1,...,p_K) \equiv \mathbb{E}\left[\delta(\rho_{k,1}-p_1,...,\rho_{K,1}-p_K)\right].$$

- Then Sanov's theorem states that

$$\ln P_n(p_1,...,p_K) \underset{n\to\infty}{\sim} -n\sum_{k=1}^{K} p_k \ln\left(\frac{p_k}{\pi_k}\right),$$

if $\sum_{k=1}^{K} p_k = 1$ and $-\infty$ otherwise.

# Sanov's Theorem and Coarse Graining of 2D Fields

- Consider a square lattice, $N^2$ sites, in a 2D area $|\mathscr{D}|$. On each lattice site, we have a random variable chosen among $\{\sigma_k\}_{1\leq k\leq K}$ with probability $1/K$.
- $p_k(\mathbf{r})$ is the local probability (coarse-graining) to observe a value of $s$ equal to $\sigma_k$ in an infinitesimal area $\mathrm{d}\mathbf{r}$ around $\mathbf{r}$.
- Using $n = N^2\mathrm{d}\mathbf{r}/|\mathscr{D}|$ and $\pi_k = 1/K$, Sanov's theorem gives

$$\ln P_N[p_1(\mathbf{r}),\dots,p_k(\mathbf{r})] \underset{N\to\infty}{\sim} -\frac{N^2\mathrm{d}\mathbf{r}}{|\mathscr{D}|}\sum_{k=1}^{K} p_k \ln(Kp_k),$$

if $\sum_{k=1}^{K} p_k(\mathbf{r}) = 1$ and $-\infty$ otherwise.
- The probability $P_N[p_1,\dots,p_K]$ to observe a field of local probabilities $(p_1(\mathbf{r}),\dots,p_K(\mathbf{r}))$ over the whole area $\mathscr{D}$ is

$$\ln P_N[p_1,\dots,p_K] \underset{N\to\infty}{\sim} -\frac{N^2}{|\mathscr{D}|}\sum_{k=1}^{K}\int_{\mathscr{D}} p_k \ln(Kp_k)\,\mathrm{d}\mathbf{r},$$

if for all $\mathbf{r}$ $\sum_{k=1}^{K} p_k(\mathbf{r}) = 1$ and $-\infty$ otherwise.

# Entropy and Sanov's Theorem

- The ensemble of the realizations giving a specified local probabilities $(p_1(\mathbf{r}), ..., p_K(\mathbf{r}))$ defines a macrostate $\Gamma_N[p_1, ..., p_K]$.
- We consider $\Omega_N[p_1, ..., p_K] = \sharp[p_1, ..., p_K]$ the number of microscopic configurations that correspond to this macrostate. This number of configurations is $\Omega_N = P_N K^{N^2}$ (using that the number of networks in the ensemble is $K^{N^2}$).
- Using that $\sum_{k=1}^{K} p_k(\mathbf{r}) = 1$, we get

$$\mathscr{S}[p_1, ..., p_K] \equiv \lim_{N\to\infty} \frac{1}{N^2} \log \Omega_N[p_1, ..., p_K] = -\frac{1}{|\mathscr{D}|} \sum_{k=1}^{K} \int_{\mathscr{D}} p_k(\mathbf{r}) \ln(p_k(\mathbf{r}))\, d\mathbf{r},$$

if for all $\mathbf{r}$ $\sum_{k=1}^{K} p_k(\mathbf{r}) = 1$ and $-\infty$ otherwise.

# Adding Constraints: Example of the Vorticity Distribution

- For a given realization, $N_k$ is the total number of sites with the value $\sigma_k$. $\Omega_N(\mathbf{A})$ is the number of realizations with $N_k = N^2 A_k / |\mathscr{D}|$.
- This ensemble was used to define the microcanonical ensemble.
- The constraints $N_k = N^2 A_k / |\mathscr{D}|$ are equivalent to $\int_{\mathscr{D}} p_k \, d\mathbf{r} = A_k$. Then

$$S(\mathbf{A}) = \sup_{\left\{ p \middle| \int_{\mathscr{D}} p_k \, d\mathbf{r} = A_k \text{ and } \sum_{k=1}^{K} p_k(\mathbf{r}) = 1 \right\}} \left\{ \mathscr{S}[p_1, ..., p_K] \right\}$$

- Using Lagrange multipliers, we prove that the supremum is reached for $p_k = A_k / |\mathscr{D}|$ and

$$S(\mathbf{A}) = \lim_{N\to\infty} \frac{1}{N^2} \log \Omega_N(\mathbf{A}) = -\sum_{k=1}^{K} \frac{A_k}{|\mathscr{D}|} \log\left(\frac{A_k}{|\mathscr{D}|}\right).$$

(could be deduced from $\Omega_N(\mathbf{A}) = N^2! / \Pi_{k=1}^{K} N_k!$, and the use of Stirling's formula).

# Energy Constraint and Long Range Interaction

- We define the average vorticity for the macrostate $[p_1,...,p_K]$

$$\bar{\omega}(\mathbf{r}) = \sum_{k=1}^{K} \sigma_k p_k(\mathbf{r}).$$

- Because the energy has long range interactions, one can prove that for any realization

$$\mathscr{E}[\omega_N] \equiv \frac{1}{2N^2}\sum_{i,j=1}^{N} G_{ij}\omega_i\omega_j = \frac{1}{2N^2}\sum_{i,j=1}^{N} G_{ij}\bar{\omega}_i\bar{\omega}_j + o\left(\frac{1}{N}\right).$$

- The microstate energies concentrate close to a single value: the energy of the average vorticity.
- As a consequence, the energy constraint can be imposed as a constraint on the macrostate (this is a justification for this case of a mean field approximation).

# Robert-Sommeria-Miller (RSM) Theory

## The most probable vorticity field (for $D(\sigma)=\frac{1}{2}\delta(\sigma+1)+\frac{1}{2}\delta(\sigma-1)$)

- A probabilistic description of the vorticity field $\omega$: $p(\mathbf{x})$ is the local probability to have $\omega(\mathbf{x})=1$ at point $\mathbf{x}$.
- A measure of the number of microscopic field $\omega$ corresponding to a probability $p$ (Liouville and Sanov theorems):

$$\textit{Macrostate entropy}: \mathscr{S}_2[p] \equiv -\frac{1}{|\mathscr{D}|}\int_{\mathscr{D}} d\mathbf{r}\,[p\log p+(1-p)\log(1-p)]$$

- The microcanonical RSM variational problem (MVP):

$$S\left(E,\frac{1}{2},\frac{1}{2}\right)=\sup_{\{p|\mathscr{N}[p]=1\}}\left\{\mathscr{S}_2[p]\mid\mathscr{E}[\overline{\omega}]=E\right\}\ (\text{MVP}).$$

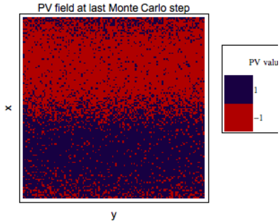
- Critical points are steady solutions of the 2D Euler equations:

$$\overline{\omega}=\tanh(\beta\psi).$$

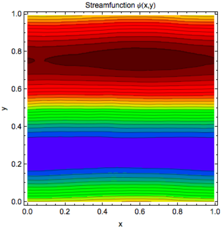
# A Typical Vorticity Field for the Microcanonical Measure

## A two vorticity level case: $\omega \in \{-1,1\}$, $E = 0.6E_{max}$, $N\times N = 128\text{x}128$

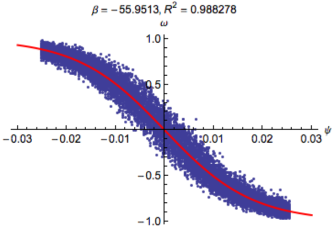
- Creutz's algorithm: a generalization of Metropolis-Hasting's algorithm that samples microcanonical measures.




Vorticity field




Stream function




Coarse-grained-vorticity vs stream function

# Outline

# Equilibrium Statistical Mechanics for Geophysical Flows

## The Robert-Sommeria-Miller theory

- Statistical mechanics of the Potential Vorticity mixing: emergence from *random initial conditions*, **stability**, predictability of the flow organization.
- Gulf Stream and Kuroshio currents as statistical equilibria.
- Ocean mesoscale vortices as statistical equilibria.

# The Simplest Model: the 1-1/2 Layer Quasi-Geostrophic Model

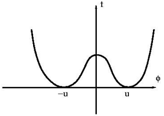
We describe Jupiter's troposphere by the Quasi Geostrophic model (one and half layer):

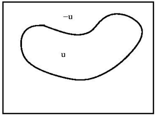
$$\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = 0 \; ; \; \mathbf{v} = \mathbf{e}_z \times \nabla \psi \; ; \; q = \Delta \psi - \frac{\psi}{R^2} - h(y).$$

# Variational Problem for The Statistical Equilibria
## (The case of the 1-1/2 layer Quasi Geostrophic model)

Variational problem: limit $R \to 0$. ($\phi = \psi/R^2$).

$$\begin{cases} \min\left\{F_R[\phi] \,|\, \text{with } A[\phi] \text{ given}\right\}, \\ \text{with } F_R[\phi] = \int_D d\mathbf{r} \left[\frac{R^2(\nabla\phi)^2}{2} + f(\phi) - R\phi h_0(y)\right] \text{ and } A[\phi] = \int_D d\mathbf{r}\, \phi. \end{cases}$$




The function $f$ : two minima

Phase coexistence

- An analogy with first order phase transitions.
- Modica (90'), function with bounded variations.

# Reduction to a One Dimensional Variational Problem

## An isoperimetrical problem balanced by the effect of the deep flow

- A variational problem for the jet shape (interface)

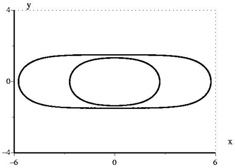
$$F_R\left[\phi_R\right] = 2Re_c L - 2Ru\int_{A_1} d\mathbf{r}\, h_0(y) + o\left(R\right).$$

- Laplace equation:

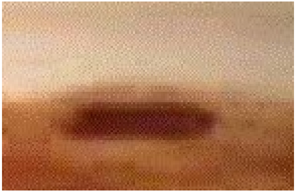
$$\frac{e_c}{r} = -u\left(\alpha_1 - h_0\left(y\right)\right).$$

# Jovian Vortex Shape: Phase Coexistence

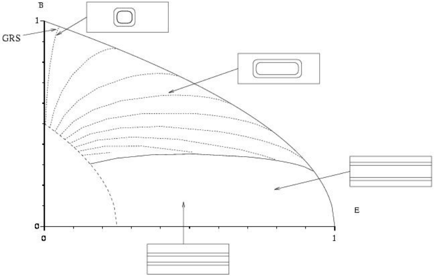
## An isoperimetrical problem balanced by the effect of the deep flow




Left: analytic results.
Below left: the Great Red Spot and a White Ovals.
Below right: Brown Barges.



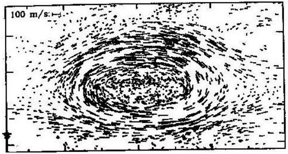
# A Phase Diagram for Jovian Vortices and Jets



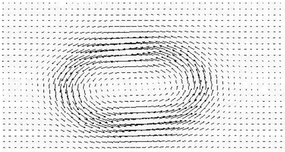

$E$ is the energy and $B$ measures the asymmetry of the initial PV distribution

# Great Red Spot of Jupiter

## Real flow and statistical mechanics predictions (1-1/2 layer QG model)




Observation data (Voyager)

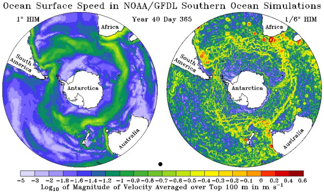


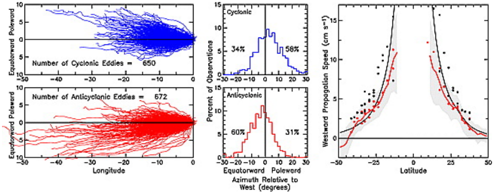
Statistical equilibrium

- A very good agreement. A simple model, analytic description, from theory to observation + New predictions.
- F. BOUCHET and J. SOMMERIA 2002 ***JFM*** **(QG model).**

# Ocean Rings (Mesoscale Ocean Vortices)

## Gulf Stream rings - Agulhas rings - Meddies - etc ...




Hallberg–Gnanadesikan

Chelton and co. - GRL 2007

- Both cyclonic and anticyclonic rings drift westward with a velocity $\tilde{\beta} R^2$.
- Statistical mechanics explains the ring qualitative shape, and their observed drifts.
  A. Venaille, and F. Bouchet, JPO, 2011.

# What about Dynamics?

- Are microcanonical measures invariant measures of the 2D Euler or Quasi-Geostrophic equations?

F. Bouchet and M. Corvellec, J. Stat. Mech. 2010, Invariant measures of the 2D Euler and Vlasov equations.

- What about weakly forced and dissipated systems?

# 2D Stochastic Navier-Stokes Eq. and 2D Euler Steady States

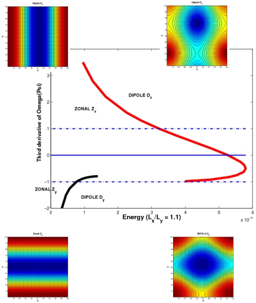
$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s \quad (1)$$

- Time scale separation: magenta terms are small.
- At first order, the dynamics is nearly a 2D Euler dynamics. The flow self organizes and converges towards steady solutions of the Euler Eq.:

$$\mathbf{u}.\nabla\omega = 0 \text{ or equivalently } \omega = f(\psi)$$

where the Stream Function $\psi$ is given by: $\mathbf{u} = \mathbf{e}_z \times \nabla\psi$.

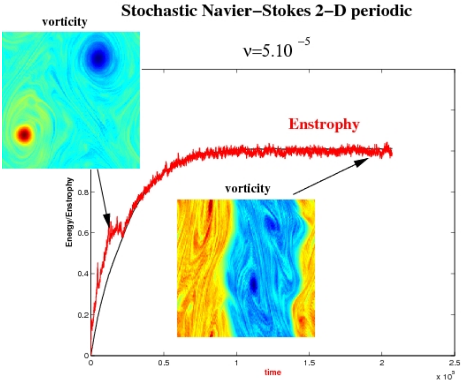
# Statistical Equilibria for the 2D-Euler Eq. (torus)




A second order phase transition.

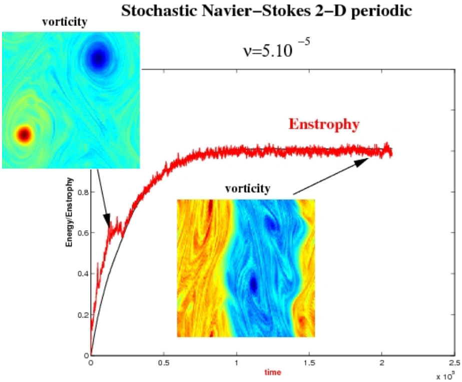
- Z. Yin, D. C. Montgomery, and H. J. H. Clercx, Phys. Fluids (2003)
- F. Bouchet, and E. Simonnet, PRL, (2009) (Lyapunov Schmidt reduction, normal form analysis)

# Numerical Simulation of the 2D Stochastic NS Eq.



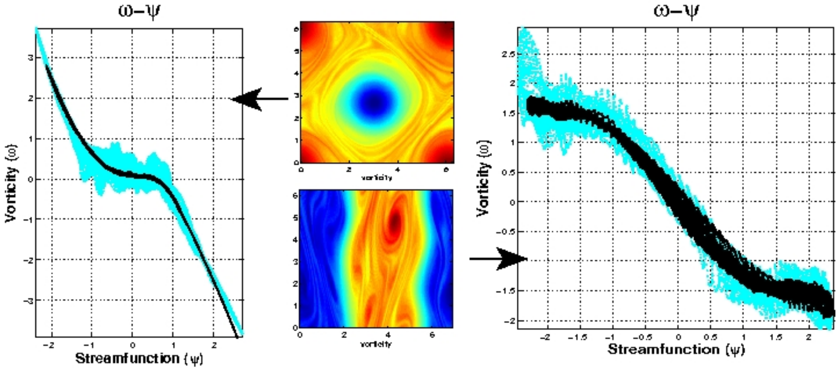

Very long relaxation times. $10^5$ turnover times.

# Numerical Simulation of the 2D Stochastic NS Eq.




Very long relaxation times. $10^5$ turnover times.

# Vorticity-Streamfunction Relation




Conclusion: we are close to steady states of the Euler Eq.

# Some Cautions and Drawbacks of this Theory

- Drawbacks: i) the theory is derived using the physically irrelevant parameter $N$ (the large deviation rate) but does not depend on it finally.
- Drawbacks: ii) the conservative dynamics may not be ergodic (see a proof in Bouchet-Corvellec).
- Drawbacks: iii) the theory is irrelevant for dynamics.

# Equilibrium: Conclusions

- We can perform equilibrium statistical mechanics in a completely standard way for the class of the 2D Euler Eq.. and the quasi-geostrophic eq..
- For the microcanonical measures (or the canonical measures), the probability of a macrostate (a local probability to observe a vorticity value) is given by a large deviation rate function.
- The most probable macrostate verifies a mean-field variational problem.
- This is relevant for some applications (which ones ?) and give a very good qualitative understanding of weakly forced and dissipated systems.

F. Bouchet, and A. Venaille, Physics Reports, 2012, Statistical mechanics of two-dimensional and geophysical flows