Non-Equilibrium Statistical Mechanics of the 2D Stochastic Navier-Stokes Equations and Geostrophic Turbulence III)

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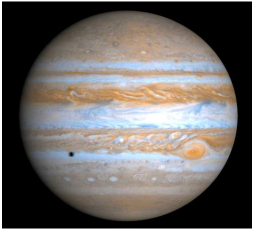
5^{th} Warsaw School of Statistical Physics.

Global Outline

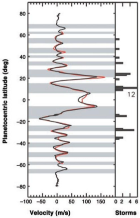
I) Introduction to geophysical fluid dynamics and the quasi-geostrophic model
II) Equilibrium Statistical mechanics of geostrophic turbulence
III) Non-equilibrium phase transitions, path integrals and instanton theory
IV) Kinetic theory (stochastic averaging) of zonal jet dynamics

Earth and Jupiter's Zonal Jets

We look for a theoretical description of zonal jets



Jupiter atmosphere



Jupiter Zonal wind (Voyager and Cassini, from Porco et al 2003)

How to theoretically predict such a velocity profile?

Far From Equilibrium Problems

- In the previous lectures, we have discussed the dynamics of phase transitions for turbulent flows in an equilibrium setup (Langevin dynamics).
- We have shown that those results are useful to predict non-equilibrium phase transitions.
- The dynamics of the transition, for instance the instanton, can then be calculated (numerically, ...).
- All these examples involve the dynamics of the largest scales of the flow only (at first approximation), and the action of the noise directly on the largest scales only.
- However, this is not always the correct phenomenology. For instance the evolution of the coherent structure may be due cumulative effects of independent noises acting on much smaller scales (entropic effects). This is what we will study now.

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s,$$

where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, $q = \omega + \beta_d y$ is the Potential Vorticity (PV), f_s is a random force, α is the Rayleigh friction coefficient.

- Quasi-Gestrophic models: the basic models for midlatitude large scale turbulence.

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1. Stochastic averaging and jet formation in geostrophic turbulence.
   - The stochastic quasi-geostrophic equations.
   - Stochastic averaging (with C. Nardini and T. Tangarife).
   - Validity of this approach, and the main technical points.
2. Inviscid relaxation of the 2D Euler equations
   - Irreversibility in turbulence
   - Large Time Asymptotics of the linearized 2D Euler Eq.
   - The Kolmogorov flow

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The Barotropic Quasi-Geostrophic Equations

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$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu_d\Delta\omega-\lambda\omega+\sqrt{2\varepsilon}f_s,$$

  where $\omega=(\nabla\wedge\mathbf{v}).\mathbf{e}_z$ is the vorticity, $q=\omega+\beta y$ is the Potential Vorticity (PV), f_s is a random Gaussian field with correlation $\langle f_S(\mathbf{x},t)f_S(\mathbf{x}',t')\rangle=C(\mathbf{x}-\mathbf{x}')\delta(t-t')$, ε is the average energy input rate, λ is the Rayleigh friction coefficient.
- 4 parameters: ε, λ, β and L
- 2 independent non-dimensional parameters: spatial scale unit such that $L=2\pi$, temporal scale such that the average total energy is one.

Energy Balance

$$\frac{d\mathbb{E}(E)}{dt} = -2\lambda\mathbb{E}(E) - \nu_d\mathbb{E}(Z) + \varepsilon$$

- Then, in the turbulent regime, where viscous energy dissipation is negligible

$$\mathbb{E}_S(E) \simeq \frac{\varepsilon}{2\lambda}$$

- We will work with the time scale unit that $\frac{\varepsilon}{2\lambda} = 1$.

The Barotropic Quasi-Geostrophic Equations

- The non-dimensional version of the barotropic QG equation
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s,$$

with $q=\omega+\beta' y$

- The relation with the dimensional parameters is:

$$\alpha=L^2\sqrt{\frac{2\lambda^3}{\varepsilon}}.$$

$$\beta'=L^3\beta\sqrt{\frac{2\lambda}{\varepsilon}}=\left(\frac{L}{L_{Rhines}}\right)^2$$

- Spin up or spin down time $=1/\alpha\ll 1=$ jet inertial time scale.

The 2D Stochastic Navier-Stokes Equations

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega + \sqrt{\nu} f_s$$

- Some recent mathematical results: Bricmont, Debussche, Hairer, Kuksin, Kupiainen, Mattingly, Shirikyan, Sinai, ...
  - Existence of a stationary measure μ_ν. Existence of $lim_{\nu\to 0}\ \mu_\nu$,
  - In this limit, almost all trajectories are solutions of the 2D Euler equations.

Kuksin, S. B., & Shirikyan, A. (2012). Mathematics of two-dimensional turbulence. Cambridge University Press.

- We would like to describe the invariant measure:
  - Is it concentrated close to steady solutions of the 2D Euler (quasi-geostrophic) equations?
  - Can we describe the dynamics among these states?

The Barotropic Quasi-Geostrophic Equations

The limit of weak forces and dissipation

- The non-dimensional version of the barotropic QG equation
- Quasi-Geostrophic equations with random forces

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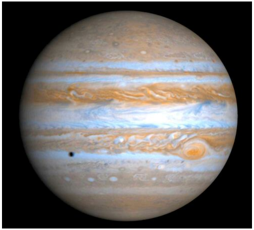
Weak Fluctuations around Jupiter's Zonal Jets



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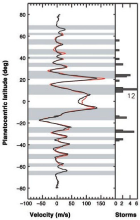

Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003)

We will treat those weak fluctuations perturbatively.

Jet Formation in the Barotropic QG Model

In the weak forces and dissipation limit

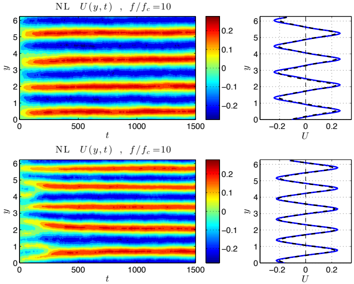

Figure by P. Ioannou (Farrell and Ioannou)

The Barotropic Quasi-Geostrophic Equations

- At leading order, the inertial equations is an Hamiltonian system.
- Quasi-Geostrophic equations with no forces and dissipation

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = 0$$

with $q = \omega + \beta' y$.
- It conserves energy

$$E = \frac{1}{2}\int_{\mathscr{D}} \mathbf{v}^2 \mathrm{d}\mathbf{x},$$

enstrophy, and an infinite number of Casimir invariants.
- We want to use that zonal jets of the inertial dynamics are attractors.
- Orbital stability? How is this possible for Hamiltonian systems?

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Averaging out the Turbulence

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s.$$

- $P[q]$ is the PDF for the Potential Vorticity field q (a functional). Fokker–Planck equation:

$$\frac{\partial P}{\partial t}=\int \mathrm{d}\mathbf{r}\,\frac{\delta}{\delta q(\mathbf{r})}\left\{\left[\mathbf{v}.\nabla q-\nu\Delta\omega+\alpha\omega+\int \mathrm{d}\mathbf{r}'\,C(\mathbf{r},\mathbf{r}')\frac{\delta}{\delta q(\mathbf{r})}\right]P\right\}.$$

- Time scale separation. We decompose into slow (zonal flows) and fast variables (eddy turbulence)

$$q_z(y)=\langle q\rangle\equiv\frac{1}{2\pi}\int_{\mathscr{D}}\mathrm{d}x\,q(x,y)\text{ and }q(x,y)=q_z(y)+\sqrt{\alpha}q_m(x,y).$$

- Stochastic reduction (Van Kampen, Gardiner, ...) using the time scale separation.
- We average out the turbulent degrees of freedom.

A New Fokker–Planck Equation for the Zonal Jets

The averaged equation describes the evolution of slow variables only

- $R[q_z]$ is the PDF to observe the Zonal Potential Vorticity q_z.

$$\frac{1}{\alpha}\frac{\partial R}{\partial t} = \int \mathrm{d}y_1 \, \frac{\delta}{\delta q_z(y_1)} \left\{ \left[\frac{\partial}{\partial y} \mathbb{E}_{q_z} \left\langle v_{m,y} q_m \right\rangle + \omega_z(y_1) - \frac{\nu}{\alpha} \frac{\partial^2 q_z}{\partial y^2}(y_1) + \right.\right.$$
$$\left.\left. + \int \mathrm{d}y_2 \, C_z(y_1, y_2) \frac{\delta}{\delta q_z(y_2)} \right] R \right\}.$$

- This new Fokker–Planck equation is equivalent to the stochastic dynamics

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = -\frac{\partial}{\partial y}\mathbb{E}_{q_z}\left\langle v_{m,y} q_m \right\rangle - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2} + \eta_z,$$

with $\langle \eta_z(y,t)\eta_z(y',t')\rangle = C_z(y,y')\delta(t-t')$.

The Deterministic Part and the Quasilinear Approximation

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F\left[q_z\right] - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}.$$

- $F\left[q_z\right] = -\frac{\partial}{\partial y}\mathbb{E}_{q_z}\left\langle v_{m,y} q_m\right\rangle$. The average of the Reynolds stress is over the statistics of the quasilinear inertial dynamics:

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = f_s$$

and

$$\left\langle v_{m,y} q_m\right\rangle = \frac{1}{L_y}\int \mathrm{d}y\, \mathbb{E}_{q_z}\left[v_{m,y} q_m\right].$$

- We identify SSST by Farrell and Ioannou (JAS, 2003); quasilinear theory by Bouchet (PRE, 2004); CE2 by Marston, Conover and Schneider (JAS, 2008); Sreenivasan and Young (JAS, 2011).

Dynamics of the Relaxation to the Averaged Zonal Flows

The turbulence has been averaged out

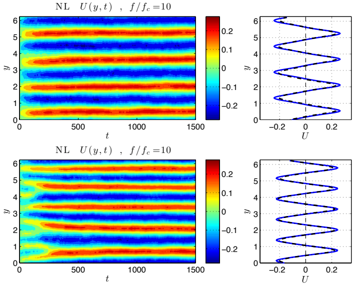


Figure by P. Ioannou (Farrell and Ioannou)

- Extremely efficient numerical simulation of the averaged jet dynamics.

The Stochastic Dynamics of the Zonal Jet

The turbulence has been averaged out

- We can now go further. What is the effect of the noise term?

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F\left[q_z\right] - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2} + \eta_z .$$

- $R\left[q_z\right]$ is the PDF to observe the Zonal Potential Vorticity q_z.

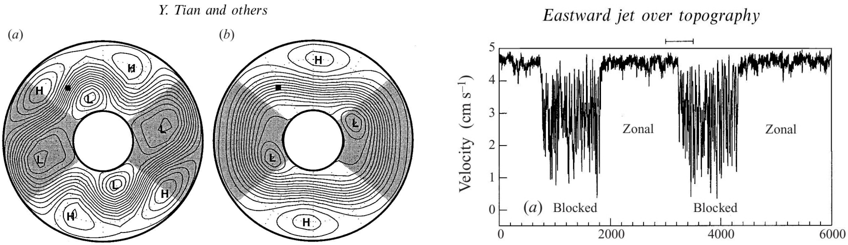
$$\frac{1}{\alpha}\frac{\partial R}{\partial t} = \int \mathrm{d}y_1\, \frac{\delta}{\delta q_z(y_1)}\left\{\left[\frac{\partial}{\partial y}\mathbb{E}_{q_z}\left\langle v_{m,y} q_m\right\rangle + \omega_z(y_1) - \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}(y_1) + \right.\right.$$
$$\left.\left. + \int \mathrm{d}y_2\, C_z(y_1,y_2)\frac{\delta}{\delta q_z(y_2)}\right] R\right\}.$$

- This equation describes the zonal jet statistics and not only the mean zonal flow.
- This statistics can be nearly Gaussian, but can also be strongly non-Gaussian.

Rare Transitions in Real Flows?

Rotating tank experiments (Quasi Geostrophic dynamics)

Transitions between blocked and zonal states:



Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

Do multiple attractors and rare transitions exist for geostrophic turbulence?

Theory based on non-equilibrium statistical mechanics?

Multiple Attractors Do Exist for the Barotropic QG Model

Two attractors for the same set of parameters

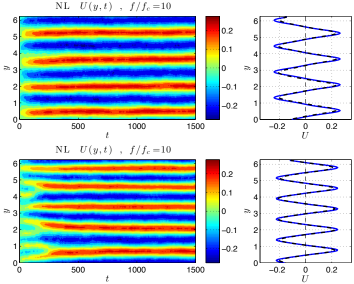


Figure by P. Ioannou (Farrell and Ioannou)

- Two attractors for the mean zonal flow for one set of parameters.
- What is the dynamics for the transition? What is the rate?

Work in Progress : Zonal Flow Instantons

Onsager Machlup formalism (50'). Statistical mechanics of histories

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F[q_z] - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2} + \eta_z .$$

- Path integral representation of transition probabilities:

$$P(q_{z,0}, q_{z,T}, T) = \int_{q(0)=q_{z,\mathbf{0}}}^{q(T)=q_{z,T}} \mathscr{D}[q_z] \exp\left(-\mathscr{S}[q_z]\right) \text{ with}$$

$$\mathscr{S}[q_z] = \frac{1}{2}\int_{\mathbf{0}}^{T} \mathrm{d}t \int \mathrm{d}y_{\mathbf{1}}\mathrm{d}y_{\mathbf{2}} \left[\frac{\partial q_z}{\partial t} - F[q_z] + \omega_z - \frac{\nu}{\alpha}\frac{\partial^{\mathbf{2}} q_z}{\partial y^{\mathbf{2}}}\right](y_{\mathbf{1}}) C_Z(y_{\mathbf{1}}, y_{\mathbf{2}}) \left[\frac{\partial q_z}{\partial t} - F[q_z] + \omega_z - \frac{\nu}{\alpha}\frac{\partial^{\mathbf{2}} q_z}{\partial y^{\mathbf{2}}}\right](y_{\mathbf{2}}).$$

- Instanton (or Freidlin-Wentzel theory): the most probable path with fixed boundary conditions

$$S(q_{z,0}, q_{z,T}, T) = \min_{\left\{q_z \middle| q_z(0)=q_{z,0} \text{ and } q_z(T)=q_{z,T}\right\}} \left\{\mathscr{S}[q_z]\right\}.$$

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The Real Issue was to Cope with UltraViolet Divergences

We will prove that they are no such divergences

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = f_s$$

- We need to prove that the Gaussian process has an invariant measure, even when $\nu = 0$, and $\alpha = 0$.
- This is true because the linearized Quasi-Geostrophic or Euler dynamics is non-normal.
- $g(\mathbf{r},\mathbf{r}',t) = \mathbb{E}_{q_z}\left(q_m(\mathbf{r},t)\,q_m(\mathbf{r}',t)\right)$ solves a Lyapunov equation

$$\frac{\partial g}{\partial t} + L_U^{0(1)}g + L_U^{0(2)}g = e_{kl} + C.C.\ \text{ with } \mathrm{e}_{kl}(x,y) = \mathrm{e}^{i(kx+ly)}$$

That can be solved in terms of the linearized Hamiltonian Eq.:

$$g_{kl}(\mathbf{r}_1,\mathbf{r}_2,t) = \int_0^t \mathrm{e}^{-t_1 L_U^0}\left[\mathrm{e}_{kl}\right](x_1,y_1)\mathrm{e}^{-t_1 L_U^0}\left[\mathrm{e}_{kl}^*\right](x_2,y_2)\,\mathrm{d}t_1 + \mathrm{C.C.}.$$

Inviscid Damping of the Linearized Hamiltonian Eq.

This proves that they are no ultraviolet divergences

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = 0 \text{ with } q_m(t=0) = q_m(0)$$

- The velocity field for the linearized Hamiltonian equation decreases algebraically at large times

$$v_{m,x}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,x,\infty}(y)}{t}\exp(-ikU(y)t) \text{ and } v_{m,y}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,y,\infty}(y)}{t^2}\exp(-ikU(y)t).$$

F. Bouchet and H. Morita, 2010, Physica D.

- This result proves that the velocity-velocity two point correlation functions of the Gaussian process have an asymptotic solution independent on dissipation.

Stat. Mech. of Zonal Jets: Conclusion

- Stochastic averaging for the barotropic Quasi-Geostrophic equation leads to a non-linear Fokker-Planck equation.
- This Fokker-Planck equation predicts the Reynolds stress and jet statistics. Related to Quasilinear theory and SSST.
- For some parameters, multiple attractors are observed.
- Path integral, instanton and large deviation theories can predict rare transitions between attractor

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Irreversibility in Fluid Mechanics and in Turbulence

Do we need viscosity to explain irreversible behavior of turbulent flows ?

- In many fluid mechanics or turbulence textbooks, it is stated, for example, that "Viscosity, whatever small, is necessary to explain the irreversible behavior of turbulent flows".
- Based on "D'Alembert's Paradox" (Euler and Lagrange theorems) (about potential flows) and Prandtl boundary layer analysis.
- The reversibility paradox of very small Reynolds number flows.

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- Irreversibility of turbulent flows should be explained independently of microscopic irreversible phenomena.
- Today the case of 2D turbulent flows and the irreversible behavior of the 2D Euler equations

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The 2D Euler Eq.: a Hamiltonian Reversible Dynamical System

- 2D Euler equations

$$\frac{\partial \Omega}{\partial t} + \mathbf{V}.\nabla\Omega = 0$$

- Vorticity $\Omega = (\nabla \wedge \mathbf{V}).\mathbf{e}_z$. $\Omega = \Delta\psi$
- The 2D Euler Eq. are symmetric under time reversal symmetry:

$$\Omega(\mathbf{r},t) \to \Omega(\mathbf{r},t) \text{ and } \mathbf{V}(\mathbf{r},t) \to -\mathbf{V}(\mathbf{r},-t)$$

- The 2D Euler Eq. has a irreversible macroscopic behavior: relaxation of the largest scales towards equilibrium

The 2D Euler Eq.: a Hamiltonian Reversible Dynamical System

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Nonlinear Landau Damping

Clément Mouhot, and Cédric Villani, 2010

- Vlasov equation (dynamics of electrons in a plasma). μ-space density $f(x,p,t)$:

$$\frac{\partial f}{\partial t}+p\frac{\partial f}{\partial x}-\frac{dV}{dx}\frac{\partial f}{\partial p}=0.$$

- Hamiltonian and time reversible. A transport equation by a non-divergent flow, like the 2D Euler equations.
- Base state: a steady state $f=f_0(p)$. Understanding of the linearized equation by Landau (1946)
- Proof of the irreversible convergence, for large times, of f (weak topology) and ρ (strong topology) towards homogeneous densities (Mouhot, and Villani, 2010)

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The 2D Linearized Euler Eq.

- 2D Euler equations

$$\frac{\partial \Omega}{\partial t} + \mathbf{V}.\nabla \Omega = 0$$

- Base state : a stable steady state $\mathbf{v}_0 = U(y)\mathbf{e}_x$, with vorticity ω_0: $\mathbf{v}_0.\nabla \omega_0 = 0$
- The 2D Euler equation, linearized close to $\mathbf{v}_0$, $\Omega = \omega_0(y) + \omega$ and $\mathbf{V} = \mathbf{v} + U(y)\mathbf{e}_x$

$$\frac{\partial \omega}{\partial t} - U''(y)\, v_y + U(y)\frac{\partial \omega}{\partial x} = 0$$

The Case of a Constant Shear in a Channel

Easily solvable (trivial) – For pedagogical purpose

$$\frac{\partial \omega}{\partial t} - U''(y)\, v_y + U(y)\frac{\partial \omega}{\partial x} = 0$$

- $U(y) = sy$. $-l \leq y \leq l$. Then $\omega_0' = -U''(y) = 0$. A drastic simplification.

$$\frac{\partial \omega}{\partial t} + sy\frac{\partial \omega}{\partial x} = 0$$

- Fourier series for the spatial variable $\omega(x,y,t) = \omega(y,t)\exp(ikx)$:

$$\frac{\partial \omega}{\partial t} + iksy\omega = 0 \text{ then } \omega(y,t) = \omega(y,0)\exp(-iksyt)$$

The solution for the vorticity is trivial in that case

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Easily solvable (trivial) – For pedagogical purpose

$$\frac{\partial \omega}{\partial t} - U''(y)\, v_y + U(y)\frac{\partial \omega}{\partial x} = 0$$

- $U(y) = sy$. $-l \leq y \leq l$. Then $\omega_0' = -U''(y) = 0$. A drastic simplification.

$$\frac{\partial \omega}{\partial t} + sy\frac{\partial \omega}{\partial x} = 0$$

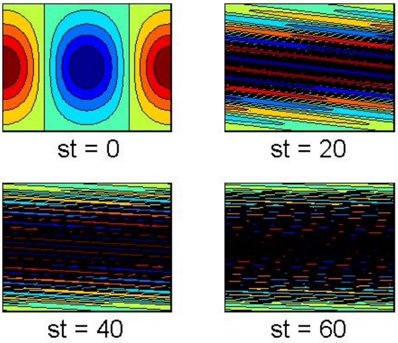
- Fourier series for the spatial variable $\omega(x,y,t) = \omega(y,t)\exp(ikx)$:

$$\frac{\partial \omega}{\partial t} + iksy\omega = 0 \text{ then } \omega(y,t) = \omega(y,0)\exp(-iksyt)$$

The solution for the vorticity is trivial in that case

Vorticity Evolution in the Case of Constant Shear

Deterministic evolution - The Orr mechanism - Base flow $U(y) = sy$



Evolution of the perturbation vorticity $\omega(t)$, advected by a constant shear s

Velocity Asymptotics in the Case of Constant Shear

The Orr mechanism - Base flow $U(y) = sy$

$$\omega(y,t) = \omega(y,0)\exp(-iksyt)$$

- We look at the solution for the velocity $\mathbf{v}(y,t)$:

$$\mathbf{v}(y,t) = \int dy\, \mathbf{G}_k(y,y')\omega(y',0)\exp(-iksy't)$$

- We have an oscillating integral. For large times:

$$v_x(y,t) \underset{t\to\infty}{\sim} \frac{v_{x,\infty}(y)}{t}\exp(-iksyt) \text{ and } v_y(y,t) \underset{t\to\infty}{\sim} \frac{v_{y,\infty}(y)}{t^2}\exp(-iksyt)$$

The velocity decreases algebraically
Orr mechanism-Case (1969)

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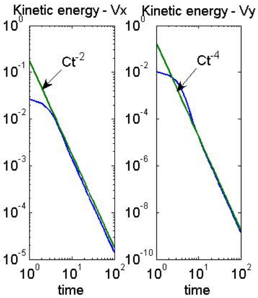
The Case of a Constant Shear in a Channel

Deterministic evolution - The Orr mechanism - Base flow $U(y) = sy$



Evolution of the perturbation kinetic energy for the transverse and longitudinal components of the velocity $\mathbf{v}(t)$

- The shear acts as an effective dissipation (Phase mixing)

The Linearized Euler Eq. close to Shear Flows

- Base flow : $\mathbf{v}_0(\mathbf{r}) = U(y)\mathbf{e}_x$. The linearized Euler equation:

$$\frac{\partial \omega}{\partial t} + ikU(y)\omega - ik\psi U''(y) = 0 \qquad (1)$$

with $\omega(x,y,t) = \omega(y,t)\exp(ikx)$ and $\omega = \frac{d^2\psi}{dy^2} - k^2\psi$

- Laplace transform: $\phi(y,c,\varepsilon) = \int_0^\infty dt\, \Psi(y,t)\exp(ik(c+i\varepsilon)t)$

$$\left(\frac{d^2}{dy^2} - k^2\right)\phi - \frac{U''(y)}{U(y)-c-i\varepsilon}\phi = \frac{\omega(y,0)}{ik(U(y)-c-i\varepsilon)} \qquad (2)$$

- This is the celebrated Rayleigh equation. A one century old classical problem in fluid mechanics, applied mathematics and mathematics. Rayleigh (1842-1919)

- Large time asymptotic is related to the limit $\varepsilon \to 0$
- Singularity of the equation : critical layer $U(y_c) = c$

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Asymptotic Behavior of the Linearized Euler Eq.

Historical results

1) Base flows without stationary points: for any y, $U'(y) \neq 0$ (monotonic profile)

- Rayleigh (1880) Mode equation
- Case (Phys. Fluid. 1960) Algebraic laws for the velocity field in the case of constant shear (wrong)
- Rosencrans and Sattinger (J. Math. Phys 1966) $v \underset{t \to 0}{=} \mathcal{O}(1/t)$ (Laplace transform tools)
- Brown,Stewartson (JFM 1980), Lundgren (Phys. Fluid. 1982) Ansatz for the asymptotic expansion, for large time
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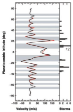
Base Flows with Stationary Streamlines

Most of geophysical flows have points such that $U'(y_0) = 0$



Most of geophysical jets have stationary streamlines (they do not verify the Rayleigh or Rayleigh-Kuo stability criteria)

- Stationary streamlines: $U'(y_0) = 0$. Velocity extrema - No shear - No Orr mechanism
- The Case velocity asymptotics (Brown and Stewartson asymptotic expansion) is not self-consistent
- $\mathbf{v}(y,t) = \int dy\, \mathbf{G}_k(y,y')\omega(y',0)\exp(-ikU(y)t)$. Stationary phase approximation: $v \underset{t\to\infty}{\propto} C/\sqrt{t}$
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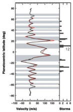

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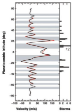

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Asymptotic Behavior of the Linearized Euler Eq.

Base flow with stationary streamlines: $U'(y_0) = 0$

- Mathematical methods : Laplace transform and detailed analysis of the singularities due to the critical layers and stationary streamlines
- By contrast with what was previously believed, we can deal with the difficulty related to the stationary streamlines

Theory : a) Asymptotic oscillatory vorticity field

$$\omega(y,t) \underset{t\to\infty}{\sim} \omega_\infty(y)\exp(ikU(y)t) + \mathcal{O}\left(\frac{1}{t^\alpha}\right)$$

b) DEPLETION OF THE VORTICITY PERTURBATION:
For any stationary streamline of the flow (y_0 such that $U'(y_0) = 0$)

$$\omega_\infty(y_0) = 0$$

- + Prediction of the asymptotic vorticity $\omega_\infty(y)$

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Outline

1. Stochastic averaging and jet formation in geostrophic turbulence.
   - The stochastic quasi-geostrophic equations.
   - Stochastic averaging (with C. Nardini and T. Tangarife).
   - Validity of this approach, and the main technical points.
2. Inviscid relaxation of the 2D Euler equations
   - Irreversibility in turbulence
   - Large Time Asymptotics of the linearized 2D Euler Eq.
   - The Kolmogorov flow

An Example: the Kolmogorov Flow

- $U(y) = \cos(y)$ in the doubly periodic domain $(0, 2\pi/\delta)x(0, 2\pi)$; δ is the aspect ratio
- Two stationary streamlines $U'(y_0) = 0$, for $y_0 = 0$ or $y_0 = \pi$
- Usual criteria for stability (Rayleigh, Arnold) do not apply
- The Kolmogorov flow is stable for $\delta > 1$ (Lyapunov stability), spectrally and linearly stable (easily proved)
- This flow has no neutral modes

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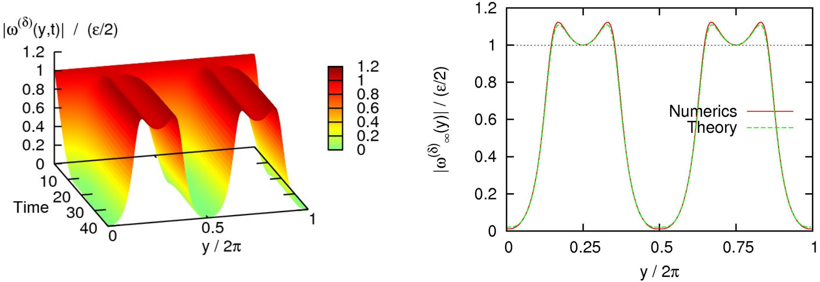
Asymptotic Behavior of the Linearized Euler Eq.

Base flow with stationary streamlines: depletion of the vorticity perturbation at the stationary streamlines



Evolution of the perturbation vorticity $\omega(t)$, advected by a shear flow $U(y) = \cos(y)$ with stationary points in $y = 0$ and $y = \pi$

Asymptotic Behavior of the Linearized Euler Eq.

Base flow with stationary streamlines : the velocity field

Theorem: algebraically decaying asymptotic velocity field

$$v_x(y,t) \underset{t\to\infty}{\sim} \frac{v_{x,\infty}(y)}{t}\exp\left(-ikU(y)t\right) \tag{3}$$

$$v_y(y,t) \underset{t\to\infty}{\sim} \frac{v_{y,\infty}(y)}{t^2}\exp\left(-ikU(y)t\right) \tag{4}$$

- What about stationary streamlines? They should give contributions of order $1/t^{1/2}$!
- No contribution from the stationary streamlines thanks to the depletion of the vorticity perturbation at stationary streamlines

Asymptotic Behavior of the Linearized Euler Eq.

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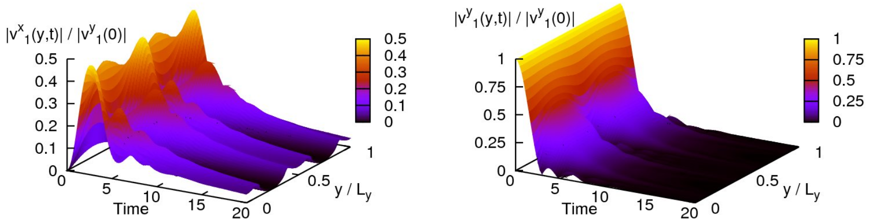
Asymptotic Behavior of the Linearized Euler Eq.

Base flow with stationary points: the velocity field



Evolution of the perturbation velocity, components $v_x(t)$ and $v_y(t)$, advected by a constant shear flow $U(y)$ with stationary streamlines

Asymptotic Behavior of the Linearized Euler Eq.

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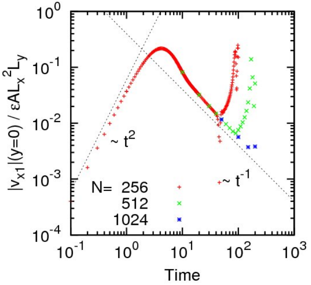

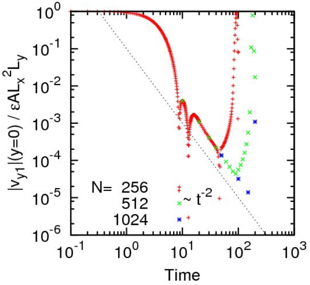

Evolution of the perturbation velocity, components $v_x(t)$ and $v_y(t)$, advected by a constant shear flow $U(y)$ with stationary streamlines

The velocity perturbation converges to zero (asymptotic stability) even without dissipation

Asymptotic Behavior of the Linearized Euler Eq.: Conclusions

- Asymptotically oscillating vorticity fields
- Algebraic decay of the velocity field with $1/t$ or $1/t^2$ laws, whatever the cases (except at stationary streamlines).
- All cases of base flow with any type of shear have been treated
- Depletion of the vorticity perturbation at the stationary streamlines
- Axisymmetric vortices should behave the same way
- The perturbation converges (weak topology) towards a Young measure

Summary and Perspectives

- Non-equilibrium statistical mechanics and large deviations can be applied to geophysical turbulence and climate.

Ongoing projects and perspectives:

- Large deviations and non-equilibrium free energies for particles with long range interactions (with K. Gawedzki).
- Microcanonical measures for the Shallow Water equations (with M. Potters and A. Venaille) and for the 3D axisymmetric Euler equations (with S. Thalabard).
- Instantons for zonal jets in the quasi-geostrophic dynamics (with C. Nardini, T. Tangarife and O. Zaboronski).

F. Bouchet, and A. Venaille, Physics Reports, 2012, Statistical mechanics of two-dimensional and geophysical flows

F. Bouchet, C. Nardini and T. Tangarife
http://hal.archives-ouvertes.fr/hal-00819779.