Atmosphere dynamics and the statistical mechanics of the two-dimensional stochastic Navier-Stokes equations and geostrophic turbulence II)

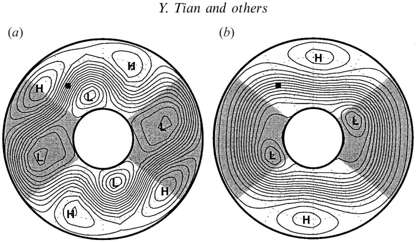
F. BOUCHET (CNRS) – ENS-Lyon and CNRS

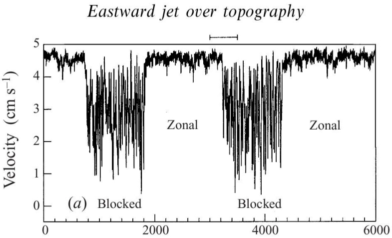
February 2014 - KU Leuven
Instituut voor Theoretische Fysica

Global Outline

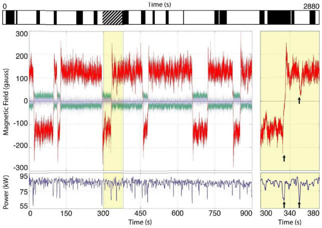
I) Introduction to climate and statistical mechanics, atmosphere dynamics and the equilibrium statistical mechanics of quasi-geostrophic models
II) Non-equilibrium phase transitions, path integrals and instanton theory
III) Kinetic theory (stochastic averaging) of zonal jet dynamics

Transitions between blocked and zonal states

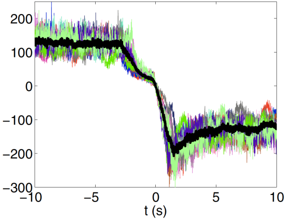




Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)



Magnetic field timeseries



Zoom on reversal paths

(VKS experiment)

In turbulent flows, transitions from one attractor to another often occur through a predictable path.

Outline

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Kramers' Equation and Overdamped Dynamics

- Langevin dynamics

$$\begin{cases} \frac{dx}{dt} & = & p \\ \frac{dp}{dt} & = & -\frac{dV}{dx}(x) - \alpha p + \sqrt{2\alpha k_B T}\eta(t) \end{cases} \quad \text{with } \mathbb{E}\left(\eta(t)\eta(t')\right) = \delta(t-t')$$

  Kramers' equation describes the evolution of the Probability Density Function (PDF).

- Overdamped Langevin dynamics

$$\frac{dx}{dt} = -\frac{dV}{dx}(x) + \sqrt{2k_B T}\eta(t)$$

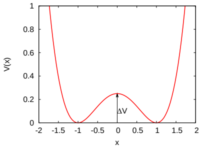
- Stationary PDF:

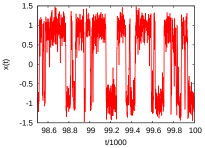
$$P_S(x) = Ce^{-\frac{V(x)}{k_B T}}$$

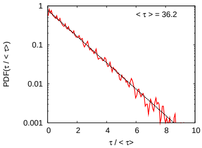
Kramers' Problem: a Pedagogical Example for Bistability

Historical example: Computation by Kramer of Arrhenius' law for a bistable mechanical system with stochastic noise

$$\frac{dx}{dt} = -\frac{dV}{dx}(x) + \sqrt{2k_B T}\eta(t) \text{ Rate : } \lambda = \frac{1}{\tau}\exp\left(-\frac{\Delta V}{k_B T}\right).$$







The problem was solved by Kramers (30'). Modern approach: path integral formulation (instanton theory, physicists) or large deviation theory (Freidlin-Wentzell, mathematicians).

Kramers' Problem: Kinetic Approach

$$\frac{dx}{dt} = -\frac{dV}{dx}(x) + \sqrt{2k_B T}\eta(t). \text{ Rate}: \; \lambda = \frac{1}{\tau}\exp\left(-\frac{\Delta V}{k_B T}\right).$$

- P_{-1} is the probability for the particle to be in the basin of attraction of x_{-1}.
- Time scales $\tau_i = \left(\frac{d^2V}{dx^2}(x_i)\right)^{-1}$. If $\lambda \ll 1/\max_i(\tau_i)$, we expect a sequence of uncorrelated jumps (Markovian).
- Then if $k_B T \ll \Delta V$, we have for $t \gg \max_i(\tau_i)$, the kinetic eq.

$$\frac{dP_{-1}}{dt} = \lambda(1 - P_{-1}) - \lambda P_{-1} = 1 - 2\lambda P_{-1}.$$

- The transition probabilities. $P(x_{-1}, T; x_1, 0)$ is the solution $P_{-1}(T)$ with initial conditions $P_1(0) = 1$.

$$P(x_{-1}, T; x_1, 0) = \frac{1}{2}\left(1 - e^{-2\lambda T}\right).$$

$$P(x_{-1}, T; x_1, 0) \underset{\max_i(\tau_i) \ll T \ll 1/\lambda}{\simeq} \lambda T = \frac{T}{\tau}\exp\left(-\frac{\Delta V}{k_B T}\right).$$

Outline

Gaussian White Noise

- What is a Gaussian white noise?
- We consider a Gaussian vector $\eta = \{\eta_i\}_{0 \leq i \leq N}$ with zero mean $\mathbb{E}(\eta_i) = 0$ and covariance $\mathbb{E}(\eta_i \eta_j) = \delta_{ij}$. Its PDF is

$$P(\eta) = \frac{1}{(2\pi)^{N/2}} \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{N} \eta_i^2} \prod_{i=1}^{N} \mathrm{d}\eta_i .$$

- A Gaussian stochastic process $\eta(t)$ with correlation function

$$\langle \eta(t)\eta(t') \rangle = \delta(t - t').$$

has a Probability Density Functional

$$P_{WN}[\eta] = \mathrm{e}^{-\frac{1}{2}\int_0^T \eta^2(t)\,\mathrm{d}t} \mathscr{D}[\eta] .$$

- The notation $\mathscr{D}[\eta]$ hides the mathematical difficulties related to the continuous time limit.

Probability Measure Over Paths

- What is the probability for the path $\{x(t)\}_{0\leq t\leq T}$, solution of

$$\frac{dx}{dt} = -\frac{dV}{dx}(x) + \sqrt{2k_B T}\eta(x,t).$$

- We start from the white noise probability

$$P_{WN}[\eta] = \mathrm{e}^{-\frac{1}{2}\int_0^T \eta^2(t)\,\mathrm{d}t}\mathscr{D}[\eta].$$

- We make a change of variables in order to get the probability for a path $\{x(t)\}_{0\leq t\leq T}$. It is

$$P_P[x] = e^{-\frac{1}{4k_B T}\int_0^T\left[\dot{x}+\frac{dV}{dx}(x)\right]^2(t)\,\mathrm{d}t} J[\eta\,|x]\,\mathscr{D}[x],$$

where $J[\eta\,|x]$ is the Jacobian of the change of variables.

- If we assume Ito convention, then $\mathrm{J}[\eta\,|x] = 1$, and

$$P_P[x] = e^{-\frac{1}{4k_B T}\int_0^T\left[\dot{x}+\frac{dV}{dx}(x)\right]^2(t)\,\mathrm{d}t}\mathscr{D}[x].$$

Path Integrals for ODE – Onsager Machlup (50')

- Path integral representation of transition probabilities:

$$P(x_T, T; x_0, 0) = \int_{x(0)=x_0}^{x(T)=x_T} e^{-\frac{\mathscr{A}_T[x]}{2k_B T}} \mathscr{D}[x]$$

$$\text{with } \mathscr{A}_T[x] = \int_0^T \mathscr{L}[x, \dot{x}] \; \mathrm{d}t \text{ and } \mathscr{L}[x, \dot{x}] = \frac{1}{2}\left[\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}(x)\right]^2.$$

- The most probable path from x_0 to x_T is the minimizer of

$$A_T(x_0, x_T) = \min_{\{x(t)\}} \left\{ \mathscr{A}_T[x] \,|\, x(0) = x_0 \text{ and } x(T) = x_T \right\}.$$

- We may consider the low temperature limit, using a saddle point approximation (WKB), Then we obtain the large deviation result

$$\log P(x_T, T; x_0, 0) \underset{\frac{k_B T}{\Delta V} \to 0}{\sim} -\frac{A_T(x_0, x_T)}{2k_B T}.$$

Relaxation Paths Minimize the Action

$$\mathscr{A}_T[x] = \int_0^T \mathscr{L}[x,\dot{x}]\;\mathrm{d}t \text{ and } \mathscr{L}[x,\dot{x}] = \frac{1}{2}\left[\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}(x)\right]^2.$$

- A relaxation path $\{x_r(t)\}_{0\leq t\leq T}$ is a solution of

$$\dot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

  Then we see that

$$\mathscr{A}_T[x_r] = 0.$$

- Interpretation: if one follows the deterministic dynamics, no noise is needed and the cost is zero.
- Because for any path $\mathscr{A}_T[x_r] \geq 0$, any relaxation path minimizes the action.

Fluctuation Paths and Instantons

- The most probable path from an attractor of the system x_0 to a state x is called a fluctuation path. It solves

$$A_\infty(x_0,x) = \min_{\{x(t)\}} \left\{ \mathscr{A}_\infty[x] \,|\, x(-\infty) = x_0 \text{ and } x(0) = x \right\}.$$

- When the WKB limit is justified (low temperature), most of the paths leading to a rare fluctuation x concentrate close to the fluctuation path. The probability to observe x is

$$P(x) \sim C e^{-\frac{A_\infty(x)}{2k_B T}}.$$

- In bistable systems (more than one attractor), fluctuation paths from one attractor x_1 to a saddle point x_s play an important role. They lead to a change of basin of attraction. They are called instantons. The transition rate is

$$P(x_{-1}, T; x_1, 0) \sim C e^{-\frac{A(x_{-1},x_1)}{2k_B T}},$$

with $A(x_{-1},x_1) = \min_{\{x(t)\}} \left\{ \mathscr{A}_\infty[x] \,|\, x(-\infty) = x_1 \text{ and } x(+\infty) = x_s \right\}$.

Outline

Time Reversal and Action Symmetry

$$\mathscr{A}_T[x] = \int_0^T \mathscr{L}[x,\dot{x}]\ \mathrm{d}t \text{ with } \mathscr{L}[x,\dot{x}] = \frac{1}{2}\left[\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\right]^2.$$

- We consider a path $x = \{x(t)\}_{0\le t\le T}$ and the reversed path $R[x] = \{x(T-t)\}_{0\le t\le T}$. We have

$$\mathscr{L}[R[x,\dot{x}]] = \frac{1}{2}\left[-\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\right]^2 = \frac{1}{2}\left[\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\right]^2 - 2\dot{x}\frac{\mathrm{d}V}{\mathrm{d}x}.$$

- Then, using $\dot{x}\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}t}V(x)$,

$$\mathscr{A}_T[R[x]] = \mathscr{A}_T[x] + 2V(x(T)) - 2V(x(0))$$

Action Symmetry and Detailed Balance

- We start from

$$P(x_T,T;x_0,0)=\int_{x(0)=x_0}^{x(T)=x_T}\mathrm{e}^{-\frac{\mathscr{A}_T[x]}{2k_BT}}\mathscr{D}[x].$$

- We perform the change of variables $x\to R[x]$ in the path integral. We use

$$\mathscr{A}_T[R[x]]=\mathscr{A}_T[x]+2V(x(T))-2V(x(0))$$

- Then

$$P(x_T,T;x_0,0)=\mathrm{e}^{-\frac{V(x_T)}{k_BT}+\frac{V(x_0)}{k_BT}}\int_{x(0)=x_T}^{x(T)=x_0}\mathrm{e}^{-\frac{\mathscr{A}_T[x]}{2k_BT}}\mathscr{D}[x].$$

- Using that the stationary distribution is $P_s(x)=C\mathrm{e}^{-\frac{V(x)}{k_BT}}$ we thus conclude

$$P(x_T,T;x_0,0)P_S(x_0)=P(x_0,T;x_T,0)P_S(x_T).$$

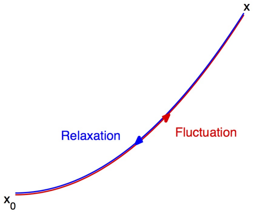
This is the statement of detailed balance.

Time Reversed Relaxation Paths Minimize the Action

We have the symmetry relation

$$\mathscr{A}_T[R[x]] = \mathscr{A}_T[x] + 2V(x(T)) - 2V(x(0))$$

We conclude that the time reversed relaxation paths also minimizes the action.



- The minimizer of the action from an attractor of the system to any point of its basin of attraction is the reversed of the relaxation path.
- This is an extended Onsager-Machlup relation. For time reversible systems, the most probable way to get a fluctuation is through the reversal of the relaxation path from this fluctuation.

Instantons are Time Reversed Relaxation Paths

- We have the symmetry relation

$$\mathscr{A}_T\left[R\left[x\right]\right] = \mathscr{A}_T\left[x\right] + 2V(x(T)) - 2V(x(0))$$

- Using this equation, we can conclude that instantons are time reversed relaxation paths from a saddle to an attractor. Then we obtain the large deviation result

$$\log P\left(x_1, T; x_{-1}, 0\right) \underset{\frac{k_B T}{\Delta V} \to 0}{\sim} -\frac{\Delta V}{k_B T}.$$

- The computation of the prefactor is more tricky

$$P(x_{-1}, T; x_1, 0) \underset{t \ll 1/\lambda}{\simeq} \frac{T}{\tau} \exp\left(-\frac{\Delta V}{k_B T}\right) \text{ with } \tau = 2\pi \left(\frac{d^2 V}{dx^2}(x_0)\frac{d^2 V}{dx^2}(x_{-1})\right)^{-1/2}.$$

This is the subject of Langer theory (70'), see also Caroli, Caroli, and Roulet, J. Stat. Phys., 1981, for a computation through path integrals.

Time Reversal and Action Symmetry: Conclusions

- We consider a path $x = \{x(t)\}_{0 \leq t \leq T}$ and its reversed path $R[x] = \{x(T-t)\}_{0 \leq t \leq T}$. We have

$$\mathscr{A}_T[R[x]] = \mathscr{A}_T[x] + 2V(x(T)) - 2V(x(0)).$$

- This implies detailed balance.
- This implies that the most probable path to reach a state x (a fluctuation) is the time reversal of a relaxation path starting from x (dissipation).
- This is a generalized Onsager-Machlup relation, that explains quite easily and naturally fluctuation-dissipation relations.
- For dynamics symmetric by time reversal, instantons are time reversed relaxation paths.

Kramers' Equation Without the Overdamped Assumption

- Langevin dynamics

$$\begin{cases} \frac{dx}{dt} & = & p \\ \frac{dp}{dt} & = & -\frac{dV}{dx}(x) - \alpha p + \sqrt{2\alpha k_B T}\eta(t) \end{cases} \quad \text{with } \mathbb{E}\left(\eta(t)\eta(t')\right) = \delta(t-t')$$

- We note a state $X = (x, p)$. Transition probabilities and path integrals:

$$P(X_T, T; X_0, 0) = \int_{X(0)=X_0}^{X(T)=X_T} e^{-\frac{\mathscr{A}_T[X]}{2k_B T}} \mathscr{D}[X]$$

$$\text{with } \mathscr{A}_T[X] = \int_0^T \mathscr{L}\left[X, \dot{X}\right] \, dt.$$

$$\text{and } \mathscr{L}\left[X, \dot{X}\right] = \begin{cases} \frac{1}{2\alpha}\left[\dot{p} + \frac{dV}{dx}(x) + \alpha p\right]^2 & \text{if } \dot{x} = p \\ -\infty & \text{otherwise} \end{cases}.$$

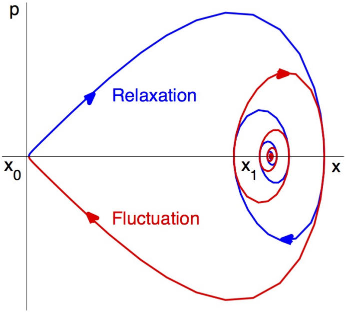
- Temporal symmetry of the Hamiltonian system

$$\begin{cases} \frac{dx}{dt} & = & p \\ \frac{dp}{dt} & = & -\frac{dV}{dx}(x). \end{cases}$$

- We define the involution $I[(x,p)] = (x,-p)$ (velocity inversion). The reversibility of the Hamiltonian equations means that if $X =$ (x(t),p(t)) is a solution, then the reversed path $X_r = I(x(T-t), p(t-T))$ is also a solution.
- Action symmetry: we easily check that

$$\mathscr{A}_T[X_r] = \mathscr{A}_T[X] + 2V(X(T)) - 2V(X(0)).$$

$$\mathscr{A}_T[X_r] = \mathscr{A}_T[X] + 2V(X(T)) - 2V(X(0)).$$



We need to invert the velocity in order to define the relaxation path.

Outline

Langevin Dynamics In a General Framework

$$\frac{\partial q}{\partial t} = \mathscr{F}[q](\mathbf{r}) - \alpha \int_{\mathscr{D}} C(\mathbf{r},\mathbf{r}') \frac{\delta \mathscr{G}}{\delta q(\mathbf{r}')}[q]\,\mathrm{d}\mathbf{r}' + \sqrt{2\alpha\gamma}\eta,$$

- Assumptions: i) $\mathscr{F}$ verifies a Liouville theorem

$$\nabla.\mathscr{F} \equiv \int_{\mathscr{D}} \frac{\delta \mathscr{F}}{\delta q(\mathbf{r})}\,\mathrm{d}\mathbf{r} = 0 \quad \left(\text{Generalization of } \nabla.\mathscr{F} \equiv \sum_{i=1}^{N} \frac{\partial \mathscr{F}}{\partial q_i} = 0 \right),$$

- ii) The potential $\mathscr{G}$ is a conserved quantity of $\frac{\partial q}{\partial t} = \mathscr{F}[q](\mathbf{r})$:

$$\int_{\mathscr{D}} \mathscr{F}[q](\mathbf{r}) \frac{\delta \mathscr{G}}{\delta q(\mathbf{r})}[q]\,\mathrm{d}\mathbf{r} = 0.$$

- iii) η a Gaussian process, white in time, with covariance

$$\mathbb{E}\left[\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\right] = C(\mathbf{r},\mathbf{r}')\delta(t-t').$$

- For most classical Langevin dynamics:

$$\mathscr{F}[q](\mathbf{r}) = \{q, \mathscr{H}\} \text{ and } \mathscr{G} = \mathscr{H}.$$

Time Reversal and Action Duality

$$\mathscr{A}[q,T] = \int_0^T \mathscr{L}\left[q(t), \frac{\partial q}{\partial t}(t)\right] \mathrm{d}t, \text{ with}$$

$$\mathscr{L}\left[q, \frac{\partial q}{\partial t}\right] = \frac{1}{2\alpha}\int_{\mathscr{D}}\int_{\mathscr{D}} P(\mathbf{r}',t)C^{-1}(\mathbf{r},\mathbf{r}')P(\mathbf{r}',t)\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}' \text{ with } P(\mathbf{r},t) \equiv \frac{\partial q}{\partial t} - \mathscr{F}[q](\mathbf{r}) + \alpha\int_{\mathscr{D}} C(\mathbf{r},\mathbf{r_2})\frac{\delta \mathscr{G}}{\delta q(\mathbf{r_2})}[q]\mathrm{d}\mathbf{r_2}.$$

- Consider any involution $I[q]$ (such that $I^2 = \mathrm{Id}$). Then

$$\mathscr{A}[q,T] = \mathscr{A}_r[q_r,T] + 2(\mathscr{G}[q(T)] - \mathscr{G}[q(0)]),$$

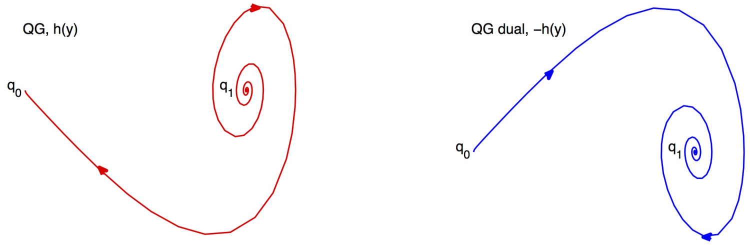
$$\text{with } \mathscr{L}_r\left[q, \frac{\partial q}{\partial t}\right] = \frac{1}{2\alpha}\int_{\mathscr{D}}\int_{\mathscr{D}} P_r(\mathbf{r}',t)C^{-1}(\mathbf{r},\mathbf{r}')P_r(\mathbf{r}',t)\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}' \text{ with } P_r(\mathbf{r},t) \equiv \frac{\partial q}{\partial t} - \mathscr{F}_r[q](\mathbf{r}) + \alpha\int_{\mathscr{D}} C_r(\mathbf{r},\mathbf{r_2})\frac{\delta \mathscr{G}_r}{\delta q(\mathbf{r_2})}[q]\mathrm{d}\mathbf{r_2}.$$

$$\text{with } \mathscr{F}_r = -I\mathrm{o}\mathscr{F}\mathrm{o}I,\ C_r = I^+CI, \text{ and } \mathscr{G}_r[q] = \mathscr{G}[I[q]],$$

where I^+ is the adjoint of I for the L^2 scalar product.

- The action is equal, up to the potential values, to the action of a reversed path in a conjugated dynamics.

Relaxation Paths and Fluctuation Paths



The fluctuation paths of the direct dynamics are the reversed of the relaxation paths of the dual dynamics, and vice versa (temporal symmetry breaking).

Time Reversal and Action Duality: Conclusions

- We consider a path $q = \{q(t)\}_{0\leq t\leq T}$ and its reversed path $q_r = \{I[q(T-t)]\}_{0\leq t\leq T}$. We have

$$\mathscr{A}_T[q_r] = \mathscr{A}_T[q] + 2V(q(T)) - 2V(q(0)).$$

- Transition probabilities of the direct process are related to transition probabilities of the dual process (a generalization of detailed balance).
- This implies that the most probable path to reach a state x (a fluctuation) is the time reversal of a relaxation path starting from $I[x]$ for the dual process (dissipation).
- This is a generalized Onsager-Machlup relation, that justifies generalization of fluctuation-dissipation relations.
- Instantons are the time reversed relaxation paths of the dual process.

Outline

Langevin Dynamics for the Quasi-Geostrophic Eq.

$$\frac{\partial q}{\partial t} = \mathbf{v}\left[q-h\right].\nabla q - \alpha \int_{\mathscr{D}} C(\mathbf{r},\mathbf{r}') \frac{\delta \mathscr{G}}{\delta q(\mathbf{r}')}\left[q\right] d\mathbf{r}' + \sqrt{2\alpha\gamma}\eta,$$

- Assumptions: i) $\mathscr{F} = -\mathbf{v}\left[q-h\right].\nabla q$ verifies a Liouville theorem.
- ii) The potential $\mathscr{G}$ is a conserved quantity of $\frac{\partial q}{\partial t} = \mathscr{F}\left[q\right](\mathbf{r})$ with

$$\mathscr{G} = \mathscr{C} + \beta \mathscr{E},$$

with a Casimir functionals

$$\mathscr{C}_c = \int_{\mathscr{D}} d\mathbf{r}\, c(q),$$

and energy

$$\mathscr{E} = -\frac{1}{2}\int_{\mathscr{D}} d\mathbf{r}\left[q - H\cos(2y)\right]\psi = \frac{1}{2}\int_{\mathscr{D}} d\mathbf{r}\, \nabla\psi^2.$$

The Dual Quasi-Geostrophic Dynamics

- For the 2D-Euler or quasi-geostrophic equations, the potential vorticity (vorticity) fields are changed through $q \to -q$ by a time reversal. Hence $I[q] = -q$.
- The direct dynamics for the Quasi-Geostrophic dynamics is

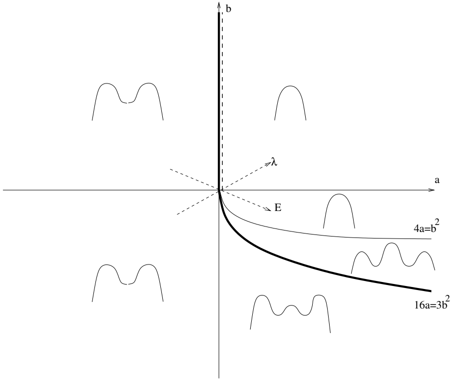
$$\frac{\partial q}{\partial t} = \mathbf{v}[q-h].\nabla q,$$

while the dual one is

$$\frac{\partial q}{\partial t} = \mathbf{v}[q+h].\nabla q.$$

- The dual dynamics is the one where h is replaced by $-h$.
- The Langevin Quasi-Geostrophic dynamics does not satisfy detailed balance. The Langevin 2D Euler dynamics satisfies detailed balance if $c(\omega) = -c(\omega)$.

Outline



Tricritical point corresponding to the normal form
$s(m) = -m^6 - \frac{3b}{2} m^4 - 3am^2.$

A Quasi-Geostrophic Potential with A Tricritical Point

$$\mathscr{G} = (1-\varepsilon)\frac{1}{2}\int_{\mathscr{D}} \mathrm{d}\mathbf{r}\,[q - H\cos(2y)]\,\psi + \int_{\mathscr{D}} \mathrm{d}\mathbf{r}\left[\frac{q^2}{2} - a_4\frac{q^4}{4} + a_6\frac{q^6}{4}\right] \text{ with } h(y) = H\cos(2y).$$

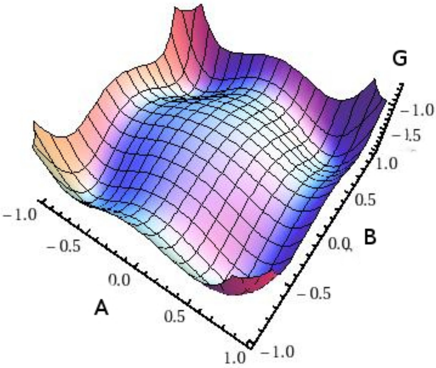
- There is a tricritical transition (transition from first order to second order) close to $\varepsilon = 0$ and $a_4 = 0$ for small H.
- Close to the transition the stochastic dynamics can be reduced to a two-degrees of freedom stochastic dynamics, which is a gradient dynamics with potential

$$G(A,B) = -\frac{H^2}{3} + \varepsilon\left[A^2+B^2\right] - \frac{3a_4}{2}\left[A^2+B^2\right]^2 + \frac{a_6}{6}\gamma\left[A^2+B^2\right]^3 + \frac{5\pi}{144}a_6H^2\left(A^2-B^2\right)^2.$$

- And the potential vorticity field is

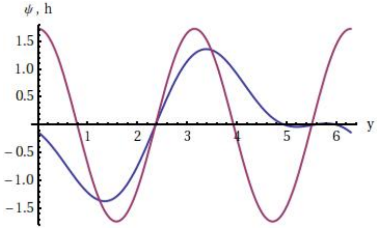
$$q(y) \simeq A\cos(y) + B\sin(y).$$

The Reduced Potential

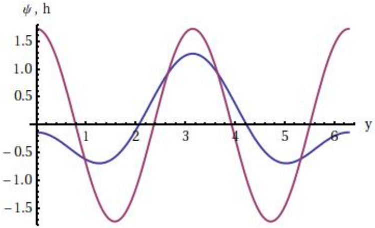


The reduced potential

One Attractor and One Saddle

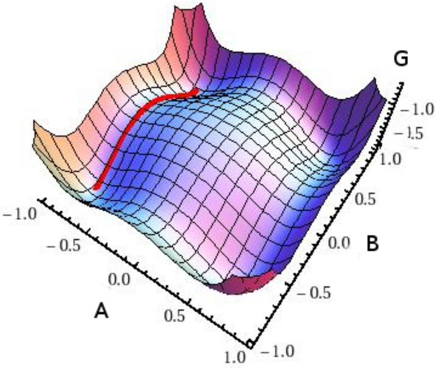


The streamfunction for one of the attractors and the topography (red).



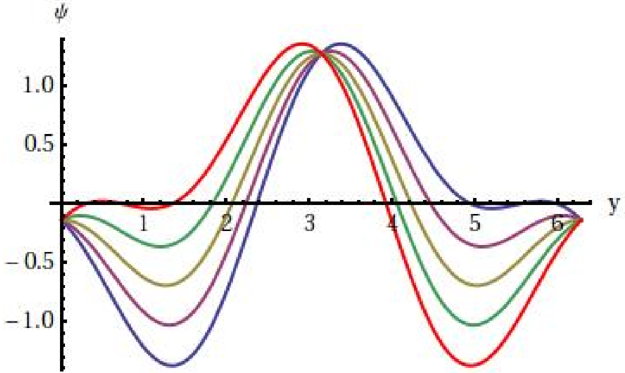
A saddle.

The Reduced Potential and the Instanton



The reduced potential and one instanton/relaxation path.

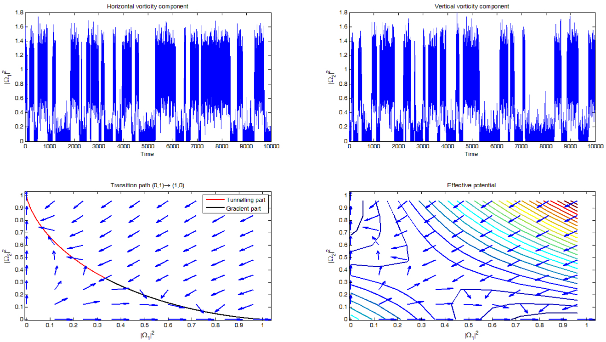
The Instanton/Relaxation Dynamics



5 streamfunction for an instanton/relaxation path (attractor, intermediate, saddle, intermediate, attractor).

Quasi-Geostrophic Tricritical Point

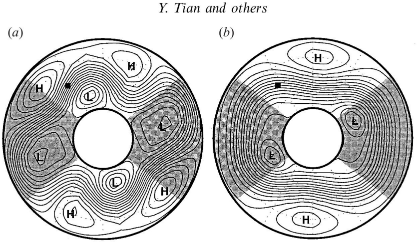
- For this turbulent dynamics, we can predict the phase diagram (a tricritial point). For a range of parameter, we have first order phase transitions.
- Using large deviations, we can compute transition probabilities.
- We can compute the transition rate between two attractors.
- Most transitions concentrate close to the optimal one, it is describe by an instanton that is easily computed.
- Sufficiently close to the tricritical point, the dynamics reduces to a two degrees of freedom stochastic dynamics.

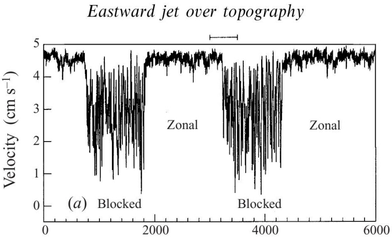


The reduced potential and one instanton/relaxation path.

Outline

Transitions between blocked and zonal states:





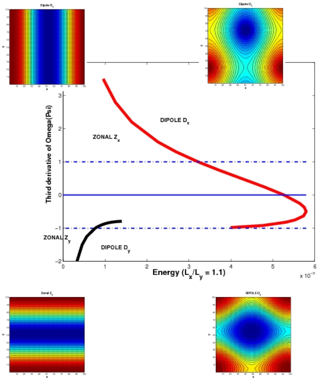
Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

2D Stochastic Navier-Stokes Eq. and 2D Euler Steady States

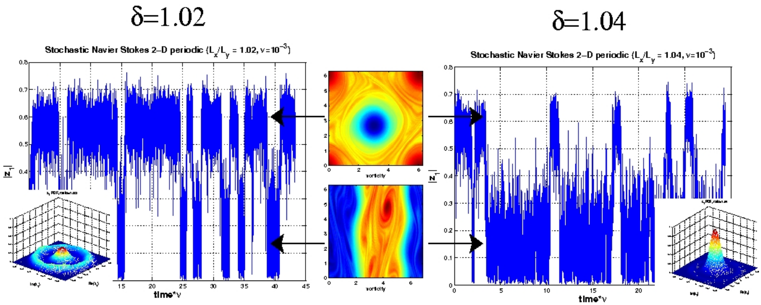
$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s$$

- Time scale separation: magenta terms are small.

Statistical Equilibria for the 2D-Euler Eq. (doubly periodic)



A second order phase transition.



Order parameter : $z_1 = \int dxdy \exp(iy)\omega(x,y)$.

For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$

F. Bouchet and E. Simonnet, PRL, 2009.

The Action of the 2D Stochastic Navier-Stokes

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s \text{ with } \left\langle f_s(\mathbf{x},t), f_s(\mathbf{x}',t')\right\rangle = C(\mathbf{x}-\mathbf{x}')\delta(t-t').$$

$$\mathscr{S}[T,x] = \frac{1}{2}\int_0^T \mathrm{d}t \int_{\mathscr{D}} \mathrm{d}\mathrm{x}\mathrm{d}\mathrm{x}'\, p(\mathrm{x},\mathbf{t})C(\mathrm{x}-\mathrm{x}')p(\mathrm{x}',\mathbf{t}),$$

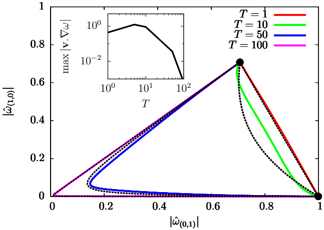
$$\textit{with } p = \frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega + \alpha\omega - \nu\Delta\omega.$$

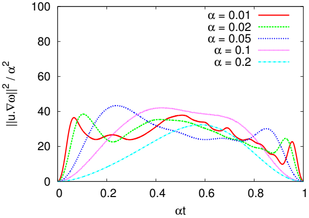
- We can compute explicitly and study the stability of many instantons (parallel flows to parallel flows, spatial white noise, Laplacian eigenmodes, etc.).
- Definition: $C_{\mathbf{k}} = \int_{\mathscr{D}} \mathrm{d}\mathrm{x} \exp(i\mathbf{k}.\mathrm{x})\, C(\mathrm{x})$. If $C_{\mathbf{k}} = 0$ for some $\mathbf{k}$, the force is called degenerate, non-degenerate otherwise.

- We discretize action integral both in time and space (time using the central differencing scheme, and space using pseudo-spectral decomposition)
- Fix the initial and final states throughout the minimization
- Newton or quasi-Newton methods are prohibitively expensive to implement (Hessian)
- We implement a gradient method or steepest descent method:
- Then iteratively minimize an initial guess (simultaneously over space and time) in the direction of the anti-gradient:

$$\omega^{n+1} = \omega^{n} - c_n \frac{\delta S(\omega^{n})}{\delta \omega^{n}}$$

Instantons from Dipole to Parallel Flows



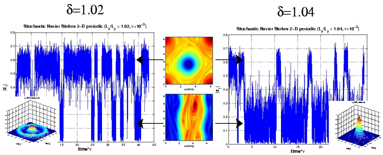


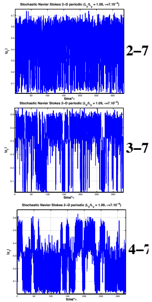
Comparison of numerical instantons with analytical ones

Instantons are close to the set of attractors

- In the limit of weak forces and dissipations, instantons follows the set of attractors of the 2D Euler equations.

Degenerate Forces Prevent Bistability





Order parameter : $z_1 = \int dxdy \exp(iy)\omega(x,y)$.

For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$.

The 2D Stochastic Navier-Stokes Eq. and Freidlin–Wentzell Framework

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s.$$

- Time scale separation: magenta terms are small.
- At first order, the dynamics is nearly a 2D Euler dynamics. The flow self organizes and converges towards steady solutions of the Euler Eq.:

$$\mathbf{v}.\nabla \omega = 0 \text{ or equivalently } \omega = f(\psi)$$

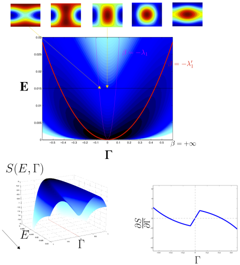
  where the Stream Function ψ is given by: $\mathbf{v} = \mathbf{e}_z \times \nabla \psi$.
- It looks like an underdamped dynamics, but the right hand side actually has an infinite number of attractors.
- The 2D Navier-Stokes equations does not enter in the the Freidlin–Wentzell framework.

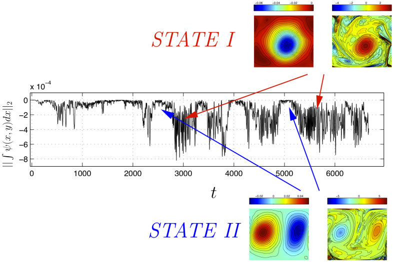
$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = 0$$

- If $\omega(\mathbf{x},t)$ is a solution of the 2D Euler Eq., then for any $\lambda > 0$, $\lambda \omega(\mathbf{x},\lambda t)$ is also a solution (nonlinearity is homogeneous of degree 2).
- Then any steady solutions ω is connected to zero through the path $s\omega(st)$, $0 \leq s \leq 1$.
- Any two steady states ω_0 and ω_1 are connected through a continuous path $\Omega(s)$, $0 \leq s \leq 1$ among the set of steady state.
- The set of steady states of the 2D Euler equations is connected (please see section 3.3).

F. BOUCHET, and H. TOUCHETTE, 2012, Non-classical large deviations for a noisy system with non-isolated attractors, *J. Stat. Mech.*, P05028.

Outline

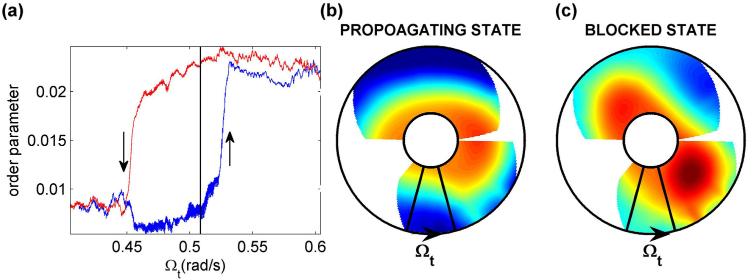




Simulations by E. Simonnet

A. VENAILLE, and F. BOUCHET, 2011, J. Stat. Phys.; M. CORVELLEC and F. BOUCHET, 2012, condmat.

Bistability in a Rotating Tank Experiment
Rotating tank with a single-bump topography

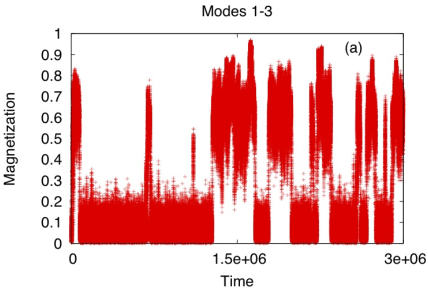


Bistability (hysteresis) in rotating tank experiments

M. MATHUR, and J. SOMMERIA, to be submitted to J. Geophys. Res., M. MATHUR, J. SOMMERIA, E. SIMONNET, and F. BOUCHET, in preparation.

Non-Equilibrium Phase Transitions for the Stochastic Vlasov Eq.

with a theoretical prediction based on non-equilibrium kinetic theory



Time series for the order parameter for the 1D stochastic Vlasov Eq.

C. NARDINI, S. GUPTA, S. RUFFO, T. DAUXOIS, and F. BOUCHET, 2012, J. Stat. Mech., L01002, and 2012 J. Stat. Mech., P12010.

Outline

- A huge number of Hamiltonian PDEs have connected attractors

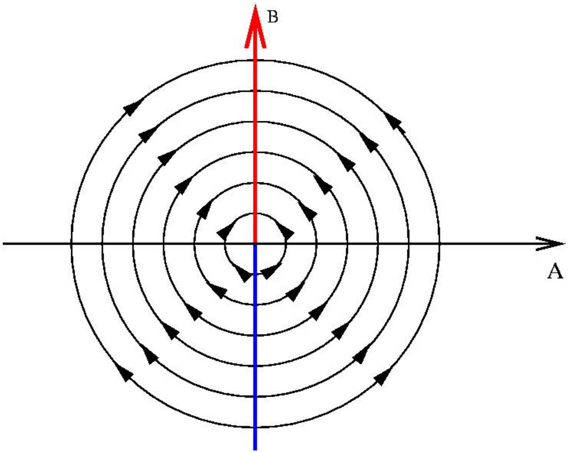
$$\left\{ \begin{array}{lcl} \frac{dA}{dt} & = & -AB \\ \frac{dB}{dt} & = & A^2 \end{array} \right.$$

- A quadratic nonlinearity. Conservation of energy

$$E = A^2 + B^2$$

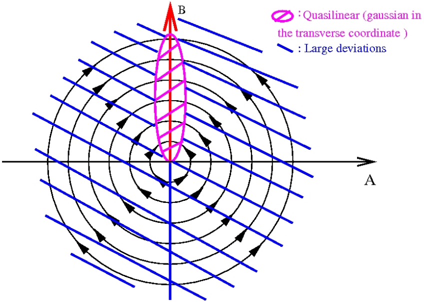
- A connected set of steady states. For any B, $A = 0$ is an equilibrium

Phase Space of the A-B Model



$$\begin{cases} dA & = & (-AB-\nu A)\,dt+\sqrt{\nu}\sigma_1 dW_1 \\ dB & = & \left(A^2-\nu B\right)dt+\sqrt{\nu}\sigma_2 dW_2 \end{cases}$$

- Stationary measure in the limit $\nu \to \infty$



- First step of the adiabatic treatment : understand the evolution of the rapid variable A, for a fixed value of the slow variable B.
- At first order, for small ν, A is a Ornstein–Uhlenbeck process. $dA = (-AB - \nu A)\,dt + \sqrt{\nu}\sigma_1 dW_1$. Locally Gaussian :

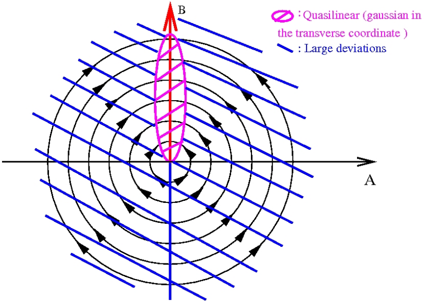
$$P(A) = C(B)\exp\left(-\frac{BA^2}{\nu\sigma_1^2}\right)$$

$$P(A,B) = C_1 \exp\left(-\frac{BA^2}{\nu\sigma_1^2}\right) B^{\frac{\sigma_1^2}{\sigma_2^2}+\frac{1}{2}} \exp\left(-\frac{B^2}{\sigma_2^2}\right) \quad ; \quad P(E) = C_1 E^{\frac{\sigma_1^2}{\sigma_2^2}} \exp\left(-\frac{E^2}{\sigma_2^2}\right)$$

- A non trivial distribution
- The PDF is not concentrated. The weak forces and dissipation do not select a single equilibrium energy E.

$$\begin{cases} dA = (-AB - \nu A)\,dt + \sqrt{\nu}\sigma_1 dW_1 \\ dB = \left(A^2 - \nu B\right) dt + \sqrt{\nu}\sigma_2 dW_2 \end{cases}$$

- Stationary measure in the limit $\nu \to \infty$



$$dx = f(x)dt + \sqrt{\nu}dW$$

- Hypothesis: the deterministic dynamics has isolated attractors. Large deviation results:

$$P(X) \sim \exp\left(-\frac{V(X)}{\nu}\right) \text{ to mean } \lim_{\nu \to 0} \nu \log P = -V$$

$$\text{with } V(X) = \inf_{t>0} \inf_{\{x(t) | x(0) \in 0 \text{ and } x(t)=X\}} L[x]$$

$$\text{and } L[x] = \frac{1}{2}\int_0^t ds\, (\dot{x} - f(x))^2$$

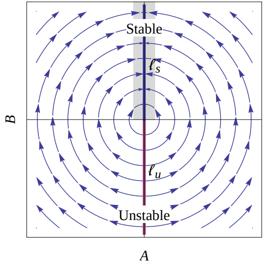
- Because of the connected attractors, the AB model does not fulfill the hypothesis of the Freidlin–Wentzell theorems

Non Classical Rate for the Large Deviations of the A-B Model

- Large deviation result:

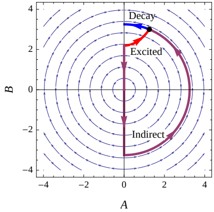
$$P(A,B) \sim \exp\left(-\frac{V(A,B)}{\sqrt{\nu}}\right) \text{ to mean } \lim_{\nu\to 0}\sqrt{\nu}\log P = -V$$

with $V(A,B)=0$ if $A=0, B>0$ and $V(A,B)=\frac{2\sqrt{2}}{3}\left(A^2+B^2\right)^{3/4}$ otherwise



$$L[x] = \frac{1}{2}\int_0^t \mathrm{ds}\left(\dot{x} - f(x)\right)^2$$

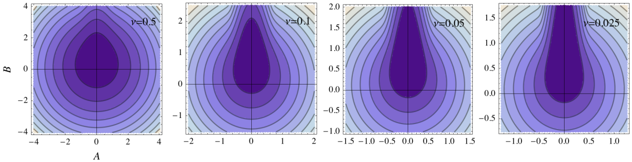
- The action is zero for paths along the set of steady states and along a deterministic trajectory.



$$P(x = x_1, t = 0; x = x_2, t = T) = \int_{x(0)=x_1}^{x(T)=x_2} \mathscr{D}[x] \exp\left(-\frac{1}{2\nu} L[x]\right)$$

$$\lim_{\nu\to 0}\sqrt{\nu}\log P(A,B) = -V$$

with $V(A,B)=0$ if $A=0, B>0$ and $V(A,B)=\frac{2\sqrt{2}}{3}\left(A^2+B^2\right)^{3/4}$ otherwise



Non-Eq. Phase Transitions and Instantons: Conclusions

- We predicted and observed non-equilibrium phase transitions for the 2D Navier-Stokes equations and in experiments.
- We can numerically compute instantons for simple turbulent flows.
- The 2D Navier-Stokes equations does not enter in the Freidlin-Wentzell framework.
- In the inertial limit, instantons follow the connected set of attractors.
- There is no large deviations for transitions between attractors for non-degenerate forces (no bistability).

F. BOUCHET, and H. TOUCHETTE, 2012, Non-classical large deviations for a noisy system with non-isolated attractors, *J. Stat. Mech.*, P05028., F. Bouchet, J. Laurie, E. Simonnet, and O. Zaboronski, to be submitted to PRL, J. Laurie and F. Bouchet, to be submitted to Phys. Rev. E.