Large Deviations, Instantons, and the Statistical Mechanics of Atmosphere Jets

F. BOUCHET (CNRS) – ENS-Lyon and CNRS

January 2013 - Statistical mechanics and climate - Reading

Collaborators

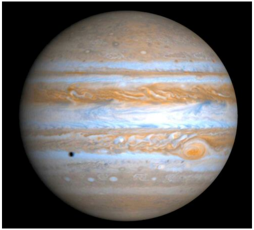
- Equilibrium statistical mechanics of ocean jets and vortices: A. Venaille (PHD student, now in post doc in Princeton)
- Equilibrium statistical mechanics of the Great Red Spot of Jupiter: J. Sommeria (LEGI-Coriolis, Grenoble)
- Random changes of flow topology in the 2D Navier-Stokes equations: E. Simonnet (INLN-Nice) (ANR Statocean)
- Asymptotic stability and inviscid damping of the 2D-Euler equations: H. Morita (Tokyo university) (ANR Statflow)

Collaborators

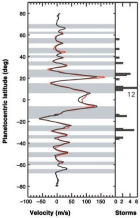
- Invariant measures of the 2D Euler and Vlasov equations: M. Corvellec (PHD student, INLN Nice, CNLS Los Alamos and ENS-Lyon)
- Instantons and large deviations for the 2D Navier-Stokes equations: J. Laurie (Post-doc ANR Statocean), O. Zaboronski (Warwick Univ.)
- Large deviations for systems with connected attractors: H. Touchette (Queen Mary Univ, London)
- Stochastic Averaging and Jet Formation in Geostrophic Turbulence: C. Nardini and T. Tangarife (ENS-Lyon)

Earth and Jupiter's Zonal Jets

We look for a theoretical description of zonal jets



Jupiter atmosphere



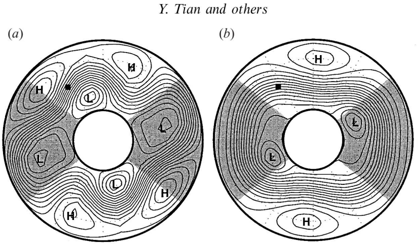
Jupiter Zonal wind (Voyager and Cassini, from Porco et al 2003)

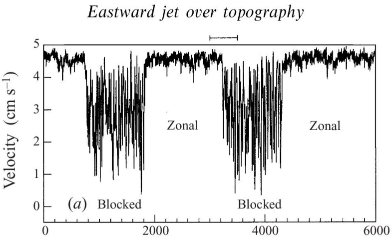
How to theoretically predict such velocity profile?

Phase Transitions in Rotating Tank Experiments

The rotation as an ordering field (Quasi Geostrophic dynamics)

Transitions between blocked and zonal states





Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s,$$

where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, $q = \omega + \beta_d y$ is the Potential Vorticity (PV), f_s is a random force, α is the Rayleigh friction coefficient.

- Quasi-Gestrophic models: the basic models for midlatitude large scale turbulence.

The 2D Stochastic-Navier-Stokes (SNS) Equations

- The simplest model for two dimensional turbulence
- Navier Stokes equations with random forces

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{\sigma} f_s$$

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- An academic model with experimental realizations (Sommeria and Tabeling experiments, rotating tanks, magnetic flows, and so on). Analogies with geophysical flows (Quasi Geostrophic and Shallow Water layer models)

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Equilibrium: the 2D Euler Equations

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- Conservative dynamics - Hamiltonian (non canonical) and time reversible

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The Main Issues for Physicists

- What makes the dynamics of geophysical flows so peculiar?
- Why does the large scales of geophysical flows self-organize?
- Can we predict the statistics of the large scales of geophysical flows?
- Can we predict phase transitions for geophysical turbulent flows and their statistics?

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Non-Equilibrium Stat. Mech.

1. Stochastic averaging technics (kinetic theory in a stochastic framework).
2. Large deviation for transition probabilities for rare events (through path integrals).
3. Tools from field theory in statistical physics.

The Main Mathematical Challenges

- Can we define microcanonical measures for the 2D Euler and Quasi-Geostrophic equations?
- Are those microcanonical measures invariant measures?
- Can we treat close to equilibria dynamics in the framework of kinetic theory?
- What are the properties of the kinetic equations?
- Would the Freidlin-Wentzell theory generalize to the stochastic Navier-Stokes equations?

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- Conservative dynamics - Hamiltonian (non canonical) and time reversible
- Invariants:

$$\text{Energy: } E[\omega] = \frac{1}{2}\int_{\mathscr{D}} d^2x\, \mathbf{v}^2 == -\frac{1}{2}\int_{\mathscr{D}} d^2x\, \omega\psi$$

$$\text{Casimir's functionals: } \mathscr{C}_s[\omega] = \int_{\mathscr{D}} d^2x\, s(\omega)$$

$$\text{Vorticity distribution: } D(\sigma) = \frac{dA}{d\sigma} \text{ with } A(\sigma) = \int_D d^2x\, \chi_{\{\omega(\mathbf{x})\leq\sigma\}}$$

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Consequences of Multiple Invariants of 2D flows I)

Multiple steady solutions – Multiple stable steady solutions

- Any non degenerate minimum of a conserved quantity is a stable steady solution (think to mechanics and energy).
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PV-Psi Relation for Jupiter's Great Red Spot

From Dowling and Ingersoll empirical analysis (1989)

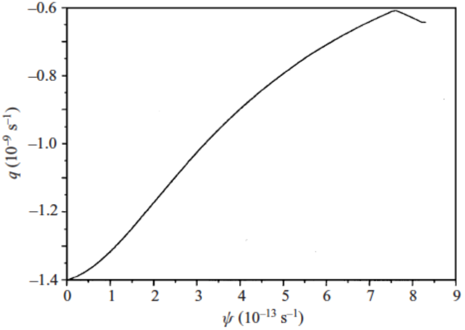

FIGURE 12. Potential vorticity q^*_{SW} versus stream function Ψ^*_{SW} from the determination of $q^*_{SW}(B^*_e)$ by Dowling & Ingersoll (1989) (their table 1) in the Great Red Spot (for $R = 2200\,\mathrm{km}$ $q: 10^{-9}\,\mathrm{s}^{-1}$ and $\Psi: 10^{13}\,\mathrm{m\,s}^{-1}$). These SW potential vorticity and stream function are proportional to the QG ones in the QG limit. We observe that this function is in reasonable agreement with the tanh-like relation of the two-PV-level Gibbs states.

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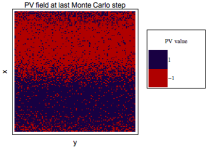
Microstates for the 2D Euler Eq.

The case with 2 levels of vorticity (for pedagogical purpose)

- We discuss the case $D(\sigma)=\frac{1}{2}\delta(\sigma+1)+\frac{1}{2}\delta(\sigma-1)$, $(\omega(\mathbf{r})\in\{-1,1\}$ with $\pm$ 1 values occupying equal areas).
- Vorticity points on a lattice of size NxN (used for instance as weight in a finite elements approximations of 2D fields)

$$X_N=\left\{\omega=\left(\omega_{ij}\right)_{1\leq i,j\leq N}\,|\,\forall i,j\,\omega_{ij}\in\{-1,1\},\sum_{i,j=1}^{N^2}\omega_{ij}=0\right\}.$$

- $\omega\in X_N$: microstate. X_N is the set of microstates



Vorticity on a NxN lattice.

Microcanonical Measures for the 2D Euler Eq.

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$$\Gamma_N(E_0,\Delta E) = \{\omega \in X_N \mid E_0 \leq \mathscr{E}[\omega] \leq E_0 + \Delta E\},\ \Omega_N(E_0,\Delta E) = \sharp\{\Gamma_N(E_0,\Delta E)\}$$

- Finite dimensional approximate measures : equiprobability of all microstates with given energy

$$< \mu_N(E_0,\Delta E), \mathscr{A}[\omega] >= \frac{1}{\Omega_N(E_0,\Delta E)} \sum_{\omega\in\Gamma_{\mathbf{N}}(E_\mathbf{0},\Delta E)} \mathscr{A}[\omega].$$

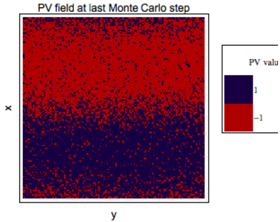
- Microcanonical measures for the 2D Euler equations:

$$\mu(E_0) = \lim_{N\to\infty} \mu_N(E_0,\Delta E) \text{ and } S(E_0) = \lim_{N\to\infty} \frac{1}{N^2} \ln(\Omega_N(E_0,\Delta E)).$$

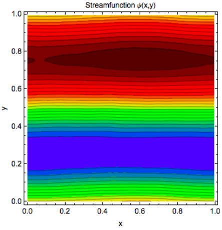
A Typical Vorticity Field for the Microcanonical Measure

A two vorticity level case: $\omega \in \{-1,1\}$, $E = 0.6E_{max}$, Nx$N = 128$x128

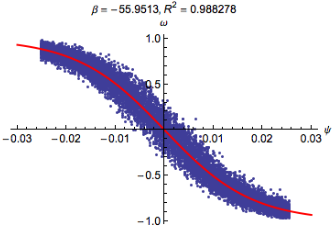
- Creutz's algorithm: a generalization of Metropolis-Hasting's algorithm that samples microcanonical measures.



Vorticity field



Stream function



Coarse-grained-vorticity vs stream function

How to Deal with the Microcanonical Measure

- Finite dimensional approximate measures : equiprobability of all microstates with given energy

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- The limit $N \to \infty$ is rather simple.
- The 2D-Euler has a mean-filed behavior. The microcanonical measure is a Young measure, with local probabilities which are determined by maximization of a mean-field entropy. This is a large deviations result, proven by generalization of Sanov's theorem.

Outline

Macrostates Through Coarse-Graining

- Coarse-graining: we divide the $N \times N$ lattice into $(N/n) \times (N/n)$ boxes (n^2 sites per box).
- These boxes are centered on sites (In, Jn). (I, J) label the boxes $(0 \leq I, J \leq N/n - 1)$.
- $F_{IJ}^{\pm}$ is the frequency to find the value ± 1 in the box (I, J) $(F_{IJ}^{+} + F_{IJ}^{-} = 1)$

$$F_{IJ}^{\pm}(\omega) = \frac{1}{n^2} \sum_{(i,j) \in (I,J)} \delta_d(\omega_{ij} - (\pm 1)).$$

- A macrostate $P^N = \left\{ p_{IJ}^{\pm} \right\}_{0 \leq I, J \leq N/n - 1}$, is the set of all microstates $\left\{ \omega^N \in X_N \middle| \text{ for all } I, J, F_{IJ}^{\pm}(\omega^N) = p_{IJ}^{\pm} \right\}$.
- Macrostate entropy = logarithm of the cardinal of the macrostate

$$S_N[p^N] = \frac{1}{N^2} \log \sharp \left(P^N \right).$$

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Large Deviations for the Microcanonical Measure

- The probability to observe a macrostate p is

$$\mathscr{P}[p] \underset{N \gg n \gg 1}{\sim} e^{N\mathscr{S}[p/p_{eq}]}, \text{ with}$$

$$\mathscr{S}[p/p_{eq}] \equiv -\int_{\mathscr{D}} d\mathbf{r} \left[p \log\left(\frac{p}{p_{eq}}\right) + (1-p)\log\left(\frac{1-p}{1-p_{eq}}\right)\right].$$

- This was obtained through a generalization of Sanov's theorem.

Robert-Sommeria-Miller (RSM) Theory

The most probable vorticity field (for $D(\sigma) = \frac{1}{2}\delta(\sigma+1) + \frac{1}{2}\delta(\sigma-1)$)

- A probabilistic description of the vorticity field ω: $p(\mathbf{x})$ is the local probability to have $\omega(\mathbf{x}) = 1$ at point $\mathbf{x}$.
- A measure of the number of microscopic field ω corresponding to a probability p (Liouville and Sanov theorems):

$$Mean - field\ entropy : \mathscr{S}[p] \equiv -\int_{\mathscr{D}} d\mathbf{r}\, [p \log p + (1-p)\log(1-p)] .$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0) = \sup_{\{p \mid \mathscr{N}[p]=1\}} \left\{ \mathscr{S}[p] \mid \bar{\mathscr{E}}[\overline{\omega}] = E_0 \right\} \text{ (MVP)}.$$

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$$\overline{\omega}=\tanh(\beta\psi).$$

Robert-Sommeria-Miller (RSM) Theory

The most probable vorticity field (for $D(\sigma)=\frac{1}{2}\delta(\sigma+1)+\frac{1}{2}\delta(\sigma-1)$)

- A probabilistic description of the vorticity field ω: $p(\mathbf{x})$ is the local probability to have $\omega(\mathbf{x})=1$ at point $\mathbf{x}$.
- A measure of the number of microscopic field ω corresponding to a probability p (Liouville and Sanov theorems):

$$\textit{Mean}-\textit{field entropy}:\ \mathscr{S}[p]\equiv-\int_{\mathscr{D}}d\mathbf{r}\,\left[p\log p+(1-p)\log(1-p)\right].$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0)=\sup_{\{p|\mathscr{N}[p]=1\}}\left\{\mathscr{S}[p]\mid\bar{\mathscr{E}}[\overline{\omega}]=E_0\right\}\ (\text{MVP}).$$

- Critical points are steady solutions of the 2D Euler equations:

$$\overline{\omega}=\tanh(\beta\psi).$$

Large Deviations for the Microcanonical Measure

- The probability to observe a macrostate p is

$$\mathscr{P}\left[p\right] \underset{N \gg n \gg 1}{\sim} e^{N\mathscr{S}\left[p/p_{eq}\right]}, \text{ with}$$

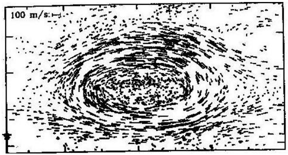
$$\mathscr{S}\left[p/p_{eq}\right] \equiv -\int_{\mathscr{D}} d\mathbf{r} \left[p \log\left(\frac{p}{p_{eq}}\right) + (1-p)\log\left(\frac{1-p}{1-p_{eq}}\right)\right].$$

- This was obtained through a generalization of Sanov's theorem.

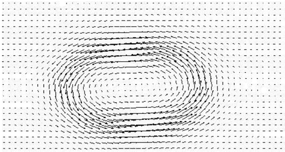
Outline

Great Red Spot of Jupiter

Real flow and statistical mechanics predictions (1-1/2 layer QG model)



Observation data (Voyager)

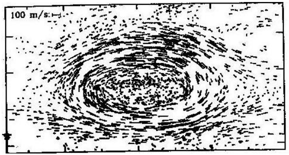


Statistical equilibrium

- A very good agreement. A simple model, analytic description, from theory to observation + New predictions.
- F. BOUCHET and J. SOMMERIA 2002 *JFM* **(QG model)**

Great Red Spot of Jupiter

Real flow and statistical mechanics predictions (1-1/2 layer QG model)



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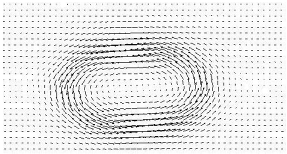

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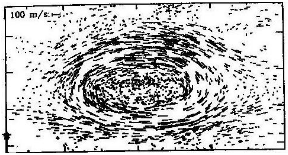

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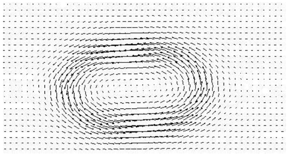
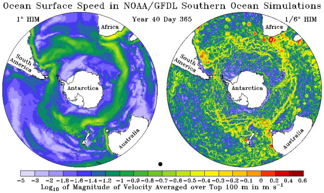


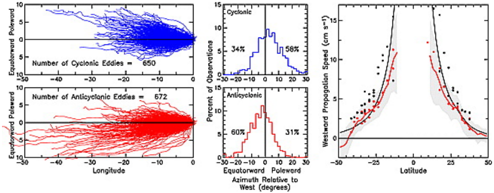
Statistical equilibrium

- A very good agreement. A simple model, analytic description, from theory to observation + New predictions.
- F. BOUCHET and J. SOMMERIA 2002 ***JFM* (QG model)**

Ocean Rings (Mesoscale Ocean Vortices)

Gulf Stream rings - Agulhas rings - Meddies - etc ...



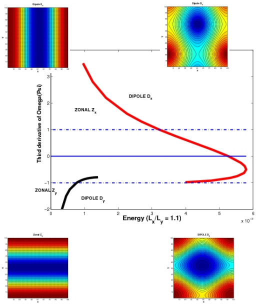


Hallberg–Gnanadesikan

Chelton and co. - GRL 2007

- Both cyclonic and anticyclonic rings drift westward with a velocity $\tilde{\beta} R^2$
- Statistical mechanics explains the ring qualitative shape, and their observed drifts.
  A. Venaille, and F. Bouchet, JPO, 2011

Statistical Equilibria for the 2D-Euler Eq. (torus)



A second order phase transition.

- Z. Yin, D. C. Montgomery, and H. J. H. Clercx, Phys. Fluids (2003)
- F. Bouchet, and E. Simonnet, PRL, (2009) (Lyapunov Schmidt reduction, normal form analysis)

What about Dynamics?

- Are microcanonical measures invariant measures of the 2D Euler or Quasi-Geostrophic equations?

F. Bouchet and M. Corvellec, J. Stat. Mech. 2010, Invariant measures of the 2D Euler and Vlasov equations.

- What about weakly forced and dissipated systems?

2D Stochastic Navier-Stokes Eq. and 2D Euler Steady States

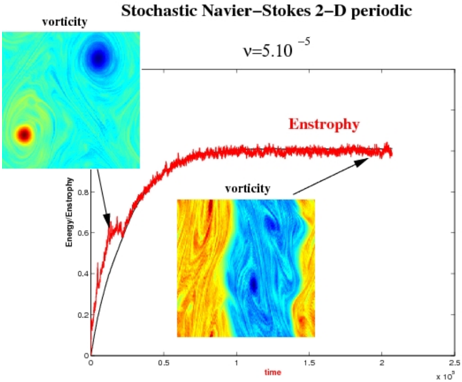
$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s \tag{1}$$

- Time scale separation: magenta terms are small.
- At first order, the dynamics is nearly a 2D Euler dynamics. The flow self organizes and converges towards steady solutions of the Euler Eq.:

$$\mathbf{u}.\nabla \omega = 0 \text{ or equivalently } \omega = f(\psi)$$

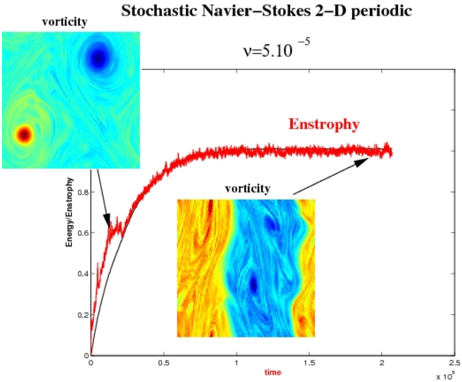
where the Stream Function ψ is given by: $\mathbf{u} = \mathbf{e}_z \times \nabla \psi$.

Numerical Simulation of the 2D Stochastic NS Eq.



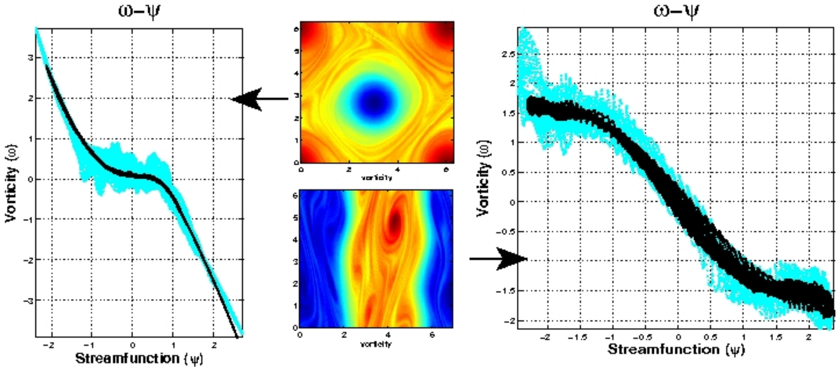
Very long relaxation times. 10^5 turnover times.

Numerical Simulation of the 2D Stochastic NS Eq.



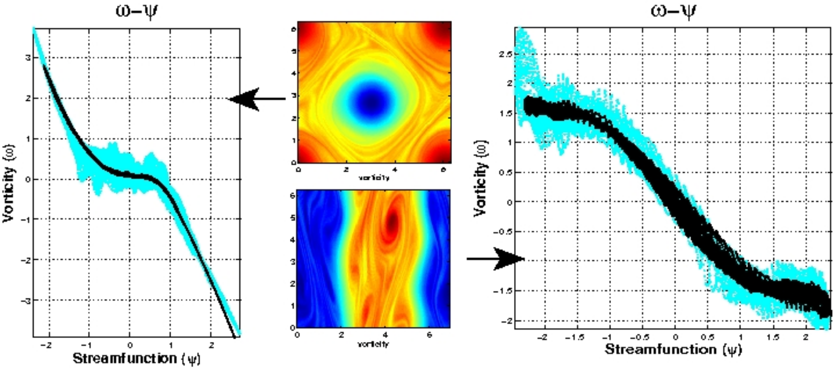
Very long relaxation times. 10^5 turnover times.

Vorticity-Streamfunction Relation



Conclusion: we are close to steady states of the Euler Eq.

Vorticity-Streamfunction Relation



Conclusion: we are close to steady states of the Euler Eq.

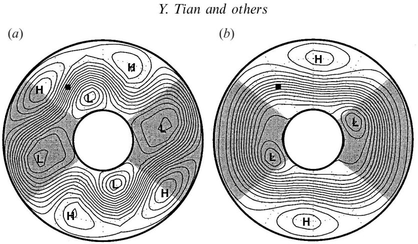
Equilibrium: Conclusions

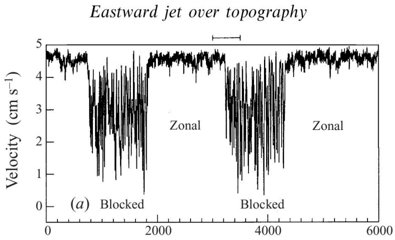
- We can perform equilibrium statistical mechanics in a completely standard way for the class of the 2D Euler eq. and the quasi-geostrophic model class.
- For the microcanonical measures (or the canonical measures), the probability of a macrostate (a local probability to observe a vorticity value) is given by a large deviation rate function.
- The most probable macrostate verifies a mean-field variational problem.
- This is relevant for some applications (which ones ?) and give a very good qualitative understanding of weakly forced and dissipated systems.

Outline

Non-Equilibrium Phase Transitions in Real Flows
Rotating tank experiments (Quasi Geostrophic dynamics)

Transitions between blocked and zonal states:

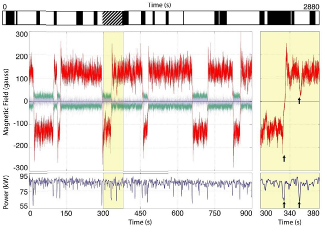




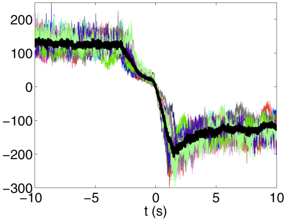
Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

Random Transitions in Turbulence Problems

Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)



Magnetic field timeseries



Zoom on reversal paths

(VKS experiment)

In turbulent flows, transitions from one attractor to another often occur through a predictable path

2D Stochastic Navier-Stokes Eq. and 2D Euler Steady States

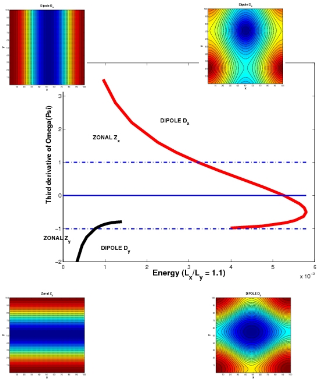
$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s$$

- Time scale separation: magenta terms are small.
- At first order, the dynamics is nearly a 2D Euler dynamics. The flow self organizes and converges towards steady solutions of the Euler Eq.:

$$\mathbf{v}.\nabla\omega = 0 \text{ or equivalently } \omega = f(\psi)$$

  where the Stream Function ψ is given by: $\mathbf{v} = \mathbf{e}_z \times \nabla\psi$.
- Steady states of the Euler equation will play a crucial role. Degeneracy : what does select f ?

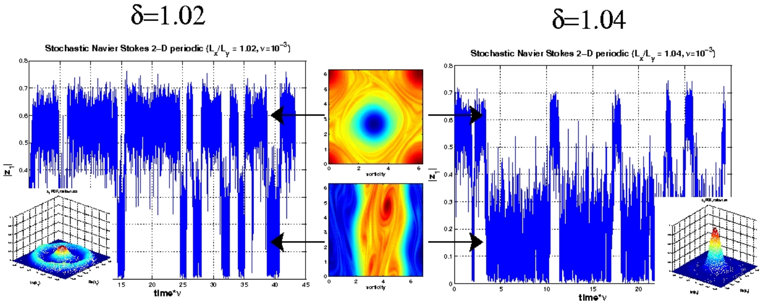
Steady States for the 2D-Euler Eq. (doubly periodic)



A second order phase transition.

Non-Equilibrium Phase Transition

The time series and PDF of the Order Parameter

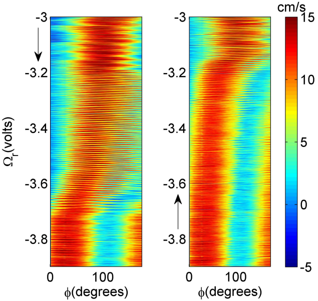


Order parameter : $z_1 = \int dxdy \, \exp(iy)\omega(x,y)$.

For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$

F. Bouchet and E. Simonnet, PRL, 2009.

Bistability in Rotating Tank Experiments
M. Mathur, J. Sommeria (LEGI)



Bistability (hysteresis) in rotating tank experiments

Outline

1. Equilibrium statistical mechanics
   - The microcanonical measure
   - Sanov's theorem and the mean field variational problem
   - Applications of equilibrium statistical mechanics
2. Non-equilibrium phase transitions and large deviations
   - Random changes of flow topology in the 2D stochastic Navier–Stokes Eq. (F. B., E. Simonnet and H. Morita)
   - Large deviations and path integrals
   - Instantons for the 2D stochastic Navier–Stokes equations (F.B. and J. Laurie)
3. Stochastic averaging and jet formation in geostrophic turbulence
   - The stochastic quasi-geostrophic equations
   - Stochastic averaging
   - Bistability for zonal jet dynamics

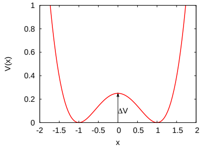
Path Integrals And Large Deviations In Non-Equilibrium Statistical Mechanics

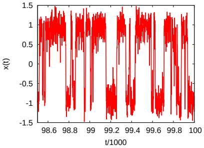
- Aim: Entropy and free energy are extremely useful in equilibrium statistical mechanics: they encode all the statistics of the system. How to compute similar quantity for out of equilibrium systems?
- Answer: Large deviations for ensembles of dynamical paths = out of equilibrium and dynamical free energies. How to compute these?

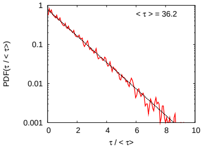
Kramer's Problem: a Pedagogical Example for Bistability

Historical example: Computation by Kramer of the Arrhenius law for a bistable mechanical system with stochastic noise

$$\frac{dx}{dt} = -\frac{dV}{dx}(x) + \sqrt{2D}\eta(t) \ \text{Rate}: \ \lambda = A\exp\left(-\frac{\Delta V}{RT}\right) \ \text{with } RT \propto 2D.$$







The problem was solved by Kramers (30'). Modern approach: path integral formulation (instanton theory, physicists) or large deviation theory (Freidlin-Wentzell, mathematicians).

Path Integrals for ODE – Onsager Machlup (50')

$$\frac{dx}{dt} = f(x) + \sqrt{2D}\eta(x,t).$$

- Path integral representation of transition probabilities:

$$P(x_0,x_T,T) = P(x=x_0,t=0;x=x_T,t=T) = \int_{x(0)=x_0}^{x(T)=x_T} \mathscr{D}[x]\exp\left(-\frac{\mathscr{S}[T,x]}{2D}\right)$$

$$\text{with } \mathscr{S}[T,x] = \frac{1}{2}\int_0^T \mathrm{d}t\left\{[\dot{x}-f(x)]^2 - 2Df'(x)\right\}.$$

- Instanton: the most probable path with fixed boundary conditions

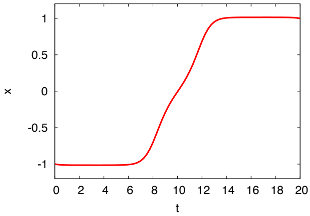
$$S(T,x_0,x_T) = \min_{x(t)}\left\{\mathscr{S}[T,x]\,|x(0)=x_0 \text{ and } x(T)=x_T\right\}.$$

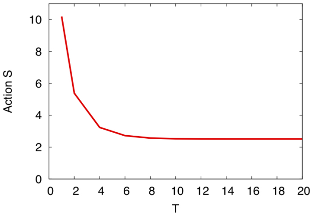
- Saddle point approximation (WKB) gives large deviations results:

$$\log P(x_0,x_T,T) \underset{D\to 0}{\sim} -\frac{S(T,x_0,x_T)}{2D}.$$

What to Do with Path Integrals ?

- Solving the equations in the saddle point approximation using theory or numerical optimization (gradient methods).
- Transition rates and transition trajectories are given by minima and minimizers of the action.
- It explains why most transition trajectories concentrate close to a single one (instanton trajectory).





Path Integrals And Turbulence Problems

- It has never been developed before
- Aim: compute extremely rare but essential events like transitions paths between attractors and transition rates
- This is unfeasible using conventional tools (direct numerical simulation)
- The main issue: Is it feasible for turbulence problems? For which class of models (in terms of complexity)?
- The route to follow:

1. Determine attractors
2. Study instantons between attractors

Outline

The Action of the 2D Stochastic Navier-Stokes

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s \text{ with } \left\langle f_s(\mathbf{x},t), f_s(\mathbf{x}',t')\right\rangle = C(\mathbf{x}-\mathbf{x}')\delta(t-t')$$

$$\mathscr{S}[T,x] = \frac{1}{2}\int_0^T \mathrm{d}t \int_{\mathscr{D}} \mathrm{d}\mathrm{x}\mathrm{d}\mathrm{x}'\, p(\mathrm{x},\mathbf{t})C(\mathrm{x}-\mathrm{x}')p(\mathrm{x}',\mathbf{t})$$

$$\textit{with } p = \frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega + \alpha\omega - \nu\Delta\omega$$

- We can compute explicitly and study the stability of many instantons (parallel to parallel flows, spatial white noise, Laplacian eigenmodes, etc.)
- Definition: $C_{\mathbf{k}} = \int_{\mathscr{D}} \mathrm{d}\mathrm{x} \exp(i\mathbf{k}.\mathrm{x})\, C(\mathrm{x})$. If $C_{\mathbf{k}} = 0$ for some $\mathbf{k}$, the force is called degenerate, non-degenerate otherwise.

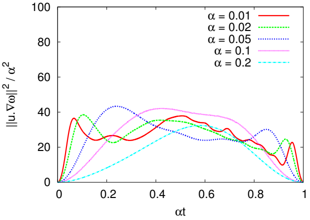
Algorithm for Action Minimization

A variational approach

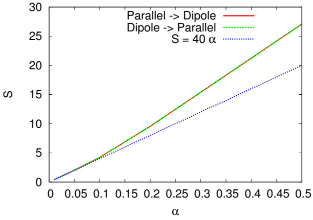
- We discretize action integral both in time and space (time using the central differencing scheme, and space using pseudo-spectral decomposition)
- Fix the initial and final states throughout the minimization
- Newton or quasi-Newton methods are prohibitively expensive to implement (Hessian)
- We implement a gradient method or steepest descent method:
- Then iteratively minimize an initial guess (simultaneously over space and time) in the direction of the anti-gradient:

$$\omega^{n+1} = \omega^n - c_n \frac{\delta S(\omega^n)}{\delta \omega^n}$$

Instanton from Dipole to Parallel Flows



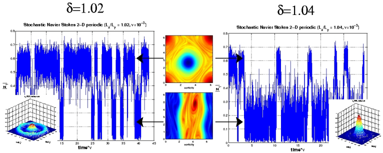
Instanton are close to the set of attractors

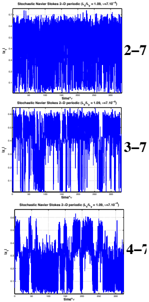


Scaling of S with α shows no large deviation

- $\log P(\omega_0, \omega_T, T) \underset{\alpha \to 0}{\sim} -\frac{S(T,\omega_0,\omega_T)}{2\alpha}$. This is not a large number. The stationary distribution or the transition probabilities are not concentrated. No large deviation. No bistability.

Degenerate Forces Prevent Bistability





Order parameter : $z_1 = \int dxdy \exp(iy)\omega(x,y)$.

For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$.

The 2D Stochastic Navier-Stokes Eq. and Freidlin–Wentzell Framework

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s \qquad (2)$$

- Time scale separation: magenta terms are small.
- At first order, the dynamics is nearly a 2D Euler dynamics. The flow self organizes and converges towards steady solutions of the Euler Eq.:

$$\mathbf{v}.\nabla \omega = 0 \text{ or equivalently } \omega = f(\psi)$$

  where the Stream Function ψ is given by: $\mathbf{v} = \mathbf{e}_z \times \nabla \psi$.
- It looks like an underdamped dynamics, but the right hand side actually has attractors.
- The 2D Navier-Stokes equations does not enter in the Freidlin–Wentzell framework.

The Set of Attractors of the 2D Euler Eq. is Connected

A trivial consequence of the 2D Euler equation scale invariance

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = 0$$

- If $\omega(\mathbf{x},t)$ is a solution of the 2D Euler Eq., then for any $\lambda > 0$, $\lambda \omega(\mathbf{x},\lambda t)$ is also a solution (nonlinearity is homogeneous of degree 2).
- Then any steady solutions ω is connected to zero through the path $s\omega(st)$, $0 \leq s \leq 1$.
- Any two steady states ω_0 and ω_1 are connected through a continuous path $\Omega(s)$, $0 \leq s \leq 1$ among the set of steady state.
- The set of steady states of the 2D Euler equations is connected.

F. BOUCHET, and H. TOUCHETTE, 2012, Non-classical large deviations for a noisy system with non-isolated attractors, *J. Stat. Mech.*, P05028.

Non-Eq. Phase Transitions and Instantons: Conclusions

- We predicted and observed non-equilibrium phase transitions for the 2D Navier-Stokes equations and in experiments.
- We can numerically compute instantons for simple turbulent flows.
- The 2D Navier-Stokes equations does not enter in the Freidlin-Wentzell framework.
- In the inertial limit, instantons follow the connected set of attractors.
- There is no large deviations for transitions between attractors for non-degenerate forces (no bistability).

F. BOUCHET, and H. TOUCHETTE, 2012, Non-classical large deviations for a noisy system with non-isolated attractors, *J. Stat. Mech.*, P05028., F. Bouchet, J. Laurie, E. Simonnet, and O. Zaboronski, to be submitted to PRL, J. Laurie and F. Bouchet, to be submitted to Phys. Rev. E.

Outline

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s,$$

where $\omega=(\nabla\wedge\mathbf{v}).\mathbf{e}_z$ is the vorticity, $q=\omega+\beta_d y$ is the Potential Vorticity (PV), f_s is a random force, α is the Rayleigh friction coefficient.

- Turbulence : time scale separation.
- Spin up or spin down time $=1/\alpha\ll 1=$ jet inertial time scale.

Outline

Averaging out the Turbulence

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s.$$

- $P_T[q]$ is the PDF for the Potential Vorticity field q (a functional)

$$\frac{\partial P_T}{\partial t}=\int \mathrm{d}\mathbf{r}\,\frac{\delta}{\delta q(\mathbf{r})}\left\{\left[\mathbf{v}.\nabla q-\nu\Delta\omega+\alpha\omega+\int \mathrm{d}\mathbf{r}'\,C(\mathbf{r},\mathbf{r}')\frac{\delta}{\delta q(\mathbf{r})}\right]P_T\right\}.$$

- Time scale separation. We decompose into slow (zonal flows) and fast variables (eddy turbulence)

$$Q_0(y)=\frac{1}{2\pi}\int_{\mathscr{D}}\mathrm{d}x\,q \text{ and } q=Q_0+\sqrt{\alpha}\delta q.$$

- Stochastic reduction (Van Kampen, Gardiner, ...) using the time scale separation.
- We average out the turbulent degrees of freedom.

A New Fokker–Planck Equation for the Zonal Jets

- $P_Z[Q_0]$ is the PDF to observe the Zonal Potential Vorticity Q_0.

$$\frac{\partial P_Z}{\partial \tau} = \int \mathrm{d}y_1 \frac{\delta}{\delta Q_0(y_1)} \left\{ \left[\frac{\partial}{\partial y} \langle \delta v_z \delta q \rangle + \Omega_0(y_1) + \frac{\nu}{\alpha} \Delta Q_0(y_1) + \right.\right.$$
$$\left.\left. + \int \mathrm{d}y_2\, C_Z(y_1, y_2) \frac{\delta}{\delta Q_0(y_2)} \right] P_Z \right\}.$$

- This new Fokker–Planck equation is equivalent to the stochastic dynamics

$$\frac{1}{\alpha}\frac{\partial Q_0}{\partial t} = -\frac{\partial}{\partial y} \langle \delta v_y \delta q \rangle - \Omega_0 - \frac{\nu}{\alpha} \Delta Q_0 + \eta_z,$$

with $\langle \eta_Z(y,t) \eta_Z(y',t') \rangle = C_Z(y,y') \delta(t-t')$.

The Deterministic Part and the Quasilinear Approximation

$$\frac{1}{\alpha}\frac{\partial Q_0}{\partial t} = -F[Q_0] - \Omega_0 - \frac{\nu}{\alpha}\Delta Q_0.$$

- $F[Q_0] = -\frac{\partial}{\partial y}\langle \delta v_y \delta q \rangle$. The average of the Reynolds stress is over the quasilinear dynamics statistics:

$$\partial_t \delta q + U(y)\frac{\partial \delta q}{\partial x} + \delta v_y \frac{\partial \Omega_0}{\partial y} = \nu \Delta \delta \omega - \alpha \delta \omega + \sqrt{2\alpha} f_s$$

and

$$\langle \delta v_y \delta q \rangle = \frac{1}{L_z}\int \mathrm{d}z \, \mathbb{E}[\delta v_y \delta q].$$

- We identify SSST by Farrell and Ioannou (JAS, 2003); quasilinear theory by Bouchet (PRE, 2004); CE2 by Marston, Conover and Schneider (JAS, 2008); Sreenivasan and Young (JAS, 2011).

Dynamics of the Relaxation to the Averaged Zonal Flows

The turbulence has been averaged out

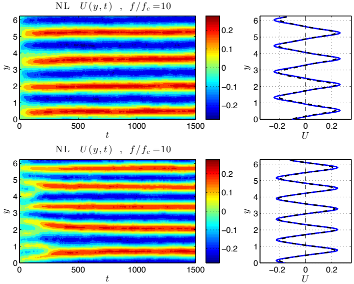


Figure by P. Ioannou (Farrell and Ioannou)

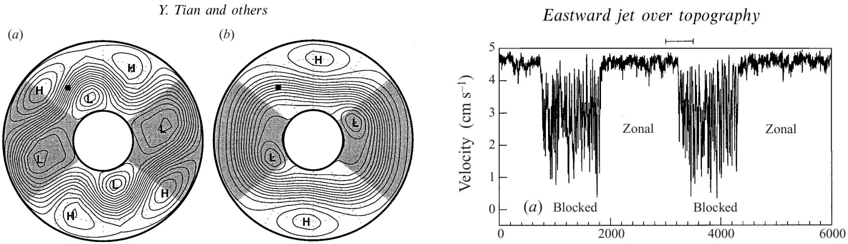
- We can now go further. What is the effect of the noise term ?

$$\frac{1}{\alpha}\frac{\partial Q_0}{\partial t} = -F[Q_0] - \Omega_0 - \frac{\nu}{\alpha}\Delta Q_0 + \eta_z.$$

Rare Transitions in Real Flows

Rotating tank experiments (Quasi Geostrophic dynamics)

Transitions between blocked and zonal states:



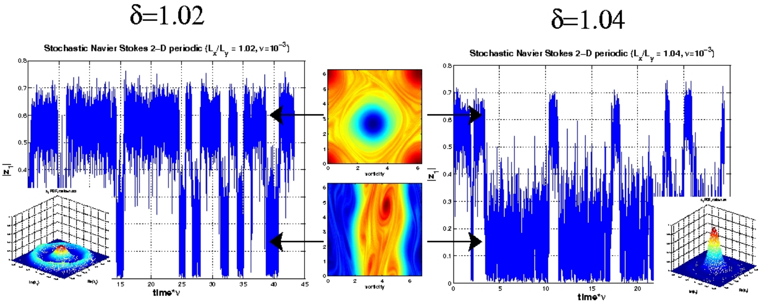
Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

Can such multiple attractors and rare transitions exists for geostrophic turbulence?

Theory based on non-equilibrium statistical mechanics?

Rare Transitions for the 2D Navier-Stokes Eq.

The time series and PDF of the Order Parameter



Order parameter : $z_1 = \int dxdy\, \exp(iy)\omega(x,y)$.

For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$.

F. Bouchet and E. Simonnet, PRL, 2009.

Outline

Multiple Attractors Do Exist for the Barotropic QG Model

Two attractors for the same set of parameters

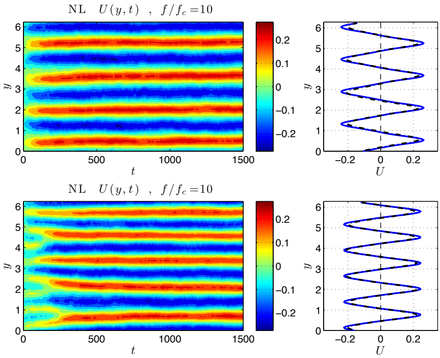


Figure by P. Ioannou (Farrell and Ioannou)

- Two attractors for the mean zonal flow for one set of parameters.

Work in Progress : Zonal Flow Instantons

Onsager Machlup formalism (50'). Statistical mechanics of histories

$$\frac{\partial Q_0}{\partial \tau} = -F[Q_0] - \Omega_0 + \frac{\nu}{\alpha}\Delta Q_0 + \eta_z.$$

- Path integral representation of transition probabilities:

$$P(Q_0, Q_T, T) = \int_{q(0)=Q_0}^{q(T)=Q_T} \mathscr{D}[Q] \exp(-\mathscr{S}[Q]) \text{ with}$$

$$\mathscr{S}[Q] = \frac{1}{2}\int_0^T dt \int dy_1 dy_2 \left[\frac{\partial Q}{\partial t} + F[\Omega] + \Omega + \frac{\nu}{\alpha}\Delta Q\right](y_1) C_z(y_1, y_2) \left[\frac{\partial Q}{\partial t} + F[Q] + \Omega + \frac{\nu}{\alpha}\Delta Q\right](y_2).$$

- Instanton: the most probable path with fixed boundary conditions

$$S(T, \Omega_0, \Omega_T) = \min_{\{\omega | \omega(0)=\Omega_0 \text{ and } \omega(T)=\Omega_T\}} \{\mathscr{S}[\omega]\}.$$

Stat. Mech. of Zonal Jets: Conclusion

- Stochastic averaging for the barotropic Quasi-Geostrophic equation leads to a non-linear Fokker-Planck equation.
- This Fokker-Planck equation predicts the Reynolds stress and jet statistics. Related to Quasilinear theory and SSST.
- For some parameters, multiple attractors are observed.
- Path integral, instanton and large deviations theories can predict the rare transition between attractors.

Summary and Perspectives

- Non-equilibrium statistical mechanics and large deviations can be applied to geophysical turbulence and climate.

Ongoing projects and perspectives:

- Large deviations and non-equilibrium free energies for particles with long range interactions (with K. Gawedzki).
- Microcanonical measures for the Shallow Water equations (with M. Potters and A. Venaille) and for the 3D axisymmetric Euler equations (with S. Thalabard).
- Instantons for zonal jets in the quasi-geostrophic dynamics (with C. Nardini and T. Tangarife).

F. Bouchet, and A. Venaille, Physics Reports, 2012, Statistical mechanics of two-dimensional and geophysical flows