Stochastic averaging, large deviations, and random transitions for the dynamics of jets and vortices in 2D and geostrophic turbulence

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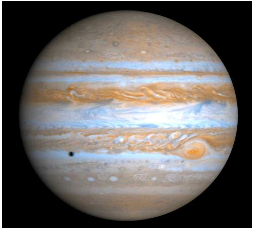
March 2013 - Fukuoka (IUTAM symposium)

Collaborators

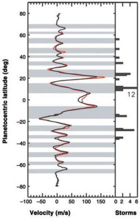
- Stochastic Averaging and Jet Formation in Geostrophic Turbulence: C. Nardini and T. Tangarife (ENS-Lyon)
- Instantons and large deviations for the 2D Navier-Stokes equations: J. Laurie (Post-doc ANR Statocean) and O. Zaboronski (Warwick Univ.)
- Phase Transitions for the 2D Navier-Stokes equations: E. Simonnet (INLN-Nice) (ANR Statocean) and H. Morita (Kyoto)
- Phase Transitions in rotating tank experiments: M. Mathur and J. Sommeria (LEGI-Grenoble) (ANR Statocean)

Earth and Jupiter's Zonal Jets

We look for a theoretical description of zonal jets



Jupiter's atmosphere



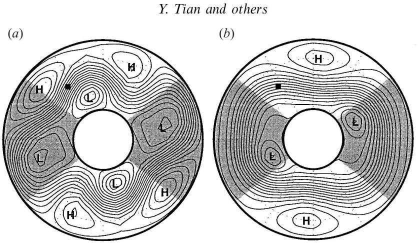
Jupiter's Zonal wind (Voyager and Cassini, from Porco et al 2003)

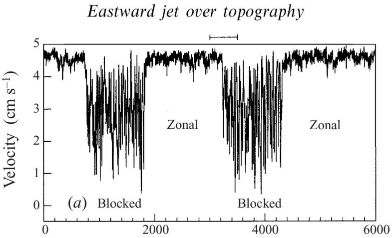
How to theoretically predict such velocity profile?

Phase Transitions in Rotating Tank Experiments

The rotation as an ordering field (Quasi Geostrophic dynamics)

Transitions between blocked and zonal states





Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

Non-Equilibrium Stat. Mech.

1. Stochastic averaging technics (kinetic theory in a stochastic framework).
2. Large deviation for transition probabilities for rare events (through path integrals).
3. Tools from field theory and statistical physics.

The Barotropic Quasi-Geostrophic Equations

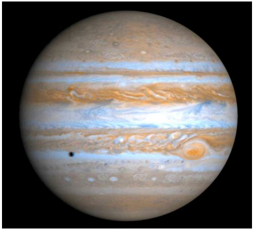
- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s,$$

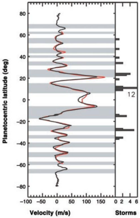
where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, $q = \omega + \beta_d y$ is the Potential Vorticity (PV), f_s is a random force with autocorrelation function $C(\mathbf{r},\mathbf{r}')\delta(t-t')$, α is the Rayleigh friction coefficient.

- Turbulence and time scale separation.
- Spin up or spin down time $= 1/\alpha \ll 1 =$ jet inertial time scale.

Weak Fluctuations around Jupiter's Zonal Jets



Jupiter's atmosphere



Jupiter's Zonal wind (Voyager and Cassini, from Porco et al 2003)

We will treat those weak perturbations perturbatively.

Averaging out the Turbulence

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu\Delta\omega-\alpha\omega+\sqrt{2\alpha}f_s.$$

- $P[q]$ is the PDF for the Potential Vorticity field q (a functional). Fokker–Planck equation:

$$\frac{\partial P}{\partial t}=\int\mathrm{d}\mathbf{r}\,\frac{\delta}{\delta q(\mathbf{r})}\left\{\left[\mathbf{v}.\nabla q-\nu\Delta\omega+\alpha\omega+\int\mathrm{d}\mathbf{r}'\,C(\mathbf{r},\mathbf{r}')\frac{\delta}{\delta q(\mathbf{r})}\right]P\right\}.$$

- Time scale separation. We decompose into slow (zonal flows) and fast variables (eddy turbulence)

$$q_z(y)=\frac{1}{2\pi}\int_{\mathscr{D}}\mathrm{d}x\,q \text{ and } q=q_z+\sqrt{\alpha}q_m.$$

- Stochastic reduction (Van Kampen, Gardiner, ...) using the time scale separation.
- We average out the turbulent degrees of freedom.

A New Fokker–Planck Equation for the Zonal Jets

- $R[q_z]$ is the PDF to observe the Zonal Potential Vorticity q_z.

$$\frac{1}{\alpha}\frac{\partial R}{\partial t} = \int \mathrm{d}y_1 \frac{\delta}{\delta q_z(y_1)}\left\{\left[\frac{\partial}{\partial y}\left\langle v_{m,y} q_m\right\rangle + \omega_z(y_1) - \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}(y_1) + \right.\right.$$
$$\left.\left. + \int \mathrm{d}y_2\, C_z(y_1,y_2)\frac{\delta}{\delta q_z(y_2)}\right] R\right\}.$$

- This new Fokker–Planck equation is equivalent to the stochastic dynamics

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = -\frac{\partial}{\partial y}\left\langle v_{m,y} q_m\right\rangle - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2} + \eta_z,$$

with $\langle \eta_z(y,t)\eta_z(y',t')\rangle = C_z(y,y')\delta(t-t')$.

The Deterministic Part and the Quasilinear Approximation

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F\left[q_z\right] - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}.$$

- $F\left[q_z\right] = -\frac{\partial}{\partial y}\left\langle v_{m,y} q_m \right\rangle$. The average of the Reynolds stress is over the statistics of the quasilinear dynamics:

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = \nu \Delta q_m - \alpha \omega_m + \sqrt{2\alpha} f_s$$

and

$$\left\langle v_{m,y} q_m \right\rangle = \frac{1}{L_y}\int \mathrm{d}y \, \mathbb{E}_{q_z}\left[v_{m,y} q_m\right].$$

- We identify SSST by Farrell and Ioannou (JAS, 2003); quasilinear theory by Bouchet (PRE, 2004); CE2 by Marston, Conover and Schneider (JAS, 2008); Sreenivasan and Young (JAS, 2011).

Dynamics of the Relaxation to the Averaged Zonal Flows

The turbulence has been averaged out

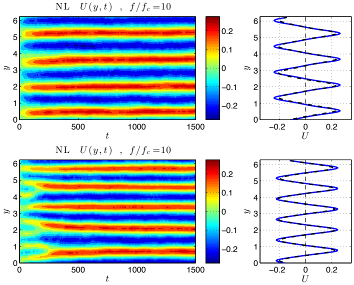


Figure by P. Ioannou (Farrell and Ioannou)

- Extremely efficient numerical simulation of the average jet dynamics.

The Real Issue was to Cope with UltraViolet Divergences

We have proven that they are no such divergences

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = \nu \Delta q_m - \alpha \omega_m + \sqrt{2\alpha} f_s$$

- We need to prove that the Guassian process has an invariant measure which is well behaved in the limit $\nu \to 0$, and $\alpha \to 0$.
- This is true because the linearized Quasi-Geostrophic or Euler dynamics is non-normal.
- The result is based on asymptotics of the linearized equations:

$$v_{m,x}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,x,\infty}(y)}{t} \exp\left(-ikU(y)t\right) \text{ and } v_{m,y}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,y,\infty}(y)}{t^2} \exp\left(-ikU(y)t\right).$$

F. Bouchet and H. Morita, 2010, Physica D.

Dynamics of the Relaxation to the Averaged Zonal Flows

The turbulence has been averaged out

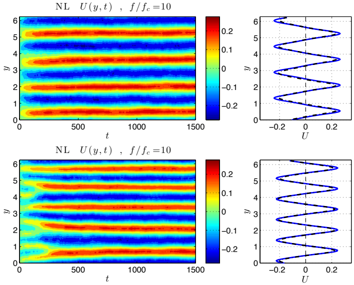


Figure by P. Ioannou (Farrell and Ioannou)

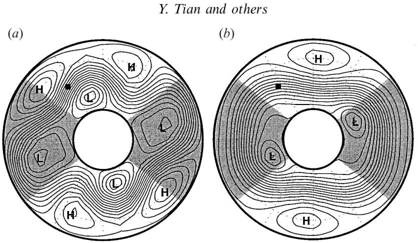
- We can now go further. What is the effect of the noise term?

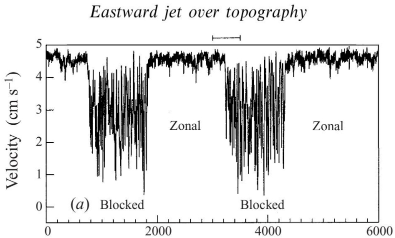
$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t}=F\left[q_z\right]-\omega_z+\frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}+\eta_z.$$

Rare Transitions in Real Flows?

Rotating tank experiments (Quasi Geostrophic dynamics)

Transitions between blocked and zonal states:





Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

Can such multiple attractors and rare transitions exist for geostrophic turbulence?

Theory based on non-equilibrium statistical mechanics?

Multiple Attractors Do Exist for the Barotropic QG Model

Two attractors for the same set of parameters

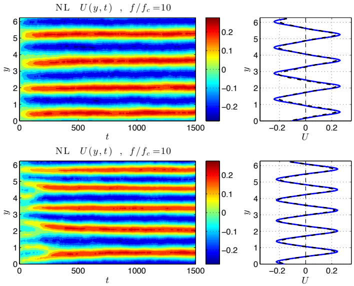


Figure by P. Ioannou (Farrell and Ioannou)

- Two attractors for the mean zonal flow for one set of parameters.
- What is the dynamics for the transition? What is the rate?

Work in Progress : Zonal Flow Instantons

Onsager Machlup formalism (50'). Statistical mechanics of histories

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t}=F[q_z]-\omega_z+\frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}+\eta_z.$$

- Path integral representation of transition probabilities:

$$P(q_{z,0},q_{z,T},T)=\int_{q(0)=q_{z,0}}^{q(T)=q_{z,T}}\mathscr{D}[q_z]\exp\left(-\mathscr{S}[q_z]\right)\text{ with}$$

$$\mathscr{S}[q_z]=\frac{1}{2}\int_0^T \mathrm{d}t\int \mathrm{d}y_1\mathrm{d}y_2\left[\frac{\partial q_z}{\partial t}-F[q_z]+\omega_z-\frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}\right](y_1)C_Z(y_1,y_2)\left[\frac{\partial q_z}{\partial t}-F[q_z]+\omega_z-\frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}\right](y_2).$$

- Instanton (or Freidlin-Wentzel theory): the most probable path with fixed boundary conditions

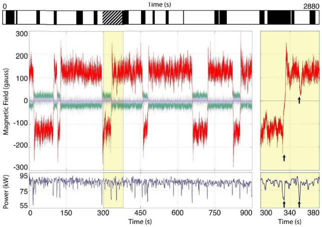
$$S(q_{z,0},q_{z,T},T)=\min_{\left\{q_z\middle|q_z(0)=q_{z,0}\text{ and }q_z(T)=q_{z,T}\right\}}\left\{\mathscr{S}[q_z]\right\}.$$

Stat. Mech. of Zonal Jets: Conclusion

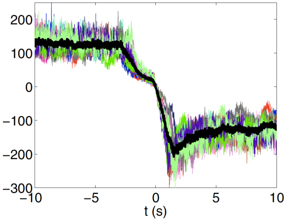
- Stochastic averaging for the barotropic Quasi-Geostrophic equation leads to a non-linear Fokker-Planck equation.
- This Fokker-Planck equation predicts the Reynolds stress and jet statistics. Related to Quasilinear theory and SSST.
- For some parameters, multiple attractors are observed.
- Path integral, instanton and large deviations theories can predict the rare transitions between attractors.

Random Transitions in Turbulence Problems

Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)



Magnetic field timeseries



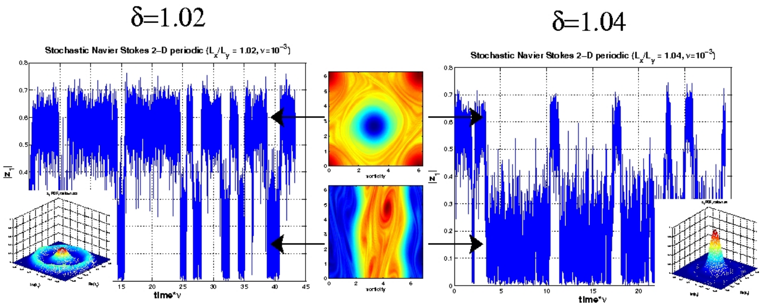
Zoom on reversal paths

(VKS experiment)

In turbulent flows, transitions from one attractor to another often occur through a predictable path.

Non-Equilibrium Phase Transition for the 2D Navier-Stokes Eq., Path Integrals and Instantons

The time series and PDF of the Order Parameter



Order parameter : $z_1 = \int dxdy \exp(iy)\omega(x,y)$.

For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$

F. Bouchet and E. Simonnet, PRL, 2009.

The Action of the 2D Stochastic Navier-Stokes

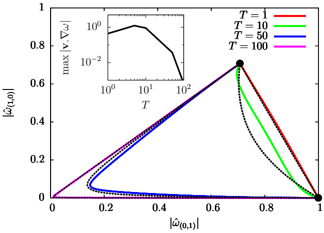
$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s \text{ with } \left\langle f_s(\mathbf{x},t), f_s(\mathbf{x}',t') \right\rangle = C(\mathbf{x}-\mathbf{x}')\delta(t-t').$$

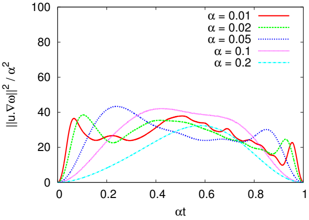
$$\mathscr{S}[T,x] = \frac{1}{2}\int_0^T \mathrm{d}t \int_{\mathscr{D}} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}' \, p(\mathbf{x},\mathbf{t}) C(\mathbf{x}-\mathbf{x}') p(\mathbf{x}',\mathbf{t}),$$

$$\textit{with } p = \frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega + \alpha \omega - \nu \Delta \omega.$$

- We can compute explicitly and study the stability of many instantons (parallel to parallel flows, spatial white noise, Laplacian eigenmodes, etc.).

Instantons from Dipole to Parallel Flows





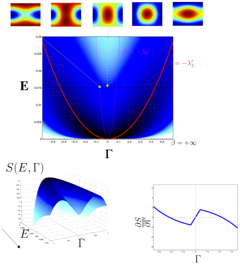
Comparison of numerical instanton with analytical ones

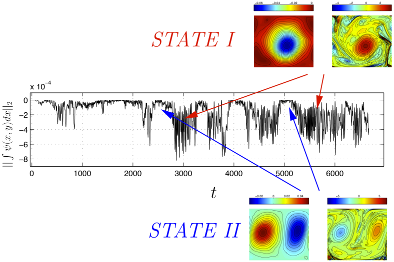
Instanton are close to the set of attractors

- In the limit of weak forces and dissipations, instantons follows the set of attractors of the 2D Euler equations.

Bistability in the 2D Navier–Stokes Eq. in a Channel

“Predicted” from equilibrium statistical mechanics

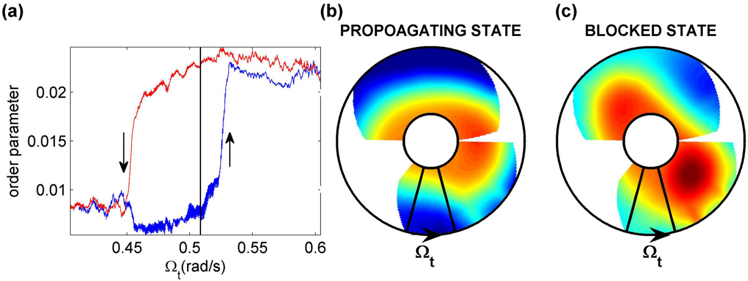




Simulations by E. Simonnet

A. VENAILLE, and F. BOUCHET, 2011, J. Stat. Phys.; M. CORVELLEC and F. BOUCHET, 2012, condmat.

Bistability in Rotating Tank Experiments
Rotating tank with a single-bump topography



Bistability (hysteresis) in rotating tank experiments

M. MATHUR, and J. SOMMERIA, to be submitted to J. Geophys. Res., M. MATHUR, J. SOMMERIA, E. SIMONNET, and F. BOUCHET, in preparation.

Summary and Perspectives

- Non-equilibrium statistical mechanics, field theory and large deviations techniques can be applied to geophysical turbulence.
- A theory for the statistics of jets and vortices in statistically stationary states, and for non-equilibrium phase transitions.

Ongoing projects and perspectives:

- Microcanonical measures for the Shallow Water equations (with M. Potters and A. Venaille) and for the 3D axisymmetric Euler equations (with S. Thalabard).
- Instantons for zonal jets in the quasi-geostrophic dynamics (with C. Nardini, T. Tangarife and O. Zaboronski).

F. Bouchet, and A. Venaille, Physics Reports, 2012, Statistical mechanics of two-dimensional and geophysical flows