Abrupt transitions and large deviations in geophysical turbulent flows

F. BOUCHET (CNRS) – ENS-Lyon and CNRS

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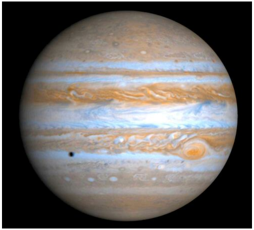
Collaborators

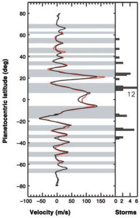
- Random changes of flow topology in the 2D Navier-Stokes equations: E. Simonnet (INLN-Nice) (ANR Statocean)
- Asymptotic stability and inviscid damping of the 2D-Euler equations: H. Morita (Tokyo university) (ANR Statflow)
- Large deviations, instantons non-equilibrium phase transition for quasi-geostrophic turbulence: J. Laurie (Post-doc ANR Statocean), O. Zaboronski (Warwick Univ.)
- Stochastic Averaging and Jet Formation in Geostrophic Turbulence: C. Nardini and T. Tangarife (ENS-Lyon)
- Phase transitions in rotating tank experiments: J. Sommeria (LEGI-Grenoble) and M. Mathur (Post-doc ANR Statocean, now in India)

Jupiter's Zonal Jets

We look for a theoretical description of zonal jets



Jupiter's atmosphere

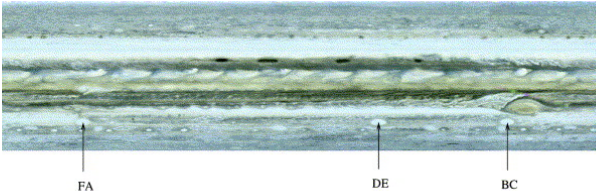


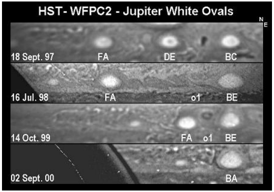
Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003)

How to theoretically predict such a velocity profile?

Has One of Jupiter's Jets Been Lost ?

What is the probability of this event ?



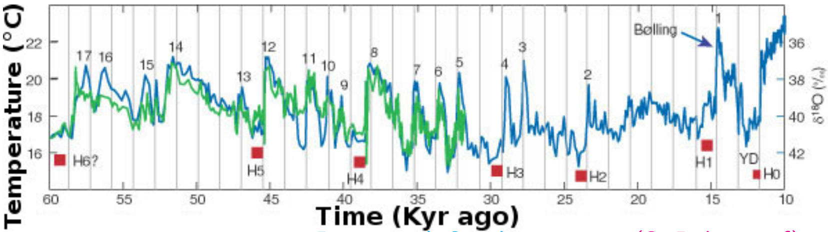


Jupiter's white ovals (see Youssef and Markus 2005)

The white ovals appeared in 1939-1940 (Rogers 1995). Following an instability of the zonal jet ?

Abrupt Climate Changes

Long times matter



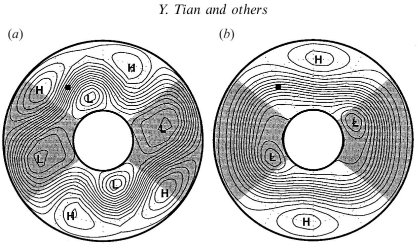
Temperature versus time: Dansgaard–Oeschger events (S. Rahmstorf)

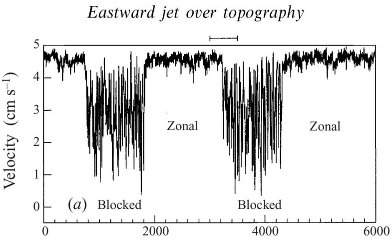
- What is the dynamics and probability of abrupt climate changes?
- Predict attractors, transition pathways and probabilities.
- Study a hierarchy of models of ocean circulation and of turbulent atmospheres.

Phase Transitions in Rotating Tank Experiments

The rotation as an ordering field (Quasi Geostrophic dynamics)

Transitions between blocked and zonal states

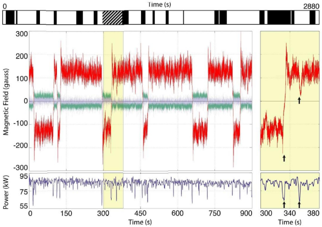




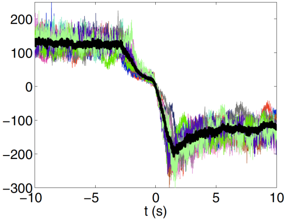
Y. Tian and col, J. Fluid. Mech. (2001) (groups of H. Swinney and M. Ghil)

Random Transitions in Turbulence Problems

Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)



Magnetic field timeseries



Zoom on reversal paths

(VKS experiment)

In turbulent flows, transitions from one attractor to another often occur through a predictable path.

- Compute attractors, transition pathways and probabilities.

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\sigma} f_s,$$

with $q = \omega + \beta y$.

- $\beta = 0$: the two–dimensional stochastic Navier-Stokes equations.

The Main Scientific Issues

- How to characterize and predict the attractors of turbulent geophysical flows?
- In case of multiple attractors, can we compute their relative probability?
- Can we compute the transition pathways and the transition probabilities?

Large Deviation Theory and Statistical Mechanics

- Probability of an order parameter $p[q]$ and large deviations

$$p[q](\mathbf{x},\sigma,t) = \langle \delta\left(q(\mathbf{x},t)-\sigma\right)\rangle$$

$$\mathscr{P}[p] \underset{\varepsilon\ll 1}{\sim} C e^{-\frac{\mathscr{F}[p]}{\varepsilon}}.$$

- For equilibrium systems, $\mathscr{F}$ is the free energy, and $\varepsilon = k_B T/N$.
- Computing $\mathscr{F}$ "solves" the dynamics (most probable state, fluctuations, phase transitions).
- The large deviation function $\mathscr{F}$ can be computed from the dynamics (Macroscopic fluctuation theory, instanton theory).
- Large deviation theory extends statistical mechanics tools to **non-equilibrium** systems.

The Main Mathematical Questions

- How to characterize and predict attractors in turbulent geophysical flows?
- When is Freidlin–Wentzell theory relevant for turbulent flows?
- Large deviation results beyond Freidlin–Wentzell theory?

Outline

1. The 2D Navier–Stokes Eqs
   - The equilibrium statistical mechanics
   - Non equilibrium phase transitions
   - Other close to equilibrium bifurcations in turbulent flows (F.B., M. Mathur, E. Simonnet, and J. Sommeria)
2. Large deviations for Quasi-Geostrophic Langevin dynamics.
   - Langevin dynamics, time reversal symmetry and large deviations.
   - Instantons for Langevin quasi-geostrophic dynamics (F.B., J. Laurie, and O. Zaboronski).
   - Non-Equilibrium Instantons for the 2D Navier–Stokes equations (F.B. and J. Laurie)
3. Stochastic averaging and jet formation in geostrophic turbulence.
   - The stochastic quasi-geostrophic equations.
   - Stochastic averaging (with C. Nardini and T. Tangarife).
   - Validity of this approach, and the main technical points.

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The 2D Euler Equations

- 2D Euler equations:

$$\frac{\partial \omega}{\partial t} + \mathbf{v}[\omega].\nabla\omega = 0,$$

Vorticity $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$. Stream function ψ: $\mathbf{v} = \mathbf{e}_z \times \nabla\psi$, $\omega = \Delta\psi$.

- Conservative dynamics - Hamiltonian (non canonical) and time reversible.
- Invariants:

Energy: $E[\omega] = \frac{1}{2}\int_{\mathscr{D}} \mathbf{v}^2\, d\mathbf{r} = -\frac{1}{2}\int_{\mathscr{D}} \omega\psi\, d\mathbf{r},$

Casimir's functionals: $\mathscr{C}_s[\omega] = \int_{\mathscr{D}} s(\omega)\, d\mathbf{r},$

Vorticity distribution: $D(\sigma) = \frac{dA}{d\sigma}$ with $A(\sigma) = \int_D \chi_{\{\omega(\mathbf{x}) \le \sigma\}}\, d\mathbf{r}.$

Equilibrium Large Deviation: Macrostate Entropy

The most probable vorticity field (Miller–Robert–Sommeria theory)

- A probabilistic description of the vorticity field ω: $\rho(\mathbf{x},\sigma)$ is the local probability to have $\omega(\mathbf{x}) = \sigma$ at point $\mathbf{x}$.
- A measure of the number of microscopic field ω corresponding to a probability ρ (Liouville and Sanov theorems):

$$\textit{Macrostate entropy} : \mathscr{S}[\rho] \equiv -\int_{\mathscr{D}} \mathrm{d}\mathbf{r}\mathrm{d}\sigma\, \rho \log \rho \, .$$

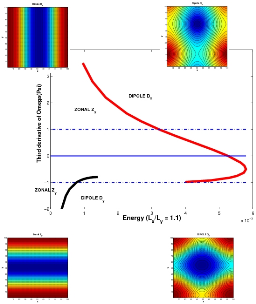
- The microcanonical variational problem (MVP):

$$S(E) = \sup_{\{\rho \mid \mathscr{N}[\rho]=1\}} \left\{ \mathscr{S}_2[\rho] \mid \mathscr{E}[\overline{\omega}] = E \text{ and } D(\sigma) = d(\sigma) \right\} \text{ (MVP)}.$$

- Critical points are steady solutions of the 2D Euler equations:

$$\overline{\omega} = f_d(\beta\psi).$$

Statistical Equilibria for the 2D-Euler Eq. (torus)



A second order phase transition.

- Z. Yin, D. C. Montgomery, and H. J. H. Clercx, Phys. Fluids (2003)
- F. Bouchet, and E. Simonnet, PRL, (2009) (Lyapunov Schmidt reduction, normal form analysis).

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The 2D Stochastic-Navier-Stokes (SNS) Equations

- The simplest model for two dimensional turbulence.
- Navier Stokes equations with random forces

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{\sigma} f_s,$$

where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, f_s is a random force, α is the Rayleigh friction coefficient.

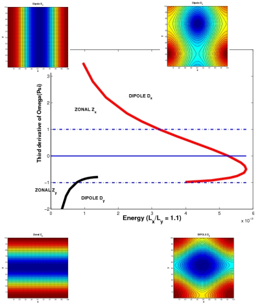
The 2D Stochastic Navier-Stokes Equations

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega + \sqrt{\nu} f_s.$$

- Some recent mathematical results: Bricmont, Glatt-Holtz, Hairer, Kuksin, Kupiainen, Mattingly, Shirikyan, Sinai;
  - Existence of a stationary measure μ_ν. Existence of $lim_{\nu\to 0}\ \mu_\nu$,
  - In this limit, almost all trajectories are solutions of the 2D Euler equations.

Kuksin, S. B., & Shirikyan, A. (2012). Mathematics of two-dimensional turbulence. Cambridge University Press.

Statistical Equilibria for the 2D-Euler Eq. (torus)



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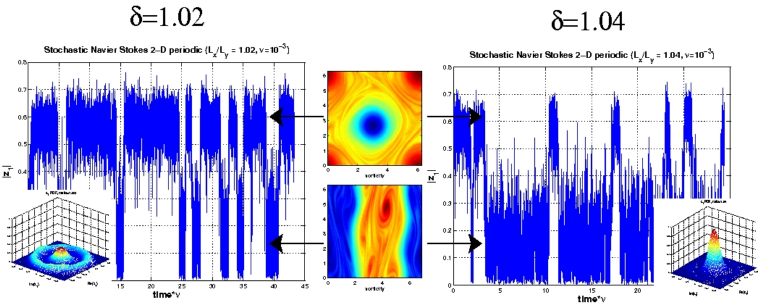
Non-Equilibrium Phase Transition (2D Navier–Stokes Eq.)

The time series and PDF of the Order Parameter



Order parameter : $z_1 = \int dxdy \, \exp(iy)\omega(x,y)$.

For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$

F. Bouchet and E. Simonnet, PRL, 2009.

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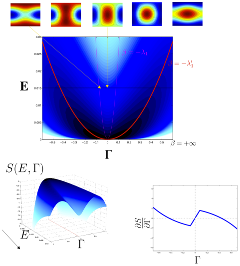
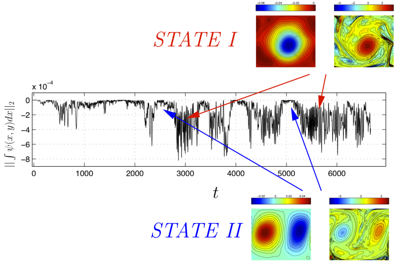
Bistability in the 2D Navier–Stokes Eq. in a Channel

"Predicted" from equilibrium statistical mechanics



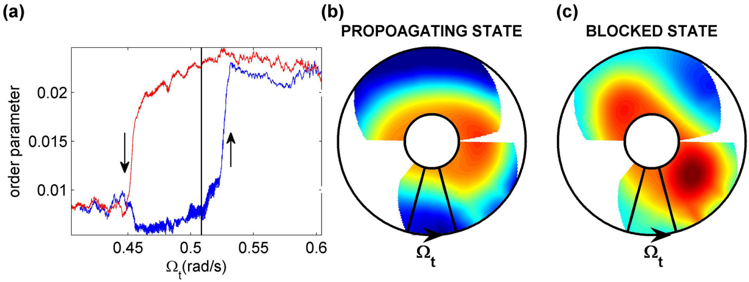


Simulations by E. Simonnet

A. VENAILLE, and F. BOUCHET, 2011, J. Stat. Phys.; M. CORVELLEC and F. BOUCHET, 2012, condmat.

Bistability in a Rotating Tank Experiment

Rotating tank with a single-bump topography

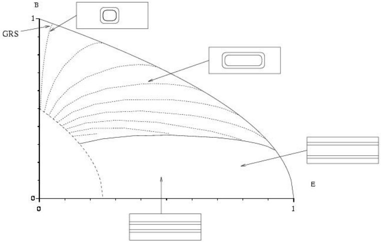


Bistability (hysteresis) in rotating tank experiments

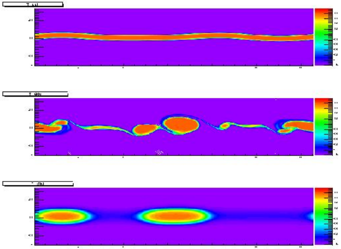
M. MATHUR, and J. SOMMERIA, to be submitted to J. Geophys. Res., M. MATHUR, J. SOMMERIA, E. SIMONNET, and F. BOUCHET, in preparation.

Jet-Vortices Phase Transition on Jupiter

Phase diagram for a 1-1/2 QG Jupiter model



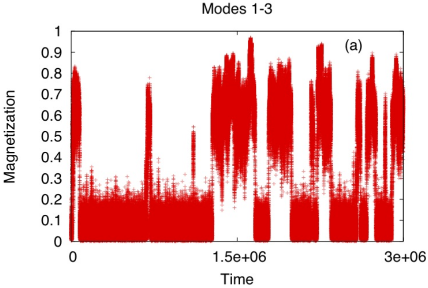
Jupiter's phase diagram



Transition between a jet and oval vo

Non-Equilibrium Phase Transitions for the Stochastic Vlasov Eq.

with a theoretical prediction based on non-equilibrium kinetic theory



Time series for the order parameter for the 1D stochastic Vlasov Eq.

C. NARDINI, S. GUPTA, S. RUFFO, T. DAUXOIS, and F. BOUCHET, 2012, J. Stat. Mech., L01002, and 2012 J. Stat. Mech., P12010.

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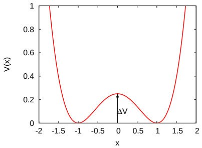
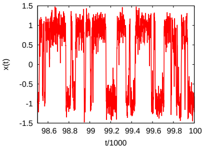
Outline

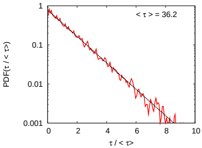
Kramers' Problem: a Pedagogical Example for Bistability

Historical example: Computation by Kramers of Arrhenius' law for a bistable mechanical system with stochastic noise

$$\frac{dx}{dt} = -\frac{dV}{dx}(x) + \sqrt{2k_B T_e}\eta(t) \text{ Rate : } \lambda = \frac{1}{\tau}\exp\left(-\frac{\Delta V}{k_B T_e}\right).$$







The problem was solved by Kramers (30'). Modern approach: path integral formulation (instanton theory, physicists) or large deviation theory (Freidlin-Wentzell, mathematicians).

Path Integrals for ODE – Onsager Machlup (50')

- Path integral representation of transition probabilities:

$$P(x_t,T;x_0,0)=\int_{x(0)=x_0}^{x(T)=x_T} e^{-\frac{\mathscr{A}_T[x]}{2k_BT_e}}\mathscr{D}[x]$$

$$\text{with } \mathscr{A}_T[x]=\int_0^T \mathscr{L}[x,\dot{x}]\,\mathrm{d}t \text{ and } \mathscr{L}[x,\dot{x}]=\frac{1}{2}\left[\dot{x}+\frac{\mathrm{d}V}{\mathrm{d}x}(x)\right]^2.$$

- The most probable path from x_0 to x_T is the minimizer of

$$A_T(x_0,x_T)=\min_{\{x(t)\}}\left\{\mathscr{A}_T[x]\,|x(0)=x_0 \text{ and } x(T)=x_T\right\}.$$

- We may consider the low temperature limit, using a saddle point approximation (WKB), Then we obtain the large deviation result

$$\log P(x_T,T;x_0,0)\underset{\frac{k_BT_e}{\Delta V}\to 0}{\sim}-\frac{A_T(x_0,x_T)}{2k_BT_e}.$$

Time Reversal and Action Duality

- We consider a path $x = \{x(t)\}_{0\le t\le T}$ and its reversed path $I[x] = x_r = \{x(T-t)\}_{0\le t\le T}$. We have

$$\mathscr{A}_T[x_r] = \mathscr{A}_T[x] + 2V(x(T)) - 2V(x(0)).$$

- Transition probabilities of the direct process are related to transition probabilities of the dual process (a generalization of detailed balance).
- This implies that the most probable path to reach a state x (a fluctuation) is the time reversal of a relaxation path starting from $I[x]$ for the dual process (dissipation).
- This is a generalized Onsager-Machlup relation, that justifies generalization of fluctuation-dissipation relations.
- Instantons (the minimizers of the variational problem) are the time reversed relaxation paths of the dual process.

Instantons are Time Reversed Relaxation Paths

- We have the symmetry relation

$$\mathscr{A}_T[R[x]] = \mathscr{A}_T[x] + 2V(x(T)) - 2V(x(0))$$

- Using this equation, we can conclude that instantons are time reversed relaxation paths from a saddle to an attractor. Then we obtain the large deviation result

$$\log P(x_1, T; x_{-1}, 0) \underset{\frac{k_B T_e}{\Delta V} \to 0}{\sim} -\frac{\Delta V}{k_B T_e}.$$

- The computation of the prefactor is more tricky

$$P(x_{-1}, T; x_1, 0) \underset{t \ll 1/\lambda}{\simeq} \frac{T}{\tau} \exp\left(-\frac{\Delta V}{k_B T_e}\right) \text{ with } \tau = 2\pi \left(\frac{d^2 V}{dx^2}(x_0) \frac{d^2 V}{dx^2}(x_{-1})\right)^{-1/2}.$$

This is the subject of Langer theory (70'), see also Caroli, Caroli, and Roulet, J. Stat. Phys., 1981, for a computation through path integrals.

Outline

Langevin Dynamics In a General Framework

$$\frac{\partial q}{\partial t}=\mathscr{F}[q](\mathbf{r})-\alpha\int_{\mathscr{D}}C(\mathbf{r},\mathbf{r}')\frac{\delta\mathscr{G}}{\delta q(\mathbf{r}')}[q]\,\mathrm{d}\mathbf{r}'+\sqrt{2\alpha\gamma}\eta,$$

- Assumptions: i) $\mathscr{F}$ verifies a Liouville theorem

$$\nabla.\mathscr{F}\equiv\int_{\mathscr{D}}\frac{\delta\mathscr{F}}{\delta q(\mathbf{r})}\,\mathrm{d}\mathbf{r}=0\quad\left(\text{Generalization of }\nabla.\mathscr{F}\equiv\sum_{i=1}^{N}\frac{\partial\mathscr{F}}{\partial q_i}=0\right),$$

- ii) The potential $\mathscr{G}$ is a conserved quantity of $\frac{\partial q}{\partial t}=\mathscr{F}[q](\mathbf{r})$:

$$\int_{\mathscr{D}}\mathscr{F}[q](\mathbf{r})\frac{\delta\mathscr{G}}{\delta q(\mathbf{r})}[q]\,\mathrm{d}\mathbf{r}=0.$$

- iii) η a Gaussian process, white in time, with covariance

$$\mathbb{E}\left[\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\right]=C(\mathbf{r},\mathbf{r}')\delta(t-t').$$

- For most classical Langevin dynamics:

$$\mathscr{F}[q](\mathbf{r})=\{q,\mathscr{H}\}\text{ and }\mathscr{G}=\mathscr{H}.$$

Langevin Dynamics for the Quasi-Geostrophic Eq.

$$\frac{\partial q}{\partial t} = \mathbf{v}\left[q-h\right].\nabla q - \alpha \int_{\mathscr{D}} C(\mathbf{r},\mathbf{r}') \frac{\delta \mathscr{G}}{\delta q(\mathbf{r}')}\left[q\right] \mathrm{d}\mathbf{r}' + \sqrt{2\alpha\gamma}\eta .$$

- Assumptions: i) $\mathscr{F} = -\mathbf{v}\left[q-h\right].\nabla q$ verifies a Liouville theorem.
- ii) The potential $\mathscr{G}$ is a conserved quantity of $\frac{\partial q}{\partial t} = \mathscr{F}\left[q\right](\mathbf{r})$ with

$$\mathscr{G} = \mathscr{C} + \beta \mathscr{E},$$

with a Casimir functionals

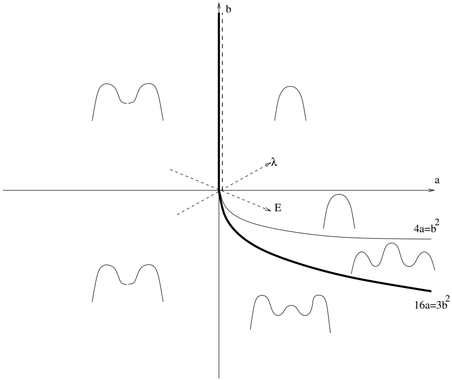
$$\mathscr{C}_c = \int_{\mathscr{D}} \mathrm{d}\mathbf{r}\, c(q),$$

and energy

$$\mathscr{E} = -\frac{1}{2}\int_{\mathscr{D}} \mathrm{d}\mathbf{r}\, \left[q - H\cos(2y)\right]\psi = \frac{1}{2}\int_{\mathscr{D}} \mathrm{d}\mathbf{r}\, \nabla\psi^2 .$$

Tricritical Points

Bifurcation from a second order to a first order phase transition



Tricritical point corresponding to the normal form

$$s(m) = -m^6 - \frac{3b}{2} m^4 - 3am^2.$$

A Quasi-Geostrophic Potential with A Tricritical Point

$$\mathscr{G}=(1-\varepsilon)\frac{1}{2}\int_{\mathscr{D}}\mathrm{d}\mathbf{r}\,[q-H\cos(2y)]\,\psi+\int_{\mathscr{D}}\mathrm{d}\mathbf{r}\left[\frac{q^2}{2}-a_4\frac{q^4}{4}+a_6\frac{q^6}{4}\right]\text{ with } h(y)=H\cos(2y).$$

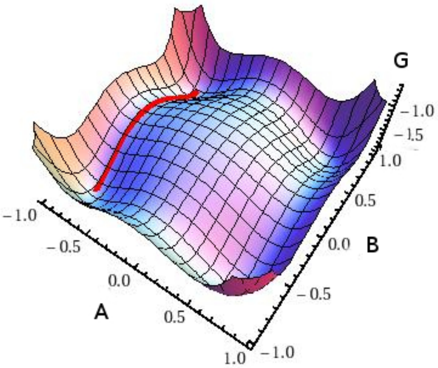
- There is a tricritical transition (transition from first order to second order) close to $\varepsilon=0$ and $a_4=0$ for small H.
- Close to the transition the stochastic dynamics can be reduced to a two-degrees of freedom stochastic dynamics, which is a gradient dynamics with potential

$$G(A,B)=-\frac{H^2}{3}+\varepsilon\left[A^2+B^2\right]-\frac{3a_4}{2}\left[A^2+B^2\right]^2+\frac{a_6}{6}\gamma\left[A^2+B^2\right]^3+\frac{5\pi}{144}a_6H^2\left(A^2-B^2\right)^2.$$

- And the potential vorticity field is

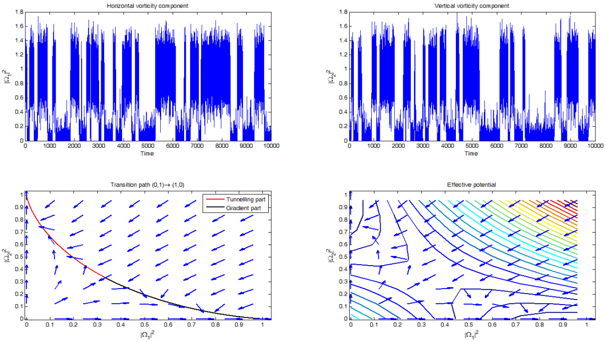
$$q(y)\simeq A\cos(y)+B\sin(y).$$

The Reduced Potential and the Instanton



The reduced potential and one instanton/relaxation path.

Bistability for the Langevin Quasi-Geostrophic Eq.



The reduced potential and one instanton/relaxation path.

The Enstrophy Measure for the 2D Navier-Stokes Eq.

A special case of Langevin dynamics

- The 2D Navier-Stokes equation $\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega + \sqrt{\varepsilon\nu} f_l$ with

$$\mathbb{E}\left[f_l(\mathbf{r},t)f_l(\mathbf{r}',t)\right] = -\Delta_{\mathbf{r}}\left[C_l \exp\left(-\frac{(\mathbf{r}-\mathbf{r}')^2}{2l^2}\right)\right] \underset{l\to 0}{\to} -\Delta_{\mathbf{r}}\left[\delta\left(\mathbf{r}-\mathbf{r}'\right)^2\right].$$

- In the limit, we expect an enstrophy measure

$$\mathscr{P}[\omega] \sim \exp\left(-\frac{1}{2\varepsilon}\int d\mathbf{r}\,\omega^2\right).$$

.

- The enstrophy is indeed the Freidlin-Wentzell quasipotential in the limit $l \to 0$.

$$\lim_{l\to 0} \mathscr{F}_l[\omega] = \frac{1}{2}\int d\mathbf{r}\,\omega^2.$$

Z. BRZENIAK AND S. CERRAI AND M. FREIDLIN, preprint, 2014

Conclusion for Phase Transitions of the Langevin Quasi-Geostrophic Eq.

- For this turbulent dynamics, we can predict the phase diagram (a tricritical point). For a range of parameter, we have first order phase transitions.
- Using large deviations, we can compute transition probabilities.
- We can compute the transition rate between two attractors.
- Most transitions concentrate close to the optimal one, it is describe by an instanton that is easily computed.
- Sufficiently close to the tricritical point, the dynamics reduces to a two degrees of freedom stochastic dynamics.

F. Bouchet, J. Laurie and O. Zaboronsky, J. Stat. Phys., 2014

Outline

1. The 2D Navier–Stokes Eqs
   - The equilibrium statistical mechanics
   - Non equilibrium phase transitions
   - Other close to equilibrium bifurcations in turbulent flows (F.B., M. Mathur, E. Simonnet, and J. Sommeria)
2. Large deviations for Quasi-Geostrophic Langevin dynamics.
   - Langevin dynamics, time reversal symmetry and large deviations.
   - Instantons for Langevin quasi-geostrophic dynamics (F.B., J. Laurie, and O. Zaboronski).
   - Non-Equilibrium Instantons for the 2D Navier–Stokes equations (F.B. and J. Laurie)
3. Stochastic averaging and jet formation in geostrophic turbulence.
   - The stochastic quasi-geostrophic equations.
   - Stochastic averaging (with C. Nardini and T. Tangarife).
   - Validity of this approach, and the main technical points.

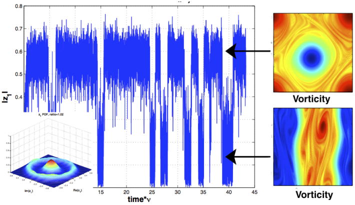
2D Stochastic Navier-Stokes Eq. and 2D Euler Steady States

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = \nu\Delta\omega - \alpha\omega + \sqrt{2\alpha} f_s$$

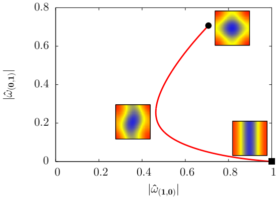
- This is no more a Langevin dynamics.
- Time scale separation: magenta terms are small.

Instantons: Maximum Likelihood Paths

- Most trajectories that lead to a rare event follow the easiest path.
- Large deviation theory: instantons as minimum action paths.



2D Navier-Stokes equations (time: 10 000) (PRL)



Numerical instanton (time of order 1) (J. Stat. Phys.)

- Goal: predict attractors, transition pathways and probabilities.
- Instanton computations will predict them when it is not possible to do that using direct numerical simulations.

Instanton in Turbulent Flows: Conclusions

- For some restricted classes of force spectrum (Langevin dynamics), we can solve completly the problem (compute the large deviation functionnals, fluctuation paths, transition probabilities, instantons, and so on).
- This is usually not the case. Then we have partial answers only. We can 1) rely on equilibrium large deviation and test empirically their interest for slightly non equilibrium situations 2) compute instantons numerically 3) We have few more cases with explicit instanton solutions.
- A lot is still to be understood.
- More can be done theoretically in the inertial limit.

Outline

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=\nu\Delta\omega-\alpha\omega+\sqrt{2\sigma}f_s,$$

with $q=\omega+\beta y$.

The Inertial Limit

- The non-dimensional version of the barotropic QG equation.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s,$$

with $q = \omega + \beta' y$.

- Spin up or spin down time $= 1/\alpha \gg 1 =$ jet inertial time scale.

Jet Formation in the Barotropic QG Model

In the inertial (weak forces and dissipation) limit

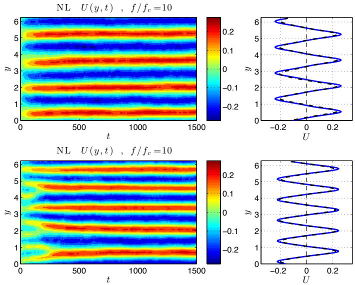
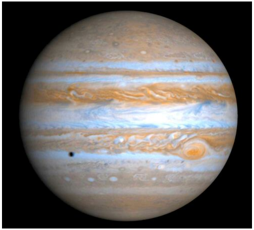
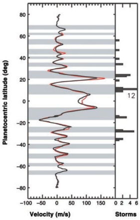


Figure by P. Ioannou (Farrell and Ioannou).

Weak Fluctuations around Jupiter's Zonal Jets



Jupiter's atmosphere.



Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003).

We will treat those weak fluctuations perturbatively (inertial limit).

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Averaging out the Turbulence

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s.$$

- $P[q]$ is the PDF for the Potential Vorticity field q (a functional). Fokker–Planck equation:

$$\frac{\partial P}{\partial t} = \int \mathrm{d}\mathbf{r} \, \frac{\delta}{\delta q(\mathbf{r})} \left\{ \left[\mathbf{v}.\nabla q - \nu \Delta \omega + \alpha \omega + \int \mathrm{d}\mathbf{r}' \, C(\mathbf{r},\mathbf{r}') \frac{\delta}{\delta q(\mathbf{r})} \right] P \right\}.$$

- Time scale separation. We decompose into slow (zonal flows) and fast variables (eddy turbulence)

$$q_z(y) = \langle q \rangle \equiv \frac{1}{2\pi} \int_{\mathscr{D}} \mathrm{d}x \, q \text{ and } q = q_z + \sqrt{\alpha} q_m.$$

- Stochastic reduction (Van Kampen, Gardiner, ...) using the time scale separation.
- We average out the turbulent degrees of freedom.

A New Fokker–Planck Equation for the Zonal Jets

- $R[q_z]$ is the PDF to observe the Zonal Potential Vorticity q_z:

$$\frac{1}{\alpha}\frac{\partial R}{\partial t}=\int \mathrm{d}y_1\,\frac{\delta}{\delta q_z(y_1)}\left\{\left[\frac{\partial}{\partial y}\mathbb{E}_{q_z}\left\langle v_{m,y}q_m\right\rangle+\omega_z(y_1)-\frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}(y_1)+\right.\right.$$
$$\left.\left.+\int \mathrm{d}y_2\,C_z(y_1,y_2)\frac{\delta}{\delta q_z(y_2)}\right]R\right\}.$$

- This new Fokker–Planck equation is equivalent to the stochastic dynamics

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t}=-\frac{\partial}{\partial y}\mathbb{E}_{q_z}\left\langle v_{m,y}q_m\right\rangle-\omega_z+\frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}+\eta_z,$$

with $\langle\eta_z(y,t)\eta_z(y',t')\rangle=C_z(y,y')\delta(t-t')$.

The Deterministic Part and the Quasilinear Approximation

Deterministic quasilinear dynamics

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F[q_z] - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}.$$

- $F[q_z] = -\frac{\partial}{\partial y}\mathbb{E}_{q_z}\langle v_{m,y} q_m \rangle$. The average of the Reynolds stress is over the Ornstein-Uhlenbeck process for the linearized dynamics close to the current zonal flow $U(y)$ and vorticity profile q_z, with random forces:

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = \nu\Delta q_m - \alpha\omega_m + f_s.$$

- We identify SSST by Farrell and Ioannou (JAS, 2003); quasilinear theory by Bouchet (PRE, 2004); CE2 by Marston, Conover and Schneider (JAS, 2008); Sreenivasan and Young (JAS, 2011).

Dynamics of the Relaxation to the Averaged Zonal Flows

Deterministic quasilinear dynamics

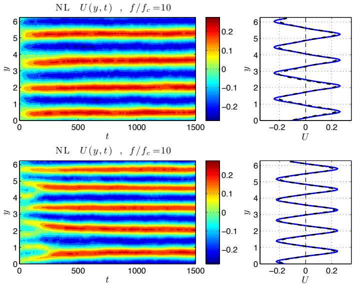
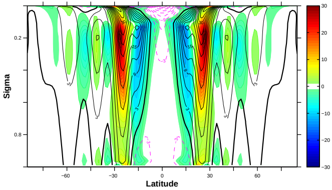


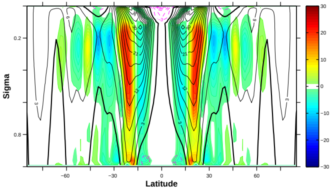
Figure by P. Ioannou (Farrell and Ioannou).

Troposphere Dynamics and the Quasilinear Approximation

Comparison of quasilinear approximation and DNS for the primitive equations



Full equations (DNS).



Quasilinear approximation.

Zonal wind and momentum convergence for the primitive equations.

Farid Ait Chaalal and Tapio Schneider (Caltech and ETH Zurich).

- The qualitative structure of a fast rotating Earth troposphere is well approximated by quasilinear dynamics.

The Stochastic Dynamics of the Zonal Jet

Beyond the deterministic quasilinear approximation: the noise term

- We can now go further. What is the effect of the noise term?

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F\left[q_z\right] - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2} + \eta_z.$$

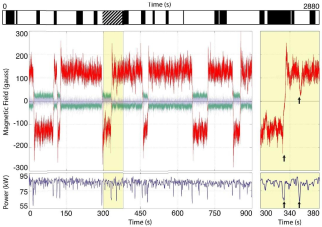
- $R\left[q_z\right]$ is the PDF to observe the Zonal Potential Vorticity q_z:

$$\frac{1}{\alpha}\frac{\partial R}{\partial t} = \int \mathrm{d}y_1 \, \frac{\delta}{\delta q_z(y_1)} \left\{ \left[\frac{\partial}{\partial y} \mathbb{E}_{q_z} \left\langle v_{m,y} q_m \right\rangle + \omega_z(y_1) - \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}(y_1) + \right.\right.$$
$$\left.\left. + \int \mathrm{d}y_2 \, C_z(y_1,y_2) \frac{\delta}{\delta q_z(y_2)} \right] R \right\}.$$

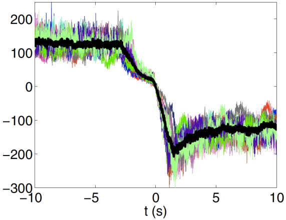
- This equation describes the zonal jet statistics and not only the mean zonal flow.
- This statistics can be nearly Gaussian, but can also be strongly non-Gaussian.

Random Transitions in Turbulence Problems

Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)



Magnetic field timeseries



Zoom on reversal paths

(VKS experiment)

In turbulent flows, transitions from one attractor to another often occur through a predictable path.

- Compute attractors, transition pathways and probabilities.

Multiple Attractors Do Exist for the Barotropic QG Model

Two attractors for the same set of parameters

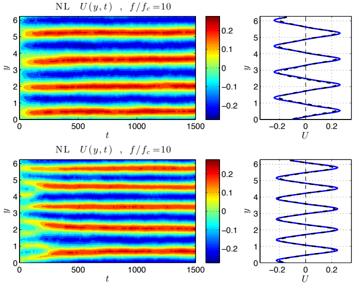


Figure by P. Ioannou (Farrell and Ioannou).

- Two attractors for the mean zonal flow for one set of parameters.
- What is the dynamics for the transition? What is the transition rate?

Work in Progress : Zonal Flow Instantons

Onsager Machlup formalism (50'). Statistical mechanics of histories

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F[q_z] - \omega_z + \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2} + \eta_z.$$

- Path integral representation of transition probabilities:

$$P(q_{z,0}, q_{z,T}, T) = \int_{q(0)=q_{z,0}}^{q(T)=q_{z,T}} \mathscr{D}[q_z]\exp\left(-\mathscr{S}[q_z]\right) \text{ with}$$

$$\mathscr{S}[q_z] = \frac{1}{2}\int_0^T \mathrm{d}t \int \mathrm{d}y_1 \mathrm{d}y_2 \left[\frac{\partial q_z}{\partial t} - F[q_z] + \omega_z - \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}\right](y_1) C_Z(y_1, y_2) \left[\frac{\partial q_z}{\partial t} - F[q_z] + \omega_z - \frac{\nu}{\alpha}\frac{\partial^2 q_z}{\partial y^2}\right](y_2).$$

- Instanton (or Freidlin-Wentzel theory): the most probable path with fixed boundary conditions

$$S(q_{z,0}, q_{z,T}, T) = \min_{\left\{q_z \middle| q_z(0)=q_{z,0} \text{ and } q_z(T)=q_{z,T}\right\}} \left\{\mathscr{S}[q_z]\right\}.$$

Outline

The Real Issue was to Cope with UltraViolet Divergences

We have proven that they are no such divergences

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = \nu \Delta q_m - \alpha \omega_m + \sqrt{2} f_s$$

- We need to prove that the Gaussian process has an invariant measure which is well behaved in the limit $\nu \to 0$, and $\alpha \to 0$.
- This is true because of inviscid damping of the Quasi-Geostrophic or Euler dynamics.
- The result is based on asymptotics of the linearized equations:

$$v_{m,x}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,x,\infty}(y)}{t} \exp(-ikU(y)t) \text{ and } v_{m,y}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,y,\infty}(y)}{t^2} \exp(-ikU(y)t).$$

F. Bouchet and H. Morita, 2010, Physica D. (Related to Landau-Damping and the recent result of Bedrossian and Masmoudi).

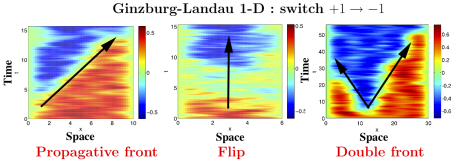
Stat. Mech. of Zonal Jets: Conclusions

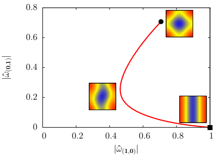
- Stochastic averaging for the barotropic Quasi-Geostrophic equation leads to a non-linear Fokker-Planck equation.
- This Fokker-Planck equation predicts the Reynolds stress and jet statistics. Related to Quasilinear theory and SSST.
- For some parameters, multiple attractors are observed.
- Path integral, instanton and large deviation theories can predict rare transitions between attractors.

F. Bouchet, C. Nardini and T. Tangarife, 2013 J. Stat. Phys.

Numerical Computation of Rare Events and Large Deviations

Computation of least action paths (instantons) and/or multilevel splitting





Multilevel-splitting: Ginzburg-Landau transitions (with E. Simonnet and J. Rolland)

2D Navier-Stokes instantons (with J. Laurie)

- Rare events and their probability can now be computed numerically in complex dynamical systems.

Summary and Perspectives

- Non-equilibrium statistical mechanics and large deviation theory apply to geophysical turbulence.

Ongoing projects and perspectives:

- Large deviations and non-equilibrium free energies for particles with long range interactions (with K. Gawedzki, and C. Nardini).
- Microcanonical measures for the Shallow Water equations (with M. Potters and A. Venaille).
- Instantons for zonal jets in the quasi-geostrophic dynamics (with T. Tangarife, E. Van-den-Eijnden, and O. Zaboronski).
- Rare events, large deviations, and extreme heat waves in the atmosphere (with J. Wouters).

F. Bouchet, and A. Venaille, Physics Reports, 2012

F. Bouchet, C. Nardini and T. Tangarife, J. Stat. Phys., 2013

F. Bouchet, J. Laurie and O. Zaboronsky, J. Stat. Phys., 2014