Statistical mechanics and kinetic theory of the 2D Euler and stochastic Navier-Stokes equations

F. BOUCHET – ENS-Lyon and CNRS

Kinetic Theory and Fluid Mechanics" – March 2012 – Lyon

Outline

I) Introduction
II) Equilibrium Statistical mechanics of the 2D Euler equations
II) Kinetic theory of the 2D Euler and Navier-Stokes equations
IV) Non-equilibrium phase transitions and large deviations in the 2D Navier-Stokes equations

Introduction

1. Equilibrium statistical mechanics
   - Microcanonical measures of the 2D Euler Eq.
   - Sanov's theorem and the mean field variational problem
2. Applications of equilibrium statistical mechanics
   - Jupiter's Great Red Spot (F.B. and J. Sommeria)
   - Equilibrium statistical mechanics of large scale ocean dynamics (A. Venaille and F.B.)
3. Young measures and invariant measures to the 2D Euler equations
   - How to prove the invariance of the microcanonical measure ?
   - Invariant Young measures (F.B. and Marianne C.)

Outline

Statistical Mechanics for 2D and Geophysical Flows

- Equilibrium mechanics is not relevant for turbulent flows, except for few exceptions
- 2D-Euler or some classes of models for geophysical flows are proper conservative systems (no anomalous dissipation of energy)
- Statistical equilibrium: A very old idea, some famous contributions Onsager (1949), Joyce and Montgomery (1970), Caglioti Marchioro Pulvirenti Lions (1990), Robert (1990), Miller (1990), Robert et Sommeria (1991), Eyink and Spohn (1994), Kiessling and Lebowitz (1994), Bodineau and Guionnet (1999), Boucher, Ellis and Turkington (1999)

Equilibrium: the 2D Euler Equations

- 2D Euler equations:

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = 0$$

Vorticity $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$. Stream function ψ: $\mathbf{v} = \mathbf{e}_z \times \nabla\psi$, $\omega = \Delta\psi$

- Conservative dynamics - Hamiltonian (non canonical) and time reversible
- Invariants:

Energy: $$\mathscr{E}[\omega] = \frac{1}{2}\int_{\mathscr{D}} d^2x\, \mathbf{v}^2 = -\frac{1}{2}\int_{\mathscr{D}} d^2x\, \omega\psi$$

Casimir's functionals: $$\mathscr{C}_s[\omega] = \int_{\mathscr{D}} d^2x\, s(\omega)$$

Vorticity distribution: $$D(\sigma) = \frac{dA}{d\sigma} \text{ with } A(\sigma) = \int_D d^2x\, \chi_{\{\omega(\mathbf{x})\leq\sigma\}}$$

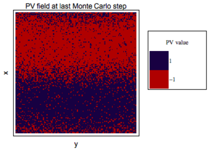
Microstates for the 2D Euler Eq.

The case with 2 levels of vorticity (for pedagogical purpose)

- We discuss the case $D(\sigma)=\frac{1}{2}\delta(\sigma+1)+\frac{1}{2}\delta(\sigma-1)$, ($\omega(\mathbf{r})\in\{-1,1\}$ with $\pm$ 1 values occupying equal areas).
- Vorticity points on a lattice of size NxN (used for instance as weight in a finite elements approximations of 2D fields)

$$X_N=\left\{\omega=\left(\omega_{ij}\right)_{1\leq i,j\leq N}\,|\,\forall i,j\,\omega_{ij}\in\{-1,1\},\sum_{i,j=1}^{N^2}\omega_{ij}=0\right\}$$

- $\omega\in X_N$: microstate. X_N is the set of microstates



Vorticity on a NxN lattice

Microcanonical Measures for the 2D Euler Eq.

The case with 2 levels of vorticity (for pedagogical purpose)

- Vorticity points on a lattice of size NxN

$$X_N = \left\{ \omega = (\omega_{ij})_{1 \le i,j \le N} \mid \forall i,j\, \omega_{i,j} \in \{-1,1\}, \sum_{i,j=1}^{N^2} \omega_{ij} = 0 \right\}$$

$$\Gamma_N(E_0,\Delta E) = \{\omega \in X_N \mid E_0 \le \mathscr{E}[\omega] \le E_0 + \Delta E\}, \; \Omega_N(E_0,\Delta E) = \sharp\{\Gamma_N(E_0,\Delta E)\}$$

- Finite dimensional approximate measures : equiprobability of all microstates with given energy

$$< \mu_N(E_0,\Delta E), \mathscr{A}[\omega] >= \frac{1}{\Omega_N(E_0,\Delta E)} \sum_{\omega \in \Gamma_N(E_0,\Delta E)} \mathscr{A}[\omega].$$

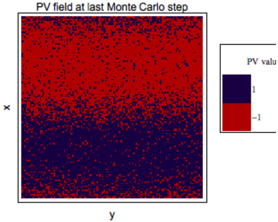
- Microcanonical measures for the 2D Euler equations:

$$\mu(E_0) = \lim_{N\to\infty} \mu_N(E_0,\Delta E) \text{ and } S(E_0) = \lim_{N\to\infty} \frac{1}{N^2} \ln(\Omega_N(E_0,\Delta E)).$$

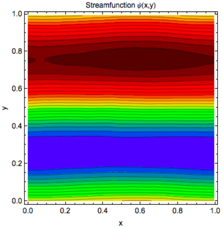
A Typical Vorticity Field for the Microcanonical Measure

A two vorticity level case: $\omega \in \{-1,1\}$, $E = 0.6E_{max}$, $N x N = 128x128$

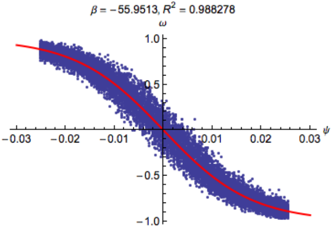
- Creutz's algorithm: a generalization of Metropolis-Hasting's algorithm that samples microcanonical measures.



Vorticity field



Stream function



Coarse-grained-vorticity vs stream function

How to Deal with the Microcanonical Measure

- Finite dimensional approximate measures : equiprobability of all microstates with given energy

$$< \mu_N(E_0, \Delta E), \mathscr{A}[\omega] >= \frac{1}{\Omega_N(E_0, \Delta E)} \sum_{\omega \in \Gamma_{\mathbf{N}}(\mathbf{E_0}, \Delta E)} \mathscr{A}[\omega] .$$

- Microcanonical measures for the 2D Euler equations:

$$\mu(E_0) = \lim_{N \to \infty} \mu_N(E_0, \Delta E) \text{ and } \mathrm{S}(E_0) = \lim_{N \to \infty} \frac{1}{N^2} \ln(\Omega_N(E_0, \Delta E))$$

- The limit $N \to \infty$ is rather simple
- The 2D-Euler has a mean-filed behavior. The microcanonical measure is a Young measure, with local probabilities which are determined by maximization of a mean-field entropy. This is a large deviations result, proven by generalization of Sanov's theorem

Outline

Macrostates Through Coarse-Graining

- Coarse-graining: we divide the $N \times N$ lattice into $(N/n) \times (N/n)$ boxes (n^2 sites per box)
- These boxes are centered on sites (In, Jn). (I, J) label the boxes $(0 \leq I, J \leq N/n - 1)$
- $F_{IJ}^{\pm}$ is the frequency to find the value ± 1 in the box (I, J) $(F_{IJ}^{+} + F_{IJ}^{-} = 1)$

$$F_{IJ}^{\pm}(\omega) = \frac{1}{n^2} \sum_{(i,j) \in (I,J)} \delta_d(\omega_{ij} - (\pm 1))$$

- A macrostate $P^N = \left\{p_{IJ}^{\pm}\right\}_{0 \leq I,J \leq N/n-1}$, is the set of all microstates $\left\{\omega^N \in X_N \,\middle|\, \text{for all } I, J,\, F_{IJ}^{\pm}(\omega^N) = p_{IJ}^{\pm}\right\}$
- Macrostate entropy = logarithm of the cardinal of the macrostate

$$S_N[p^N] = \frac{1}{N^2} \log \sharp\left(P^N\right)$$

Macrostates Through Coarse-Graining

- Coarse-graining: we divide the $N \times N$ lattice into $(N/n) \times (N/n)$ boxes (n^2 sites per box)
- These boxes are centered on sites (In, Jn). (I, J) label the boxes $(0 \leq I, J \leq N/n - 1)$
- $F_{IJ}^{\pm}$ is the frequency to find the value ± 1 in the box (I, J) $(F_{IJ}^{+} + F_{IJ}^{-} = 1)$

$$F_{IJ}^{\pm}(\omega) = \frac{1}{n^2} \sum_{(i,j)\in(I,J)} \delta_d(\omega_{ij} - (\pm 1))$$

- A macrostate $P^N = \{p_{IJ}^{\pm}\}_{0 \leq I,J \leq N/n-1}$, is the set of all microstates $\left\{ \omega^N \in X_N \,\middle|\, \text{for all } I, J, F_{IJ}^{\pm}(\omega^N) = p_{IJ}^{\pm} \right\}$
- Macrostate entropy = logarithm of the cardinal of the macrostate

$$S_N[p^N] = \frac{1}{N^2} \log \sharp \left(P^N\right)$$

Macrostates Through Coarse-Graining

- Coarse-graining: we divide the $N \times N$ lattice into $(N/n) \times (N/n)$ boxes (n^2 sites per box)
- These boxes are centered on sites (In, Jn). (I, J) label the boxes $(0 \leq I, J \leq N/n - 1)$
- $F_{IJ}^{\pm}$ is the frequency to find the value ± 1 in the box (I, J) $(F_{IJ}^{+} + F_{IJ}^{-} = 1)$

$$F_{IJ}^{\pm}(\omega) = \frac{1}{n^2} \sum_{(i,j)\in(I,J)} \delta_d(\omega_{ij} - (\pm 1))$$

- A macrostate $P^N = \left\{ p_{IJ}^{\pm} \right\}_{0 \leq I,J \leq N/n-1}$, is the set of all microstates $\left\{ \omega^N \in X_N \middle| \text{ for all } I, J, F_{IJ}^{\pm}(\omega^N) = p_{IJ}^{\pm} \right\}$
- Macrostate entropy = logarithm of the cardinal of the macrostate

$$S_N[p^N] = \frac{1}{N^2} \log \sharp \left(P^N \right)$$

Asymptotic Entropy with No Energy Constraint

- Macrostate entropy = logarithm of the cardinal of the macrostate

$$S_N[p^N] = \frac{1}{N^2}\log\sharp\left(P^N\right) = \frac{1}{N^2}\log\sharp\left\{\omega^N \in X_N \,\middle|\, \text{for all } I, J,\, F_{IJ}^{\pm}(\omega^N) = p_{IJ}^{\pm}\right\}$$

- Boltzmann (combinatorics and the Stirling formula) proved

$$S_N[P^N] \underset{N \gg n \gg 1}{\sim} \begin{cases} \mathscr{S}_N[P^N] \equiv -\frac{n^2}{N^2}\sum_{I,J}\left(p_{IJ}^{+}\log p_{IJ}^{+} + p_{IJ}^{-}\log p_{IJ}^{-}\right) & \text{if } p_{IJ}^{+} + p_{IJ}^{-} = 1 \\ -\infty & \text{otherwise,} \end{cases}$$

- Analogy with Sanov's theorem (this is a large deviation result with N^2 the large deviation parameter)

Asymptotic Entropy with No Energy Constraint

- Macrostate entropy = logarithm of the cardinal of the macrostate

$$S_N[p^N] = \frac{1}{N^2}\log\sharp\left(P^N\right) = \frac{1}{N^2}\log\sharp\left\{\omega^N \in X_N \,\middle|\, \text{for all } I,J,\, F_{IJ}^{\pm}(\omega^N) = p_{IJ}^{\pm}\right\}$$

- Boltzmann (combinatorics and the Stirling formula) proved

$$S_N[P^N] \underset{N\gg n\gg 1}{\sim} \begin{cases} \mathscr{S}_N[P^N] \equiv -\frac{n^2}{N^2}\sum_{I,J}\left(p_{IJ}^{+}\log p_{IJ}^{+} + p_{IJ}^{-}\log p_{IJ}^{-}\right) & \text{if } p_{IJ}^{+}+p_{IJ}^{-}=1 \\ -\infty & \text{otherwise,} \end{cases}$$

- Analogy with Sanov's theorem (this is a large deviation result with N^2 the large deviation parameter)

Energy Fluctuations over the Macrostate p^N

$$\mathscr{E}[\omega] = -\frac{1}{2}\int_{\mathscr{D}} d^2\mathbf{x}\,\omega\psi = \frac{1}{2}\int_{\mathscr{D}}\int_{\mathscr{D}} d^2\mathbf{x}\,d^2\mathbf{x}'\,G(\mathbf{x},\mathbf{x}')\omega(\mathbf{x})\omega(\mathbf{x}')$$

$$\mathscr{E}_N[\omega] = \frac{1}{2N^4}\sum_{i,j=0}^{N-1}\sum_{i',j'=0}^{N-1} G_{ij,i'j'}\,\omega_{i'j'}\,\omega_{ij}$$

- Not all microstates $\omega \in P^N$ have the same energy. The energy constraint can thus not be recast as a simple constraint on the macrostate P^N
- We use $G_{IJ,I'J'}$ the average value of the coupling constants $G_{ij,i'j'}$ over the box (I,J)

$$G_{ij,i'j'} = G_{IJ,I'J'} + o\left(\frac{1}{n}\right) \text{ then } \mathscr{E}_N[\omega] = \frac{1}{2N^4}\sum_{i,j=0}^{N-1}\sum_{i',j'=0}^{N-1} G_{IJ,I'J'}\,\omega_{i'j'}\,\omega_{ij} + o\left(\frac{1}{n}\right)$$

Energy Fluctuations over the Macrostate p^N

$$\mathscr{E}[\omega] = -\frac{1}{2}\int_{\mathscr{D}} d^2\mathbf{x}\,\omega\psi = \frac{1}{2}\int_{\mathscr{D}}\int_{\mathscr{D}} d^2\mathbf{x}\,d^2\mathbf{x}'\,G(\mathbf{x},\mathbf{x}')\omega(\mathbf{x})\omega(\mathbf{x}')$$

$$\mathscr{E}_N[\omega] = \frac{1}{2N^4}\sum_{i,j=0}^{N-1}\sum_{i',j'=0}^{N-1} G_{ij,i'j'}\,\omega_{i'j'}\,\omega_{ij}$$

- Not all microstates $\omega \in P^N$ have the same energy. The energy constraint can thus not be recast as a simple constraint on the macrostate P^N
- We use $G_{IJ,I'J'}$ the average value of the coupling constants $G_{ij,i'j'}$ over the box (I,J)

$$G_{ij,i'j'} = G_{IJ,I'J'} + o\left(\frac{1}{n}\right) \text{ then } \mathscr{E}_N[\omega] = \frac{1}{2N^4}\sum_{i,j=0}^{N-1}\sum_{i',j'=0}^{N-1} G_{IJ,I'J'}\,\omega_{i'j'}\,\omega_{ij} + o\left(\frac{1}{n}\right)$$

Energy Fluctuations over the Macrostate p^N

$$\mathscr{E}[\omega] = -\frac{1}{2}\int_{\mathscr{D}} d^2\mathbf{x}\,\omega\psi = \frac{1}{2}\int_{\mathscr{D}}\int_{\mathscr{D}} d^2\mathbf{x}\,d^2\mathbf{x}'\,G(\mathbf{x},\mathbf{x}')\omega(\mathbf{x})\omega(\mathbf{x}')$$

$$\mathscr{E}_N[\omega] = \frac{1}{2N^4}\sum_{i,j=0}^{N-1}\sum_{i',j'=0}^{N-1} G_{ij,i'j'}\,\omega_{i'j'}\,\omega_{ij}$$

- Not all microstates $\omega \in P^N$ have the same energy. The energy constraint can thus not be recast as a simple constraint on the macrostate P^N
- We use $G_{IJ,I'J'}$ the average value of the coupling constants $G_{ij,i'j'}$ over the box (I,J)

$$G_{ij,i'j'} = G_{IJ,I'J'} + o\left(\frac{1}{n}\right) \text{ then } \mathscr{E}_N[\omega] = \frac{1}{2N^4}\sum_{i,j=0}^{N-1}\sum_{i',j'=0}^{N-1} G_{IJ,I'J'}\,\omega_{i'j'}\,\omega_{ij} + o\left(\frac{1}{n}\right)$$

Macrostate Energy

$$\mathscr{E}_N[\omega] = \frac{1}{2N^4} \sum_{i,j=0}^{N-1} \sum_{i',j'=0}^{N-1} G_{IJ,I'J'} \omega_{i'j'} \omega_{ij} + o\left(\frac{1}{n}\right)$$

- Coarse-grained vorticity is defined as an average over boxes

$$\overline{\omega}_{IJ} = \frac{1}{n^2} \sum_{(i,j)\in(I,J)} \omega_{ij} = p_{IJ}^{+} - p_{IJ}^{-}$$

- Macrostate energy

$$\bar{\mathscr{E}}_N[\overline{\omega}] = \frac{n^2}{2N^2} \sum_{I,J} \sum_{I',J'} G_{IJ,I'J'} \overline{\omega}_{IJ} \overline{\omega}_{I'J'}$$

- For any microstate $\omega \in P^N$

$$\mathscr{E}_N[\omega] = \bar{\mathscr{E}}_N[\overline{\omega}] + o\left(\frac{1}{n}\right)$$

- More precisely, for large n the distribution of the microstate energies concentrate close to the macrostate energy

Macrostate Energy

$$\mathscr{E}_N[\omega] = \frac{1}{2N^4}\sum_{i,j=0}^{N-1}\sum_{i',j'=0}^{N-1} G_{IJ,I'J'}\omega_{i'j'}\omega_{ij} + o\left(\frac{1}{n}\right)$$

- Coarse-grained vorticity is defined as an average over boxes

$$\overline{\omega}_{IJ} = \frac{1}{n^2}\sum_{(i,j)\in(I,J)} \omega_{ij} = p_{IJ}^{+} - p_{IJ}^{-}$$

- Macrostate energy

$$\bar{\mathscr{E}}_N[\overline{\omega}] \equiv \frac{n^2}{2N^2}\sum_{I,J}\sum_{I',J'} G_{IJ,I'J'}\overline{\omega}_{IJ}\overline{\omega}_{I'J'}$$

- For any microstate $\omega \in P^N$

$$\mathscr{E}_N[\omega] = \bar{\mathscr{E}}_N[\overline{\omega}] + o\left(\frac{1}{n}\right)$$

- More precisely, for large n the distribution of the microstate energies concentrate close to the macrostate energy

Entropy of a Macrostate with Energy Constraint

- A new macrostate (P^N, E_0): the set of $\omega \in P^N$ with $E_0 \leq \mathscr{E}_N[\omega] \leq E_0 + \Delta E$. Or $(P^N, E_0) = P^N \cap \Gamma_N(E, \Delta E)$

$$\Omega_N(E_0, \Delta E) = \sharp \{\Gamma_N(E_0, \Delta E)\} \text{ and } \mathrm{S}(E_0) = \lim_{N\to\infty} \frac{1}{N^2} \ln(\Omega_N(E_0, \Delta E))$$

- The Boltzmann entropy of (P^N, E_0) is $\frac{1}{N^2} \log \sharp (P^N, E_0)$

$$S_N[(P^N, E_0)] \underset{N \gg n \gg 1}{\sim} \begin{cases} \mathscr{S}_N[P^N] & \text{if } p_{IJ}^{+} + p_{IJ}^{-} = 1 \text{ and } \mathscr{E}_N[\overline{\omega_{IJ}^N}] = E_0 \\ -\infty & \text{otherwise} \end{cases}$$

- Because of the exponential concentration, for $N \gg n \gg 1$, the ensemble Boltzmann entropy and the Boltzmann entropy of the most probable macrostate are equal

$$S(E_0) = \max_{\{p | \mathscr{N}[p]=1\}} \left\{ -\int_{\mathscr{D}} d\mathbf{r} \, [p \log p + (1-p) \log(1-p)] \mid \bar{\mathscr{E}}[\bar{\omega}] = E_0 \right\}$$

Entropy of a Macrostate with Energy Constraint

- A new macrostate (P^N, E_0): the set of $\omega \in P^N$ with $E_0 \leq \mathscr{E}_N[\omega] \leq E_0 + \Delta E$. Or $(P^N, E_0) = P^N \cap \Gamma_N(E, \Delta E)$

$$\Omega_N(E_0, \Delta E) = \sharp\{\Gamma_N(E_0, \Delta E)\} \text{ and } S(E_0) = \lim_{N\to\infty} \frac{1}{N^2} \ln(\Omega_N(E_0, \Delta E))$$

- The Boltzmann entropy of (P^N, E_0) is $\frac{1}{N^2} \log \sharp (P^N, E_0)$

$$S_N[(P^N, E_0)] \underset{N \gg n \gg 1}{\sim} \begin{cases} \mathscr{S}_N[P^N] & \text{if } p_{IJ}^+ + p_{IJ}^- = 1 \text{ and } \mathscr{E}_N[\overline{\omega_{IJ}^N}] = E_0 \\ -\infty & \text{otherwise} \end{cases}$$

- Because of the exponential concentration, for $N \gg n \gg 1$, the ensemble Boltzmann entropy and the Boltzmann entropy of the most probable macrostate are equal

$$S(E_0) = \max_{\{p | \mathscr{N}[p]=1\}} \left\{ -\int_{\mathscr{D}} d\mathbf{r} \, [p \log p + (1-p) \log(1-p)] \mid \bar{\mathscr{E}}[\bar{\omega}] = E_0 \right\}$$

Entropy of a Macrostate with Energy Constraint

- A new macrostate (P^N, E_0): the set of $\omega \in P^N$ with $E_0 \leq \mathscr{E}_N[\omega] \leq E_0 + \Delta E$. Or $(P^N, E_0) = P^N \cap \Gamma_N(E, \Delta E)$

  $\Omega_N(E_0, \Delta E) = \sharp\{\Gamma_N(E_0, \Delta E)\}$ and $S(E_0) = \lim_{N\to\infty} \frac{1}{N^2} \ln(\Omega_N(E_0, \Delta E))$

- The Boltzmann entropy of (P^N, E_0) is $\frac{1}{N^2} \log \sharp (P^N, E_0)$

$$S_N[(P^N, E_0)] \underset{N \gg n \gg 1}{\sim} \begin{cases} \mathscr{S}_N[P^N] & \text{if } p_{IJ}^{+} + p_{IJ}^{-} = 1 \text{ and } \mathscr{E}_N[\overline{\omega_{IJ}^{N}}] = E_0 \\ -\infty & \text{otherwise} \end{cases}$$

- Because of the exponential concentration, for $N \gg n \gg 1$, the ensemble Boltzmann entropy and the Boltzmann entropy of the most probable macrostate are equal

$$S(E_0) = \max_{\{p \mid \mathscr{N}[p]=1\}} \left\{ -\int_{\mathscr{D}} d\mathbf{r}\, [p \log p + (1-p)\log(1-p)] \mid \bar{\mathscr{E}}[\bar{\omega}] = E_0 \right\}$$

Robert-Sommeria-Miller (RSM) Theory

The most probable vorticity field (for $D(\sigma)=\frac{1}{2}\delta(\sigma+1)+\frac{1}{2}\delta(\sigma-1)$)

- A probabilistic description of the vorticity field ω: $p(\mathbf{x})$ is the local probability to have $\omega(\mathbf{x})=1$ at point $\mathbf{x}$
- A measure of the number of microscopic field ω corresponding to a probability p (Liouville and Sanov theorems):

$$\textit{Mean-field entropy}: \mathscr{S}[p] \equiv -\int_{\mathscr{D}} d\mathbf{r}\,[p\log p + (1-p)\log(1-p)]$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0) = \sup_{\{p \mid \mathscr{N}[p]=1\}} \left\{ \mathscr{S}[p] \mid \bar{\mathscr{E}}[\overline{\omega}] = E_0 \right\} \text{ (MVP)}.$$

- Critical points are steady solutions of the 2D Euler equations:

$$\overline{\omega} = \tanh(\beta\psi)$$

Robert-Sommeria-Miller (RSM) Theory

The most probable vorticity field (for $D(\sigma) = \frac{1}{2}\delta(\sigma+1) + \frac{1}{2}\delta(\sigma-1)$)

- A probabilistic description of the vorticity field ω: $p(\mathbf{x})$ is the local probability to have $\omega(\mathbf{x}) = 1$ at point $\mathbf{x}$
- A measure of the number of microscopic field ω corresponding to a probability p (Liouville and Sanov theorems):

$$Mean-field\ entropy: \mathscr{S}[p] \equiv -\int_{\mathscr{D}} d\mathbf{r}\,[p\log p + (1-p)\log(1-p)]$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0) = \sup_{\{p \mid \mathscr{N}[p]=1\}} \left\{ \mathscr{S}[p] \mid \bar{\mathscr{E}}[\overline{\omega}] = E_0 \right\} \text{ (MVP)}.$$

- Critical points are steady solutions of the 2D Euler equations:

$$\overline{\omega} = \tanh(\beta\psi)$$

Robert-Sommeria-Miller (RSM) Theory

The most probable vorticity field (for $D(\sigma) = \frac{1}{2}\delta(\sigma+1) + \frac{1}{2}\delta(\sigma-1)$)

- A probabilistic description of the vorticity field ω: $p(\mathbf{x})$ is the local probability to have $\omega(\mathbf{x}) = 1$ at point $\mathbf{x}$
- A measure of the number of microscopic field ω corresponding to a probability p (Liouville and Sanov theorems):

$$\mathit{Mean-field\ entropy}: \mathscr{S}[p] \equiv -\int_{\mathscr{D}} d\mathbf{r}\, [p \log p + (1-p)\log(1-p)]$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0) = \sup_{\{p \mid \mathscr{N}[p]=1\}} \left\{ \mathscr{S}[p] \mid \bar{\mathscr{E}}[\overline{\omega}] = E_0 \right\} \text{ (MVP)}.$$

- Critical points are steady solutions of the 2D Euler equations:

$$\overline{\omega} = \tanh(\beta\psi)$$

Robert-Sommeria-Miller (RSM) Theory

The most probable vorticity field (for $D(\sigma) = \frac{1}{2}\delta(\sigma+1) + \frac{1}{2}\delta(\sigma-1)$)

- A probabilistic description of the vorticity field ω: $p(\mathbf{x})$ is the local probability to have $\omega(\mathbf{x}) = 1$ at point $\mathbf{x}$
- A measure of the number of microscopic field ω corresponding to a probability p (Liouville and Sanov theorems):

$$\mathit{Mean-field\ entropy}: \mathscr{S}[p] \equiv -\int_{\mathscr{D}} d\mathbf{r}\,[p\log p + (1-p)\log(1-p)]$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0) = \sup_{\{p|\mathscr{N}[p]=1\}} \left\{ \mathscr{S}[p] \mid \bar{\mathscr{E}}[\overline{\omega}] = E_0 \right\} \text{ (MVP)}.$$

- Critical points are steady solutions of the 2D Euler equations:

$$\overline{\omega} = \tanh(\beta\psi)$$

Robert-Sommeria-Miller (RSM) Theory

The most probable vorticity field (for $D(\sigma) = \frac{1}{2}\delta(\sigma+1) + \frac{1}{2}\delta(\sigma-1)$)

- A probabilistic description of the vorticity field ω: $p(\mathbf{x})$ is the local probability to have $\omega(\mathbf{x}) = 1$ at point $\mathbf{x}$
- A measure of the number of microscopic field ω corresponding to a probability p (Liouville and Sanov theorems):

$$Mean-field\ entropy: \mathscr{S}[p] \equiv -\int_{\mathscr{D}} d\mathbf{r}\, [p \log p + (1-p)\log(1-p)]$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0) = \sup_{\{p \mid \mathscr{N}[p]=1\}} \left\{ \mathscr{S}[p] \mid \bar{\mathscr{E}}[\overline{\omega}] = E_0 \right\} \text{ (MVP)}.$$

- Critical points are steady solutions of the 2D Euler equations:

$$\overline{\omega} = \tanh(\beta\psi)$$

Robert-Sommeria-Miller (RSM) Theory

With a general vorticity distribution

- A probabilistic description of the vorticity field q: $\rho(\mathbf{x},\sigma)$ is the local probability to have $\omega(\mathbf{x})=\sigma$ at point $\mathbf{x}$
- A measure of the number of microscopic field q corresponding to a probability ρ:

$$\text{Boltzmann-Gibbs Entropy:} \quad \mathscr{S}[\rho] \equiv -\int_{\mathscr{D}} d\mathbf{x} \int_{-\infty}^{+\infty} d\sigma\, \rho \log \rho$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0,d) = \sup_{\{\rho | N[\rho]=1\}} \left\{ \mathscr{S}[\rho] \mid E[\overline{q}] = E_0 \,, D[\rho] = d \right\} \text{(MVP)}.$$

- Critical points are steady flows of the 2D Euler Eq.:

$$\omega = f(\psi)$$

Robert-Sommeria-Miller (RSM) Theory

With a general vorticity distribution

- A probabilistic description of the vorticity field q: $\rho(\mathbf{x},\sigma)$ is the local probability to have $\omega(\mathbf{x}) = \sigma$ at point $\mathbf{x}$
- A measure of the number of microscopic field q corresponding to a probability ρ:

Boltzmann-Gibbs Entropy: $\mathscr{S}[\rho] \equiv -\int_{\mathscr{D}} d\mathbf{x} \int_{-\infty}^{+\infty} d\sigma\, \rho \log \rho$

- The microcanonical RSM variational problem (MVP):

$$S(E_0,d) = \sup_{\{\rho | N[\rho]=1\}} \left\{ \mathscr{S}[\rho] \mid E[\overline{q}] = E_0 \, , D[\rho] = d \right\} \text{(MVP)}.$$

- Critical points are steady flows of the 2D Euler Eq.:

$$\omega = f(\psi)$$

Robert-Sommeria-Miller (RSM) Theory

With a general vorticity distribution

- A probabilistic description of the vorticity field q: $\rho(\mathbf{x},\sigma)$ is the local probability to have $\omega(\mathbf{x}) = \sigma$ at point $\mathbf{x}$
- A measure of the number of microscopic field q corresponding to a probability ρ:

Boltzmann-Gibbs Entropy: $$\mathscr{S}[\rho] \equiv -\int_{\mathscr{D}} d\mathbf{x} \int_{-\infty}^{+\infty} d\sigma\, \rho \log \rho$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0,d) = \sup_{\{\rho | N[\rho]=1\}} \left\{ \mathscr{S}[\rho] \mid E[\overline{q}] = E_0 \,, D[\rho] = d \right\} \text{ (MVP)}.$$

- Critical points are steady flows of the 2D Euler Eq.:

$$\omega = f(\psi)$$

Robert-Sommeria-Miller (RSM) Theory

With a general vorticity distribution

- A probabilistic description of the vorticity field q: $\rho(\mathbf{x},\sigma)$ is the local probability to have $\omega(\mathbf{x}) = \sigma$ at point $\mathbf{x}$
- A measure of the number of microscopic field q corresponding to a probability ρ:

$$\text{Boltzmann-Gibbs Entropy:} \quad \mathscr{S}[\rho] \equiv -\int_{\mathscr{D}} d\mathbf{x} \int_{-\infty}^{+\infty} d\sigma\, \rho \log \rho$$

- The microcanonical RSM variational problem (MVP):

$$S(E_0,d) = \sup_{\{\rho | N[\rho]=1\}} \left\{ \mathscr{S}[\rho] \mid E[\overline{q}] = E_0 \;, D[\rho] = d \right\} \text{(MVP)}.$$

- Critical points are steady flows of the 2D Euler Eq.:

$$\omega = f(\psi)$$

Robert-Sommeria-Miller (RSM) Theory

With a general vorticity distribution

- A probabilistic description of the vorticity field q: $\rho(\mathbf{x},\sigma)$ is the local probability to have $\omega(\mathbf{x})=\sigma$ at point $\mathbf{x}$
- A measure of the number of microscopic field q corresponding to a probability ρ:

$$\text{Boltzmann-Gibbs Entropy:} \quad \mathscr{S}[\rho] \equiv -\int_{\mathscr{D}} d\mathbf{x} \int_{-\infty}^{+\infty} d\sigma\, \rho \log \rho$$

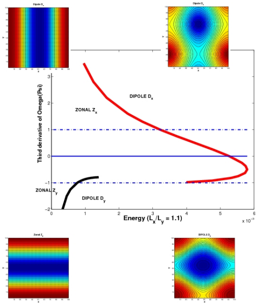
- The microcanonical RSM variational problem (MVP):

$$S(E_0,d) = \sup_{\{\rho \mid N[\rho]=1\}} \left\{ \mathscr{S}[\rho] \mid E[\overline{q}] = E_0 \,, D[\rho] = d \right\} \text{ (MVP)}.$$

- Critical points are steady flows of the 2D Euler Eq.:

$$\omega = f(\psi)$$

Statistical Equilibria for the 2D-Euler Eq. (torus)



A second order phase transition.

- Z. Yin, D. C. Montgomery, and H. J. H. Clercx, Phys. Fluids (2003)
- F. Bouchet, and E. Simonnet, PRL, (2009) (Lyapunov Schmidt reduction, normal form analysis)

Young Measures

- Young measures are product measures: the probability distribution of the vorticity field at an arbitrary number of points $\{\mathbf{r}_k\}$ is given by the product of the independent measures $\rho(\sigma, \mathbf{r}_k)$ at each point $\mathbf{r}_k$
- The set of vorticity fields $\omega(\mathbf{r})$ is a special class of Young measures with $\rho(\sigma, \mathbf{r}) = \delta(\sigma - \omega(\mathbf{r}))$
- The set of microcanonical measures is a special class of Young measures

$$\rho_{\beta,\alpha}(\sigma, \mathbf{r}) = \frac{1}{Z(\beta\psi(\mathbf{r}))} e^{\beta\sigma\psi(\mathbf{r}) - \alpha(\sigma)}$$

Young Measures

- Young measures are product measures: the probability distribution of the vorticity field at an arbitrary number of points $\{\mathbf{r}_k\}$ is given by the product of the independent measures $\rho(\sigma,\mathbf{r}_k)$ at each point $\mathbf{r}_k$
- The set of vorticity fields $\omega(\mathbf{r})$ is a special class of Young measures with $\rho(\sigma,\mathbf{r}) = \delta(\sigma - \omega(\mathbf{r}))$
- The set of microcanonical measures is a special class of Young measures

$$\rho_{\beta,\alpha}(\sigma,\mathbf{r}) = \frac{1}{Z(\beta\psi(\mathbf{r}))} e^{\beta\sigma\psi(\mathbf{r}) - \alpha(\sigma)}$$

Young Measures

- Young measures are product measures: the probability distribution of the vorticity field at an arbitrary number of points $\{\mathbf{r}_k\}$ is given by the product of the independent measures $\rho(\sigma, \mathbf{r}_k)$ at each point $\mathbf{r}_k$
- The set of vorticity fields $\omega(\mathbf{r})$ is a special class of Young measures with $\rho(\sigma, \mathbf{r}) = \delta(\sigma - \omega(\mathbf{r}))$
- The set of microcanonical measures is a special class of Young measures

$$\rho_{\beta,\alpha}(\sigma, \mathbf{r}) = \frac{1}{Z(\beta\psi(\mathbf{r}))} e^{\beta\sigma\psi(\mathbf{r}) - \alpha(\sigma)}$$

Outline

1. Equilibrium statistical mechanics
   - Microcanonical measures of the 2D Euler Eq.
   - Sanov's theorem and the mean field variational problem
2. Applications of equilibrium statistical mechanics
   - Jupiter's Great Red Spot (F.B. and J. Sommeria)
   - Equilibrium statistical mechanics of large scale ocean dynamics (A. Venaille and F.B.)
3. Young measures and invariant measures to the 2D Euler equations
   - How to prove the invariance of the microcanonical measure ?
   - Invariant Young measures (F.B. and Marianne C.)

Equilibrium Statistical Mechanics for Geophysical Flows

The Robert-Sommeria-Miller theory

- Statistical mechanics of the Potential Vorticity mixing: emergence from *random initial conditions*, **stability**, predictability of the flow organization
- Gulf Stream and Kuroshio currents as statistical equilibria
- Ocean mesoscale vortices as statistical equilibria

Equilibrium Statistical Mechanics for Geophysical Flows

The Robert-Sommeria-Miller theory

- Statistical mechanics of the Potential Vorticity mixing: emergence from *random initial conditions*, **stability**, predictability of the flow organization
- Gulf Stream and Kuroshio currents as statistical equilibria
- Ocean mesoscale vortices as statistical equilibria

The Simplest Model: the 1-1/2 Layer Quasi-Geostrophic Model

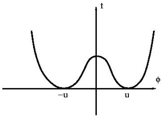
We describe Jupiter's troposphere by the Quasi Geostrophic model (one and half layer):

$$\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = 0 \; ; \; \mathbf{v} = \mathbf{e}_z \times \nabla \psi \; ; \; q = \Delta \psi - \frac{\psi}{R^2} - h(y)$$

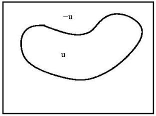
Variational Problem for The Statistical Equilibria
(The case of the 1-1/2 layer Quasi Geostrophic model)

Variational problem: limit $R \to 0$. $(\phi = \psi / R^2)$.

$$\begin{cases} \min \left\{ F_R[\phi] \,|\, \text{with } A[\phi] \text{ given} \right\} \\ \text{with } F_R[\phi] = \int_D d\mathbf{r} \left[\frac{R^2 (\nabla\phi)^2}{2} + f(\phi) - R\phi\, h_0(y) \right] \text{ and } A[\phi] = \int_D d\mathbf{r}\, \phi. \end{cases}$$



The function f : two minima



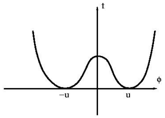
Phase coexistence

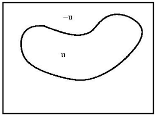
- An analogy with first order phase transitions.
- Modica (90'), function with bounded variations.

Variational Problem for The Statistical Equilibria
(The case of the 1-1/2 layer Quasi Geostrophic model)

Variational problem: limit $R \to 0$. ($\phi = \psi / R^2$).

$$\left\{ \begin{array}{c} \min \left\{ F_R[\phi] \,|\, \text{with } A[\phi] \text{ given} \right\} \\ \text{with } F_R[\phi] = \int_D d\mathbf{r} \left[\frac{R^2 (\nabla\phi)^2}{2} + f(\phi) - R\phi\, h_0(y) \right] \text{ and } A[\phi] = \int_D d\mathbf{r}\, \phi . \end{array} \right.$$





The function f : two minima

Phase coexistence

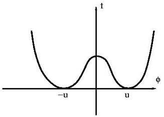
- An analogy with first order phase transitions.
- Modica (90'), function with bounded variations.

Variational Problem for The Statistical Equilibria

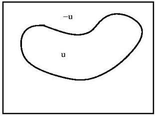
(The case of the 1-1/2 layer Quasi Geostrophic model)

Variational problem: limit $R \to 0$. $(\phi = \psi / R^2)$.

$$\begin{cases} \min \left\{ F_R[\phi] \,|\, \text{with } A[\phi] \text{ given} \right\} \\ \text{with } F_R[\phi] = \int_D d\mathbf{r} \left[\frac{R^2 (\nabla \phi)^2}{2} + f(\phi) - R \phi h_0(y) \right] \text{ and } A[\phi] = \int_D d\mathbf{r}\, \phi . \end{cases}$$



The function f : two minima



Phase coexistence

- An analogy with first order phase transitions.
- Modica (90'), function with bounded variations.

Reduction to a One Dimensional Variational Problem

An isoperimetrical problem balanced by the effect of the deep flow

- A variational problem for the jet shape (interface)

$$F_R[\phi_R] = 2Re_c L - 2Ru \int_{A_1} d\mathbf{r}\, h_0(y) + o(R). \qquad (1)$$

- Laplace equation:

$$\frac{e_c}{r} = -u(\alpha_1 - h_0(y)). \qquad (2)$$

Reduction to a One Dimensional Variational Problem

An isoperimetrical problem balanced by the effect of the deep flow

- A variational problem for the jet shape (interface)

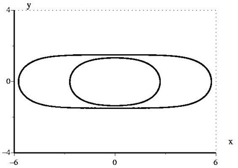
$$F_R[\phi_R] = 2Re_c L - 2Ru \int_{A_1} d\mathbf{r}\, h_0(y) + o(R). \qquad (1)$$

- Laplace equation:

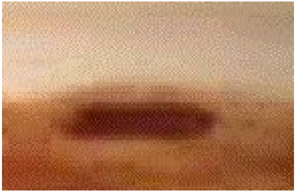
$$\frac{e_c}{r} = -u\left(\alpha_1 - h_0(y)\right). \qquad (2)$$

Jovian Vortex Shape: Phase Coexistence

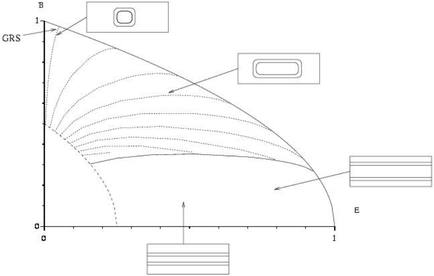
An isoperimetrical problem balanced by the effect of the deep flow



Left: analytic results.
Below left: the Great Red Spot and a White Ovals.
Below right: Brown Barges.



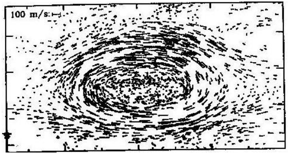
A Phase Diagram for Jovian Vortices and Jets



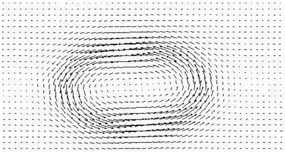
E is the energy and B measures the asymmetry of the initial PV distribution

Great Red Spot of Jupiter

Real flow and statistical mechanics predictions (1-1/2 layer QG model)



Observation data (Voyager)

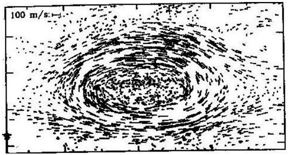


Statistical equilibrium

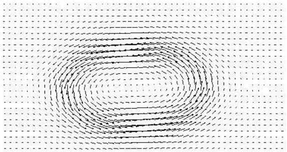
- A very good agreement. A simple model, analytic description, from theory to observation + New predictions.
- F. BOUCHET and J. SOMMERIA 2002 *JFM* **(QG model)**

Great Red Spot of Jupiter

Real flow and statistical mechanics predictions (1-1/2 layer QG model)



Observation data (Voyager)

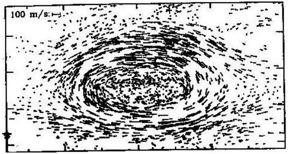


Statistical equilibrium

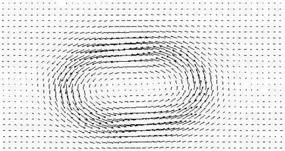
- A very good agreement. A simple model, analytic description, from theory to observation + New predictions.
- F. BOUCHET and J. SOMMERIA 2002 *JFM* **(QG model)**

Great Red Spot of Jupiter

Real flow and statistical mechanics predictions (1-1/2 layer QG model)



Observation data (Voyager)



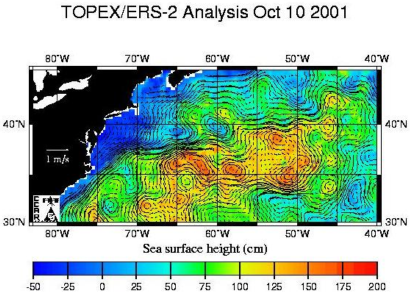
Statistical equilibrium

- A very good agreement. A simple model, analytic description, from theory to observation + New predictions.
- F. BOUCHET and J. SOMMERIA 2002 ***JFM*** **(QG model)**

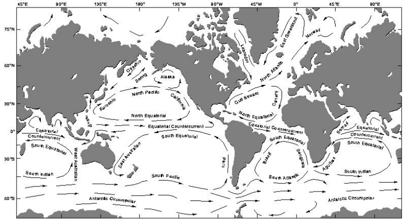
Outline

1. Equilibrium statistical mechanics
   - Microcanonical measures of the 2D Euler Eq.
   - Sanov's theorem and the mean field variational problem
2. Applications of equilibrium statistical mechanics
   - Jupiter's Great Red Spot (F.B. and J. Sommeria)
   - Equilibrium statistical mechanics of large scale ocean dynamics (A. Venaille and F.B.)
3. Young measures and invariant measures to the 2D Euler equations
   - How to prove the invariance of the microcanonical measure ?
   - Invariant Young measures (F.B. and Marianne C.)

Is the Map of Ocean Currents a Statistical Equilibrium ?



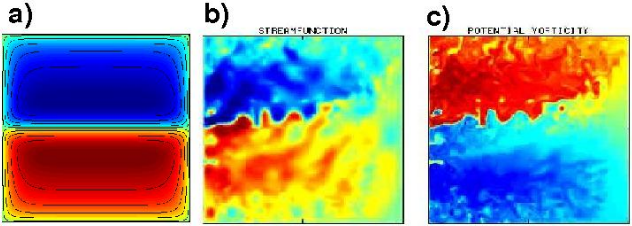
North Atlantic sea height



Map of ocean currents

Strong Eastward Jets are Statistical Equilibria

Statistical equilibria of the QG 1-1/2 layer in a closed basin

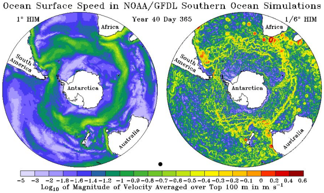


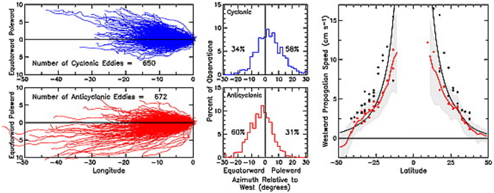
- The states with negative PV to the north (eastward jet), and positive PV to the south (westward jet) are equivalent.
- The beta effect $h(y) = \tilde{\beta} y$ breaks the symmetry between westward and eastward jets.

A. Venaille, and F. Bouchet, J. Phys. Oceanography, 2011.

Ocean Rings (Mesoscale Ocean Vortices)

Gulf Stream rings - Agulhas rings - Meddies - etc ...





Hallberg–Gnanadesikan

Chelton and co. - GRL 2007

- Both cyclonic and anticyclonic rings drift westward with a velocity $\tilde{\beta} R^2$
- Statistical mechanics explains the ring qualitative shape, and their observed drifts.
  A. Venaille, and F. Bouchet, JPO, 2011

Outline

1. Equilibrium statistical mechanics
   - Microcanonical measures of the 2D Euler Eq.
   - Sanov's theorem and the mean field variational problem
2. Applications of equilibrium statistical mechanics
   - Jupiter's Great Red Spot (F.B. and J. Sommeria)
   - Equilibrium statistical mechanics of large scale ocean dynamics (A. Venaille and F.B.)
3. Young measures and invariant measures to the 2D Euler equations
   - How to prove the invariance of the microcanonical measure ?
   - Invariant Young measures (F.B. and Marianne C.)

Invariance of Microcanonical Measures

- For finite dimensional Hamiltonian systems, invariance of microcanonical measures is trivial (Liouville theorem, conservation of the phase space volume)
- The 2D Euler Eq. are a Hamiltonian system with Lie-Poisson brackets for the vorticity variables. This provides detailed Liouville theorems.
- A uniform discretization of the vorticity field is thus a good starting point
- However proving invariance of the limit measure is not trivial, but contrast with finite dimensional Hamiltonian systems

Why Classical Route Fails ?

- A classical route (Bourgain, Non Linear Schrödinger equations) is to use finite dimensional approximate dynamics with invariant measures and to study the limit measure
- This route does not work for the 2D Euler Eq., because of the multiplicity of invariants
- There exist N^2-dimensional approximations of the 2D Euler equations with N conserved Casimirs (Zeitlin–Gallagher). But statistical mechanics of this model seems intractable
- From a statistical mechanics point of view, the good framework is local discretization of vorticity field (mean field behavior). However, there is then no finite dimensional approximation with conservation laws and natural invariant measures
- Another route : direct a-posteriori proof of the invariance of the microcanonical measures

Outline

1. Equilibrium statistical mechanics
   - Microcanonical measures of the 2D Euler Eq.
   - Sanov's theorem and the mean field variational problem
2. Applications of equilibrium statistical mechanics
   - Jupiter's Great Red Spot (F.B. and J. Sommeria)
   - Equilibrium statistical mechanics of large scale ocean dynamics (A. Venaille and F.B.)
3. Young measures and invariant measures to the 2D Euler equations
   - How to prove the invariance of the microcanonical measure ?
   - Invariant Young measures (F.B. and Marianne C.)

Hopf's Equation for the 2D Euler Eq.

- 2D Euler equations

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla \omega = 0$$

- Characteristic functional and moment generating functional

$$F[\lambda, t] = \left\langle \mathrm{e}^{i \int \lambda(\mathbf{r})\omega(\mathbf{r},t)\mathrm{d}\mathbf{r}} \right\rangle \text{ and } H[\lambda, t] = \log F[\lambda, t]$$

- Hopf's equation: each realization is a solution to the 2D Euler equations

$$\frac{\partial F}{\partial t} + i \iint \mathrm{d}\mathbf{r}' \, \mathrm{d}\mathbf{r} \; \nabla \lambda(\mathbf{r}) \cdot \mathbf{G}(\mathbf{r}, \mathbf{r}') \frac{\delta^2 F}{\delta \lambda(\mathbf{r}) \delta \lambda(\mathbf{r}')} = 0,$$

where G is the Laplacian Green function.

Hopf's Equation for the 2D Euler Eq.

- 2D Euler equations

$$\frac{\partial \omega}{\partial t} + \mathbf{v}.\nabla\omega = 0$$

- Characteristic functional and moment generating functional

$$F[\lambda,t] = \left\langle \mathrm{e}^{i\int \lambda(\mathbf{r})\omega(\mathbf{r},t)\mathrm{d}\mathbf{r}} \right\rangle \text{ and } H[\lambda,t] = \log F[\lambda,t]$$

- Hopf's equation: each realization is a solution to the 2D Euler equations

$$\frac{\partial F}{\partial t} + i\iint \mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{r}\ \nabla\lambda(\mathbf{r})\cdot\mathbf{G}(\mathbf{r},\mathbf{r}')\frac{\delta^2 F}{\delta\lambda(\mathbf{r})\delta\lambda(\mathbf{r}')} = 0,$$

where G is the Laplacian Green function.

Young Measures

- Young measures are product measures: the probability distribution of the vorticity field at an arbitrary number of points $\{\mathbf{r}_k\}$ is given by the product of the independent measures $\rho(\sigma,\mathbf{r}_k)$ at each point $\mathbf{r}_k$
- The set of deterministic vorticity fields $\omega(\mathbf{r})$ is a special class of Young measures with $\rho(\sigma,\mathbf{r}) = \delta(\sigma-\omega(\mathbf{r}))$
- The set of microcanonical measures is a special class of Young measures

$$\rho_{\beta,\alpha}(\sigma,\mathbf{r}) = \frac{1}{Z(\beta\psi(\mathbf{r}))}\mathrm{e}^{\beta\sigma\psi(\mathbf{r})-\alpha(\sigma)}$$

The 2D Euler Dynamics of Young Measures

- The cumulant-generating function of ω at point $\mathbf{r}$

$$h(\lambda,\mathbf{r}) = \log f(\lambda,\mathbf{r}) \text{ with } f(\lambda,\mathbf{r}) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma \; \mathrm{e}^{i\lambda\sigma}\rho(\sigma,\mathbf{r})$$

- Lemma (a consequence of the law of large numbers):

1. For Young measures, the velocity field is independent of the vorticity field
2. At each point, it has a Dirac distribution functions

$$P(\mathbf{v},\mathbf{r}) = \delta(\mathbf{v}-\bar{\mathbf{v}}) \text{ with } \bar{\omega}(\mathbf{r}) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma \; \sigma\rho(\sigma,\mathbf{r}) = \frac{\partial h}{\partial\lambda}(0,\mathbf{r})$$

- Then, for Young measures, the evolution of the moment generating functional is equivalent to

$$\frac{\partial h}{\partial t} + \bar{\mathbf{v}}\cdot\nabla h = 0$$

The 2D Euler Dynamics of Young Measures

- The cumulant-generating function of ω at point $\mathbf{r}$

$$h(\lambda,\mathbf{r}) = \log f(\lambda,\mathbf{r}) \text{ with } f(\lambda,\mathbf{r}) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma \, \mathrm{e}^{i\lambda\sigma} \rho(\sigma,\mathbf{r})$$

- Lemma (a consequence of the law of large numbers):

1. For Young measures, the velocity field is independent of the vorticity field
2. At each point, it has a Dirac distribution functions

$$P(\mathbf{v},\mathbf{r}) = \delta(\mathbf{v} - \bar{\mathbf{v}}) \text{ with } \bar{\omega}(\mathbf{r}) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma \, \sigma\rho(\sigma,\mathbf{r}) = \frac{\partial h}{\partial \lambda}(0,\mathbf{r})$$

- Then, for Young measures, the evolution of the moment generating functional is equivalent to

$$\frac{\partial h}{\partial t} + \bar{\mathbf{v}} \cdot \nabla h = 0$$

The 2D Euler Dynamics of Young Measures

- The cumulant-generating function of ω at point $\mathbf{r}$

$$h(\lambda,\mathbf{r}) = \log f(\lambda,\mathbf{r}) \text{ with } f(\lambda,\mathbf{r}) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma \; \mathrm{e}^{i\lambda\sigma}\rho(\sigma,\mathbf{r})$$

- Lemma (a consequence of the law of large numbers):

1. For Young measures, the velocity field is independent of the vorticity field
2. At each point, it has a Dirac distribution functions

$$P(\mathbf{v},\mathbf{r}) = \delta(\mathbf{v}-\bar{\mathbf{v}}) \text{ with } \bar{\omega}(\mathbf{r}) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma \; \sigma\rho(\sigma,\mathbf{r}) = \frac{\partial h}{\partial \lambda}(0,\mathbf{r})$$

- Then, for Young measures, the evolution of the moment generating functional is equivalent to

$$\frac{\partial h}{\partial t} + \bar{\mathbf{v}}\cdot\nabla h = 0$$

Invariant Young Measures

$$\frac{\partial h}{\partial t} + \bar{\mathbf{v}} \cdot \nabla h = 0$$

- A Young measure is invariant if h is invariant over any streamline of the average velocity $\bar{\mathbf{v}}$.
- A class of invariant Young measures. If h depends on the streamfunction only $h = h(\sigma, \bar{\psi}(\mathbf{r}))$. (equivalently $\rho = \rho(\sigma, \bar{\psi}(\mathbf{r})))$ and verify a self-consistency relation

$$\bar{\omega} = \Delta \bar{\psi} = \int d\sigma\, \sigma \rho(\sigma, \bar{\psi}(\mathbf{r}))$$

- All Young measures built on steady solutions of the 2D Euler equations are invariant Young measures
- Microcanonical measures is a subset of the set of invariant measures

Invariant Young Measures

$$\frac{\partial h}{\partial t} + \bar{\mathbf{v}} \cdot \nabla h = 0$$

- A Young measure is invariant if h is invariant over any streamline of the average velocity $\bar{\mathbf{v}}$.
- A class of invariant Young measures. If h depends on the streamfunction only $h = h(\sigma, \bar{\psi}(\mathbf{r}))$. (equivalently $\rho = \rho(\sigma, \bar{\psi}(\mathbf{r}))$) and verify a self-consistency relation

$$\bar{\omega} = \Delta \bar{\psi} = \int d\sigma\, \sigma \rho(\sigma, \bar{\psi}(\mathbf{r}))$$

- All Young measures built on steady solutions of the 2D Euler equations are invariant Young measures
- Microcanonical measures is a subset of the set of invariant measures

Invariant Young Measures

- All Young measures built on steady solutions of the 2D Euler equations are invariant Young measures
- Microcanonical measures is only a small subset of the set of invariant measures
- The 2D Euler equations are not ergodic (in this sense)
- Need for understanding of the stability of those invariant measures

Invariant Young Measures

- All Young measures built on steady solutions of the 2D Euler equations are invariant Young measures
- Microcanonical measures is only a small subset of the set of invariant measures
- The 2D Euler equations are not ergodic (in this sense)
- Need for understanding of the stability of those invariant measures

Invariant Young Measures

- All Young measures built on steady solutions of the 2D Euler equations are invariant Young measures
- Microcanonical measures is only a small subset of the set of invariant measures
- The 2D Euler equations are not ergodic (in this sense)
- Need for understanding of the stability of those invariant measures

Dynamics of the 2D Euler and Young Measures

Large Time Asymptotics

- Perturbations of the 2D Euler equations close to parallel flows converge, for large times, to invariant Young measures
- Weak perturbations of the Vlasov equation close to homogeneous dynamical equilibrium converge towards invariant Young measures
- Two conjectures:

1. Weak perturbations of the 2D Euler equations close to dynamical equilibria converge to invariant Young measures
2. The 2D Euler equations converge to invariant Young measures

Dynamics of the 2D Euler and Young Measures

Large Time Asymptotics

- Perturbations of the 2D Euler equations close to parallel flows converge, for large times, to invariant Young measures
- Weak perturbations of the Vlasov equation close to homogeneous dynamical equilibrium converge towards invariant Young measures
- Two conjectures:

1. Weak perturbations of the 2D Euler equations close to dynamical equilibria converge to invariant Young measures
2. The 2D Euler equations converge to invariant Young measures

Dynamics of the 2D Euler and Young Measures

Large Time Asymptotics

- Perturbations of the 2D Euler equations close to parallel flows converge, for large times, to invariant Young measures
- Weak perturbations of the Vlasov equation close to homogeneous dynamical equilibrium converge towards invariant Young measures
- Two conjectures:

1. Weak perturbations of the 2D Euler equations close to dynamical equilibria converge to invariant Young measures
2. The 2D Euler equations converge to invariant Young measures

The program of equilibrium statistical mechanics

1. Has the limit measure $\mu_{m,n}$ a simpler expression ? Mean field behavior ?
   Sanov's theorem justifies the microcanonical RSM variational problem and relates $\mu_{m,n}$ to Young measures.
   J. Michel and R. Robert, Comm. Math. Phys. (1994),
   R.S. Boucher, C. Ellis and B. Turkington, Annals Prob. (1999).
2. Is $\mu_{m,n}$ an invariant measure of the 2D Euler equations ?
   Dynamics of Young measures: Formal proof.
   F. Bouchet and M. Corvellec J. Stat. Mech. (2011).
3. Are the 2D Euler equations ergodic ?
   No they are not. The set of invariant Young measure is much larger that the set of microcanonical measures. F. Bouchet and M. Corvellec J. Stat. Mech. (2011).

The program of equilibrium statistical mechanics

1. Has the limit measure $\mu_{m,n}$ a simpler expression ? Mean field behavior ?
   Sanov's theorem justifies the microcanonical RSM variational problem and relates $\mu_{m,n}$ to Young measures.
   J. Michel and R. Robert, Comm. Math. Phys. (1994),
   R.S. Boucher, C. Ellis and B. Turkington, Annals Prob. (1999).
2. Is $\mu_{m,n}$ an invariant measure of the 2D Euler equations ?
   Dynamics of Young measures: Formal proof.
   F. Bouchet and M. Corvellec J. Stat. Mech. (2011).
3. Are the 2D Euler equations ergodic ?
   No they are not. The set of invariant Young measure is much larger that the set of microcanonical measures. F. Bouchet and M. Corvellec J. Stat. Mech. (2011).

The program of equilibrium statistical mechanics

1. Has the limit measure $\mu_{m,n}$ a simpler expression ? Mean field behavior ?
   Sanov's theorem justifies the microcanonical RSM variational problem and relates $\mu_{m,n}$ to Young measures.
   J. Michel and R. Robert, Comm. Math. Phys. (1994),
   R.S. Boucher, C. Ellis and B. Turkington, Annals Prob. (1999).
2. Is $\mu_{m,n}$ an invariant measure of the 2D Euler equations ?
   Dynamics of Young measures: Formal proof.
   F. Bouchet and M. Corvellec J. Stat. Mech. (2011).
3. Are the 2D Euler equations ergodic ?
   No they are not. The set of invariant Young measure is much larger that the set of microcanonical measures. F. Bouchet and M. Corvellec J. Stat. Mech. (2011).

Summary

Messages :

- We can built microcanonical measures for the 2D Euler equations and similar models
- They are Young measures, with local probabilities maximizing a mean-field variational problem (large deviation result)
- Jupiter vortices, ocean vortices and ocean eastward jets as statistical equilibria
- The dynamics and dynamical stability of Young measures seems an essential problem to understand

F. Bouchet, and A. Venaille, Physics Reports, 2011, Statistical mechanics of two-dimensional and geophysical flows

Publications

1. F. Bouchet, Physica D, 2008 Simplified variational problems for the statistical equilibria of 2D flows.
2. F. Bouchet and E. Simonnet, PRL (March 2009), Random changes of flow topology in 2D and geophysical turbulence.
3. A. Venaille and F. Bouchet, PRL (March 2009), Phase transitions, ensemble inequivalence and Fofonoff flows.
4. F. Bouchet and H. Morita, Physica D (April 2010), Asymptotic stability of the 2D Euler and of the 2D linearized Euler equations.
5. A. Venaille and F. Bouchet, J. Phys. Oceanography. Are strong mid-basin eastward jets (Gulf Stream, Kuroshio) statistical equilibria?
6. F. Bouchet and A. Venaille, Physics Reports: Statistical mechanics of two dimensional and geophysical flows.
7. F. Bouchet and M. Corvellec, J. Stat. Mech. Invariant measures of the 2D Euler and Vlasov equations.