EQUILIBRIUM STATISTICAL MECHANICS EXPLANATION OF OCEANIC JETS AND RINGS

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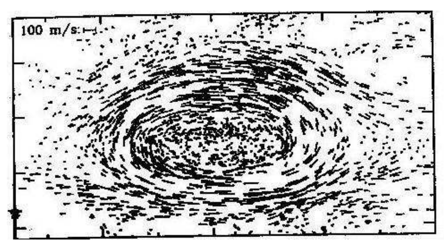
Statistical Mechanics of Large Scale Geophysical Flows

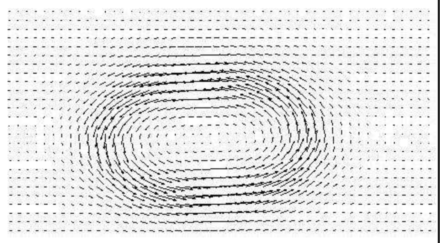
Large scale statistics of turbulent flows

In many applications of fluid dynamics, one of the most important problem is the prediction of the very high Reynolds' large-scale flows. The highly turbulent nature of such flows, for instance ocean circulation or atmosphere dynamics, renders a probabilistic description desirable. A statistical mechanics explanation of the self-organization of geophysical flows has been proposed by Robert-Sommeria and Miller (RSM).

The RSM theory has been successfully applied to the Jupiter's troposphere (Bouchet Sommeria), Figure: velocity field of Jupiter's Red Great Spot.

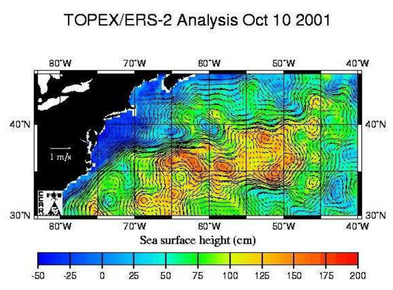


Observation (Voyager)

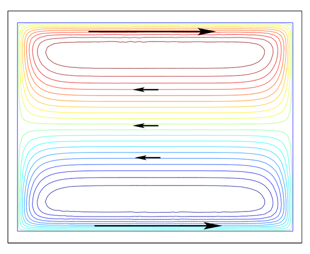

Statistical Equilibrium

Toward ocean applications

We show in the following that oceanic rings and midlatitudes eastward jets can actually be explained in the framework of RSM theory.



Robust coherent structures in the ocean: midlatitude eastward jets (here the Gulf Stream) and rings.


So far the only statistical equilibrium found is the Fofonoff flow
Salmon-Holloway and Bretherton-Haidvogel

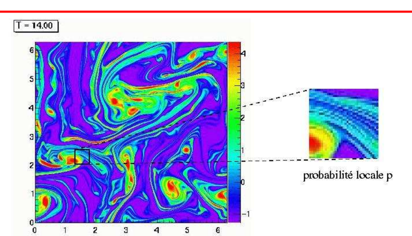
The 1-1/2 Layer Quasi-Geostrophic Model

$$\frac{\partial q}{\partial t} + \mathbf{u}.\nabla q = 0 \quad \text{with} \quad q = -\Delta\psi + \frac{\psi}{R^2} - h(y) \quad \text{and} \quad \mathbf{u} = -\mathbf{e}_z \wedge \nabla\psi$$

where q is the potential vorticity (PV), $\mathbf{u}$ the velocity field, ψ the stream function, and h the equivalent topography induced by the deep zonal flow.

Conservation laws : Energy : $E = \frac{1}{2}\int_D d\mathbf{r}\left(\mathbf{u}^2 + \frac{\psi^2}{R^2}\right)$ Casimirs : $C_f(q) = \int_D d\mathbf{r}\, f(q)$

Statistical Mechanics of the Quasi-Geostrophic Model



Left: Typical potential vorticity (PV) field, from a numerical simulations with initially only two PV values (red and blue).

Let $p(\mathbf{r})$ be the local probability to have one of the two initial PV values (red or blue above). The **Entropy** counts the number of states corresponding to a given p (Robert and Sommeria (1991), Miller (1991)) : $S = -\int_D [\, p(\mathbf{r}) \ln p(\mathbf{r}) + (1-p(\mathbf{r})) \ln(1-p(\mathbf{r}))\,] d\mathbf{r}$.

The entropy maximum is **the most probable state for a given energy and PV distribution** after complete PV mixing.

Statistical Equilibria : $\max\{S \,|\, \text{with } E = E_0 \text{ and } A = A_0\}$ where A is the area occupied by one of the potential vorticity levels, E is the energy and S the equilibrium entropy.

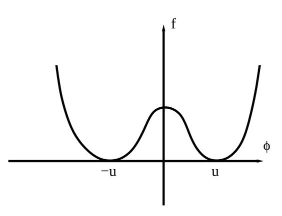
- Critical points: a **stationary state** given by $q = -\Delta\psi + \frac{\psi}{R^2} - h(y) = f_{\alpha,\beta}(\psi)$
- Analytical results in the limit of small Rossby deformation radius: $R \to 0$.

Coexistence of Two Phases Separated by an Interface

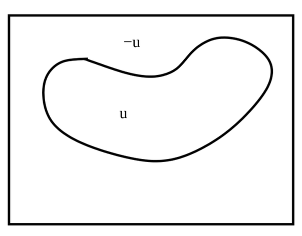
Entropy maximization is equivalent to the variational problem:

$$\left\{ \begin{array}{l} \min\{F_R[\phi] \,|\, \text{with } A[\phi] \text{ given}\} \\ \text{with } F_R[\phi] = \int_D d\mathbf{r} \left[\frac{R^2(\nabla\phi)^2}{2} + f(\phi) - R\phi h_0(y)\right] \text{ and } A[\phi] = \int_D d\mathbf{r}\, \phi \end{array} \right.$$

This describes a **first order phase transition** (analogous to a gaz buble in a liquid)

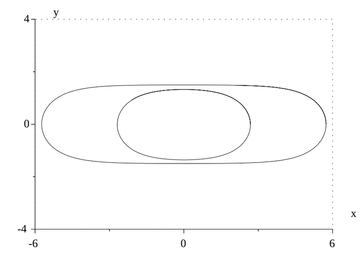

f has two minima

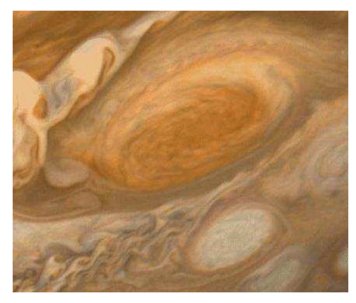
ϕ thus takes the two values where f reaches its minima, in two subdomains (phase separation) separated by an interface (see right figure)



- Asymptotic expansion ($R \to 0$), variational problem for the curve formed by the jet: $\min\left\{F_R[\phi_R] = 2Re_cL - 2Ru\int_{A_1} d\mathbf{r}\, h_0(y) + o(R)\right\}$
- Laplace equation: link between the curvature radius r and the free energy difference: $\frac{e_c}{r} = -u\,(\alpha_1 - h_0(y))$
- **Conclusion : without topography, the interface should minimize its length, for a fixed area. The topography will slightly change this picture.**

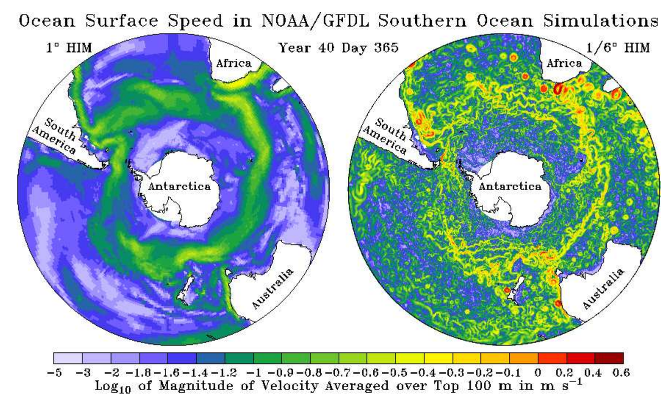
Jovian Vortices as Statistical Equilibria


Statistical Equilibria

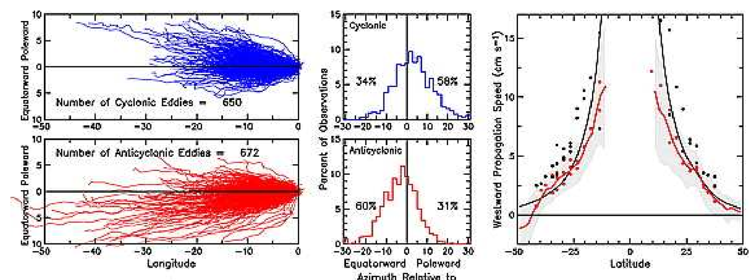

Great Red Spot and White Oval BC


A Brown Barge

Ocean Mesoscale Vortices as Statistical Equilibria

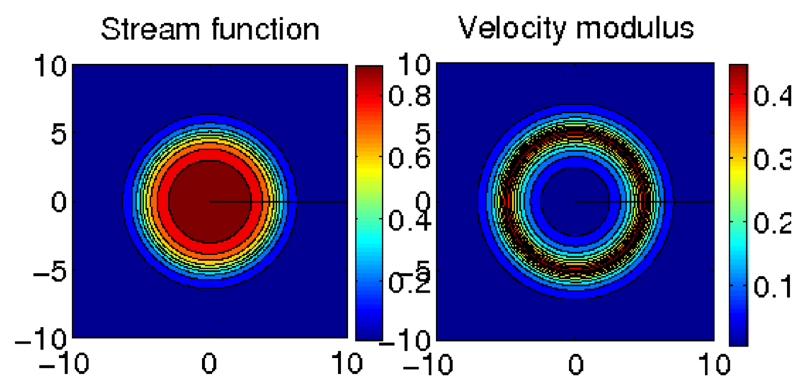


Rings are everywhere
Hallberg-Gnanadesikan - JPO 2006


Westward drift observations
Chelton and co. - GRL 2007

- A large part of mesoscale variability is explained by rings of size $100-200$ km.
- Both cyclonic and anticyclonic rings drift westward with a velocity $\tilde{\beta}R^2$.

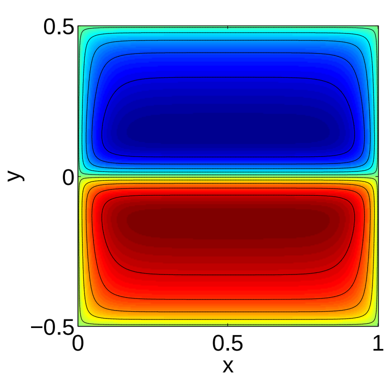
Without topography: equilibria are rings (like bubbles in usual thermodynamics).



- In a domain with translational symmetry (channel), using Noether's theorem we obtain a new invariant: $L = \int d^2x\, yq$.
- Statistical equilibria with this new invariant: $q = (s')^{-1}(-\beta\psi + \gamma y)$
- Steady solution to the QG in a reference frame drifting with constant westward velocity $V_\gamma = -\tilde{\beta}R^2\mathbf{e}_x$
- **Conclusion: equilibrium statistical mechanics explains the formation of circular rings and their westward drift**

Ocean Midlatitude Jets as Statistical Equilibria

Left: equilibrium stream function. Without beta effect, the states with positive PV to the north (eastward jet), and positive PV to the south (westward jet) are equivalent. The beta effect breaks this symmetry.



- Statistical equilibria of the QG 1-1/2 layer in a closed basin $h(y) = 0$:

$$F_R[\phi_R] = 2Re_cL \quad \text{and} \quad \frac{e_c}{r} = u\alpha_1$$

- Statistical equilibria of the QG 1-1/2 layer in a closed basin $h(y) = \beta y$:

$$F_R[\phi_R] = 2Re_cL - 2Ru\int_{A_+} dl\, \beta y \qquad F_{Eastward} > F_{Westward}$$

- **Conclusion 1 : With a beta effect, global statistical equilibria are the ones with westward jets. The bad ones !**
- Are the eastward jets metastable (local equilibria) in presence of a deep zonal flow ? ($\psi_2 = ay$ and then $h(y) = \beta y - \psi_2/R^2 = \tilde{\beta}y$)

$$F_R[\phi_R] = 2Re_cL - 2Ru\int_{A_+} dl\, \tilde{\beta}y \qquad \delta^2\mathcal{F} \geq \left[-2u\tilde{\beta} + Re\left(\frac{k\pi}{L_x}\right)^2\right]\int dx\, (\delta l)^2 .$$

- We have local free energy minima (for all δl, $\delta^2\mathcal{F} \geq 0$) if $\tilde{\beta} < \tilde{\beta}_c = \frac{1}{2}\frac{e}{u}\frac{\pi^2}{L_x^2}R$.
- **Conclusion 2: With a low effective beta effect, eastward jets are local statistical equilibria.**
- **Conclusion 3: In the ocean case, eastward jets are probably marginally unstable, from a statistical mechanics point of view.**

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- F. BOUCHET and J. SOMMERIA, 2002, *J. Fluid. Mech.*, **464** 465-207
- A. VENAILLE and F. BOUCHET, 2009, *Phys. Rev. Lett.*, **102** 104501
- F. BOUCHET and A. VENAILLE, 2009, *Peyresq Non Linear Lectures*, World Scientific
- A. VENAILLE, and F. BOUCHET, *submitted to Journal of Physical Oceanography*