

Baroclinic Instability of Frontal Geostrophic Currents over a Slope

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ABSTRACT

The Phillips problem of baroclinic instability is generalized in a frontal geostrophic model. The configuration used here is a two-layer flow (with quasigeostrophic upper-layer current) over a sloping bottom. Baroclinic instability in the frontal model has a single unstable mode, corresponding to isobaths and isopycnals sloping in the same direction, contrary to the quasigeostrophic model, which has two unstable modes. In physical terms, this is explained by the absence of relative vorticity in the lower (frontal) layer. Indeed, the frontal geostrophic model can be related to the quasigeostrophic model in the limit of very small thickness of the lower layer, implying that potential vorticity reduces to vortex stretching in this layer. This stability study is then extended to unsteady flows. In the frontal geostrophic model, a mean flow oscillation can stabilize an unstable steady flow; it can destabilize a stable steady flow only for a discrete spectrum of low frequencies. In this case, the model equations reduce to the Mathieu equation, the properties of which are well known.

1. Introduction

Baroclinic instability of intense flows has long since been recognized as one of the major sources of meso-scale variability in the atmosphere and oceans. In the ocean, the instability of intense zonal jets of the general circulation (such as the Gulf Stream) has been studied at length, both experimentally and numerically (see Holland and Haidvogel 1980; Kontoyannis and Watts 1994; Flierl et al. 1999). Many other flows, deep or abyssal, with a frontal¹ structure and flowing over a sloping bottom, can be baroclinically unstable. A well-known example is the Mediterranean Water Undercurrents that originate from the watermass exchange at the Straits of Gibraltar between the Atlantic Ocean and the Mediterranean Sea. South of Portugal (near 8°W), this flow is composed of two thin jets of warm and salty water on the Iberian continental slope at 800-m and 1200-m depth; each jet is roughly 200–300 m thick with

velocity maxima in the 0.3–0.5 m s⁻¹ range. This flow structure (over a steep slope) corresponds to noticeable vertical deviations of isopycnal surfaces. As they encounter the Portimão Canyon and Cape Saint Vincent, these MWU undergo baroclinic instability and long-lived anticyclonic eddies (meddies) detach, sometimes accompanied by shallower cyclones. The MWU are also subject to low-frequency variability at their source, the Straits of Gibraltar, also observed downstream (Chérubin et al. 2003). One can wonder if a pulsating source can change the stability properties of a current, rendering it more stable than its time average (i.e., steady) analog.

The aim of this paper is therefore to investigate the baroclinic instability of both steady and unsteady currents over a sloping bottom in a frontal geostrophic model. The choice of this model (as compared with the quasigeostrophic framework) is justified by its ability to handle frontal flows over sloping topography. Though many oceanic currents are narrow jets (e.g., the MWU), we consider here (as a first step) a simplified flow and geometric configuration. A two-layer zonal channel flow is used and the mean flow velocity is chosen uniform horizontally in each layer. We neglect the planetary beta effect, and the bottom slope is constant. This problem is therefore an extension of the Phillips (1954) model, already generalized to time-dependent currents

¹ By “frontal,” we mean “associated with vertical displacement of isopycnal surfaces on the order of the current thickness.”

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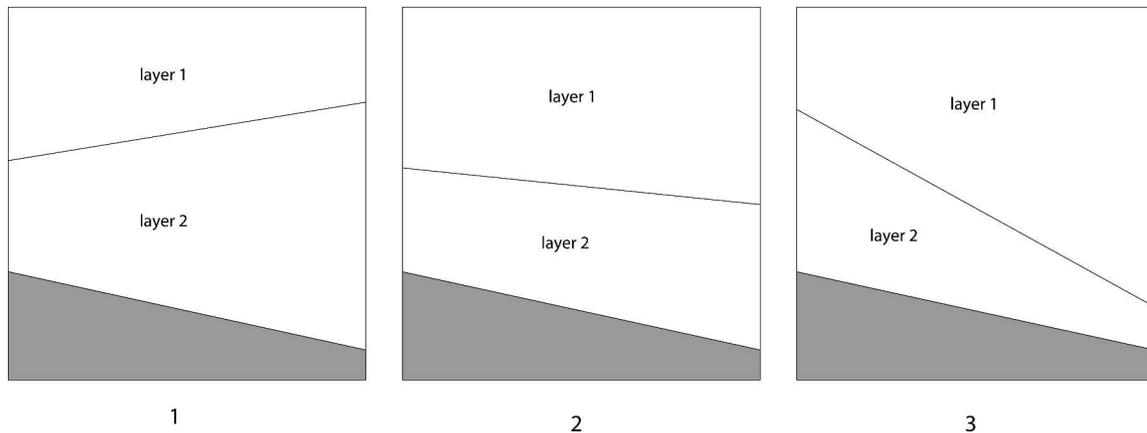


FIG. 1. Three flow/topography configurations for baroclinic instability.

by Pedlosky and Thomson (2003) for quasigeostrophic flows. We recall that in quasigeostrophic theory two unstable modes are found for this problem, one with isobaths and isopycnals sloping in opposite directions (case 1 of Fig. 1) and one with the isobaths and isopycnals sloping in the same direction (case 3). The flow and topography configuration where both layer thicknesses increase in the same direction (case 2) is stable in the quasigeostrophic model.

2. Model equations

The frontal geostrophic model is used here in a two-layer, zonal channel configuration. It includes bottom topography and allows the lower layer to be frontal, while the upper one has only minor variations in thickness. The model equations were first derived in (Swaters 1991) using an expansion in the topographic slope. For simplicity, we retain here the notations used in (Swaters 1998):

$$\nabla^2 \eta_t + J(h + \eta, h_b) + J(\eta, \nabla^2 \eta) = 0 \quad \text{and} \quad h_t + J(\eta + h_b, h) = 0, \tag{1}$$

where η is the surface elevation, h is the bottom layer thickness, and h_b is the bottom topography elevation (see Fig. 2). The pressure in the lower layer is $p = h_b + \eta + h$ (in dimensionless form). The Jacobian is defined as usual: $J(A, B) = A_x B_y - A_y B_x$. Velocity in each layer is geostrophic. Note that the surface elevation could be replaced by upper-layer pressure in the case of a rigid lid (see section 3). The channel has width L .

To transpose Phillips' problem with topography into that model, the mean flow U_1 and U_2 must be zonal and uniform in each layer and the corresponding surface elevation $\bar{\eta}$ and lower-layer thickness \bar{h} must vary linearly with latitude. Since only the velocity shear $U_1 - U_2$ is involved in the stability calculation, we can impose

$U_1 = 0$ with no loss of generality. Then we have $\bar{\eta} = 0$,

$$h_b = sy, \quad \text{and} \quad \bar{h} = h_0 + by.$$

In dimensionless form, the vertical shear is $U_1 - U_2 = b + s$. By symmetry, we can choose $s < 0$ and b can be either positive or negative, allowing all configurations described in Fig. 1.

3. Linear instability of steady flow

The stability of the stationary two-layer flow is investigated using a normal-mode perturbation approach:

$$\eta = \eta' \quad \text{and} \quad h = \bar{h} + h',$$

with

$$(\eta', h') = \Re\{[\alpha(t), \beta(t)] \exp[i(kx + ly)]\},$$

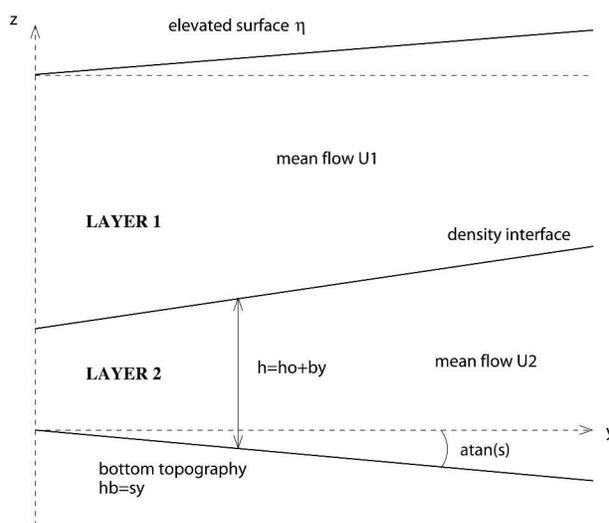


FIG. 2. Configuration and variables for the frontal geostrophic and quasigeostrophic models.

where $\Re(z)$ stands for the real part of the complex number z . The dynamical Eqs. (1) are linearized in the perturbation to yield

$$\partial_t \mathbf{X} = \mathbf{M} \mathbf{X},$$

where \mathbf{M} is a 2×2 matrix, and \mathbf{X} is a vector with components α and β . With $K^2 = k^2 + l^2$, we have

$$\mathbf{M} = ik \begin{pmatrix} \frac{s}{K^2} & \frac{s}{K^2} \\ -b & s \end{pmatrix} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}. \quad (2)$$

Setting now $[\alpha(t), \beta(t)] \sim (\alpha, \beta) \exp(-ikct)$, the linearized perturbation equation can be solved to yield

$$c = c_{\pm} = \frac{-s(K^2 + 1)}{2K^2} \pm \frac{s}{2K^2} \sqrt{\Delta},$$

with $\Delta = (K^2 + 1)^2 - 4K^2(1 + b/s)$. The vector components are then determined by

$$(K^2 - 1 \pm \sqrt{\Delta}) \alpha_{\pm} + 2\beta_{\pm} = 0.$$

The marginal stability condition is obtained by imposing Δ to vanish, rendering the eigenvalues of \mathbf{M} real:

$$(K^2 + 1)^2 - 4K^2 \left(1 + \frac{b}{s}\right) = 0. \quad (3)$$

The stability regimes are plotted in the plane $(K^2, b/s)$ on Fig. 3. We note that only one situation is unstable, that with isopycnals and isobaths sloping in the same direction. Indeed, in the frontal geostrophic model, baroclinic instability is associated with the release of gravitational potential energy due to the down slope motion of the dense current while overlying fluid moves upslope. This creates a preferred direction in the frontal geostrophic model, that described by case 3, where the reservoir of dense fluid lies upslope initially. Mathematically, the integrated perturbation energy equation [Eq. (3.5a) in Swaters (1991)] must have a positive right hand side for baroclinic instability, that is, a net upslope transport of light fluid.

4. Comparison with the quasigeostrophic model

a. Comparison of model equations and of their intrinsic relations

Since the quasigeostrophic approximation filters out fast waves, we do not write the model equations with a free surface elevation but with surface pressure π (we refer to Fig. 2 for the description of the variables in the quasigeostrophic model). Using these notations, the two-layer frontal geostrophic equations are in dimensionless form:

$$\begin{aligned} \nabla^2 \pi_t + J(h + \pi, h_b) + J(\pi, \nabla^2 \pi) &= 0 \quad \text{and} \\ h_t + J(\pi + h_b, h) &= 0. \end{aligned}$$

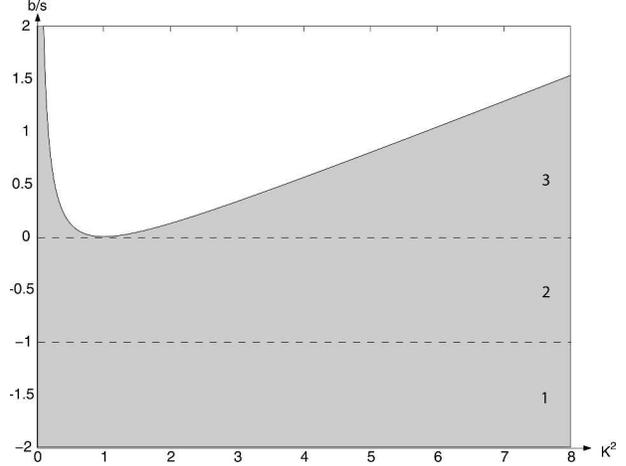


FIG. 3. Baroclinic instability of a uniform baroclinic current in the two-layer frontal geostrophic model. The flow is stable in the shaded area of the $K^2, b/s$ plane and unstable elsewhere. Cases 1, 2, and 3 correspond to the flow/topography configurations of Fig. 1.

The two-layer quasigeostrophic equations are (scaling surface pressure by the upper-layer thickness H_1)

$$(\nabla^2 \pi + h)_t + J(\pi, \nabla^2 \pi + h + h_b) = 0 \quad (4)$$

and

$$[\delta \nabla^2 (\pi + h) - h]_t + J[\pi + h + h_b, \delta \nabla^2 (\pi + h + h_b) - h] = 0, \quad (5)$$

where $\delta = H_2/H_1$ is the layer thickness ratio. Setting $\delta = 0$, Eq. (5) becomes

$$h_t + J(\pi + h_b, h) = 0,$$

the second frontal geostrophic equation, which subtracted from Eq. (4) provides the first frontal geostrophic equation.

The similarities between the two models result from their derivation from the primitive equation shallow-water (PE-SW) model. This derivation is based on an expansion in Rossby number, followed by a truncation at first order, for the quasigeostrophic (QG) model. The frontal geostrophic (FG) model is derived from the same (PE-SW) equations, by using the bottom slope as a small parameter (with similar effect as the Rossby number). In particular, both systems of equations are quadratic in barotropic pressure and in interface elevation, whereas frontal geostrophic equations for surface flows are cubic in these variables (Benilov and Reznik 1996).² Frontal geostrophic equations for surface flows become quadratic in their variables only when the barotropic mode is strong and takes over the evolution of the interface from the baroclinic mode (Benilov and

² The origin of these cubic terms lies in the degeneracy of the quasigeostrophic equation for the baroclinic mode in the frontal limit when the advecting velocity is essentially baroclinic and geostrophic.

Reznik 1996, and references therein). Such dynamics are then governed by the often-used set of equations

$$\begin{aligned} \nabla^2 \pi_t + J(\pi, \nabla^2 \pi) + \alpha \pi_x &= 0 \quad \text{and} \\ h_t + J(\pi, h) &= 0, \end{aligned} \quad (6)$$

where α is a scaled planetary beta effect (see also Dewar and Gaillard 1993). In the present frontal geostrophic model, the lower-layer thickness is advected both by upper-layer (barotropic) pressure gradients and by gradients of bottom topography. Therefore only quadratic terms are present.

The differences between the two models are

- 1) the absence of relative vorticity in the lower layer in the FG model [indeed, in the FG model, the lower-layer velocity scales as the upper one multiplied by $\delta/(1 + \delta)$] and
- 2) the necessity of a finite bottom slope in the FG model since its derivation is based on this parameter (whereas the QG model does not have this limitation).

b. Comparison of instability properties in the two models

To avoid a lengthy comparison of linearized model equations and solution properties, we follow Sakai (1989) or Pichevin (1998) and consider baroclinic instability as the resonance between two layerwise Rossby waves. The upper-layer Rossby wave has phase speed $c_1 = -(b + s)/(K^2 + 1)$ due to the slope of the density interface; the lower-layer Rossby wave has $c_2 = -(b + s) + b/(\delta K^2 + 1)$, a combination of the effects of the sloping interface and of the bottom slope (also including the mean flow). Sakai states that a necessary condition for baroclinic instability is that the two Rossby waves mutually reinforce and thus propagate at the same speed. This leads to

$$\frac{b}{s} = \frac{K^2(1 + \delta K^2)}{1 - \delta K^4}.$$

Therefore, two unstable modes are possible if $\delta \neq 0$ with either sign of b/s . When $\delta = 0$ (the frontal geostrophic case), b/s must be positive for baroclinic instability to occur. Clearly, the disappearance of relative vorticity in the lower layer renders the lower-layer phase speed solely dependent on the bottom slope. In the frontal geostrophic model, baroclinic instability can only occur in case 3 (referring to Fig. 1) where the layerwise waves propagate in the same direction. On the contrary, in case 1, the lower-layer phase speed does not change sign in the FG model (contrary to the QG case), and thus there can be no phase locking nor resonance between waves.

Another approach to compare the instability properties of the two models is to solve numerically the linearized equations of the quasigeostrophic model for

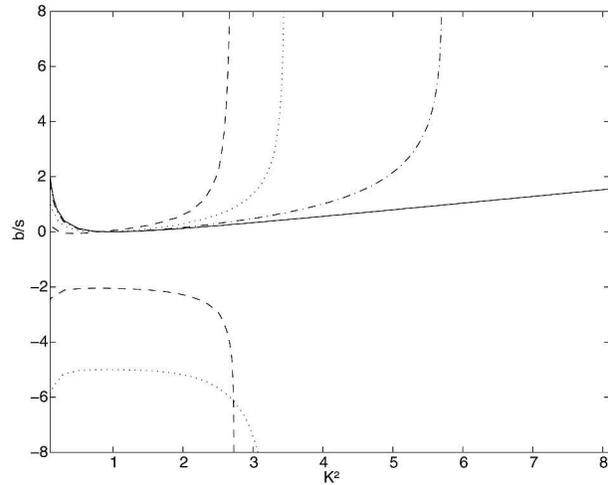


FIG. 4. Marginal stability curves in the K^2 , b/s plane for the uniform baroclinic current in a quasigeostrophic model for various values of δ : dashed lines $\delta = 1$, solid lines $\delta = 0$, $\delta = 0.25$ in dotted lines, and $\delta = 0.025$ in dash-dotted lines.

normal mode perturbations and to decrease δ . The marginal stability curves for various values of δ are superimposed on Fig. 4. Clearly, the unstable mode corresponding to interface and topography sloping in opposite directions progressively disappears when δ diminishes.

Two complementary notes to this analysis are the following:

- 1) bringing the bottom slope to zero in the FG model (though this is not allowed formally) brings growth rates of perturbations to zero also, and
- 2) the vertical structure of the unstable modes is comparable in the QG and FG models. Short waves are intensified in lower-layer thickness, and long waves are intensified in barotropic (or here surface) pressure.

5. Linear instability of oscillating flow

a. Derivation of the evolution equation for the amplitude of the perturbation

The stability of a time-dependent flow is now investigated in the neighborhood of the marginal curve (in the frontal geostrophic model), following the method described in Pedlosky and Thomson (2003). The time-dependent part of the shear is included here in \bar{h} , the lower-layer thickness:

$$\bar{h} = h_0 + [1 + f(t)]by. \quad (7)$$

The time-dependent part of the mean shear can be written (following Pedlosky and Thomson 2003 and references therein):

$$f(t) = \varepsilon^2 G + \varepsilon H \cos(\omega t), \quad (8)$$

with $\varepsilon \ll 1$. Here G is an increment of the steady shear in the vicinity of the marginal stability curve and H is the amplitude of its oscillating part. We have also introduced two time scales, τ and T , so that $\partial_t = \partial_\tau + \varepsilon \partial_T$. Unsteady flow instability will take place over the slower time scale T while the forcing of the mean flow occurs on the faster one τ . Note that Eq. (7) is comparable with Eq. (4.1) of Mooney and Swaters (1996), who developed a weakly nonlinear asymptotic analysis for marginally unstable flows for the frontal geostrophic equations (albeit without the time varying part).

To accommodate both quasigeostrophic and frontal geostrophic cases, the algebraic developments are presented hereinafter in generic form. Again the evolution of $\mathbf{X}(h', \pi')$ is given by $\partial_t \mathbf{X} = \mathbf{M}\mathbf{X}$, where the matrix \mathbf{M} is now $\mathbf{M} = \mathbf{M}_0 + \varepsilon H \cos(\omega\tau)\mathbf{M}_1 + \varepsilon^2 G\mathbf{M}_1$, with

$$\mathbf{M}_j = ik \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \quad (j = 0, 1) \quad (9)$$

and

$$\begin{aligned} a_0 &= \alpha[(s+b)(1+2\delta K^2) - b], \\ b_0 &= \alpha[(s+b)(1+\delta K^2) - b], \\ c_0 &= K^2\alpha[(s+b)\delta(K^2-1) - b], \\ d_0 &= K^2\alpha[(s+b)(\delta K^2+1) - b], \\ a_1 &= 2\delta K^2 b/\alpha, \quad b_1 = \delta K^2 b/\alpha, \\ c_1 &= [\delta(K^2-1) - 1]b/(K^2\alpha), \\ d_1 &= \delta b/\alpha, \quad \text{and} \\ \alpha &= \frac{1}{K^2(1+\delta+\delta K^2)} \end{aligned}$$

[from Eqs. (4)–(5)], the frontal geostrophic case being recovered for $\delta = 0$. The calculations can be usefully simplified by setting

$$\begin{aligned} B &= X \exp[-ik(a_0 + d_0)t/2] \quad \text{and} \\ N_0 &= M_0 - ik(a_0 + d_0)Id/2. \end{aligned}$$

Expanding $B = B_0 + \varepsilon B_1 + \varepsilon^2 B_2$ we obtain

$$\begin{aligned} \partial_\tau B_0 &= N_0 B_0, \\ \partial_T B_0 + \partial_\tau B_1 &= N_0 B_1 + H \cos(\omega\tau)M_1 B_0, \quad \text{and} \\ \partial_T B_1 + \partial_\tau B_2 &= N_0 B_2 + H \cos(\omega\tau)M_1 B_1 + GM_1 B_0, \end{aligned}$$

and the calculation follows that of Pedlosky and Thomson (2003).

At zeroth order in ε , with the marginality condition $(a_0 - d_0)^2 = -4b_0c_0$, we recover the independence of B_0 from the fast time: $\partial_\tau B_0 = 0$, and the proportionality between layerwise components of B_0 : $B_{01} = (a_0 - d_0)B_{02}/2c_0$.

At first order in ε and setting $\alpha_{11} = a_1(a_0 - d_0)/2 + b_0c_1$ and $\alpha_{12} = b_1(a_0 - d_0)/2 + b_0d_1$, we have

$$\begin{aligned} B_{11} &= \frac{-ikH}{\omega^2} [ik \cos(\omega\tau)(\alpha_{11}B_{01} + \alpha_{12}B_{02}) \\ &\quad - \omega \sin(\omega\tau)(a_1B_{01} + b_1B_{02})] \quad \text{and} \\ B_{12} &= \frac{1}{ikb_0} \left[\frac{dB_{01}}{dT} - \frac{k^2H}{\omega} \sin(\omega\tau)(\alpha_{11}B_{01} + \alpha_{12}B_{02}) \right. \\ &\quad \left. - \frac{ikB_{11}}{2}(a_0 - d_0) \right]. \end{aligned}$$

At second order in ε , the secular terms in the equation

$$\partial_T B_1 - H \cos(\omega\tau)\mathbf{M}_1 B_1 - G\mathbf{M}_1 B_0$$

must vanish in some integral sense (by averaging over the period of fast variation). Substituting the expressions for the first-order terms in the resulting equation leads to the slow time variation of the zeroth-order amplitude:

$$\begin{aligned} &\frac{d^2 B_{01}}{dT^2} + k^2[b_0c_1 - d_1(a_0 - d_0)/2] \\ &\times \left[\frac{k^2 H^2}{2\omega^2} \left(\alpha_{11} - \alpha_{12} \frac{a_0 - d_0}{2b_0} \right) + G \right] B_{01} = 0 \quad (10) \end{aligned}$$

(obviously B_{02} satisfies the same equation).

b. Interpretation in the frontal geostrophic case

In the frontal geostrophic case, Eq. (10) becomes

$$\frac{d^2 B_{01}}{dT^2} + \frac{k^2 bs}{K^2} \left(\frac{k^2 H^2}{2\omega^2} \frac{bs}{K^2} - G \right) B_{01} = 0,$$

leading to the condition for instability

$$\frac{k^2 bs}{2\omega^2 K^2} H^2 < G. \quad (11)$$

Then two cases must be investigated:

- 1) For a subcritical flow ($G < 0$), this condition cannot be met because $b/s > 0$ in the vicinity of the marginal curve. A high- or medium-frequency oscillation cannot destabilize a stable flow.
- 2) For a supercritical flow ($G > 0$), an unstable steady flow can be stabilized by a mean flow pulsation as shown by Fig. 5.

To investigate the influence of a low-frequency oscillation on the stability of a subcritical flow in the frontal geostrophic model, we have to rescale the pulsation ω as $\varepsilon\omega$ and set $\delta(T) = \varepsilon^2[G + H \cos(\omega T)]$ (as in Pedlosky and Thomson 2003). The expansion in ε leads to the well-known Mathieu equation

$$\partial_T^2 B_0 - \frac{k^2 bs}{K^2} [G + H \cos(\omega T)] B_0 = 0$$

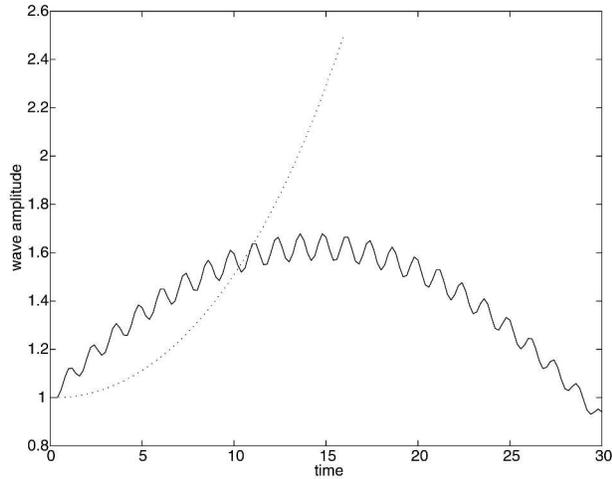


FIG. 5. Time evolution of wave amplitude obtained by integration of the Mathieu equation with $K^2 = 3$, $b = 1$, $s = 3$, and $k = 1$ for $G = 0.01$, $H = 0.8$, and $\omega = 5$; the dotted line shows the corresponding evolution of the unstable normal mode of the stationary flow.

(Bender and Orszag 1978, 560–566), showing instability for a discrete spectrum of frequencies

$$\omega = \frac{2}{n} \sigma, \quad n = 1, 2, \dots, \quad \text{and} \quad \sigma = \left(-\frac{kbs}{K^2} G \right)^{1/2},$$

when $G < 0$. This equation can be integrated numerically and its solution can be compared with that of the steady problem. Figure 6 shows the destabilization of a linearly stable steady flow by a low-frequency oscillation.

c. Comparison with the quasigeostrophic case

Here, we return to the quasigeostrophic model, using the complete expressions for a_j , b_j , c_j , and d_j given in

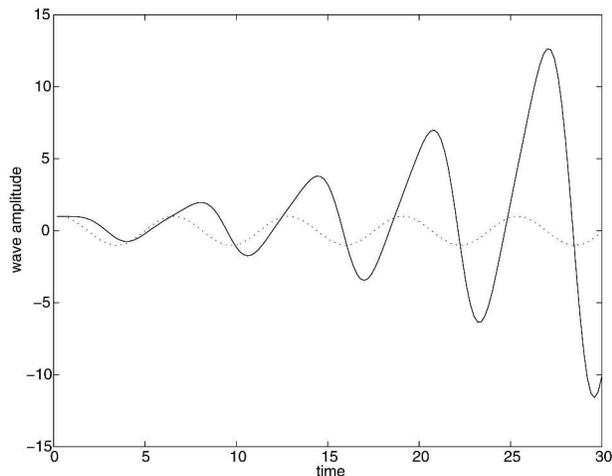


FIG. 6. As in Fig. 5 but for $G = -1$, $H = 1$, and $\omega = 1$ (first critical frequency, $n = 1$); the dotted line shows the corresponding evolution of the stable normal mode of the stationary flow.

section 5a. We study the instability of the unsteady flow in the transition regime between the quasigeostrophic and frontal geostrophic models, that is, when $\delta \rightarrow 0$.

First, since the instability occurs in the vicinity of the marginal stability curve for steady flow, we expand the marginal stability condition $(a_0 - d_0)^2 = -4b_0c_0$ in powers of $\delta \ll 1$. Keeping only linear terms in δ , we obtain

$$\begin{aligned} &\theta^2[(1 - K^2)^2 + 2\delta(2K^4 - 7K^2 - 1)] \\ &+ 2K^2\theta[-2 + \delta(K^4 + K^2 + 2)] - 4K^2\delta = 0, \end{aligned}$$

with $\theta = s/b$. Moreover, Fig. 4 shows that, in the limit of small wavenumbers, marginal stability occurs for both signs of b/s when δ decreases. Neglecting terms in $K^n\delta^n$ with $(n, m) > 1$, we have

$$\theta_{1,2} \sim \frac{2K^2}{(1 - K^2)^2} [2; -\delta/2].$$

For $\delta = 0$, we recover the positive value of b/s . For finite δ , a negative value of b/s exists, the module of which grows as δ decreases (see again Fig. 4).

Now, for both signs of b/s , we inspect how the term

$$\left(b_0c_1 + d_1 \frac{d_0 - a_0}{2} \right) \left[\frac{k^2H^2}{2\omega^2} \left(\alpha_{11} + \alpha_{12} \frac{d_0 - a_0}{2b_0} \right) + G \right]$$

can change sign, again in the limit of small δ and near marginality. This corresponds to changes from stability to instability for the oscillating flow.

The first term $T_1 = b_0c_1 + d_1 d_0 - a_0/2$ reduces to

$$T_1 = \frac{b^2}{K^2} \left[\theta \left(-1 + \frac{\delta}{2} + 2\delta K^2 + \frac{\delta}{K^2} \right) + \delta K^2 \right]$$

when only linear terms in δ are kept. For $b/s > 0$, we have $\theta = 4K^2/(1 - K^2)^2 + O(\delta)$ so that $T_1 \sim -4b^2/(1 - K^2)^2 + O(\delta) < 0$. For $b/s < 0$ and small K , we have $\theta \sim -\delta/(1 - K^2)^2$ so that $T_1 = \delta K^2[1 + 1/(1 - K^2)^2] > 0$. In both cases, T_1 does not change sign near marginality and therefore does not contribute to a transition from stability to instability. The second term

$$T_2 = \frac{k^2H^2}{2\omega^2} \left(\alpha_{11} + \alpha_{12} \frac{d_0 - a_0}{2b_0} \right) + G$$

can be written

$$\begin{aligned} T_2 = &\frac{k^2H^2b^2}{2\omega^2K^2} \left[\theta(-1 + 2\delta + \delta K^2) - 2\delta K^2 + \frac{2\delta}{K^2 - 1} \right] \\ &+ G. \end{aligned}$$

For $b/s > 0$, we have $\theta = 4K^2/(1 - K^2)^2 + O(\delta)$ near the marginal curve so that $T_2 \sim G - 2k^2H^2b^2/[\omega^2(1 - K^2)^2]$, which can change sign and therefore contribute to the stabilization of an otherwise unstable steady flow as in the frontal geostrophic model. For $b/s < 0$, and

again for small K , the expression for T_2 near the marginal stability curve of the steady flow is

$$T_2 = \frac{k^2 H^2 b^2}{2\omega^2 K^2} \left[\frac{\delta K^2}{(1 - K^2)^2} + \frac{2\delta}{(K^2 - 1)} - 2\delta K^2 \right] + G$$

or

$$T_2 = \frac{-\delta k^2 H^2 b^2}{\omega^2 K^2} + G$$

for long waves, so that stabilization of an unstable steady flow by a small oscillatory component requires increasingly low frequencies as δ goes to zero.

In summary, parametric instability in the quasigeostrophic model is allowed near the marginal stability curve for $b/s > 0$ as $\delta \rightarrow 0$, while it requires increasingly low frequencies of the oscillatory component for $b/s < 0$ in the long wave limit, again as $\delta \rightarrow 0$. In that case, rescaling of ω must be performed as in section 5b. When $\delta = 0$, parametric instability leads to stabilization of supercritical flows with $b/s > 0$ as shown previously.

6. Conclusions and perspectives

A generalized Phillips problem (the baroclinic instability of a horizontally uniform, vertically sheared current in a two-layer fluid) was studied in a frontal geostrophic model for bottom flows over topography. Contrary to the quasigeostrophic model, the frontal geostrophic model allows only one unstable mode for steady bottom flows, corresponding to like-signed isopycnic and topographic slopes. In physical terms, this corresponds to the release of available potential energy of the mean flow lying upslope. Formally, it was shown that this frontal model is the limit of the quasigeostrophic model when the lower-layer thickness becomes very small. It was also mentioned that this frontal geostrophic model retains only quadratic terms in its dynamical equations since the density interface evolution is governed by barotropic and topographic flow advection. A simple stability analysis showed that the resonance of Rossby waves vanishes for opposite-signed topographic and isopycnic slopes when the lower-layer thickness becomes negligible; this is due to the absence of relative vorticity in the lower layer. This explains why only a single unstable mode exists for steady, frontal geostrophic, bottom flows.

This study was then extended to unsteady flows having an oscillatory component. A multiple time-scale expansion was used to obtain the slow time evolution of the amplitude of the perturbation. In the frontal geostrophic model, an unstable steady flow can be stabilized by a mean flow pulsation, and a subcritical flow can be destabilized by a low-frequency oscillation for a discrete spectrum of frequencies. In the limit of small lower-layer thickness, the quasigeostrophic model also allows stabilization of otherwise unstable steady flows

by an oscillatory component for like-signed isopycnic and topographic slopes. For opposite signed slopes, effects of the oscillatory component on flow stability can only occur for increasingly low frequencies, in the long wave domain, as the lower-layer thickness decreases to zero. For subcritical flows with low-frequency pulsation, the slow time evolution of the variables is governed by a Mathieu equation.

There are interesting extensions of this work:

- 1) *The case of a frontal geostrophic surface flow.* For such flows, the dynamical equations most often retain cubic terms and lead to differential equations with nonconstant coefficients for the linear stability problem.
- 2) *The application of the present theory to the Mediterranean water outflow in the Gulf of Cadiz, a frontal current propagating on the Iberian continental slope.* In the wavelet analysis of current-meter recordings of this outflow upstream of Portimão Canyon, Serra (2004) has identified oscillations with periods ranging between 6 and 14 days. Future work will quantify how much this oscillation can modify the baroclinic instability of this flow near Portimão Canyon, and if it may control meddy formation.

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