Summer school
Special values of L-functions

Contents

1. Aim of the summer school

The summer school will be aimed at Ph. D. students and young researchers. The aim is to give an overview of Beilinson's conjectures and its refinement by Bloch and Kato on special values of L functions, relying on the abundant literature on the subject. More particularly, we will focus on the classical computations which justify the conjectures as well as on the motivic theory which constitutes the theoretical basis.

2. Abstract of courses

A. **L-functions and Galois cohomology.** (2 talks, J. Johnson-Leung)

The aim of this course is to introduce the material on L-functions and Galois cohomology which is needed to formulate the Bloch-Kato conjecture. More precisely, it will cover the following topics :

- Definition of L-functions associated to (étale realizations of) motives. Possibly, it could also cover equivariant L-functions (i.e. associated to motives with commutative coefficients).
- Definition of Fontaine's period rings.
- Cohomology of local Galois representations : definition of H^1 and its subgroup H^1_f.
- Definition of local Tamagawa numbers.

References. [Ser94], [BK90, §3-4], [FPR94, Chap. I, §3-4].

Pre-requisite. Étale cohomology.

B. **Regulators and motives.** (3 talks, J. Wildeshaus)

The first aim of this course is to present Beilinson's regulator, from rational motivic cohomology to Deligne cohomology. The second aim is to introduce the audience to the theory of motivic complexes conjectured by Beilinson and developed by Voevodsky.

B.1. *Grothendieck motives and motivic complexes.*

B.2. *Motivic complexes and homotopy theory.*

B.3. *Deligne cohomology and regulator.*

References. [VSF00], [MVW06], [EV88], [Sou86].

Pre-requisite. Basic algebraic geometry ([Har77] or [DLØ$^+$07, Levine]), intersection theory ([Ser65]), sheaf theory ([Mil80] or [DLØ$^+$07, Levine]), basics on algebraic K-theory.

C. **Conjectures on special values.** (3 talks, F. Brunault and O. Fouquet)

The aim of this course is to formulate the Tamagawa Number Conjecture (TNC) on special values of L-functions by Beilinson, Bloch-Kato, Fontaine-Perrin-Riou..., building on courses A and B. We will give examples and a survey of known results. We will also cover the equivariant refinement (ETNC) by Kato, Burns-Flach.

C.1. *Formulation of the Tamagawa Number Conjecture (TNC)..*

C.2. *Examples and known results of TNC for number fields.*

C.3. *The Equivariant Tamagawa Number Conjecture (ETNC). Examples and known results for elliptic curves.*

References. [Ram89], [DS91], [Sch91], [Nek94], [BK90], [FPR94], [BF01], [Fla04], [Bel].

Pre-requisite. Modular forms.

D. **Modular aspects.** (2 talks, O. Fouquet)

This course will highlight on recent results on the Bloch-Kato conjectures in the modular world, building on Kato's Euler system.

D.1. *Kato's Euler system associated to a modular form.*

D.2. *Gealy's work on the Bloch-Kato conjecture for modular forms.*

References. D1 : [Kat04] ; D2 : [Gea05].

Pre-requisite. Modular forms.

References

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[Ram89] Dinakar Ramakrishnan, *Regulators, algebraic cycles, and values of L-functions*, Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), Contemp. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 183–310. MR 991982 (90e:11094)

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