AROUND THE GYSIN TRIANGLE I

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Abstract. In [FSV00, chap. 5], V. Voevodsky introduces the Gysin triangle associated with a closed immersion i between smooth schemes. This triangle contains the Gysin morphism associated with i but also the residue morphism.

In [Dég04] and [Dég08b], we started a study of the Gysin triangle and especially its functoriality. In this article, we complete this study by proving notably the functoriality of the Gysin morphism of a closed immersion. This allows us to define a general Gysin morphism attached to a projective morphism between smooth schemes which we study further. As an illustration, we deduce a direct proof of duality for motives of projective smooth schemes.

Finally, this study also involves the residue morphisms. Indeed formulas with the Gysin morphisms of closed immersions have their counterpart for the corresponding residue morphisms. We exploit these formulas in a computation of the E_1-differentials of the coniveau spectral sequence analog to that of Quillen in K-theory and deduce results on the coniveau spectral sequence associated with realization functors.

Introduction

This article is an extension of previous works of the author on the Gysin triangle, [Dég04] and [Dég08b], in the setting of triangulated mixed motives. Recall that to a closed immersion $i : Z \to X$ of codimension n between smooth schemes over a perfect field k is associated a distinguished triangle

$$M(X-Z) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} M(X-Z)[1]$$

in the triangulated category $DM_{gm}^{eff}(k)$. Its construction is given in section 1.2. The original point in the study of *op. cit.* is that the well-known formulas involving the Gysin morphism i^* – for example the projection formula and the excess intersection formula for Chow groups – also correspond to formulas involving the residue morphism $\partial_{X,Z}$. Indeed, they fit in a general study of the functoriality of the Gysin triangle, which is recalled in proposition 1.19.

The main technical result which we obtain here, Theorem 1.34, is the compatibility of the Gysin morphism i^* with composition, but, as explained previously, it also gives formulas for the residue morphism. We quote it in this introduction:

Theorem. *Let X be a smooth scheme, Y (resp. Y') be a smooth closed subscheme of X of pure codimension n (resp. m). Assume the reduced scheme Z associated with $Y \cap Y'$ is smooth of pure codimension d. Put $Y_0 = Y - Z$, $Y'_0 = Y' - Z$, $X_0 = X - Y \cup Y'$.*

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Then the following diagram, with i,j,k,l,i′ the evident closed immersions, is commutative :

$$\begin{tikzcd}
M(X) \arrow[r, "j^*"] \arrow[d, "i^*"'] \arrow[dr, phantom, "(1)"] & M(Y')(m)[2m] \arrow[r, "\partial_{X,Y'}"] \arrow[d, "k^*"] \arrow[dr, phantom, "(2)"] & M(X-Y')[1] \arrow[d, "(i')^*"] \\
M(Y)(n)[2n] \arrow[r, "l^*"'] & M(Z)(d)[2d] \arrow[r, "\partial_{Y,Z}"] \arrow[d, "\partial_{Y',Z}"] \arrow[dr, phantom, "(3)"] & M(Y_0)(n)[2n+1] \arrow[d, "\partial_{X_0,Y_0}"] \\
& M(Y_0')(m)[2m+1] \arrow[r, "-\partial_{X_0,Y_0'}"'] & M(X_0)[2].
\end{tikzcd}$$

Whereas formulas (1) and (2) give the functoriality of the Gysin triangle with respect to the Gysin morphism, formula (3) is specific to the residue morphism and analog to the change of variable theorem for the residue of differential forms.

We use this result to construct the Gysin morphism $f^* : M(X) \to M(Y)(d)[2d]$ of a projective morphism $f : Y \to X$ of pure codimension d, by considering a factorization of f into a closed immersion and the projection of a projective bundle. Indeed, in the case of a projective bundle $p : P \to X$ of constant rank n, the Gysin morphism $p^* : M(X) \to M(P)(-n)[-2n]$ is given by the twist of the canonical embedding through the projective bundle isomorphism (recalled in 1.7):

$$M(P) = \bigoplus_{0 \leq i \leq n} M(X)(i)[2i].$$

The key observation (Proposition 2.2) in the general construction is that, for any section s of P/X, $s^*p^* = 1$. Then we derive easily the following properties of this general Gysin morphism[1]:

(4) For any projective morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$, $(fg)^* = g^*f^*$ (Prop. 2.9).

(5) Consider a cartesian square of smooth schemes

$$\begin{tikzcd}
T \arrow[r, "g"] \arrow[d, "q"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

such that f and g are projective of the same codimension.
Then, $f^*p_* = q_*g^*$ (Prop. 2.10).

(6) Consider a topologically cartesian square[2] of smooth schemes

$$\begin{tikzcd}
T \arrow[r, "g"] \arrow[d, "j"'] & Z \arrow[d, "i"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

such that f is projective and i is a closed immersion.
Let $h : (Y-T) \to (X-Z)$ be the morphism induced by f.
Then, $h^*\partial_{X,Z} = \partial_{Y,T}g^*$ (Prop. 2.13).

(7) Let X be a smooth scheme and $f : Y \to X$ be an étale cover. Let ${}^tf$ be the finite correspondence from X to Y given by the transpose of (the graph of) f. Then $f^* = ({}^tf)_*$ (Prop. 2.15).[3]

[1] To make these formulas clearer, we do not indicate the shifts and twists on morphisms.
[2] *i.e.* $T = (Y \times_X Z)_{red}$.
[3] The case of an arbitrary finite equidimensional morphism f requires a detailed study of the Gersten resolution and is treated in [Dég09, 7.1]

We also mention a generalization of the formula in point (5). Consider the same square but assume the morphism f (resp. g) is projective of codimension n (resp. m). Let ξ be the *excess vector bundle* over T associated with the latter square, of rank $e = n - m$. Then, $f^* p_* = \big(\mathfrak{c}_e(\xi) \boxtimes q_*\big) \circ g^*$. This formula is analog to the excess intersection formula in the Chow groups (cf [Ful98, 6.6(c)]). The reader is referred to Proposition 2.12 for more details.

A nice application of the general Gysin morphism is the construction of the duality pairings for a smooth projective scheme X of dimension n. Let $p : X \to \mathrm{Spec}(k)$ (resp. $\delta : X \to X \times_k X$) be the canonical projection (resp. diagonal embedding) of X/k. We obtain duality pairings (cf Theorem 2.18)

$$\eta : \mathbb{Z} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{\delta_*} M(X)(-n)[-2n] \otimes M(X)$$
$$\epsilon : M(X) \otimes M(X)(-n)[-2n] \xrightarrow{\delta^*} M(X) \xrightarrow{p_*} \mathbb{Z}.$$

which makes $M(X)(-n)[-2n]$ a *strong dual* of $M(X)$ in the sense of Dold-Puppe. This means that the functor $(M(X)(-n)[-2n] \otimes .)$ is both left and right adjoint to the functor $(. \otimes M(X))$ and implies the Poincaré duality isomorphism between motivic cohomology and motivic homology – the fundamental class is nothing else than the Gysin morphism p^*. Note this duality can already be deduced from Voevodsky's theorem on the existence of a monoidal functor from the category of Chow motives to the triangulated category of mixed motives (cf [FSV00, chap. 5, 2.1.4] for the effective version). But reciprocally, our result allows to recover this functor directly (cf Remark 2.19). Finally, based on an idea of [CD07], we also give another construction of the motive with compact support associated with a smooth scheme (see Definition 2.21). Such a construction already appears in [FSV00, chap. 5] – which can also be applied to the singular case. But ours gives most of the related properties without requiring resolution of singularities. It agrees with that of Voevodsky when resolution of singularities hold.

The remaining part of the article is concerned with the study of the coniveau filtration in the category of motives. In particular, we introduce the notion of a triangulated exact couple (cf Definition 3.1) which allows to study the analog of the coniveau spectral sequence directly inside the category $DM_{gm}(k)$ or rather in its category of pro-objects. We call this analog the *motivic coniveau exact couple* (Definition 3.5). Our principal result is the expression of the corresponding differentials in terms of morphisms of *generic motives* (see section 3.2.1 for a recollection on generic motives and Proposition 3.13 for the computation) – note the key argument is the formula (6) above.

Then we provide a link between this computation and the theory of cycle modules by M. Rost (cf [Ros96]). Consider a Grothendieck abelian category $\mathscr{A}$ and a cohomological functor $H : DM_{gm}(k)^{op} \to \mathscr{A}$. In [Dég08b], we attached to H a family of cycle modules basically defined as the *restriction* of H (up to a shift) to the category of generic motives (see section 4.2 for details). On the other hand, as H defines a twisted cohomology with supports, we can consider the well-known coniveau spectral sequence with coefficients in H. As an application of the previous study, we get a canonical identification of the E_1-term with Rost's cycle complexe with coefficients in the corresponding cycle modules.[4] As a corollary, we get most of the classical results of Bloch-Ogus in the case of the functor H using the theory of Rost.

[4]For an analog of this computation in K-theory, see the proof of [Qui73, 5.14].

Let us mention a nice example which has not yet been considered in the literature. Suppose k has characteristic $p > 0$. Let W be the Witt ring of k, K its fraction field. Consider a smooth scheme X. We denote by $H^*_{crys}(X/W)$ the crystalline cohomology of X defined in [Ber74]. When X is affine, we also consider the Monsky-Washnitzer cohomology $H^*_{MW}(X)$ defined in [MW68]. In the following statement, X is assumed to be proper smooth :

(8) Let $\mathcal{H}^*_{MW}$ be the Zariski sheaf on $\mathscr{S}m(k)$ associated with the presheaf H^*_{MW}. Then $\mathcal{H}^*_{MW}(X)$ is a birational invariant of X.

(9) There exists a spectral sequence
$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X, \mathcal{H}^q_{MW}) \Rightarrow H^{p+q}_{crys}(X/W) \otimes K$$
converging to the filtration $N^p H^i_{crys}(X/W)_K$ generated by the images of the Gysin morphisms
$$H^{i-2q}_{crys}(Y/W)_K \to H^i_{crys}(X/W)_K$$
for any regular alteration of a closed subscheme T of X which is of (pure) codimension $q \geq p$.

(10) When k is separably closed, for any $p \geq 0$, $H^p_{\mathrm{Zar}}(X, \mathcal{H}^p_{MW}) = A^p(X) \otimes K$, group of p-codimensional cycles modulo algebraic equivalence.

The key ingredient for this spectral sequence is the *rigid cohomology* of Berthelot (e.g. [Ber97]) together with its realization $H_{rig} : DM_{gm}(k)^{op} \to K{-}vs$ defined in [CD07]. Remark that point (8) and (9) were already known using the results of [CTHK97][5] but point (10) is new. In fact, we give axioms on a functor H as above so that property (10) holds in the general case – when H is represented by a mixed Weil theory in the sense of [CD07], these axioms can be derived from the usual properties of the non positive cohomology groups (cf Cor. 4.19).

We finish this introduction by mentioning a more general work of the author on the Gysin triangle in an abstract situation (cf [Dég08a]). However, the direct arguments used in this text, notably with the identification of the relevant part of motivic cohomology with Chow groups, make it a clear and usable reference. In fact, it is used in the recent work of Barbieri-Viale and Kahn (cf [BVK08]). Moreover, the computation of the E_1-differentials of the coniveau spectral sequence is used in [Dég09].

The paper is organized as follows. Section 1 contains reminders on the Gysin triangle together with the main technical result (Theorem 1.34). In section 2, we define the Gysin morphism of any projective morphism between smooth schemes and deduce the Poincaré duality pairing. In section 3, we recall the coniveau filtration on a smooth scheme and associate with it the *motivic coniveau exact couple*. The section ends up with the computation of the differentials associated with that exact couple in terms of morphisms of generic motives – recollections on these are given in subsection 3.2.1. Lastly, section 4 relates this computation with the theory of cycle modules through cohomological realizations and gives the analog of Bloch-Ogus results in this setting.

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[5]In the case of rigid cohomology, the only non evident property required by [CTHK97] is Nisnevich excision but this follows from the étale descent theorem proved by [CT03].

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Notations and conventions

We fix a base field k which is assumed to be perfect. The word scheme will stand for any separated k-scheme of finite type, and we will say that a scheme is smooth when it is smooth over the base field. The category of smooth schemes is denoted by $\mathscr{S}m(k)$. Throughout the paper, when we talk about the codimension of a closed immersion, the rank of a projective bundle or the relative dimension of a morphism, we assume it is constant.

Given a vector bundle E over X, and P the associated projective bundle with projection $p : P \to X$, we will call *canonical line bundle* on P the canonical invertible sheaf λ over P characterized by the property that $\lambda \subset p^{-1}(E)$. Similarly, we will call *canonical dual line bundle* on P the dual of λ.

We say that a morphism is *projective* if it admits a factorization into a closed immersion followed by the projection of a projective bundle.[6]

We let $DM_{gm}(k)$ (resp. $DM_{gm}^{eff}(k)$) be the category of geometric motives (resp. effective geometric motives) introduced in [FSV00, chap. 5]. For the result of section 1, we work in the category $DM_{gm}^{eff}(k)$. If X is a smooth scheme, we denote by $M(X)$ the effective motive associated with X in $DM_{gm}^{eff}(k)$. From section 2 to the end of the article, we work in the category $DM_{gm}(k)$. Then $M(X)$ will be the motive associated with X in the category $DM_{gm}(k)$ (through the canonical functor $DM_{gm}^{eff}(k) \to DM_{gm}(k)$).

For a morphism $f : Y \to X$ of smooth schemes, we will simply put $f_* = M(f)$. Moreover for any integer r, we sometimes put $\mathbb{Z}((r)) = \mathbb{Z}(r)[2r]$ in large diagrams. When they are clear from the context (for example in diagrams), we do not indicate twists or shifts on morphisms.

Contents

[6] Beware this is not the convention of [EGA2] unless the aim of the morphism admits an ample line bundle.

1. The Gysin triangle

1.1. **Relative motives.**

Definition 1.1. We call closed (resp. open) pair any couple (X, Z) (resp. (X, U)) such that X is a smooth scheme and Z (resp. U) is a closed (resp. open) subscheme of X.

Let (X, Z) be an arbitrary closed pair. We will say (X, Z) is smooth if Z is smooth. For an integer n, we will say that (X, Z) has codimension n if Z has (pure) codimension n in X.

A morphism of open or closed pairs $(Y, B) \to (X, A)$ is a couple of morphisms (f, g) which fits into the commutative diagram of schemes

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ A \arrow[r, hook] & X. \end{tikzcd}$$

If the pairs are closed, we require also that this square is topologically cartesian[7].

We add the following definitions :

- The morphism (f, g) is said to be cartesian if the above square is cartesian as a square of schemes.
- A morphism (f, g) of closed pairs is said to be excisive if f is étale and g_{red} is an isomorphism.
- A morphism (f, g) of smooth closed pairs is said to be transversal if it is cartesian and the source and target have the same codimension.

We will denote conventionally open pairs as fractions (X/U).

Definition 1.2. Let (X, Z) be a closed pair. We define the relative motive $M_Z(X)$ — sometimes denoted by $M(X/X - Z)$ — associated with (X, Z) to be the class in $DM_{gm}^{eff}(k)$ of the complex

$$... \to 0 \to [X - Z] \to [X] \to 0 \to ...$$

where $[X]$ is in degree 0.

Relative motives are functorial with respect to morphisms of closed pairs. In fact, $M_Z(X)$ is functorial with respect to morphisms of the associated open pair $(X/X - Z)$. For example, if $Z \subset T$ are closed subschemes of X, we get a morphism $M_T(X) \to M_Z(X)$.

If $j : (X - Z) \to X$ denotes the complementary open immersion, we obtain a canonical distinguished triangle in $DM_{gm}^{eff}(k)$:

$$M(X - Z) \xrightarrow{j_*} M(X) \to M_Z(X) \to M(X - Z)[1]. \tag{1.2.a}$$

[7] *i.e.* cartesian as a square of topological spaces ; in other words, $B_{red} = (A \times_X Y)_{red}$.

Remark 1.3. The relative motive in $DM_{gm}^{eff}(k)$ defined here corresponds under the canonical embedding to the relative motive in $DM_{-}^{eff}(k)$ defined in [Dég04, def. 2.2].

The following proposition sums up the basic properties of relative motives. It follows directly from [Dég04, 1.3] using the previous remark. Note moreover that in the category $DM_{gm}^{eff}(k)$, each property is rather clear, except **(Exc)** which follows from the embedding theorem [FSV00, chap. 5, 3.2.6] of Voevodsky.

Proposition 1.4. *Let (X, Z) be a closed pair. The following properties of relative motives hold:*

(Red) Reduction*: If we denote by Z_0 the reduced scheme associated with Z then:*
$$M_Z(X) = M_{Z_0}(X)\,.$$

(Exc) Excision*: If $(f, g) : (Y, T) \to (X, Z)$ is an excisive morphism then $(f, g)_*$ is an isomorphism.*

(MV) Mayer-Vietoris *: If $X = U \cup V$ is an open covering of X then we obtain a canonical distinguished triangle of shape:*
$$\begin{aligned} M_{Z\cap U\cap V}(U \cap V) &\xrightarrow{M(j_U)-M(j_V)} M_{Z\cap U}(U) \oplus M_{Z\cap V}(V) \\ &\xrightarrow{M(i_U)+M(i_V)} M_Z(X) \longrightarrow M_{Z\cap U\cap V}(U \cap V)\,[1]. \end{aligned}$$
The morphism i_U, i_V, j_U, j_V stands for the obvious cartesian morphisms of closed pairs induced by the corresponding canonical open immersions.

(Add) Additivity*: Let Z' be a closed subscheme of X disjoint from Z. Then the morphism induced by the inclusions*
$$M_{Z\sqcup Z'}(X) \to M_Z(X) \oplus M_{Z'}(X)$$
is an isomorphism.

(Htp) Homotopy*: Let $\pi : (\mathbb{A}^1_X, \mathbb{A}^1_Z) \to (X, Z)$ denote the cartesian morphism induced by the projection. Then π_* is an isomorphism.*

1.2. **Purity isomorphism.**

1.5. Consider an integer $i \geq 0$. Recall that the i-th twisted motivic complex over k is defined according to Voevodsky as the *Suslin's singular simplicial complex* of the cokernel of the natural map of sheaves with transfers $\mathbb{Z}^{tr}(\mathbb{A}^i_k - 0) \to \mathbb{Z}^{tr}(\mathbb{A}^i_k)$, shifted by $2i$ degrees on the left (cf [SV00] or [FSV00]). Motivic cohomology of a smooth scheme X in degree $n \in \mathbb{Z}$ and twists i is defined following Beilinson's idea as the Nisnevich hypercohomology groups of this complex which we denote by $H^n_{\mathcal{M}}(X; \mathbb{Z}(i))$. Moreover, there is a natural pairing of complexes $\mathbb{Z}(i) \otimes \mathbb{Z}(j) \to \mathbb{Z}(i + j)$ (cf [SV00]) which induces the product on motivic cohomology.

Recall there exists[8] a canonical isomorphism
$$\epsilon_X : CH^i(X) \xrightarrow{\sim} H^{2i}_{\mathcal{M}}(X; \mathbb{Z}(i)) \tag{1.5.a}$$
which is functorial with respect to pullbacks and compatible with products.

According to [FSV00, chap. 5, 3.2.6], we also get an isomorphism
$$H^n_{\mathcal{M}}(X; \mathbb{Z}(i)) \simeq \mathrm{Hom}_{DM_{gm}^{eff}(k)}(M(X)\,, \mathbb{Z}(i)[n]) \tag{1.5.b}$$

[8] Following Voevodsky, this isomorphism is obtained from the Nisnevich hypercohomology spectral sequence of the complex $\mathbb{Z}(i)$ once we have observed that $\mathrm{H}^q(\mathbb{Z}(i)) = 0$ if $q > i$ and $\mathrm{H}^i(\mathbb{Z}(i))$ is canonically isomorphic with the i-th Milnor unramified cohomology sheaf $\mathcal{K}^M_i$. The compatibility with product and pullback then follows from a careful study (cf for example [Dég02, 8.3.4]).

where $\mathbb{Z}(i)$ on the right hand side stands (by the usual abuse of notation) for the i-th Tate geometric motive. In what follows, we will identify cohomology classes in motivic cohomology with morphisms in $DM_{gm}^{eff}(k)$ according to this isomorphism.

Thus cup-product on motivic cohomology corresponds to a product on morphisms that we describe now. Let X be a smooth scheme, $\delta : X \to X \times_k X$ be the diagonal embedding and $f : M(X) \to \mathcal{M}$, $g : M(X) \to \mathcal{N}$ be two morphisms with target a geometric motive. We define the *exterior product* of f and g, denoted by $f \boxtimes_X g$ or simply $f \boxtimes g$, as the composite

$$M(X) \xrightarrow{\delta_*} M(X) \otimes M(X) \xrightarrow{f \otimes g} \mathcal{M} \otimes \mathcal{N}. \tag{1.5.c}$$

In the case where $\mathcal{M} = \mathbb{Z}(i)[n]$, $\mathcal{N} = \mathbb{Z}(j)[m]$, identifying the tensor product $\mathbb{Z}(i)[n] \otimes \mathbb{Z}(j)[m]$ with $\mathbb{Z}(i+j)[n+m]$ by the canonical isomorphism, the above product corresponds exactly to the cup-product on motivic cohomology.

According to the isomorphism (1.5.a), motivic cohomology admits Chern classes. Thus, applying the isomorphism (1.5.b), we attach to any vector bundle E on a smooth scheme X and any integer $i \geq 0$, the following morphism in $DM_{gm}^{eff}(k)$

$$\mathfrak{c}_i(E) : M(X) \to \mathbb{Z}(i)[2i] \tag{1.5.d}$$

which corresponds under the preceding isomorphisms to the i-th Chern class of E in the Chow group. For short, we call this morphism the *i-th motivic Chern class* of E.

Remark 1.6. According to our construction, any formula in the Chow group involving pullbacks and intersections of Chern classes induces a corresponding formula for the morphisms of type (1.5.d).

1.7. We finally recall the projective bundle theorem (cf [FSV00, chap. 5, 3.5.1]). Let P be a projective bundle of rank n over a smooth scheme X, λ its canonical dual line bundle and $p : P \to X$ the canonical projection. The projective bundle theorem of Voevodsky says that the morphism

$$M(P) \xrightarrow{\sum_{i \leq n} \mathfrak{c}_1(\lambda)^i \boxtimes p_*} \bigoplus_{i=0}^{n} M(X)((i)) \tag{1.7.a}$$

is an isomorphism.

Thus, we can associate with P a family of split monomorphisms indexed by an integer $r \in [0, n]$ corresponding to the decomposition of its motive :

$$\mathfrak{l}_r(P) : M(X)(r)[2r] \to \oplus_{i \leq n} M(X)(i)[2i] \to M(P)\,. \tag{1.7.b}$$

The following lemma will be a key point in the theory of the Gysin morphism:

Lemma 1.8. *Consider the notations introduced above.*

Let $x \in CH^n(P)$ be a cycle class and $x_i \in CH^{n-i}(P)$ be cycle classes such that

$$x = \sum_{i=0}^{n} p^*(x_i).c_1(\lambda)^i. \tag{1.8.a}$$

Consider an integer $i \in [0, n]$ and the following morphisms in $DM_{gm}^{eff}(k)$

$$\mathfrak{x} : M(X) \to \mathbb{Z}(n)[2n]$$
$$\mathfrak{x}_i : M(X) \to \mathbb{Z}(n-i)[2(n-i)]$$

associated respectively with x and x_i through the isomorphisms (1.5.a) *and* (1.5.b).

Then we get the equality of morphisms $M(X)\,(i)[2i] \to \mathbb{Z}(r)[2r]$ in $DM_{gm}^{eff}(k)$:

$$\mathfrak{x} \circ \mathfrak{l}_i(P) = \mathfrak{x}_i(i)[2i].$$

Proof. Taking care of Remark 1.6, the equality (1.8.a) induces the following equality of morphisms $M(P) \to \mathbb{Z}(r)[2r]$:

$$\mathfrak{x} = \sum_{i=0}^{r} \mathfrak{c}_1(\lambda)^i \boxtimes (\mathfrak{x}_i \circ p_*) = \textstyle\sum_{i=0}^{r} \big[\mathfrak{x}_i(i)[2i]\big] \circ \mathfrak{c}_1(\lambda)^i \boxtimes p_*.$$

The second equality follows from the definition of the exterior cup product (formula (1.5.b)). Thus, the definition of $\mathfrak{l}_i(P)$ and the formula (1.7.a) for the projective bundle isomorphism on motives allow to conclude. □

Remark 1.9. Note in particular that we deduce from the preceding lemma the following weak form of the cancellation theorem of Voevodsky [Voe02]: for any smooth scheme X and any non negative integers (n, i) such that $i \leq n$, the morphism

$$\mathrm{Hom}_{DM_{gm}^{eff}(k)}(M(X), \mathbb{Z}(n-i)[2(n-i)]) \to \mathrm{Hom}_{DM_{gm}^{eff}(k)}(M(X)(i)[2i], \mathbb{Z}(n)[2n]),$$
$$\phi \mapsto \phi(i)[2i]$$

is an isomorphism.

Lemma 1.10. *Let X be a smooth scheme and E/X be a vector bundle. Consider the projective completion P of E/X, the closed pair (P, X) corresponding to the canonical section of P/X and the complement open immersion $j : U \to P$. Then the distinguished triangle* (1.2.a) *associated with* (P, X)

$$M(U) \xrightarrow{j_*} M(P) \xrightarrow{\pi_P} M_X(P) \to M(U)[1] \tag{1.10.a}$$

is split.

Proof. Recall $P = \mathbb{P}(E \oplus \mathbb{A}^1_X)$. Let $\nu : \mathbb{P}(E) \to P$ be the embedding associated with the monomorphism of vector bundles $E \to E \oplus \mathbb{A}^1_X$. The closed immersion i factors through the open immersion $j : U \to P$. Let us denote finally by L the canonical line bundle on $\mathbb{P}(E)$ and by s_0 its zero section. Then, according to [EGA2, §8], there exists an isomorphism of schemes $\epsilon : L \to U$ such that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\epsilon"] & U \arrow[d, "j"] \\ \mathbb{P}(E) \arrow[u, "s_0"] \arrow[r, "\nu"] & P. \end{tikzcd}$$

Thus the morphism j_* is isomorphic in $DM_{gm}^{eff}(k)$ to the morphism

$$\nu_* : M(\mathbb{P}(E)) \to M(P)$$

which is a split monomorphism according to the respective projective bundle isomorphisms for $\mathbb{P}(E)/X$ and P/X. □

1.11. Consider a smooth closed pair (X, Z). Let $N_Z X$ (resp. $B_Z X$) be the normal bundle (resp. blow-up) of (X, Z) and $P_Z X$ be the projective completion of $N_Z X$. We denote by $B_Z(\mathbb{A}^1_X)$ the blow-up of $\mathbb{A}^1_X$ with center $\{0\} \times Z$. It contains as a closed subscheme the trivial blow-up $\mathbb{A}^1_Z = B_Z(\mathbb{A}^1_Z)$. We consider the closed pair $(B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z)$ over $\mathbb{A}^1_k$. Its fiber over 1 is the closed pair (X, Z) and its fiber over 0 is $(B_Z X \cup P_Z X, Z)$. Thus we can consider the following deformation diagram :

$$(X, Z) \xrightarrow{\bar{\sigma}_1} (B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z) \xleftarrow{\bar{\sigma}_0} (P_Z X, Z). \tag{1.11.a}$$

This diagram is functorial in (X, Z) with respect to cartesian morphisms of closed pairs. Note finally that, on the closed subschemes of each closed pair, $\bar{\sigma}_0$ (resp. $\bar{\sigma}_1$) is the 0-section (resp. 1-section) of $\mathbb{A}^1_Z/Z$.

The existence statement in the following proposition appears already in [Dég08b, 2.2.5] but the uniqueness statement is new :

Proposition 1.12. *Let n be a natural integer.*
There exist a unique family of isomorphisms of the form

$$\mathfrak{p}_{(X,Z)} : M_Z(X) \to M(Z)(n)[2n]$$

indexed by smooth closed pairs of codimension n such that :

(1) *for every cartesian morphism $(f, g) : (Y, T) \to (X, Z)$ of smooth closed pairs of codimension n, the following diagram is commutative :*

$$\begin{tikzcd} M_T(Y) \arrow[r, "(f,g)_*"] \arrow[d, "\mathfrak{p}_{(Y,T)}"'] & M_Z(X) \arrow[d, "\mathfrak{p}_{(X,Z)}"] \\ M(T)(n)[2n] \arrow[r, "g_*(n)[2n]"] & M(Z)(n)[2n]. \end{tikzcd}$$

(2) *Let X be a smooth scheme and P be the projective completion of a vector bundle E/X of rank n. Consider the closed pair (P, X) corresponding to the 0-section of E/X. Then $\mathfrak{p}_{(P,X)}$ is the inverse of the following morphism*

$$M(X)(n)[2n] \xrightarrow{\mathfrak{l}_n(P)} M(P) \xrightarrow{\pi_P} M_X(P)\,.$$

where $\mathfrak{l}_n(P)$ is the monomorphism of (1.7.b) *and π_P is the epimorphism of the split distinguished triangle* (1.10.a).

Proof. Uniqueness : Consider a smooth closed pair (X, Z) of codimension n.

Applying property (1) to the deformation diagram (1.11.a), we obtain the commutative diagram :

$$\begin{tikzcd} M(X,Z) \arrow[r, "\bar\sigma_{1*}"] \arrow[d, "\mathfrak{p}_{(X,Z)}"'] & M\big(B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z\big) \arrow[d, "\mathfrak{p}_{(B_Z(\mathbb{A}^1_X),\mathbb{A}^1_Z)}"] & M(P_Z X, Z) \arrow[l, "\bar\sigma_{0*}"'] \arrow[d, "\mathfrak{p}_{(P_Z X,Z)}"] \\ M(Z)(n)[2n] \arrow[r, "s_{1*}"] & M(\mathbb{A}^1_Z)(n)[2n] & M(Z)(n)[2n] \arrow[l, "s_{0*}"'] \end{tikzcd}$$

Using homotopy invariance, s_{0*} and s_{1*} are isomorphisms. Thus in this diagram, all the morphisms are isomorphisms. Now, the second property of the purity isomorphisms determines uniquely $\mathfrak{p}_{(P_Z X,Z)}$, thus $\mathfrak{p}_{(X,Z)}$ is also uniquely determined.

For the existence part, we refer the reader to [Dég08b], section 2.2. □

Remark 1.13. The second point of the above proposition appears as a normalization condition. It will be reinforced later (cf Remark 2.3).

Definition 1.14. Let (X, Z) be a smooth closed pair of codimension n. Denote by j (resp. i) the open immersion $(X - Z) \to X$ (resp. closed immersion $Z \to X$).

With the notation of the preceding proposition, the morphism $\mathfrak{p}_{(X,Z)}$ will be called the *purity isomorphism* associated with (X, Z).

Using this isomorphism, we deduce from the distinguished triangle (1.2.a) the following distinguished triangle in $DM^{eff}_{gm}(k)$, called the Gysin triangle of (X, Z)

$$M(X - Z) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} M(X - Z)\,[1].$$

The morphism $\partial_{(X,Z)}$ (resp. i^*) is called the *residue* (resp. *Gysin morphism*) associated with (X, Z) (resp. i). Sometimes we use the notation $\partial_i = \partial_{(X,Z)}$.

Example 1.15. Consider a smooth scheme X and a vector bundle E/X of rank n. Let P be the projective completion of E, λ be its canonical dual invertible sheaf and $p : P \to X$ be its canonical projection. Consider the canonical section $s : X \to P$ of P/X.

We define the Thom class of E in $CH^n(P)$ as the class

$$t(E) = \sum_{i=0}^{n} p^*(c_{n-i}(E)).c_1(\lambda)^i.$$

It corresponds according to paragraph 1.5 to a morphism $\mathfrak{t}(E) : M(P) \to \mathbb{Z}(n)[2n]$.

Consider the notations of Lemma 1.10 together with the definition of the exterior product (1.5.c). Because the triangle (1.10.a) is split and because $j^*(t(E)) = 0$, the morphism

$$\mathfrak{t}(E) \boxtimes_P p_* : M(P) \to M(X)\,(n)[2n]$$

factors uniquely through π_P:

$$M(P) \xrightarrow{\pi_P} M_X(P) \xrightarrow{\epsilon_P} M(X)\,(n)[2n].$$

Because the coefficient of $c_1(\lambda)^n$ in $t(E)$ is 1, we deduce from Lemma 1.8 that $\epsilon_P \circ \mathfrak{p}_{(P,X)}^{-1} = 1$. Thus, according to the previous definition, we obtain the following formula[9]:

$$s^* = \mathfrak{t}(E) \boxtimes_P p_*. \tag{1.15.a}$$

Remark 1.16. Our Gysin triangle agrees with that of [FSV00], chap. 5, prop. 3.5.4. Indeed, in the proof of 3.5.4, Voevodsky constructs an isomorphism which he denotes by $\alpha_{(X,Z)}$. He then uses it as we use the purity isomorphism to construct his triangle. It is not hard to check that this isomorphism $\alpha_{(X,Z)}$ satisfies the two conditions of Proposition 1.12 and thus coincides with the purity isomorphism from the uniqueness statement.

1.3. **Base change formulas.** This subsection is devoted to recall some results we obtained previously in [Dég04] and [Dég08b] about the following type of morphism :

Definition 1.17. Let (X, Z) (resp. (Y, T)) be a smooth closed pair of codimension n (resp. m). Let $(f, g) : (Y, T) \to (X, Z)$ be a morphism of closed pairs.

We define the morphism $(f, g)_!$ as the following composite :

$$M(T)(m)[2m] \xrightarrow{\mathfrak{p}_{(Y,T)}^{-1}} M(Y, T) \xrightarrow{(f,g)_*} M(X, Z) \xrightarrow{\mathfrak{p}_{(X,Z)}} M(Z)(n)[2n].$$

In the situation of this definition, let $i : Z \to X$ and $k : T \to Y$ be the obvious closed embeddings and $h : (Y - T) \to (X - Z)$ be the restriction of f. Then we obtain from our definitions the following commutative diagram :

$$\begin{tikzcd} M(Y-T) \arrow[r] \arrow[d] & M(Y) \arrow[r, "j^*"] \arrow[d, "f_*"] \arrow[dr, phantom, "(1)"] & M(T)(m)[2m] \arrow[r, "\partial_{Y,T}"] \arrow[d, "(f,g)_!"] \arrow[dr, phantom, "(2)"] & M(Y-T)\,[1] \arrow[d, "h_*"] \\ M(X-Z) \arrow[r] & M(X) \arrow[r, "i^*"] & M(Z)(n)[2n] \arrow[r, "\partial_{X,Z}"] & M(X-Z)\,[1] \end{tikzcd} \tag{1.17.a}$$

The commutativity of square (1) corresponds to a *refined projection formula*. The word refined is inspired by the terminology "refined Gysin morphism" of Fulton in [Ful98]. By contrast, the commutativity of square (2) involves motivic cohomology rather than Chow groups.

[9] This is the analog of the well-known formula in Chow theory: for any cycle class $x \in CH^*(Z)$, $s_*(x) = t(E).p^*(x)$.

1.18. Let T (resp. T') be a closed subscheme of a scheme Y with defining ideal $\mathcal{J}$ (resp. $\mathcal{J}'$). We will say that a closed immersion $i : T \to T'$ is an *exact thickening of order r in Y* if $\mathcal{J}' = \mathcal{J}^r$. We recall to the reader the following formulas obtained in [Dég04, 3.1, 3.3] :

Proposition 1.19. *Let (X, Z) and (Y, T) be smooth closed pairs of codimension n and m respectively. Let $(f, g) : (Y, T) \to (X, Z)$ be a morphism of closed pairs.*

(1) (Transversal case) *If (f, g) is transversal (which implies $n = m$) then $(f, g)_! = g_*(n)[2n]$.*
(2) (Excess intersection) *If (f, g) is cartesian, we put $e = n - m$ and $\xi = g^*N_Z X/N_T Y$. Then $(f, g)_! = \mathfrak{c}_e(\xi)\boxtimes_T g_*(m)[2m]$.*
(3) (Ramification case) *If $n = m = 1$ and the canonical closed immersion $T \to Z\times_X Y$ is an exact thickening of order r in Y, then $(f, g)_! = r.g_*(1)[2]$.*

Remark 1.20. In the article [Dég08a, 4.23], the case (3) has been generalized to any codimension $n = m$. In this generality, the integer r is simply the geometric multiplicity of $Z \times_X Y$.

Corollary 1.21. *Let X be a smooth scheme such that $X = X_1 \sqcup X_2$. Consider the open and closed immersion $\nu_i : X_i \to X$ for $i = 1, 2$.*

Then the isomorphism $(\nu_{1*}, \nu_{2*}) : M(X_1) \oplus M(X_2) \to M(X)$ admits as an inverse isomorphism the map $(\nu_1^*, \nu_2^*) : M(X) \to M(X_1) \oplus M(X_2)$.

Proof. In fact, according to the first point of the above proposition, we get the following relations for $i = 1, 2$: $\nu_i^*\nu_{i*} = 1$, $\nu_{2-i}^*\nu_{i*} = 0$. This, together with the fact (ν_{1*}, ν_{2*}) is an isomorphism, allows to conclude. □

Another application of the preceding proposition is the following projection formula:

Corollary 1.22. *Let (X, Z) be a smooth pair of codimension n and $i : Z \to X$ be the corresponding closed immersion.*

Then, $(1_Z\boxtimes_Z i_*) \circ i^* = i^*\boxtimes_X 1_X : M(X) \to M(Z) \otimes M(X)\,(n)[2n]$.

Proof. Just apply the above formula to the cartesian morphism $(X, Z) \to (X \times X, Z \times X)$ induced by the diagonal embedding of X. The only thing left to check is that $(i \times 1_X)^* = i^* \otimes 1$, which was done in [Dég08b, 2.6.1]. □

Remark 1.23. In the above statement, we have loosely identified the motive $M(Z)\otimes M(X)\,(n)[2n]$ with $(M(Z)\,(n)[2n]) \otimes M(X)$ through the canonical isomorphism. This will not have any consequences in the present article. On the contrary in [Dég08b], we must be attentive to this isomorphism which may result in a change of sign (cf remark 2.6.2 of *loc. cit.*).

Another corollary of the preceding proposition is the following analog of the self-intersection formula:

Corollary 1.24. *Let (X, Z) be a smooth closed pair of codimension n with normal bundle $N_Z X$. If i denotes the corresponding closed immersion, we obtain the following equality:*

$$i^*i_* = \mathfrak{c}_n(N_Z X)\boxtimes_Z 1_{Z*}.$$

Indeed it follows from the transversal case of the preceding proposition applied to the cartesian morphism $(i, 1_Z) : (Z, Z) \to (X, Z)$ and from the commutativity of square (1) in diagram (1.17.a).

Example 1.25. Consider a vector bundle $p : E \to X$ of rank n. Let s_0 be its zero section. According to the homotopy property in $DM_{gm}^{eff}(k)$, we get $s_{0*}p_* = 1$. Thus, the preceding corollary applied to s_0 implies the following formula:

(1.25.a) $$s_0^* = \mathfrak{c}_n(p^{-1}E) \boxtimes_E p_*.$$

Moreover, the Gysin triangle associated with s_0 together with the isomorphism s_{0*} gives the following distinguished triangle:

$$M(E^\times) \longrightarrow M(E) \xrightarrow{\mathfrak{c}_n(E) \boxtimes_X 1_{X*}} M(X)(n)[2n] \xrightarrow{\partial_{E,X} \circ s_{0*}} M(E^\times)[1]$$

which we call the *Euler triangle* of E/X.[10]

Definition 1.26. Let (X, Z) be a smooth closed pair of codimension n and $i : Z \to X$ be the corresponding closed immersion. Let $\pi : Z \to \mathrm{Spec}(k)$ be the structural morphism of Z.

We define the *motivic fundamental class* of Z in X as the following composite map:

$$\eta_X(Z) : M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\pi_*} \mathbb{Z}(n)[2n].$$

Example 1.27. Let X be a smooth scheme and $p : E \to X$ be a vector bundle of rank n. According to formula (1.25.a), the motivic fundamental class of the zero section of E/X is:

(1.27.a) $$\eta_E(X) = \mathfrak{c}_n(p^{-1}E).$$

Let P/X be the projective completion of E/X. According to formula (1.15.a), the motivic fundamental class of the canonical section of P/X is:

(1.27.b) $$\eta_P(X) = \mathfrak{t}(E).$$

Remark 1.28. If we use the cancellation theorem of Voevodsky (see [Voe02] or use more directly Remark 1.9), the Gysin map i^* induces a canonical pushout[11]:

$$i_* : H^s_{\mathcal{M}}(Z; \mathbb{Z}(t)) \to H^{s+2n}_{\mathcal{M}}(X; \mathbb{Z}(t+n)).$$

Then, through the isomorphism (1.5.b), we get the equality $\eta_X(Z) = i_*(1)$, where 1 stands for the unit of the (bigraded) cohomology ring $H^*_{\mathcal{M}}(Z; \mathbb{Z}(*))$. This motivates our terminology.

According to the computations of the previous example, the following lemma is a generalization of formulas (1.15.a) and (1.25.a):

Lemma 1.29. *Let (X, Z) be a smooth closed pair of codimension n and $i : Z \to X$ be the corresponding closed immersion. Assume that i admits a retraction $p : X \to Z$.*

Then $i^* = \eta_X(Z) \boxtimes_X p_*$.

Proof. Let $\pi : Z \to \mathrm{Spec}(k)$ be the structural morphism. According to formula (1.5.c), we deduce that $\pi_* \boxtimes_Z 1_{Z*} = 1_{Z*}$. The lemma follows from the following computation:

$$\begin{aligned} i^* &\overset{(1)}{=} [\pi_* \boxtimes_Z (p_* i_*)] \circ i^* = (\pi_* \otimes p_*)(1_{Z*} \boxtimes_Z i_*) \circ i^* \overset{(2)}{=} (\pi_* \otimes p_*)(i^* \boxtimes_X 1_{Z*}) \\ &= \eta_X(Z) \boxtimes_X p_* \end{aligned}$$

where equality (1) is justified by the preceding remark and the relation $pi = 1_Z$ whereas equality (2) is in fact Corollary 1.22. □

[10] It is the analog of the Euler long exact sequence associated with E/X in cohomology.

[11] We prove in [Dég09, lem. 3.3] that this pushout coincides through the isomorphism (1.5.a) with the usual pushout in Chow theory.

Lemma 1.30. *Let X be a smooth scheme and E/X be a vector bundle of rank n. Let s (resp. s_0) be a section (resp. the zero section) of E/X. Assume that s is transversal to s_0 and consider the cartesian square:*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "k"'] & X \arrow[d, "s"] \\ X \arrow[r, "s_0"] & E \end{tikzcd}$$

Then the motivic fundamental class of i is:

$$\eta_X(Z) = \mathfrak{c}_n(E).$$

Proof. Let π (resp. π') be the structural morphism of Z (resp. X). The lemma follows from the computation below:

$$\eta_X(Z) = \pi_* i^* = \pi'_* k_* i^* \overset{(1)}{=} \pi'_* s_0^* s_* \overset{(2)}{=} \mathfrak{c}_n(p^{-1}E) \circ s_* \overset{(3)}{=} \mathfrak{c}_n(E) \circ p_* \circ s_* = \mathfrak{c}_n(E).$$

Equality (1) follows from Proposition 1.19, equality (2) from the formula (1.27.a) and equality (3) from Remark 1.6. $\square$

Example 1.31. Let E/X be a vector bundle and $p : P \to X$ be its projective completion. Let λ be the canonical dual line bundle on P. Put $F = \lambda \otimes_P p^{-1}(E)$ as a vector bundle over P. According to our conventions, we get canonical embedding $\lambda^\vee \subset p^{-1}(E \oplus \mathbb{A}^1_X)$. Then the following composite map

$$\lambda^\vee \to p^{-1}(E \oplus \mathbb{A}^1_X) \to p^{-1}(E)$$

corresponds to a section σ of F/P. One can check that σ is transversal to the zero section s_0^F of F/P and that the following square is cartesian:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d] & P \arrow[d, "\sigma"] \\ P \arrow[r, "s_0^F"] & F \end{tikzcd}$$

where s is the canonical section of P/X. Thus the preceding corollary gives the following equality: $\eta_P(X) = \mathfrak{c}_n(F)$.[12]

1.4. **Composition of Gysin triangles.** We first establish lemmas needed for the main theorem. First of all, using the projection formula in the transversal case (cf 1.19) and the compatibility of Chern classes with pullbacks, we obtain easily the following result:

Lemma 1.32. *Let (Y, Z) be a smooth pair of codimension m and P/Y be a projective bundle of dimension n. We put $V = Y - Z$ and consider the following cartesian squares :*

$$\begin{tikzcd} P_V \arrow[r, "\nu"] \arrow[d, "p_V"'] & P \arrow[d, "p"'] & P_Z \arrow[l, "\iota"'] \arrow[d, "p_Z"] \\ V \arrow[r, "j"] & Y & Z \arrow[l, "i"'] \end{tikzcd}$$

Finally, we consider the canonical line bundle λ (resp. λ_V, λ_Z) on P (resp. P_V, P_Z).

[12]In fact, from the definition of the Thom class (Example 1.15), one can check directly the equality $c_n(F) = t(E)$ in the Chow group $CH^n(P)$: the computation we get in this example shows that our (sign) conventions are coherent.

Then, for any integer $r \in [0, n]$, the following diagram is commutative

$$\begin{tikzcd}
M(P_V) \arrow[r, "\nu_*"] \arrow[d, "\mathfrak{c}_1(\lambda_V)^r \boxtimes p_{V*}"'] & M(P) \arrow[r, "\iota^*"] \arrow[d, "\mathfrak{c}_1(\lambda)^r \boxtimes p_*"'] & M(P_Z)((m)) \arrow[r, "\partial_\iota"] \arrow[d, "\mathfrak{c}_1(\lambda_Z)^r \boxtimes p_{Z*}"'] & M(P_V)\,[1] \arrow[d, "\mathfrak{c}_1(\lambda_V)^r \boxtimes p_{V*}[1]"'] \\
M(V)((r)) \arrow[r, "j_*"] & M(Y)((r)) \arrow[r, "i^*"] & M(Z)((r+m)) \arrow[r, "\partial_i"] & M(V)((r))[1].
\end{tikzcd}$$

The next lemma will be in fact the crucial case in the proof of the next theorem.

Lemma 1.33. *Let X be a smooth scheme and E/X (resp. E'/X) a vector bundle of rank n (resp. m). Let P (resp. P') be the projective completion of E/X (resp. E'/X) and i (resp. i') its canonical section.*

We put $R = P \times_X P'$ and consider the closed immersions:

$$i : X \to P, j : P \to R, k : X \to P$$

where $j = P \times_X i'$ and $k = (i, i')$. Then $k^* = i^* j^*$.

Proof. We consider the following canonical morphisms:

$$\begin{tikzcd}
R \arrow[r, "q"] \arrow[d, "q'"'] \arrow[dr, "\pi"] & P' \arrow[d, "p'"] \\
P \arrow[r, "p"'] & X
\end{tikzcd}$$

According to Lemma 1.29, we obtain

$$i^* = \eta_P(X) \boxtimes_P p_*, \quad j^* = \eta_R(P) \boxtimes_R q'_*, \quad k^* = \eta_R(X) \boxtimes_P \pi_*.$$

Applying the first case of Proposition 1.19 to the cartesian morphism of closed pairs $(q', p') : (R, P') \to (P, X)$, we obtain the relation:

$$\eta_P(X) \circ q'_* = \eta_R(P').$$

Together with the preceding computations, it implies the following equality:

$$i^* j^* = \eta_R(P) \boxtimes_P \eta_R(P') \boxtimes_P \pi_*.$$

Thus we are reduced to prove the relation:

$$\eta_R(X) = \eta_R(P) \boxtimes_R \eta_R(P'). \tag{1.33.a}$$

Consider the notations of Example 1.31 applied to the case of E/X (resp. E'/X): we get a vector bundle F/P (resp. F'/P) of rank n (resp. m) such that:

$$\eta_P(X) = \mathfrak{c}_n(F),$$
$$\text{resp. } \eta_{P'}(X) = \mathfrak{c}_m(F').$$

Let σ (resp. σ') be the section of F/P (resp. F'/P') constructed in *loc. cit.* Consider the vector bundle over R defined as:

$$G = F \times_X F' = q'^{-1}(F) \oplus q^{-1}(F').$$

We get a section $(\sigma \times_X \sigma')$ of G/P which is transversal to the zero section s_0^G and such that the following square is cartesian:

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d] & R \arrow[d, "\sigma \times_X \sigma'"] \\
R \arrow[r, "s_0^G"] & G.
\end{tikzcd}$$

Thus, according to Lemma 1.30, we obtain:

$$\eta_R(X) = \mathfrak{c}_{n+m}(G).$$

The relation (1.33.a) now follows from Remark 1.6 and the equality

$$c_{n+m}(G) = q'^*(c_n(F)).q^*(c_m(F'))$$

in $CH^{n+m}(R)$. □

Theorem 1.34. *Consider a topologically cartesian square of smooth schemes*

$$\begin{tikzcd} Z \arrow[r, "k"] \arrow[d, "l"'] & Y' \arrow[d, "j"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

such that i,j,k,l are closed immersions of respective pure codimensions n, m, s, t. We put $d = n + t = m + s$ and let $i' : (Y - Z) \to (X - Y')$, $j' : (Y' - Z) \to (X - Y)$ be the closed immersion respectively induced by i, j.

Then the following diagram is commutative :

$$\begin{tikzcd} M(X) \arrow[r, "j^*"] \arrow[d, "i^*"'] \arrow[dr, phantom, "(1)"] & M(Y')((m)) \arrow[r, "\partial_j"] \arrow[d, "k^*"] \arrow[dr, phantom, "(2)"] & M(X - Y')\,[1] \arrow[d, "(i')^*"] \\ M(Y)((n)) \arrow[r, "l^*"'] & M(Z)((d)) \arrow[r, "\partial_l"] \arrow[d, "\partial_k"] \arrow[dr, phantom, "(3)"] & M(Y - Z)((n))[1] \arrow[d, "\partial_{i'}"] \\ & M(Y' - Z)((m))[1] \arrow[r, "-\partial_{j'}"'] & M(X - Y \cup Y')\,[2] \end{tikzcd}$$

Proof. We will simply call smooth triple the data (X, Y, Y') of a triple of smooth schemes X, Y, Y' such that Y' and Y are closed subschemes of X. Such smooth triples form a category with morphisms the commutative diagrams

$$\begin{tikzcd} \bar{Y} \arrow[r, hook] \arrow[d, "g"'] & \bar{X} \arrow[d, "f"'] & \bar{Y}' \arrow[l, hook'] \arrow[d, "g'"] \\ Y \arrow[r, hook] & X & Y' \arrow[l, hook'] \end{tikzcd}$$

made of two cartesian squares. We say in addition that the morphism (f, g, g') is *transversal* if f is transversal to Y, Y' and $Y \cap Y'$.

To such a triple, we associate a geometric motive $M(X, Y, Y')$ as the cone of the canonical map of complexes of $\mathscr{S}m^{cor}(k)$

$$\begin{tikzcd} \cdots \arrow[r] & {[X - Y \cup Y']} \arrow[r] \arrow[d] & {[X - Y']} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & {[X - Y]} \arrow[r] & {[X]} \arrow[r] & \cdots \end{tikzcd}$$

where $[X]$ and $[X - Y]$ are placed in degree 0. This motive is evidently functorial with respect to morphisms of smooth triples.

We will also use the notation $M\left(\frac{X/X-Y}{X-Y'/X-Y\cup Y'}\right)$ for this motive because it is more suggestive. By definition, it fits into the following diagram, with $\Omega = Y \cup Y'$:

$$\begin{tikzcd}
(\mathcal{D}) : M(X-\Omega) \arrow[r] \arrow[d] & M(X-Y) \arrow[r] \arrow[d] & M\left(\frac{X-Y}{X-\Omega}\right) \arrow[r] \arrow[d] & M(X-\Omega)\,[1] \arrow[d] \\
M(X-Y') \arrow[r] \arrow[d] & M(X) \arrow[r] \arrow[d] \arrow[r, phantom, shift right=3ex, "(1)"] & M\left(\frac{X}{X-Y'}\right) \arrow[r] \arrow[d] \arrow[r, phantom, shift right=3ex, "(2)"] & M(X-Y')\,[1] \arrow[d] \\
M\left(\frac{X-Y'}{X-\Omega}\right) \arrow[r] \arrow[d] & M\left(\frac{X}{X-Y}\right) \arrow[r] \arrow[d] & M\left(\frac{X/X-Y}{X-Y'/X-\Omega}\right) \arrow[r] \arrow[d] \arrow[r, phantom, shift right=3ex, "(3)"] & M\left(\frac{X-Y'}{X-\Omega}[1]\right) \arrow[d] \\
M(X-\Omega)\,[1] \arrow[r] & M(X-Y)\,[1] \arrow[r] & M\left(\frac{X-Y}{X-\Omega}\right)[1] \arrow[r] & M(X-\Omega)\,[2].
\end{tikzcd}$$

In this diagram, every square is commutative except square (3) which is anticommutative due to the fact the permutation isomorphism on $\mathbb{Z}[1] \otimes \mathbb{Z}[1]$ is equal to -1. Moreover, any line or row of this diagram is a distinguished triangle.

With the hypothesis of the theorem, the proof will consist in constructing a purity isomorphism $\mathfrak{p}_{(X,Y,Y')} : M(X,Y,Y') \to M(Z)(d)[2d]$ which satisfies the following properties :

(i) *Functoriality :* The morphism $\mathfrak{p}_{(X,Y,Y')}$ is functorial with respect to transversal morphisms of smooth triples.

(ii) *Symmetry :* The following diagram is commutative :

$$\begin{tikzcd}
M(X,Y,Y') \arrow[rr] \arrow[dr, "\mathfrak{p}_{(X,Y,Y')}"'] & & M(X,Y',Y) \arrow[dl, "\mathfrak{p}_{(X,Y',Y)}"] \\
& M(Z)(d)[2d] &
\end{tikzcd}$$

where the horizontal map is the canonical isomorphism.

(iii) *Compatibility :* The following diagram is commutative :

$$\begin{tikzcd}
M\left(\frac{X-Y'}{X-\Omega}\right) \arrow[r] \arrow[d, dashed, "\mathfrak{p}_{(X-Y',Y-Z)}"'] & M\left(\frac{X}{X-Y}\right) \arrow[r] \arrow[d, dashed, "\mathfrak{p}_{(X,Y)}"'] & M(X,Y,Y') \arrow[r] \arrow[d, dashed, "\mathfrak{p}_{(X,Y,Y')}"'] & M\left(\frac{X-Y'}{X-\Omega}\right)[1] \arrow[d, dashed, "\mathfrak{p}_{(X-Y',Y-Z)}[1]"'] \\
M(Y-Z)((n)) \arrow[r] & M(Y)((n)) \arrow[r, "j^*"] & M(Z)((d)) \arrow[r, "\partial_j"] & M(Y-Z)((n))[1]
\end{tikzcd}$$

With this isomorphism, we can deduce the three relations of the theorem by considering squares (1), (2), (3) in the above diagram and applying the evident purity isomorphism where it belongs.

We then are reduced to construct the isomorphism and to prove the above relations. The second relation is the most difficult one because we have to show that two isomorphisms in a triangulated category are equal. This forces us to be very precise in the construction of the isomorphism.

<u>Construction of the purity isomorphism for smooth triples :</u>

Consider the deformation diagram (1.11.a) for the closed pair (X, Y) and put $B = B_Y(\mathbb{A}^1_X)$, $P = P_Y X$. Put also $(U, V) = (X - Y', Y - Z)$, $B_U = B \times_X U$ and $P_V = P \times_Y V$. Note that, because $Z = (Y \times_X Y')_{red}$, we get $V = Y \times_X U$; thus B_U is the deformation space of (1.11.a) for the closed pair (U, V). By functoriality of the deformation diagram and of relative motives we obtain the following morphisms of distinguished triangles :

$$\begin{tikzcd}
M(U,V) \arrow[r] \arrow[d] & M(X,Y) \arrow[r] \arrow[d] & M\left(\frac{X/X-Y}{U/U-V}\right) \arrow[r, "+1"] \arrow[d] & {} \\
M\big(B_U,\mathbb{A}^1_U\big) \arrow[r] & M\big(B,\mathbb{A}^1_Y\big) \arrow[r] & M\left(\frac{B/B-\mathbb{A}^1_Y}{B_U/B_U-\mathbb{A}^1_V}\right) \arrow[r, "+1"] & {} \\
M(P_V,V) \arrow[r] \arrow[u] & M(P,Y) \arrow[r] \arrow[u] & M\left(\frac{P/P-Y}{P_V/P_V-V}\right) \arrow[r, "+1"] \arrow[u] & {}
\end{tikzcd}$$

According to Proposition 1.12 and homotopy invariance, the vertical maps in the first two columns are isomorphisms. As the rows in the diagram are distinguished triangles, the vertical maps in the third column also are isomorphisms.

Using Lemma 1.32 with $P = \mathbb{P}(N_Y X \oplus \mathbb{A}^1_Y)$, we can consider the following morphism of distinguished triangles :

$$\begin{tikzcd}
M(P_V,V) \arrow[r] & M(P,Y) \arrow[r] & M\left(\frac{P/P-Y}{P_V/P_V-V}\right) \arrow[r, "+1"] & {} \\
M(P_V) \arrow[r] \arrow[u] \arrow[d, equal] & M(P) \arrow[r] \arrow[u] \arrow[d, equal] & M\left(\frac{P}{P_V}\right) \arrow[r, "+1"] \arrow[u] & {} \\
M(P_V) \arrow[r] & M(P) \arrow[r] & M(P_Z)((s)) \arrow[r, "+1"] \arrow[u, "\mathfrak{p}^{-1}_{(P,P_Z)}"'] & {} \\
M(Y-Z)((n)) \arrow[r] \arrow[u, "\mathfrak{l}_n(P_V)"] & M(Y)((n)) \arrow[r] \arrow[u, "\mathfrak{l}_n(P)"] & M(Z)((d)) \arrow[r, "+1"] \arrow[u, "\mathfrak{l}_n(P_Z)"'] & {}
\end{tikzcd}$$

The triangle on the bottom is obtained by tensoring the Gysin triangle of the pair (Y,Z) with $\mathbb{Z}(n)[2n]$. From Proposition 1.12, the first two of the vertical composite arrows are isomorphisms, so the last one is also an isomorphism.

If we put together (vertically) the two previous diagrams, we finally obtain the following isomorphism of triangles :

$$\begin{tikzcd}
M(U,V) \arrow[r] \arrow[d, "\mathfrak{p}_{(X-Y',Y-Z)}"'] & M(X,Y) \arrow[r] \arrow[d, "\mathfrak{p}_{(X,Y)}"'] & M(X,Y,Y') \arrow[r] \arrow[d, "(*)"] & M(U,V)\,[1] \arrow[d] \\
M(Y-Z)((n)) \arrow[r] & M(Y)((n)) \arrow[r, "j^*"] & M(Z)((d)) \arrow[r, "\partial_j"] & M(Y-Z)((n))[1].
\end{tikzcd}$$

We define $\mathfrak{p}_{(X,Y,Z)}$ as the morphism labeled $(*)$ in the previous diagram so that property (iii) follows from the construction. The functoriality property (i) follows easily from the functoriality of the deformation diagram.

<u>The remaining relation</u>

To conclude it remains only to prove the symmetry property (ii). First of all, we remark that the above construction implies immediately the commutativity of the following diagram :

$$\begin{tikzcd}
M\left(\frac{X/X-Y}{X-Y/X-Y\cup Y'}\right) \arrow[rr] \arrow[dr, "\mathfrak{p}_{(X,Y,Y')}"'] & & M\left(\frac{X/X-Y}{X-Z/X-Y}\right) \arrow[dl, "\mathfrak{p}_{(X,Y,Z)}"] \\
& M(Z)((d)), &
\end{tikzcd}$$

where the horizontal map is induced by the evident open immersions.

Thus, it will be sufficient to prove the commutativity of the following diagram :

$$\begin{tikzcd}
M\left(\frac{X}{X-Z}\right) \arrow[rr, "\alpha_{X,Y,Z}"] \arrow[dr, "\mathfrak{p}_{(X,Z)}"'] \arrow[rr, phantom, shift right=4ex, "(*)"] & & M\left(\frac{X/X-Y}{X-Z/X-Y}\right) \arrow[dl, "\mathfrak{p}_{(X,Y,Z)}"] \\
& M(Z)((n+m)), &
\end{tikzcd}$$

where $\alpha_{X,Y,Z}$ denotes the canonical isomorphism.

From now on, we consider only the smooth triples (X, Y, Z) such that Z is a closed subscheme of Y. Using the functoriality of $\mathfrak{p}_{(X,Y,Z)}$, we remark that the diagram $(**)$ is natural with respect to morphisms $f : X' \to X$ which are transversals to Y and Z.

Consider the notations of the paragraph 1.11 and put $D_Z X = B_Z(\mathbb{A}^1_X)$ for short. We will expand these notations as follows :

$$D(X, Z) = D_Z X,\ B(X, Z) = B_Z X\ , P(X, Z) = P_Z X.$$

To (X, Y, Z), we associate the evident closed pair $(D_Z X, D_Z X|_Y)$ and the *double deformation space*

$$D(X, Y, Z) = D(D_Z X, D_Z X|_Y).$$

This scheme is in fact fibered over $\mathbb{A}^2_k$. The fiber over $(1, 1)$ is X and the fiber over $(0, 0)$ is $B(B_Z X \cup P_Z X, B_Z X|_Y \cup P_Z X|_Y)$. In particular, the $(0, 0)$-fiber contains the scheme $P(P_Z X, P_Z Y)$.

We now put $\begin{cases} D = D(X, Y, Z), & R = P(R_Z X, R_Z Y) \\ D' = D(Y, Y, Z), & P = R_Z Y. \end{cases}$

Remark also that $D(Z, Z, Z) = \mathbb{A}^2_Z$ and that $R = P \times_Z P'$ where $P' = P_Y X|_Z$.[13] From the description of the fibers of D given above, we obtain a deformation diagram of smooth triples :

$$(X, Y, Z) \to (D, D', \mathbb{A}^2_Z) \leftarrow (R, P, Z).$$

Note that these morphisms are on the smaller closed subscheme the $(0, 0)$-section and $(1, 1)$-section of $\mathbb{A}^2_Z$ over Z, denoted respectively by s_0 and s_1. Now we apply these morphisms to the diagram $(*)$ in order to obtain the following commutative diagram :

$$\begin{tikzcd}
M_Z(X) \arrow[rr] \arrow[dd, "\mathfrak{p}_{(X,Z)}"'] \arrow[dr, "\alpha_{X,Y,Z}"] & & M_{\mathbb{A}^2_Z}(D) \arrow[dd, "\mathfrak{p}_{(D,\mathbb{A}^2_Z)}"'] \arrow[dr] & & M_Z(R) \arrow[ll] \arrow[dd, "\mathfrak{p}_{(R,Z)}"'] \arrow[dr, "\alpha_{R,P,Z}"] & \\
& M(X,Y,Z) \arrow[rr] \arrow[dl, "\mathfrak{p}_{(X,Y,Z)}"] & & M\big(D, D', \mathbb{A}^2_Z\big) \arrow[dl, "\mathfrak{p}_{(D,D',Z)}"] & & M(R,P,Z) \arrow[ll] \arrow[dl, "\mathfrak{p}_{(R,P,Z)}"] \\
M(Z)((n+m)) \arrow[rr, "s_{1*}"'] & & M\big(\mathbb{A}^2_Z\big)((n+m)) & & M(Z)((n+m)). \arrow[ll, "s_{0*}"] &
\end{tikzcd}$$

The square parts of this prism are commutative. As the morphisms s_{1*} and s_{0*} are isomorphisms, the commutativity of the left triangle is equivalent to the commutativity of the right one.

Thus, we are reduced to the case of the smooth triple (R, P, Z). Now, using the canonical split epimorphism $M(R) \to M_Z(R)$, we are reduced to prove the

[13]The last property is equivalent to the identification: $N(N_Z X, N_Z Y) = N_Z Y \oplus N_Y X|_Z$.

commutativity of the diagram :

$$\begin{tikzcd} M(R) \arrow[dr] \arrow[d, "i^*"'] & \\ M(Z)((d)) & M\left(\frac{R/R-P}{R-Z/R-P}\right) \arrow[l, "\mathfrak{p}_{(R,P,Z)}"] \end{tikzcd}$$

where $i : Z \to R$ denotes the canonical closed immersion.

Using the property (iii) of the isomorphism $\mathfrak{p}_{(R,P,Z)}$, we are finally reduced to prove the commutativity of the triangle

$$\begin{tikzcd} M(R) \arrow[dr, "j^*"] \arrow[d, "i^*"'] & \\ M(Z)((d)) & M(P)((n)) \arrow[l, "k^*"] \end{tikzcd}$$

where j and k are the evident closed embeddings. This is Lemma 1.33. □

As a corollary, we get the functoriality of the Gysin morphism of a closed immersion :

Corollary 1.35. *Let $Z \xrightarrow{l} Y \xrightarrow{i} X$ be closed immersion between smooth schemes such that i is of pure codimension n.*

Then, $l^* \circ i^* = (i \circ l)^*$.

As an illustration of the formulas obtained in the preceding theorem, we prove the following result:

Proposition 1.36. *Consider a smooth closed pair (X, Z) of codimension n and $\nu : Z \to X$ the corresponding immersion.*

Consider the canonical decompositions $Z = \sqcup_{i \in I} Z_i$ and $X = \sqcup_{j \in J} X_j$ into connected components. Put $\hat{Z}_j = Z \times_X X_j$. For any index $i \in I$, let $j \in J$ be the unique element such that $Z_i \subset X_j$; we let $\nu_i^j : Z_i \to X_j$ be the immersion induced by ν and we denote by Z'_i the unique scheme such that: $\hat{Z}_j = Z_i \sqcup Z'_i$.

Consider he following commutative diagram:

$$\begin{tikzcd} M(X) \arrow[r, "\nu^*"] & M(Z)((n)) \arrow[r, "\partial_{X,Z}"] & M(X-Z)\,[1] \\ \oplus_{j \in J} M(X_j) \arrow[u, "\sim"] \arrow[r, "(\nu_{ji})_{j \in J, i \in I}"'] & \oplus_{i \in I} M(Z_i)((n)) \arrow[u, "\sim"'] \arrow[r, "(\partial_{ij})_{i \in I, j \in J}"'] & \oplus_{j \in J} M\left(X_j - \hat{Z}_j\right)[1] \arrow[u, "\sim"'] \end{tikzcd}$$

where the vertical maps are the canonical isomorphisms.

Then, for any couple $(i, j) \in I \times J$,

1. *if $Z_i \subset X_j$, $\nu_{ji} = \left(\nu_i^j\right)^*$ and $\partial_{ij} = \partial_{X_j - Z'_i, Z_i}$,*
2. *otherwise, $\nu_{ji} = 0$ and $\partial_{ij} = 0$.*

Proof. We consider the following cartesian squares made of the evident immersions:

If $Z_i \subset X_j$, otherwise,

(1.36.a)
$$\begin{tikzcd} Z_i \arrow[r, "\nu_i^j"] \arrow[d, equal] & X_j \arrow[d, "x_j"'] & \hat{Z}_j \arrow[l, "\hat{z}_j"'] \arrow[d, "\nu_i^j"] & Z_i \arrow[l] \arrow[d, equal] \\ Z_i \arrow[r, "\nu_i"'] & X & Z \arrow[l, "\nu"] & Z_i \arrow[l, "z_i"] \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X_j \arrow[d, "x_j"'] & \hat{Z}_j \arrow[l, "\hat{z}_j"'] \arrow[d, "\nu_i^j"] & \emptyset \arrow[l] \arrow[d] \\ Z_i \arrow[r, "\nu_i"'] & X & Z \arrow[l, "\nu"] & Z_i \arrow[l, "z_i"] \end{tikzcd}$$

We also consider the open and closed immersion $u_j : (X_j - \hat{Z}_j) \to (X - Z)$.

According to corollary 1.21, we obtain the following equalities:

$$\nu_{ji} = z_i^* \nu^* x_{j*}, \quad \partial_{i,j} = u_j^* \partial_{X,Z} z_{i*}.$$

Then the result follows from the following computations:

$$z_i^* \nu^* x_{j*} \overset{(a)}{=} \nu_i^* x_{j*} \overset{(b)}{=} \begin{cases} (\nu_i^j)^* & \text{if } Z_i \subset X_j, \\ 0 & \text{otherwise.} \end{cases}$$

$$u_j^* \partial_{X,Z} z_{i*} \overset{(c)}{=} \partial_{X_j, \hat{Z}_j} \hat{z}_j^* z_{i*} \overset{(d)}{=} \begin{cases} \partial_{X_j, \hat{Z}_j} (z_i^j)_* \overset{(e)}{=} \partial_{X_j - Z_i', Z_i} & \text{if } Z_i \subset X_j, \\ 0 & \text{otherwise.} \end{cases}$$

We give the following justifications for each equality:
(a) : Corollary 1.35 ($\nu_i = \nu \circ z_i$).
(b) : Proposition 1.19 applied to the first square of the respective commutative diagram of (1.36.a) corresponding to the each respective case.
(c) : Theorem 1.34 applied to the second cartesian square of (1.36.a).
(d) : Proposition 1.19 applied to the third square of the respective commutative diagram of (1.36.a) corresponding to each respective case.
(e) : Proposition 1.19. □

2. Gysin morphism

In this section, motives are considered in the category $DM_{gm}(k)$.

2.1. **Construction.**

2.1.1. *Preliminaries.*

Lemma 2.1. *Let X be a smooth scheme, P/X and Q/X be projective bundles of respective dimensions n and m. We consider λ_P (resp. λ_Q) the canonical dual line bundle on P (resp. Q) and λ'_P (resp. λ'_Q) its pullback on $P \times_X Q$. Let $p : P \times_X Q \to X$ be the canonical projection.*

Then, the morphism $\sigma : M(P \times_X Q) \longrightarrow \bigoplus_{i,j} M(X)(i+j)[2(i+j)]$ given by the formula

$$\sigma = \sum_{0 \leq i \leq n,\, 0 \leq j \leq m} \mathfrak{c}_1(\lambda'_P)^i \boxtimes \mathfrak{c}_1(\lambda'_Q)^j \boxtimes p_*$$

is an isomorphism.

Proof. As σ is compatible with pullback, we can assume using property **(MV)** of Proposition 1.4 that P and Q are trivialisable projective bundles. Using the invariance of σ under automorphisms of P or Q, we can assume that P and Q are trivial projective bundles. From the definition of σ, we are reduced to the case $X = \mathrm{Spec}(k)$. Then, σ is just the tensor product of the two projective bundle isomorphisms (cf paragraph 1.7) for P and Q. □

The following proposition is the key point in the definition of the Gysin morphism for a projective morphism.

Proposition 2.2. *Let X be a smooth scheme, $p : P \to X$ be a projective bundle of rank n and $s : X \to P$ a section of p.*

Then, the composite map $M(X)((n)) \xrightarrow{\mathfrak{l}_n(P)} M(P) \xrightarrow{s^*} M(X)((n))$ is the identity.[14]

[14]In fact, this result holds in the effective category $DM_{gm}^{eff}(k)$ as the proof will show.

Proof. In this proof, we work in the category $DM_{gm}^{eff}(k)$.

Let $\eta_P(X)$ be the motivic fundamental class associated with s (see Definition 1.26). According to Lemma 1.29, we obtain: $s^* = \eta_P(X) \boxtimes_P p_*$.

Let E/X be the vector bundle on X such that $P = \mathbb{P}(E)$. Let λ be the canonical dual line bundle on P. If we consider the line bundle $L = s^{-1}(\lambda^\vee)$ on X, the section s corresponds uniquely to a monomorphism $L \to E$ of vector bundles on P. We consider the following vector bundle on P:

$$F = \lambda \otimes p^{-1}(E/L).$$

Then the canonical morphism:

$$\lambda^\vee \to p^{-1}(E) \to p^{-1}(E/L)$$

made by the canonical inclusion and the canonical projection induces a section σ of F/P which is transversal to the zero section s_0^F of F/P and such that the following square is cartesian:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d] & P \arrow[d, "\sigma"] \\ P \arrow[r, "s_0^F"] & F. \end{tikzcd}$$

Thus, according to Lemma 1.30, we get: $\eta_P(X) = \mathfrak{c}_n(F)$.

The result now follows from the computation of the top Chern class $c_n(F)$ in $CH^n(P)$ and Lemma 1.8. $\square$

Remark 2.3. As a corollary, we obtain the following reinforcement of Proposition 1.12, more precisely of the normalization condition for the purity isomorphism :

Let X be a smooth scheme, P/X be a projective bundle of rank n, and $s : X \to P$ be a section of P/X. Then, the purity isomorphism $\mathfrak{p}_{(P,s(X))}$ is the inverse isomorphism of the composition

$$M(X)((n)) \xrightarrow{\mathfrak{l}_n(P)} M(P) \xrightarrow{(1)} M_{s(X)}(P)$$

where (1) is the canonical map.

2.1.2. *Gysin morphism of a projection.* The following definition will be a particular case of Definition 2.7.

Definition 2.4. Let X be a smooth scheme, P be a projective bundle of rank n over X and $p : P \to X$ be the canonical projection.

Using the notation of (1.7.b), we put:

$$p^* = \mathfrak{l}_n(P)(-n)[-2n] : M(X) \to M(P)\,(-n)[-2n]$$

and call it the Gysin morphism of p.

Lemma 2.5. *Let P, Q be projective bundles over a smooth scheme X of respective ranks n, m. Consider the following projections :*

$$\begin{tikzcd}[column sep=small] & P \arrow[dr, "p"] & \\ P \times_X Q \arrow[ur, "q'"] \arrow[dr, "p'"'] & & X \\ & Q \arrow[ur, "q"'] & \end{tikzcd}$$

Then, the following diagram is commutative :

$$\begin{tikzcd}[column sep=small] & M(P)((-m)) \arrow[dr, "q'^*"] & \\ M(X) \arrow[ur, "p^*"] \arrow[dr, "q^*"'] & & M(P \times_X Q)((-n-m)) \\ & M(Q)((-n)) \arrow[ur, "p'^*"'] & \end{tikzcd}$$

Proof. Indeed, using the compatibility of the motivic Chern class with pullback (cf 1.5), we see that both edge morphisms in the previous diagram are equal (up to twist and suspension) to the composite

$$M(X)((n+m)) \to \bigoplus_{i \leq n, j \leq m} M(X)((i+j)) \to M(P \times_X Q)\,,$$

where the first arrow is the obvious split monomorphism and the second arrow is the inverse isomorphism to the one constructed in Lemma 2.1. □

2.1.3. *General case.* The following lemma is all we need to finish the construction of the Gysin morphism of a projective morphism :

Lemma 2.6. *Consider a commutative diagram*

$$\begin{tikzcd} & P \arrow[dr, "p"] & \\ Y \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\ & Q \arrow[ur, "q"'] & \end{tikzcd}$$

where X and Y are smooth schemes, i (resp. j) is a closed immersion of codimension $n+d$ (resp. $m+d$), P (resp. Q) is a projective bundle over X of dimension n (resp. m) with projection p (resp. q).

Then, the following diagram is commutative

$$\text{(2.6.a)} \qquad \begin{tikzcd} & M(P))((m)) \arrow[dr, "i^*"] & \\ M(X)((n+m)) \arrow[ur, "p^*"] \arrow[dr, "q^*"'] & & M(Y)((n+m+d)). \\ & M(Q)((n)) \arrow[ur, "j^*"'] & \end{tikzcd}$$

Proof. Considering the diagonal embedding $Y \xrightarrow{(i,j)} P \times_X Q$, we divide diagram (2.6.a) into three parts:

$$\begin{tikzcd} & M(P)((m)) \arrow[d, "p'^*"] \arrow[dr, "i^*"] \arrow[dr, phantom, "(2)", near start] & \\ M(X)((n+m)) \arrow[ur, bend left, "p^*"] \arrow[dr, bend right, "q^*"'] \arrow[r, phantom, "(1)"] & M(P \times_X Q) \arrow[r, "(i,j)^*"] & M(Y)((n+m+d)). \\ & M(Q)((n)) \arrow[u, "q'^*"'] \arrow[ur, "j^*"'] \arrow[ur, phantom, "(3)", near start] & \end{tikzcd}$$

The commutativity of part (1) is Lemma 2.5. The commutativity of part (2) and that of part (3) are equivalent to the case $X = Q$, $q = 1_X$ – and thus $m = 0$.

Assume we are in this case. We introduce the following morphisms where the square (*) is cartesian and γ is the graph of the X-morphism i:

$$\begin{tikzcd} & P_Y \arrow[r, "p'"] \arrow[d, "j'"'] \arrow[dr, phantom, "(*)"] & Y \arrow[d, "j"] \\ Y \arrow[ur, "\gamma"] \arrow[r, "i"'] & P \arrow[r, "p"'] & X \end{tikzcd}$$

Note that γ is a section of p'. Thus, Proposition 2.2 gives: $\gamma^* p'^* = 1$, and we reduce the commutativity of the diagram (2.6.a) to that of the following one:

$$\begin{tikzcd} & M(P_Y)((d)) \arrow[dl, "\gamma^*"'] \arrow[d, phantom, "(4)\quad"] & M(Y)((n+d)) \arrow[l, "p'^*"'] \arrow[dl, phantom, "(5)"] \\ M(Y)((n+d)) & M(P) \arrow[l, "i^*"] \arrow[u, "j'^*"'] & M(X)((n)) \arrow[l, "p^*"] \arrow[u, "j^*"'] \end{tikzcd}$$

Then commutativity of part (4) is Corollary 1.35 and that of part (5) follows from Lemma 1.32. □

Let $f : Y \to X$ be a projective morphism between smooth schemes. Following the terminology of Fulton (see [Ful98, §6.6]), we say that f has codimension d if it can be factored into a closed immersion $Y \to P$ of codimension e followed by the projection $P \to X$ of a projective bundle of dimension $e - d$. In fact, the integer d is uniquely determined (cf *loc.cit.* appendix B.7.6). Using the preceding lemma, we can finally introduce the general definition :

Definition 2.7. Let X, Y be smooth schemes and $f : Y \to X$ be a projective morphism of codimension d.

We define the Gysin morphism associated with f in $DM_{gm}(k)$

$$f^* : M(X) \to M(Y)((d))$$

by choosing a factorisation of f into $Y \xrightarrow{i} P \xrightarrow{p} X$ where i is a closed immersion of pure codimension $n + d$ and p is the projection of a projective bundle of rank n, and putting :

$$f^* = \left[M(X)((n)) \xrightarrow{\mathfrak{l}_n(P)} M(P) \xrightarrow{i^*} M(Y)((n+d)) \right] ((-n)),$$

definition which does not depend upon the choices made according to the previous lemma.

Remark 2.8. In [Dég09, 3.11], we prove that the Gysin morphism of a projective morphism f induces the usual pushout on the part of motivic cohomology corresponding to Chow groups.

2.2. **Properties.**

2.2.1. *Functoriality.*

Proposition 2.9. *Let X, Y, Z be smooth schemes and $Z \xrightarrow{g} Y \xrightarrow{f} X$ be projective morphisms of respective codimensions m and n.*

Then, in $DM_{gm}(k)$, we get the equality : $g^* \circ f^* = (fg)^*$.

Proof. We first choose projective bundles P, Q over X, of respective dimensions s and t, fitting into the following diagram with $R = P \times_X Q$ and $Q_Y = Q \times_X Y$:

$$\begin{tikzcd}
& & Q \arrow[ddrr, bend left, "q"] & & \\
& & R \arrow[u, "p'"] \arrow[dr, "q'"] & & \\
& Q_Y \arrow[ur, "i'"] \arrow[dr, "q''"] & & P \arrow[dr, "p"] & \\
Z \arrow[uuurr, bend left, "j"] \arrow[ur, "k"] \arrow[rr, "g"] & & Y \arrow[ur, "i"] \arrow[rr, "f"] & & X.
\end{tikzcd}$$

The prime exponent of a symbol indicates that the morphism is deduced by base change from the morphism with the same symbol. We then have to prove that the following diagram of $DM_{gm}(k)$ commutes :

$$\begin{tikzcd}
 & & M(Q)((t)) \arrow[d, "p'^*"] \arrow[dddrr, bend left, "j^*"] & & \\
 & (2) & M(R)((s+t)) \arrow[dr, "i'^*"] & (3) & \\
 & M(P)((s)) \arrow[ur, "q'^*"] \arrow[dr, "i^*"] & (1) & M(Q_Y)((n+t)) \arrow[dr, "k^*"] & \\
M(X) \arrow[ur, "p^*"] \arrow[uuurr, bend left, "q^*"] & & M(Y)((n)) \arrow[ur, "q''^*"] & & M(Z)((n+m)).
\end{tikzcd}$$

The commutativity of part (1) is a corollary of Lemma 1.32, that of part (2) is Lemma 2.5 and that of part (3) follows from Lemma 2.6 and Corollary 1.35. □

2.2.2. *Projection formula and excess of intersection.* From Definition 2.7 and Proposition 1.19 we directly obtain the following proposition :

Proposition 2.10. *Consider a cartesian square of smooth schemes*

(2.10.a)
$$\begin{tikzcd}
T \arrow[r, "g"] \arrow[d, "q"'] & Z \arrow[d, "p"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

such that f and g are projective morphisms of the same codimensions.

Then, the relation $f^*p_* = q_*g^*$ holds in $DM_{gm}(k)$.

2.11. Consider now a cartesian square of shape (2.10.a) such that f (resp. g) is a projective morphism of codimension m (resp. m). Then $m \leq n$ and we call $e = n - m$ the *excess of dimension* attached with (2.10.a).

We can also associate with the above square a vector bundle ξ of rank e, called the *excess bundle*. Choose $Y \xrightarrow{i} P \xrightarrow{\pi} X$ a factorisation of f such that i is a closed immersion of codimension r and π is the projection of a projective bundle of dimension s. We consider the following cartesian squares:

$$\begin{tikzcd}
T \arrow[r, "i'"] \arrow[d, "q"'] & Q \arrow[r, "\pi'"] \arrow[d] & Z \arrow[d, "p"] \\
Y \arrow[r, "i"] & P \arrow[r, "\pi"] & X
\end{tikzcd}$$

Then N_TQ is a sub-vector bundle of $q^{-1}N_YP$ and we put $\xi = q^{-1}N_YP/N_TQ$. This definition is independent of the choice of P (see [Ful98], proof of prop. 6.6).

The following proposition is now a straightforward consequence of Definition 2.7 and the second case of Proposition 1.19 :

Proposition 2.12. *Consider the above notations.*

Then, the relation $f^*p_* = \big(\mathfrak{c}_e(\xi) \boxtimes q_*((m))\big) \circ g^*$ holds in $DM_{gm}(k)$.

2.2.3. *Compatibility with the Gysin triangle.*

Proposition 2.13. *Consider a topologically cartesian square of smooth schemes*

$$\begin{tikzcd}
T \arrow[r, "j"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\
Z \arrow[r, "i"] & X
\end{tikzcd}$$

such that f and g are projective morphisms, i and j are closed immersions. Put $U = X - Z$, $V = Y - T$ and let $h : V \to U$ be the projective morphism induced by f. Let n, m, p, q be respectively the relative codimensions of i, j, f, g.

Then the following diagram is commutative

$$\begin{tikzcd} M(V)((p)) \arrow[r] & M(Y)((p)) \arrow[r, "j^*"] & M(T)((m+p)) \arrow[r, "\partial_{Y,T}"] & M(V)((p))[1] \\ M(U) \arrow[r] \arrow[u, "h^*"] & M(X) \arrow[r, "i^*"] \arrow[u, "f^*"] & M(Z)((n)) \arrow[r, "\partial_{X,Z}"] \arrow[u, "g^*((n))"'] & M(U)[1] \arrow[u, "h^*"'] \end{tikzcd}$$

where the two lines are the obvious Gysin triangles.

Proof. Use the definition of the Gysin morphism and apply Lemma 1.32, Theorem 1.34. □

2.2.4. *Gysin morphisms and transfers in the étale case.*

2.14. In [Dég08b], paragraphs 1.1 and 1.2 we have introduced another Gysin morphism for a finite equidimensional morphism $f : Y \to X$. Indeed, the transpose of the graph of f gives a finite correspondence ${}^tf$ from X to Y which induces a morphism ${}^tf_* : M(X) \to M(Y)$ in $DM_{gm}(k)$.

Proposition 2.15. *Let X and Y be smooth schemes, and $f : Y \to X$ be an étale cover.*

Then, $f^* = {}^tf_*$.

Proof. Consider the cartesian square of smooth schemes

$$\begin{tikzcd} Y \times_X Y \arrow[r, "g"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

We first prove that ${}^tf'_* f^* = g^*\, {}^tf_*$. Choose a factorisation $Y \xrightarrow{i} P \xrightarrow{\pi} X$ of f into a closed immersion and the projection of a projective bundle. The preceding square can be divided into two squares

$$\begin{tikzcd} Y \times_X Y \arrow[r, "j"] \arrow[d, "f'"'] & P \times_X Y \arrow[r, "q"] \arrow[d, "f''"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "i"] & P \arrow[r, "\pi"] & X. \end{tikzcd}$$

The assertion then follows from the commutativity of the following diagram.

$$\begin{tikzcd} M(Y \times_X Y) & M(P \times_X Y) \arrow[l, "j^*"'] & M(Y) \arrow[l, "q^*"'] \\ M(Y) \arrow[u, "{}^tf'_*"] \arrow[r, phantom, "(1)"] & M(P) \arrow[l, "i^*"] \arrow[u, "{}^tf''_*"] \arrow[r, phantom, "(2)"] & M(X) \arrow[l, "p^*"] \arrow[u, "{}^tf_*"'] \end{tikzcd}$$

The commutativity of part (1) follows from [Dég08b], prop. 2.5.2 (case 1) and that of part (2) from [Dég08b], prop. 2.2.15 (case 3).

Then, considering the diagonal immersion $Y \xrightarrow{\delta} Y \times_X Y$, it suffices to prove in view of Proposition 2.9 that $\delta^* \circ {}^tf'_* = 1$. As Y/X is étale, Y is a connected component of $Y \times_X Y$. Thus, $M(Y)$ is a direct factor of $M(Y \times_X Y)$. Then, according to corollary 1.21, δ^* is the canonical projection on this direct factor. One can easily see that ${}^tf'_*$ is the canonical inclusion and this concludes. □

2.3. **Duality pairings, motive with compact support.**

2.16. We first recall the abstract definition of duality in monoidal categories. Let $\mathscr{C}$ be a symmetric monoidal category with product $\otimes$ and unit $\mathbf{1}$. An object X of $\mathscr{C}$ is said to be *strongly dualizable* if there exists an object X^* of $\mathscr{C}$ and two maps

$$\eta : \mathbf{1} \to X^* \otimes X, \quad \epsilon : X \otimes X^* \to \mathbf{1}$$

such that the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "X\otimes\eta"] \arrow[dr, "1_X"'] & X \otimes X^* \otimes X \arrow[d, "\epsilon\otimes X"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X^* \arrow[r, "\eta\otimes X^*"] \arrow[dr, "1_{X^*}"'] & X^* \otimes X \otimes X^* \arrow[d, "X^*\otimes\epsilon"] \\ & X^* \end{tikzcd}$$

The object X^* is called a *strong dual* of X. For any objects Y and Z of $\mathscr{C}$, we then have a canonical bijection

$$\mathrm{Hom}_{\mathscr{C}}(Z \otimes X, Y) \simeq \mathrm{Hom}_{\mathscr{C}}(Z, X^* \otimes Y).$$

In other words, $X^* \otimes Y$ is the internal Hom of the pair (X, Y) for any Y. In particular, such a dual is unique up to a canonical isomorphism. If X^* is a strong dual of X, then X is a strong dual of X^*.

Suppose $\mathscr{C}$ is a closed symmetric monoidal triangulated category. Denote by $\underline{\mathrm{Hom}}$ its internal Hom. For any objects X and Y of $\mathscr{C}$ the evaluation map

$$X \otimes \underline{\mathrm{Hom}}(X, \mathbf{1}) \to \mathbf{1}$$

tensored with the identity of Y defines by adjunction a map

$$\underline{\mathrm{Hom}}(X, \mathbf{1}) \otimes Y \to \underline{\mathrm{Hom}}(X, Y).$$

The object X is strongly dualizable if and only if this map is an isomorphism for all objects Y in $\mathscr{C}$. In this case indeed, $X^* = \underline{\mathrm{Hom}}(X, \mathbf{1})$.

2.17. Let X be a smooth projective k-scheme of pure dimension n and denote by $p : X \to \mathrm{Spec}(k)$ the canonical projection, $\delta : X \to X \times_k X$ the diagonal embedding.

Then we can define morphisms

$$\eta : \mathbb{Z} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{\delta_*} M(X)(-n)[-2n] \otimes M(X)$$
$$\epsilon : M(X) \otimes M(X)(-n)[-2n] \xrightarrow{\delta^*} M(X) \xrightarrow{p_*} \mathbb{Z}.$$

One checks easily using the properties of the Gysin morphism these maps turn $M(X)(-n)[-2n]$ into the dual of $M(X)$. We thus have obtained :

Proposition 2.18. *Let X/k be a smooth projective scheme.*

Then the couple of morphisms (η, ϵ) defined above is a duality pairing. Thus $M(X)$ is strongly dualizable with dual $M(X)(-n)[-2n]$.

Remark 2.19. Using this duality in conjunction with the isomorphism (1.5.a), we obtain for smooth projective schemes X and Y, d being the dimension of Y, a canonical map:

$$\begin{aligned} CH^d(X \times Y) &\simeq \mathrm{Hom}_{DM_{gm}^{eff}(k)}(M(X) \otimes M(Y)\,, \mathbb{Z}(d)[2d]) \\ &\to \mathrm{Hom}_{DM_{gm}(k)}(M(X) \otimes M(Y)\,, \mathbb{Z}(d)[2d]) \\ &= \mathrm{Hom}_{DM_{gm}(k)}(M(X)\,, M(Y)). \end{aligned}$$

As the isomorphism (1.5.a) is compatible with products and pullbacks, we check easily this defines a monoidal functor from Chow motives to mixed motives obtaining a new construction of the stable version of the functor which appears in [FSV00, chap. 5, 2.1.4]. Recall that the cancellation theorem of Voevodsky [Voe02] implies this is a full embedding.

Note the Gysin morphism $p^* : \mathbb{Z}(n)[2n] \to M(X)$ defines indeed a homological class η_X in $H^{\mathcal{M}}_{2n,n}(X) = \mathrm{Hom}_{DM_{gm}(k)}(\mathbb{Z}(n)[2n], M(X))$.

The duality above induces an isomorphism

$$H^{p,q}_{\mathcal{M}}(X) \to H^{\mathcal{M}}_{p-2n,q-n}(X)$$

which is by definition the cap-product by η_X. Thus our duality pairing implies the classical form of Poincaré duality and the class η_X is the fundamental class of X.

2.20. The last application of this section uses the stable version of the category of motivic complexes as defined in [CD09a, 7.15] and denoted by $DM(k)$. Remember it is a triangulated symmetric monoidal category. Moreover, there is a canonical monoidal fully faithful functor $DM_{gm}(k) \to DM(k)$ (see [CD09b, 10.1.4]). The idea of the following definition comes from [CD07, 2.6.3]:

Definition 2.21. Let X be a smooth scheme of dimension d.

We define the motive with compact support of X as the object of $DM(k)$

$$M^c(X) = \mathrm{R}\underline{\mathrm{Hom}}_{DM(k)}(M(X), \mathbb{Z}(d)[2d]).$$

This motive with compact support satisfies the following properties:

(i) For any morphism $f : Y \to X$ of relative dimension n between smooth schemes, the usual functoriality of motives induces:
$$f^* : M^c(X)(n)[2n] \to M^c(Y).$$

(ii) For any projective morphism $f : Y \to X$ between smooth schemes, the Gysin morphism of f induces:
$$f_* : M^c(Y) \to M^c(X).$$

(iii) Let $i : Z \to X$ be a closed immersion between smooth schemes, and j the complementary open immersion. Then the Gysin triangle associated with (X, Z) induces a distinguished triangle:
$$M^c(Z) \xrightarrow{i_*} M^c(X) \xrightarrow{j^*} M^c(U) \xrightarrow{\partial'_{X,Z}} M^c(Z)[1].$$

(iv) If X is a smooth k-scheme of relative dimension d, p its structural morphism and δ its diagonal embedding, the composite morphism
$$M(X) \otimes M(X) \xrightarrow{\delta^*} M(X)(d)[2d] \xrightarrow{p_*} \mathbb{Z}(d)[2d]$$
induces a map
$$\phi_X : M(X) \to M^c(X)$$
which is an isomorphism when X is projective (cf 2.18). Moreover, for any open immersion $j : U \to X$, $j^* \circ \phi_X \circ j_* = \phi_U$ (this follows easily from 2.10).

Remark 2.22. Note also that the formulas we have proved for the Gysin morphism or the Gysin triangle correspond to formulas involving the data (i), (ii) or (iii) of motives with compact support.

2.23. Consider a smooth scheme X of pure dimension d. According to Definition 2.21, as soon as $M(X)$ admits a strong dual $M(X)^\vee$ in $DM(k)$, we get a canonical isomorphism:

$$M^c(X) = M(X)^\vee(d)[2d]. \tag{2.23.a}$$

The same remark can be applied if we work in $DM(k) \otimes \mathbb{Q}$. Recall that duality is known in the following cases (it follows for example from the main theorem of [Rio05]):

Proposition 2.24. *Let X be a smooth scheme of dimension d.*

(1) *Assume k admits resolution of singularities.*
Then $M(X)$ is strongly dualizable in $DM_{gm}(k)$.
(2) *In any case, $M(X) \otimes \mathbb{Q}$ is strongly dualizable in $DM_{gm}(k) \otimes \mathbb{Q}$.*

Recall that Voevodsky has defined a motive with compact support (even without the smoothness assumption). It satisfies all the properties listed above except that (i) and (iii) requires resolution of singularities. Then according to the preceding proposition and formula (2.23.a), our definition agrees with that of Voevodsky if resolution of singularities holds over k (apply [FSV00, chap. 5, th. 4.3.7]). This implies in particular that $M^c(X)$ is in $DM_{gm}(k)$ or, in the words of Voevodsky, it is *geometric*. Moreover, we know from the second case of the preceding proposition that $M^c(X) \otimes \mathbb{Q}$ is always geometric.

3. Motivic coniveau exact couple

3.1. Definition.

3.1.1. *Triangulated exact couple.* We introduce a triangulated version of the classical exact couples.

Definition 3.1. Let $\mathscr{T}$ be a triangulated category. A triangulated exact couple is the data of bigraded objects D and E of $\mathscr{T}$ and homogeneous morphisms between them

(3.1.a)
$$\begin{tikzcd} D \arrow[rr, "(1,-1)", "\alpha"'] & & D \arrow[dl, "(0,0)", "\beta"'] \\ & E \arrow[ul, "(-1,0)", "\gamma"'] & \end{tikzcd}$$

with the bidegrees of each morphism indicated in the diagram and such that the above triangle is a distinguished triangle in each bidegree.[15]

Given such a triangulated exact couple, we will usually put $d = \beta \circ \gamma$, homogeneous endomorphism of E of bidegree $(-1, 0)$. We easily get that $d^2 = 0$, thus obtaining a complex

$$\ldots \to E_{p,q} \xrightarrow{d_{p,q}} E_{p-1,q} \to \ldots$$

Let $\mathscr{A}$ be an abelian category. A cohomological functor with values in $\mathscr{A}$ is an additive functor $H : \mathscr{T}^{op} \to \mathscr{A}$ which sends distinguished triangles to long exact sequences. For p an integer, we simply put $H^p = H \circ .[-p]$.
Apply the contravariant functor $H = H^0$ to the diagram (3.1.a), we naturally obtain a commutative diagram of bigraded objects of $\mathscr{A}$:

$$\begin{tikzcd} H(D) \arrow[dr, "(1,0)"', "\gamma^*"] & & H(D) \arrow[ll, "(-1,1)"', "\alpha^*"] \\ & H(E) \arrow[ur, "(0,0)"', "\beta^*"] & \end{tikzcd}$$

[15]Note this implies in particular the relation $D_{p,q+1} = D_{p,q}[-1]$ for any couple of integers (p, q).

This is an exact couple of $\mathscr{A}$ in the classical sense (following the convention of [McC01, th. 2.8]). Thus we can associate with this exact couple a spectral sequence:

$$E_1^{p,q} = H(E_{p,q})$$

with differentials being $H(d_{p,q}) : H(E_{p-1,q}) \to H(E_{p,q})$.

Definition 3.2. Let $\mathscr{T}$ be a triangulated category and X an object of $\mathscr{T}$.

(1) A tower $X_\bullet$ over X is the data of a sequence $(X_p \to X)_{p\in\mathbb{Z}}$ of objects over X and a sequence of morphisms over X

$$.. \to X_{p-1} \xrightarrow{j_p} X_p \to ...$$

(2) Let $X_\bullet$ be a tower over X. Suppose that for each integer p we are given a distinguished triangle

$$X_{p-1} \xrightarrow{j_p} X_p \xrightarrow{\pi_p} C_p \xrightarrow{\delta_p} X_p[1]$$

where j_p is the structural morphism of the tower $X_\bullet$.

Then we associate with the tower $X_\bullet$ and the choice of cones $C_\bullet$ a triangulated exact couple

$$D_{p,q} = X_p[-p-q], \qquad E_{p,q} = C_p[-p-q]$$

with structural morphisms

$$\alpha_{p,q} = j_p[-p-q],\ \beta_{p,q} = \pi_p[-p-q],\ \gamma_{p,q} = \delta_p[-p-q].$$

Let $H : \mathscr{T}^{op} \to \mathscr{A}$ be a cohomological functor. In the situation of this definition, we thus have a spectral sequence of E_1-term: $E_1^{p,q} = H^{p+q}(C_p)$.

We consider the case where $X_\bullet$ is bounded and exhaustive *i.e.*

$$X_p = \begin{cases} 0 & \text{if } p \ll 0 \\ X & \text{if } p \gg 0. \end{cases}$$

In this case, the spectral sequence is concentrated in a band with respect to p and we get a convergent spectral sequence

$$E_1^{p,q} = H^{p+q}(C_p) \Rightarrow H^{p+q}(X).$$

The filtration on the abutment is then given by the formula

$$Filt^r(H^{p+q}(X)) = \mathrm{Ker}\left(H^{p+q}(X) \to H^{p+q}(X_r)\right).$$

3.1.2. *Definition.* We apply the preceding formalism to the classical coniveau filtration on schemes which we now recall.

Definition 3.3. Let X be a scheme.

A *flag* on X is a decreasing sequence $(Z^p)_{p\in\mathbb{N}}$ of closed subschemes of X such that for all integer $p \geq 0$, Z^p is of codimension greater than p in X. We let $\mathcal{D}(X)$ be the set of flags of X, ordered by termwise inclusion.

We will consider a flag $(Z^p)_{p\in\mathbb{N}}$ has a $\mathbb{Z}$-sequence by putting $Z^p = X$ for $p < 0$. It is an easy fact that, with the above definition, $\mathcal{D}(X)$ is right filtering.

Recall that a pro-object of a category $\mathcal{C}$ is a (*covariant*) functor F from a left filtering category $\mathcal{I}$ to the category $\mathcal{C}$. Usually, we will denote F by the intuitive notation "$\varprojlim_{i\in\mathcal{I}}$" F_i and call it the *formal projective limit.*

Definition 3.4. Let X be a scheme. We define the coniveau filtration of X as the sequence $(F_pX)_{p\in\mathbb{Z}}$ of pro-open subschemes of X such that :

$$F_pX = \text{"}\varprojlim_{Z^* \in \mathcal{D}(X)^{op}}\text{"} (X - Z^p).$$

We denote by $j_p : F_{p-1}X \to F_pX$ the canonical pro-open immersion,

$$j_p = \text{"}\varprojlim_{Z^* \in \mathcal{D}(X)^{op}}\text{"} \big((X - Z^{p-1}) \to (X - Z^p)\big).$$

Unfortunately, this is a filtration by pro-schemes, and if we apply to it the functor M termwise, we obtain a filtration of $M(X)$ in the category pro–$DM^{eff}_{gm}(k)$. This latter category is never triangulated. Nonetheless, the definition of an exact couple still makes sense for the pro-objects of a triangulated category if we replace distinguished triangles by pro-distinguished triangles[16]. We consider the tower of pro-motives above the constant pro-motive $M(X)$

$$... \to M(F_{p-1}X) \xrightarrow{j_{p*}} M(F_pX) \to ...$$

We define the following canonical pro-cone

$$Gr^M_p(X) = \text{"}\varprojlim_{Z^* \in \mathcal{D}(X)^{op}}\text{"} M\big(X - Z^p/X - Z^{p-1}\big).$$

using Definition 1.2 and its functoriality. We thus obtain pro-distinguished triangles:

$$M(F_{p-1}X) \xrightarrow{j_{p*}} M(F_pX) \xrightarrow{\pi_p} Gr^M_p(X) \xrightarrow{\delta_p} M(F_{p-1}X)[1].$$

Definition 3.5. Consider the above notations. We define the motivic coniveau exact couple associated with X in pro$-DM^{eff}_{gm}(k)$ as

$$D_{p,q} = M(F_pX)[-p-q], \qquad E_{p,q} = Gr^M_p(X)[-p-q],$$

with structural morphisms

$$\alpha_{p,q} = j_p[-p-q],\ \beta_{p,q} = \pi_p[-p-q],\ \gamma_{p,q} = \delta_p[-p-q].$$

According to the notation which follows Definition 3.1, the differential associated with the motivic coniveau exact couple is equal to the composite map of the following diagram:

(3.5.a)
$$\begin{tikzcd} Gr^M_{p+1}(X)[-p-q-1] \arrow[r, "\delta_{p+1}"] \arrow[drr, "d_{p+1,q}"'] & M(F_pX)[-p-q] \arrow[d, equal] & \\ & M(F_pX)[-p-q] \arrow[r, "\pi_p"'] & Gr^M_p(X)[-p-q]. \end{tikzcd}$$

3.2. Computations.

3.2.1. *Recollection and complement on generic motives.* We will call *function field* any finite type field extension E/k. A *model* of the function field E will be a connected smooth scheme X/k with a given k-isomorphism between the function field of X and E. Recall the following definition from [Dég08b, 3.3.1] :

[16] *i.e.* the formal projective limit of distinguished triangles.

Definition 3.6. Consider a function field E/k and an integer $n \in \mathbb{Z}$. We define the *generic motive of* E *with weight* n as the following pro-object of $DM_{gm}(k)$:

$$M(E)(n)[n] := \underset{A \subset E,\ \mathrm{Spec}(A)\ \text{model of}\ E/k}{\text{"}\varprojlim\text{"}} M(\mathrm{Spec}(A))(n)[n].$$

We denote by $DM_{gm}^{(0)}(k)$ the full subcategory of pro $- DM_{gm}(k)$ consisting of the generic motives.

Of course, given a function field E with model X/k, the pro-object $M(E)$ is canonically isomorphic to the pro-motive made by the motives of non empty open subschemes of X.

3.7. The interest of generic motives lies in their functoriality which we now review :
(1) Given any extension of function fields $\varphi : E \to L$, we get a morphism $\varphi^* : M(L) \to M(E)$ (by covariant functoriality of motives).
(2) Consider a finite extension of function fields $\varphi : E \to L$. One can find respective models X and Y of E and L together with a finite morphism of schemes $f : Y \to X$ which induces on function fields the morphism φ through the structural isomorphisms.
For any open subscheme $U \subset X$, we put $Y_U = Y \times_X U$ and let $f_U : Y_U \to U$ be the morphism induced by f. It is finite and surjective. In particular, its graph seen as a cycle in $U \times Y_U$ defines a finite correspondence from U to Y_U, denoted by ${}^t f_U$ and called the transpose of f_U (as in 2.14). We define the *norm morphism* $\varphi_* : M(E) \to M(L)$ as the well defined pro-morphism (see [Dég08b, 5.2.9])

$$\underset{U \subset X}{\text{"}\varprojlim\text{"}} \left(M(U) \xrightarrow{({}^t f|_U)_*} M(Y_U) \right)$$

through the structural isomorphisms of the models X and Y.
(3) Consider a function field E and a unit $x \in E^\times$. Given a smooth sub-k-algebra $A \subset E$ which contains x and x^{-1}, we get a morphism $f_A : \mathrm{Spec}(A) \to \mathbb{G}_m$. Recall the canonical decomposition $M(\mathbb{G}_m) = \mathbb{Z} \oplus \mathbb{Z}(1)[1]$ and consider the associated projection $M(\mathbb{G}_m) \xrightarrow{\pi} \mathbb{Z}(1)[1]$. We associate with the unit x the morphism $\gamma_x : M(E) \to M(E)(1)[1]$ defined as

$$\underset{x, x^{-1} \in A \subset E}{\text{"}\varprojlim\text{"}} \big(M(\mathrm{Spec}(A)) \xrightarrow{f_{A*}} M(\mathbb{G}_m) \xrightarrow{\pi} \mathbb{Z}(1)[1] \big).$$

One can prove moreover that if $x \neq 1$, $\gamma_x \circ \gamma_{1-x} = 0$ and $\gamma_{1-x} \circ \gamma_x = 0$ so that any element $\sigma \in K_n^M(E)$ of Milnor K-theory defines a morphism $\gamma_\sigma : M(E) \to M(E)(n)[n]$ (see also [Dég08b, 5.3.5]).
(4) Let E be a function field and v a discrete valuation on E with ring of integers $\mathcal{O}_v$ essentially of finite type over k. Let $\kappa(v)$ be the residue field of v.
As k is perfect, there exists a connected smooth scheme X with a point $x \in X$ of codimension 1 such that $\mathcal{O}_{X,x}$ is isomorphic to $\mathcal{O}_v$. This implies X is a model of E/k. Moreover, reducing X, one can assume the closure Z of x in X is smooth so that it becomes a model of $\kappa(v)$.
For an open neighborhood U of x in X, we put $Z_U = Z \times_X U$. We define the *residue morphism* $\partial_v : M(\kappa(v))(1)[1] \to M(E)$ associated with (E, v) as the pro-morphism

$$\underset{x \in U \subset X}{\text{"}\varprojlim\text{"}} \big(M(Z_U)(1)[1] \xrightarrow{\partial_{U, Z_U}} M(U - Z_U) \big).$$

The fact this pro-morphism is well defined evidently relies on the transversal case of Proposition 1.19 (see also [Dég08b, 5.4.6]).

Remark 3.8. These morphisms satisfy a set of relations which in fact corresponds exactly to the axioms of a cycle premodule by M. Rost (cf [Ros96, (1.1)]). We refer the reader to [Dég08b, 5.1.1] for a precise statement.

3.9. Consider again the situation and notations of the point (2) in paragraph 3.7. With the Gysin morphism we have introduced before, one can give another definition for the norm morphism of generic motives.
Indeed, for any open subscheme U of X, the morphism $f_U : Y_U \to U$ is finite of relative dimension 0 and thus induces a Gysin morphism $f_U^* : M(U) \to M(Y_U)$. Using Proposition 2.10, these morphisms are natural with respect to U. Thus, we get a morphism of pro-objects

$$\text{"}\varprojlim_{U \subset X}\text{"} \big(M(U) \xrightarrow{f_U^*} M(Y_U))\big).$$

which induces through the structural isomorphisms of the models X and Y a morphism $\varphi'_* : M(E) \to M(L)$.

Lemma 3.10. *Consider the above notations. Then,* $\varphi'_* = \varphi_*$.

Proof. By functoriality, we can restrict the proof to the cases where L/E is separable or L/E is purely inseparable.

In the first case, we can choose a model $f : Y \to X$ of φ which is étale. Then the lemma follows from Proposition 2.15.

In the second case, we can assume that $L = E[\sqrt[q]{a}]$ for $a \in E$. Let $A \subset E$ be a sub-k-algebra containing a such that $X = \mathrm{Spec}(A)$ is a smooth scheme. Let $B = A[t]/(t^q - a)$. Then $Y = \mathrm{Spec}(B)$ is again a smooth scheme (over k) and the canonical morphism $f : Y \to X$ is a model of L/E. We consider its canonical factorisation $Y \xrightarrow{i} \mathbb{P}^1_X \xrightarrow{p} X$ corresponding to the parameter t, together with the following diagram made of two cartesian squares:

$$\begin{tikzcd} Y \times_X Y \arrow[r, "j"] \arrow[d] & \mathbb{P}^1_Y \arrow[r, "q"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ Y \arrow[r, "i"] & \mathbb{P}^1_X \arrow[r, "p"] & X. \end{tikzcd}$$

The scheme $Y \times_X Y$ is non reduced and its reduction is Y. Moreover, the canonical immersion $Y \to Y \times_X Y$ is an exact thickening of order q in Y (cf paragraph 1.18). Thus, the following diagram is commutative :

$$\begin{tikzcd} M(Y) \arrow[d, equal] \arrow[dr, phantom, "(1)"] & M(\mathbb{P}^1_Y) \arrow[l, "j^*"'] \arrow[dr, phantom, "(2)"] & M(Y) \arrow[l, "q^*"'] \\ M(Y) & M(\mathbb{P}^1_X) \arrow[l, "i^*"] \arrow[u, "{}^t f'_*"'] & M(X)\,. \arrow[l, "p^*"] \arrow[u, "{}^t f_*"'] \end{tikzcd}$$

Indeed, part (2) (resp. (1)) is commutative by [Dég08b, 2.2.15] (resp. [Dég08b, 2.5.2: (2)]). Thus $f^* = {}^t f_*$ and this concludes. □

3.2.2. *The graded terms.* For a scheme X, we denote by $X^{(p)}$ the set of points of X of codimension p. If x is a point of X, $\kappa(x)$ will denote its residue field. The symbol "$\prod$" denotes the product in the category of pro-motives.

Lemma 3.11. *Let X be a smooth scheme and consider the notations of Definition 3.5. Then, for all integer $p \geq 0$, the purity isomorphism of Proposition 1.12 induces* *a canonical isomorphism*

$$Gr_p^M(X) \xrightarrow{\epsilon_p} \text{"}\prod_{x \in X^{(p)}}\text{"}\; M(\kappa(x))\,(p)[2p].$$

In particular, for any point $x \in X^{(p)}$ we get a canonical projection map:

$$\pi_x : Gr_p^M(X) \to M(\kappa(x))\,(p)[2p]. \tag{3.11.a}$$

Proof. Let $\mathcal{I}_p$ be the set of pairs (Z, Z') such that Z is a reduced closed subscheme of X of codimension p and Z' is a closed subset of Z containing its singular locus. Then

$$Gr_p^M(X) \simeq \text{"}\varprojlim_{(Z,Z')\in\mathcal{I}_p}\text{"}\; M(X - Z'/X - Z)\,.$$

For any element (Z, Z') of $\mathcal{I}_p$, under the purity isomorphism, we get:
$M(X - Z'/X - Z) \simeq M(Z - Z')\,(p)[2p]$.

For any point x of X, we let $Z(x)$ be the reduced closure of x in X and $\mathcal{F}(x)$ be the set of closed subschemes Z' of $Z(x)$ containing the singular locus $Z(x)_{sing}$ of $Z(x)$. By additivity of motives, we finally get an isomorphism:

$$Gr_p^M(X) \simeq \text{"}\prod_{x\in X^{(p)}}\text{"}\;\text{"}\varprojlim_{Z'\in\mathcal{F}(x)}\text{"}\; M(Z(x) - Z')\,(p)[2p].$$

This implies the lemma because $Z(x) - Z(x)_{sing}$ is a model of $\kappa(x)$. □

3.2.3. *The differentials.*

3.12. Let X be a scheme essentially of finite type[17] over k and consider a couple $(x, y) \in X^{(p)} \times X^{(p+1)}$.

Assume that y is a specialisation of x. Let Z be the reduced closure of x in X and $\tilde{Z} \xrightarrow{f} Z$ be its normalisation. Each point $t \in f^{-1}(y)$ corresponds to a discrete valuation v_t on $\kappa(x)$ with residue field $\kappa(t)$. We denote by $\varphi_t : \kappa(y) \to \kappa(t)$ the morphism induced by f. Then, we define the following morphism of generic motives

$$\partial_y^x = \sum_{t\in f^{-1}(y)} \partial_{v_t} \circ \varphi_{t*} : M(\kappa(y))(1)[1] \to M(\kappa(x)) \tag{3.12.a}$$

using the notations of 3.7.
If y is not a specialisation of x, we put conventionally $\partial_y^x = 0$.

Proposition 3.13. *Consider the above hypothesis and notations. If X is smooth then the following diagram is commutative:*

$$\begin{tikzcd}
Gr_{p+1}^M(X) \arrow[r, "d_{p+1,-p-1}"] \arrow[d, "\pi_y"'] & Gr_p^M(X)[1] \arrow[d, "\pi_x"] \\
M(\kappa(y))(p+1)[2p+2] \arrow[r, "\partial_y^x"] & M(\kappa(x))(p)[2p+1]
\end{tikzcd}$$

where the vertical maps are defined in (3.11.a) *and* $d_{p+1,-p-1}$ *in* (3.5.a).

Of course, this proposition determines every differentials of the motivic coniveau exact couple as $d_{p,q} = d_{p,-p}[-p-q]$.

[17] For the purpose of the next proposition, we need only the case where X is smooth but the general case treated here will be used later.

Proof. According to Definition 3.5, the morphism $d_{p+1,-p-1}$ is the formal projective limit of the morphisms

$$M(X-W/X-Y) \to M(X-Y)\,[1] \to M(X-Y/X-Z)\,[1], \tag{3.13.a}$$

for large enough closed subsets $W \subset Y \subset Z$ of X such that $\mathrm{codim}_X(Z) = p$, $\mathrm{codim}_X(Y) = p+1$ and $\mathrm{codim}_X(W) = p+2$. For the proof, we will consider $W \subset Y \subset Z$ as above, assume that $y \in Y$, $x \in Z$ and study (3.13.a) for Z, Y, W large enough. To simplify the notations, we will replace X by $X - W$, which means we can substract any subset of X if it has codimension greater than $p+1$.

First of all, enlarging Y, we can assume that it contains the singular locus of Z. Because the singular locus of Y has codimension greater than $p+1$ in X, we can assume by reducing X that Y is smooth. Then, using the purity isomorphism, the composite map (3.13.a) is isomorphic to the following one:

$$M(Y)((p+1)) \xrightarrow{\partial_{X,Y}} M(X-Y)[1] \xrightarrow{i_Y^*} M(Z-Y)((p))[1]$$

where $i_Y : (Z-Y) \to (X-Y)$ is the obvious restriction of the canonical closed immersion $i : Y \to Z$.

Let Y_y (resp. Z_x) be the irreducible component of Y (resp. Z) containing y (resp. x). As Y is smooth, we can write $Y = Y_y \sqcup Y'_y$. As $(Z-Y)$ is smooth, if we put $\hat{Y}_x = Y \times_Z Z_x$ then $(Z_x - \hat{Y}_x)$ is a connected component of $(Z-Y)$. We denote by $i_x : (Z_x - \hat{Y}_x) \to (X-Y)$ the obvious restriction of i_Y. According to Proposition 1.36, the following diagram is commutative:

$$\begin{tikzcd}
M(Y)((p+1)) \arrow[r, "\partial_{X,Y}"] \arrow[d] & M(X-Y)[1] \arrow[r, "i_Y^*"] \arrow[d, equal] & M(Z-Y)((p))[1] \arrow[d] \\
M(Y_y)((p+1)) \arrow[r, "\partial_{X-Y'_y,Y_y}"] \arrow[rr, bend right, "\partial^{Z,x}_{Y,y}"'] & M(X-Y)[1] \arrow[r, "i_x^*"] & M(Z_x-\hat{Y}_x)((p))[1]
\end{tikzcd}$$

where the vertical maps are the canonical projections. The proposition is equivalent to show that the formal projective limit of the maps $\partial^{Z,x}_{Y,y}$ for Z, Y, W large enough is equal to ∂^x_y (remember we have identified X with $X - W$).

Assume that y is not a specialisation of x. Then $Y_y \cap Z_x$ has codimension greater than $p+1$ in X. Therefore, reducing X again, we can assume $Y_y \cap Z_x = \emptyset$. Thus $\hat{Y}_x = Y'_y \cap Z_x$ and we can consider the following cartesian square of closed immersions between smooth schemes:

$$\begin{tikzcd}
\emptyset \arrow[r, "k"] \arrow[d, "l"'] & Y_y \arrow[d] \\
(Z_x - \hat{Y}_x) \arrow[r] & (X - Y'_y).
\end{tikzcd}$$

Then, the relation (2) of Theorem 1.34 applied to this square gives: $\partial_{X-Y'_y,Y_y} \circ i_x^* = 0$. Thus the proposition is proved in that case.

We now consider the case where y is a specialisation of x *i.e.* $Y_y \subset Z_x$. Then $Y_y \subset \hat{Y}_x$: to simplify the notation, we can assume that $Z = Z_x$ *i.e.* Z is irreducible with generic point x. Let $f : \tilde{Z} \to Z$ be the normalization of Z. The singular locus $\tilde{Z}_{sing}$ of $\tilde{Z}$ is everywhere of codimension greater than 1 in $\tilde{Z}$. Thus, $f(\tilde{Z}_{sing})$ is everywhere of codimension greater than $p+1$ in X, and we can assume by reducing X again that $\tilde{Z}$ is smooth.

Let us denote by $\tilde{Y}$ (resp. $\tilde{Y}_y$, $\tilde{Y}'_y$) the reduced inverse image of Y (resp. Y_y, Y'_y) along f. Reducing X again, we can assume that $\tilde{Y}_y$ is smooth and $\tilde{Y}_y \cap \tilde{Y}'_y = \emptyset$. Moreover, we can assume that every connected component of $\tilde{Y}_y$ dominates Y_y (by reducing X, we can throw away the non dominant connected components). In other words, the map $g_y : \tilde{Y}_y \to Y_y$ induced by f is finite and equidimensional. Then we can consider the following topologically cartesian square:

$$\begin{tikzcd}
\tilde{Y}_y \arrow[r, "\tilde{\sigma}"] \arrow[d, "g_y"'] & (\tilde{Z} - \tilde{Y}'_y) \arrow[d] \\
Y_y \arrow[r, "\sigma"] & (X - Y'_y)
\end{tikzcd}$$

where σ and $\tilde{\sigma}$ are the obvious closed immersions and the right vertical map is induced by the composite map $\tilde{Z} \xrightarrow{f} Z \xrightarrow{i} X$. Note that taking the respective complements of $\tilde{\sigma}$ and σ in the source and target of this composite map, it induces the following one:

$$(\tilde{Z} - \tilde{Y}) \xrightarrow{h} (Z - Y) \xrightarrow{i} (X - Y).$$

Thus, applying Proposition 2.13 to the preceding square together with Proposition 2.9, we obtain the following commutative diagram:

$$\begin{tikzcd}
M(Y_y)((p+1)) \arrow[r, "\partial_{X-Y'_y,Y_y}"'] \arrow[rr, bend left, "\partial^{Z,x}_{Y,y}"] \arrow[d, equal] & M(X-Y)[1] \arrow[r, "i^*"'] & M(Z-Y)((p))[1] \arrow[d, "h^*"] \\
M(Y_y)((p+1)) \arrow[r, "g_y^*"'] & M\Big(\tilde{Y}_y\Big)((p+1)) \arrow[r, "\partial_{\tilde{Z}-\tilde{Y}'_y,\tilde{Y}_y}"'] & M\Big(\tilde{Z}-\tilde{Y}\Big)((p))[1].
\end{tikzcd}$$

Note that the set of connected components of the smooth scheme $\tilde{Y}_y$ corresponds bijectively to the set $f^{-1}(y)$. For any $t \in f^{-1}(y)$, we denote by $\tilde{Y}_t$ the corresponding connected component so that $\tilde{Y}_y = \sqcup_{t\in f^{-1}(y)} \tilde{Y}_t$. Note that $\tilde{Y}_t$ is also a connected component of $\tilde{Y}$. We put:

$$\tilde{Z}_t = \tilde{Z} - (\tilde{Y} - \tilde{Y}_t).$$

This is an open subscheme of $\tilde{Z}$ containing $\tilde{Y}_t$ and $(\tilde{Z}_t - \tilde{Y}_t) = (\tilde{Z} - \tilde{Y})$. Applying Proposition 1.36, we obtain the following commutative squares:

$$\begin{tikzcd}
M(Y_y)((p+1)) \arrow[r, "g_y^*"] \arrow[d, equal] & M\Big(\tilde{Y}_y\Big)((p+1)) \arrow[r, "\partial_{\tilde{Z}-\tilde{Y}'_y,\tilde{Y}_y}"] & M\Big(\tilde{Z}-\tilde{Y}\Big)((p))[1] \arrow[d, equal] \\
M(Y_y)((p+1)) \arrow[r, "\sum_t g_t^*"] \arrow[rr, bend right, "\tilde{\partial}^{Z,x}_{Y,y}"'] & \bigoplus_{t\in f^{-1}(y)} M\Big(\tilde{Y}_t\Big)((p+1)) \arrow[u, "\sim"'] \arrow[r, "\sum_t \partial_{\tilde{Z}_t,\tilde{Y}_t}"] & M\Big(\tilde{Z}-\tilde{Y}\Big)((p))[1]
\end{tikzcd}$$

where the middle vertical map is the canonical isomorphism. We can now identify ∂^x_y with the formal projective limit of $\tilde{\partial}^{Z,x}_{Y,y}$ for Y, W large enough (remember we have assumed $Z = Z_x$). In view of formula (3.12.a), this is justified because:

- h is birational and $(\tilde{Z} - \tilde{Y})$ is a smooth model of $\kappa(x)$.
- The closed pair $(\tilde{Z}_t, \tilde{Y}_t)$ is smooth of codimension 1 and the local ring of $\mathcal{O}_{\tilde{Z}_t,\tilde{Y}_t}$ is isomorphic (through h) to the valuation ring $\mathcal{O}_{v_t}$ corresponding to the valuation v_t on $\kappa(x)$ considered in paragraph 3.12. □

4. Cohomological realization

We fix a Grothendieck abelian category $\mathscr{A}$ and consider a cohomological functor

$$H : DM_{gm}(k)^{op} \to \mathscr{A},$$

simply called a *realization functor*.

To the realization functor H, we can associate a twisted cohomology theory such that for a smooth scheme X and a pair of integers $(n,i) \in \mathbb{Z}^2$,

$$H^n(X,i) = H\big(M(X)\,(-i)[-n]\big).$$

By the very definition, this functor is contravariant, not only with respect to morphisms of smooth schemes but also for finite correspondences. According to the construction of Definition 2.7, it is covariant with respect to projective morphisms.

4.1. **The coniveau spectral sequence.** The functor H admits an obvious extension to pro-objects $\bar{H}$: pro$-DM_{gm}(k)^{op} \to \mathscr{A}$ which sends pro-distinguished triangles to long exact sequences since right filtering colimits are exact in $\mathscr{A}$. In particular, for any function fields E/k, we define

$$\bar{H}^r(E,n) = \varinjlim_{A \subset E} H^r(\mathrm{Spec}(A), n)$$

where the limit is taken over the models of E/k.

Fix an integer $n \in \mathbb{Z}$. We apply the functor $\bar{H}(?(n))$ to the pro-exact couple of 3.5. We then obtain a converging spectral sequence which, according to Lemma 3.11, has the form:

$$E_1^{p,q}(X,n) = \bigoplus_{x \in X^{(p)}} \bar{H}^{q-p}(\kappa(x), n-p) \Rightarrow H^{p+q}(X,n). \tag{4.0.b}$$

This is the coniveau spectral sequence of X with coefficients in H.

Remark 4.1. (*Bloch-Ogus theory*) The filtration on $H^*(X,n)$ which appears on the abutment of the spectral sequence (4.0.b) is the filtration which appears originally[18] in [Gro69] and [Gro68, 1.10],

$$N^r H^*(X,n) = \mathrm{Ker}\left(H^*(X,n) \to \bar{H}(M^{(r)}(X)(n)[*])\right),$$

formed by cohomology classes which vanish on an open subset with complementary of (at least) codimension r.

One can relate this spectral sequence to the one introduced in [BO74, (3.11)]. Indeed, without referring to the duality for the cohomological theory H^*, we can obviously extend H^* to a cohomology theory with support using relative motives. This is all what we need to define the spectral sequence (3.11) of *loc. cit.* Then the later spectral sequence coincides with the spectral sequence (4.0.b).

4.2. **Cycle modules.** Cycle modules have been introduced by M. Rost in [Ros96] as a notion of "coefficient systems" suitable to define "localization complexes for varieties". We recall below this theory in a way suitable for our needs.

4.2. The first step in Rost's theory is the notion of a *cycle premodule.* Basically, it is a covariant functor from the category of function fields to the category of graded

[18] In [Gro69], the filtration is called "filtration arithmétique" and in [Gro68], "filtration par le type dimensionel". One can also find in the latter article the root of the actual terminology, filtration by niveau, which was definitively adopted after the fundamental work of [BO74].

abelian groups satisfying an enriched functoriality exactly analog to that of Milnor K-theory K_*^M. In our context, we will define[19] a cycle premodule as a functor

$$\phi : DM_{gm}^{(0)}(k)^{op} \to \mathscr{A}.$$

Usually, we put $\phi(M(E)(-n)[-n]) = \phi_n(E)$ so that ϕ becomes a graded functor on function fields. In view of the description of the functoriality of generic motives recalled in 3.7, ϕ is equipped with the following structural maps:

(1) For any extension of function fields, $\varphi : E \to L$, a *corestriction* $\varphi_* : \phi_*(E) \to \phi_*(L)$ of degree 0.
(2) For any finite extension of function fields, $\varphi : E \to L$, a *norm* $\varphi^* : \phi_*(L) \to \phi_*(E)$ of degree 0, also denoted by $N_{L/E}$.
(3) For any function field E, $\phi_*(E)$ admits a $K_*^M(E)$-graded module structure.
(4) For any valued function field (E, v) with ring of integers essentially of finite type over k and residue field $\kappa(v)$, a *residue* $\partial_v : \phi_*(E) \to \phi_*(\kappa(v))$ of degree -1.

Definition 4.3. Consider again a realization functor H. For any pair of integers (q, n), we associate with H a cycle module $\hat{H}^{q,n}$ as the restriction of the functor $\bar{H}^q(., n)$ to the category $DM_{gm}^{(0)}(k)$.

Concretely, $\hat{H}^{q,n}_{-p}(E) = \bar{H}^{q-p}(E, n-p)$. Remark that,

$$\forall a \in \mathbb{Z}, \hat{H}^{q-a,n-a}_* = \hat{H}^{q,n}_{*+a} \tag{4.3.a}$$

and this is an equality of cycle modules (up to the shift in the graduation). In our notation, the choice of the grading is somewhat redundant but it will be convenient for our needs.

4.4. Rost considers further axioms on a cycle premodule ϕ which allow to build a complex from ϕ (cf [Ros96, (2.1)]). We recall these axioms to the reader using the morphisms introduced in 3.12. We say that a cycle premodule ϕ is a *cycle module* if the following two conditions are fulfilled :

(FD) Let X be a normal scheme essentially of finite type over k, η its generic point and E its functions field. Then for any element $\rho \in \phi_i(E)$, $\phi(\partial^\eta_x)(\rho) = 0$ for all but finitely many points x of codimension 1 in X.

(C) Let X be an integral local scheme essentially of finite type over k and of dimension 2. Let η (resp. s) be its generic (resp. closed) point, and E (resp. κ) be its function (resp. residue) field. Then, for any integer $n \in \mathbb{Z}$, the morphism

$$\sum_{x \in X^{(1)}} \phi_{n-1}(\partial^x_s) \circ \phi_n(\partial^\eta_x) : \phi_n(E) \to \phi_{n-2}(\kappa),$$

well defined under (FD), is zero.

When these conditions are fulfilled, for any scheme X essentially of finite type over k, we define according to [Ros96, (3.2)] a graded complex of *cycles with coefficients in ϕ* whose i-th graded[20] p-cochains are

$$C^p(X; \phi)_i = \bigoplus_{x \in X^{(p)}} \phi_{i-p}(\kappa(x)) \tag{4.4.a}$$

[19]Indeed, when $\mathscr{A}$ is the category of abelian groups, it is proved in [Dég08b, th. 5.1.1] that such a functor defines a cycle premodule in the sense of M. Rost.

[20]This graduation follows the convention of [Ros96, §5] except for the notation. The notation $C^p(X; \phi, i)$ used by Rost would introduce a confusion with twists.

and with p-th differential equal to the well defined morphism

$$d^p = \sum_{(x,y)\in X^{(p)}\times X^{(p+1)}} \phi(\partial_x^y). \tag{4.4.b}$$

The cohomology groups of this complex are called the *Chow groups with coefficients in* ϕ and denoted by $A^*(X;\phi)$ in [Ros96]. Actually, $A^*(X;\phi)$ is bigraded according to the bigraduation on $C^*(X;\phi)$.

4.5. Consider the cycle modules $\hat{H}^{q,n}$ introduced in Definition 4.3. According to this definition, the E_1-term of the spectral sequence (4.0.b) can be written as:

$$E_1^{p,q} = C^p(X, \hat{H}^{q,n})_0$$

if we use the formula (4.4.a) for the right hand side. Moreover, according to Proposition 3.13, the differential $d_1^{p,q}$ of the spectral sequence are precisely given by the formula:

$$d_1^{p,q} = \sum_{(x,y)\in X^{(p)}\times X^{(p+1)}} \hat{H}^{q,n}_{-p}(\partial_y^x).$$

This is precisely the formula (4.4.b) for the cycle premodule $\hat{H}^{q,n}$. Note that proposition *loc. cit.* implies in particular that this morphism is well defined. In other words, we have obtained that the graded abelian group $C^*(X, \hat{H}^{q,n})_0$ together with the well defined differentials of shape (4.4.b) is a complex. We deduce from this fact the following proposition:

Proposition 4.6. *Consider the previous notations.*

(i) *For any integer $q \in \mathbb{Z}$, the cycle premodule $\hat{H}^{q,n}$ is a cycle module.*

(ii) *For any smooth scheme X and any couple (q,n) of integers, there is an equality of complexes:*

$$E_1^{*,q}(X,n) = C^*(X;\hat{H}^{q,n})_0,$$

where the left hand side is the complex made by the line of the first page of the spectral sequence (4.0.b)*.*

Proof. The point (ii) follows from the preliminary 4.5.

We prove point (i), axiom (FD). Consider a normal scheme X essentially of finite type over k. We can assume it is affine of finite type. Then there exists a closed immersion $X \xrightarrow{i} \mathbb{A}^r_k$ for an integer $r \geq 0$. According to the preliminary 4.5, for any integer $a \in \mathbb{Z}$, $C^*(\mathbb{A}^r_k;\hat{H}^{q-a,n-a})_0$ is a well defined complex. Note this complexe is also equal to $C^*(\mathbb{A}^r_k;\hat{H}^{q,n})_a$ according to (4.3.a). Thus, axiom (FD) for the cycle premodule $\hat{H}^{q,n}$ follows from the fact

$$\hat{H}^{q,n}_a(E) \subset C^r(\mathbb{A}^r_k;\hat{H}^{q,n})_a$$

and the definition of the differentials given above.

For axiom (C), we consider an integral local scheme X essentially of finite type over k and of dimension 2. We have to prove that $C^*(X;\hat{H}^{q,n})$ is a complex – the differentials are well defined according to (FD). To this aim, we can assume X is affine of finite type over k. Then, there exists a closed immersion $X \to \mathbb{A}^r_k$. From the definition given above, for any integer $a \in \mathbb{Z}$, we obtain a monomorphism

$$C^p(X;\hat{H}^{q,n})_a \to C^p(\mathbb{A}^r_k;\hat{H}^{q,n})_a = C^p(\mathbb{A}^r_k;\hat{H}^{q-a,n-a})_0$$

which is compatible with differentials. Thus the conclusion follows from the preliminary 4.5. □

Remark 4.7. This proposition gives a direct proof of the main theorem [Dég08b, 6.2.1] concerning the second affirmation.

Corollary 4.8. *Using the notations of the previous proposition, the E_2-terms of the coniveau spectral sequence* (4.0.b) *are :*

$$E_2^{p,q}(X,n) = A^p(X;\hat{H}^{q,n})_0 \Rightarrow H^{p+q}(X,n).$$

Moreover, for any couple of integers (q,n) and any smooth proper scheme X, the term $E_2^{0,q}(X,n)$ is a birational invariant of X.

The second assertion follows from [Ros96, 12.10].

Example 4.9. Consider the functor $H_{\mathcal{M}} = \mathrm{Hom}_{DM_{gm}(k)}(.,\mathbb{Z})$, corresponding to motivic cohomology. In this case, following [SV00, 3.2, 3.4], for any function field E,

$$H^q_{\mathcal{M}}(E;\mathbb{Z}(p)) = \begin{cases} 0 & \text{if } q > p \text{ or } p < 0 \\ K_p^M(E) & \text{if } q = p \geq 0 \end{cases} \tag{4.9.a}$$

In particular, from Definition 4.3, $\hat{H}^{n,n}_{\mathcal{M}} = K^M_{*+n}$. In fact, this is an isomorphism of cycle modules. For the norm, this is *loc. cit.* 3.4.1. For the residue, it is sufficient (using for example [Ros96, formula (R3f)]) to prove that for any valued function field (E,v) with uniformizing parameter π, $\partial_v(\pi) = 1$ for the cycle module $\hat{H}^{n,n}_{\mathcal{M}}$. This follows from [Dég08b, 2.6.5] as for any morphism of smooth connected schemes $f : Y \to X$, the pullback $f^* : H^0_{\mathcal{M}}(X;\mathbb{Z}) \to H^0_{\mathcal{M}}(Y,\mathbb{Z})$ is the identity of $\mathbb{Z}$.

As remarked by Voevodsky at the very beginning of his theory, the vanishing mentioned above implies that the coniveau spectral sequence for $H_{\mathcal{M}}$ satisfies $E_1^{p,q}(X,n) = 0$ if $p > n$ or $q > n$. This immediately gives that the edge morphisms of this spectral sequence induce an isomorphism $A^n(X;\hat{H}^{n,n})_0 \to H^{2n}_{\mathcal{M}}(X;\mathbb{Z}(n))$. The left hand side is $A^n(X;K^M_*)_n$ and an easy verification shows this group is $CH^n(X)$.[21]

4.10. In the sequel, we will need the following functoriality of the Chow group of cycles with coefficients in a cycle module ϕ :

- $A^*(.;\phi)$ is contravariant for flat morphisms ([Ros96, (3.5)]).
- $A^*(.;\phi)$ is covariant for proper morphisms ([Ros96, (3.4)]).
- For any smooth scheme X, $A^*(X;\phi)$ is a graded module over $CH^*(X)$ ([Dég06, 5.7 and 5.12]).
- $A^*(.;\phi)$ is contravariant for morphisms between smooth schemes ([Ros96, §12]).

Note that any morphism of cycle modules gives a transformation on the corresponding Chow group with coefficients which is compatible with the functorialities listed above. Moreover, identifying $A^p(.;K^M_*)_p$ with $CH^p(.)$, as already mentioned in the preceding example, the structures above correspond to the usual structures on the Chow group. Finally, let us recall that the maps appearing in the first three points above are defined at the level of the complexes $C^*(.;\phi)$ (introduced in 4.4).

In [BO74], the authors expressed the E_2-term of the coniveau spectral sequence as the Zariski cohomology of a well defined sheaf. We get the same result in our setting. Recall from [FSV00], chap. 5 that a sheaf with transfers is an additive functor $F : (\mathscr{S}m_k^{cor})^{op} \to \mathscr{A}b$ which induces a Nisnevich sheaf on the category of smooth schemes. This theory can obviously be extended by replacing $\mathscr{A}b$ with

[21] Of course, we recover the isomorphism already used in paragraph 1.5, but we will use this more precise form later.

any abelian category $\mathscr{A}$. Let $\mathcal{H}^q(n)$ be the presheaf on the category of smooth schemes such that $\Gamma(X;\mathcal{H}^q(n)) = A^0\big(X;\hat{H}^{q,n}\big)_0$. This group is called the *n-th twisted unramified cohomology* of X with coefficients in H.

Proposition 4.11. *Consider the notations above.*

(1) *The presheaf $\mathcal{H}^q(n)$ is a homotopy invariant Nisnevich sheaf. It has a canonical structure of a sheaf with transfers.*
(2) *There are natural isomorphisms*
$$A^p(X;\hat{H}^{q,n})_0 = H^p_{\mathrm{Zar}}(X;\mathcal{H}^q(n)).$$

Proof. The first assertion follows from [Ros96, (8.6)] and [Dég06, 6.9] while the second one follows from [Ros96, (2.6)]. □

Finally, we have obtained the following shape of the coniveau spectral sequence

$$E_2^{p,q}(X,n) = H^p_{\mathrm{Zar}}\big(X;\mathcal{H}^q(n)\big) \Rightarrow H^{p+q}(X,n). \tag{4.11.a}$$

Remark 4.12. By definition, the presheaf $H^q(?,n)$ is a presheaf with transfers. For any smooth scheme X, there is a canonical map
$$H^q(X,n) \to \Gamma(X;\mathcal{H}^q(n)).$$
One can check this map is compatible with transfers so that we get a morphism of presheaves with transfers
$$H^q(?,n) \to \mathcal{H}^q(n).$$
By definition, the fiber of this map on any function field is an isomorphism. Thus, it follows from one of the main point of Voevodsky's theory (cf [FSV00, chap. 3, 4.20]) that $\mathcal{H}^q(n)$ is the Zariski sheaf associated with $H^q(?,n)$. Thus we recover in our setting the form of the coniveau spectral sequence obtained in [BO74].

4.3. **Algebraic equivalence.** In this section, we assume $\mathscr{A}$ is the category of K-vector spaces for a given field K. We assume furthermore the following conditions on the realization functor H :

(Vanishing) For any function field E and any couple of negative integers (q,n), $\bar{H}^q(E,n) = 0$.

(Rigidity) (i) $H^0(\mathrm{Spec}(k)) = K$.
(ii) For any function field E, the canonical map $\bar{H}^0(k,0) \to \bar{H}^0(E,0)$ is an isomorphism.

The element $1 \in K = H^0(\mathrm{Spec}(k)) = H(\mathbb{Z})$ determines a natural transformation

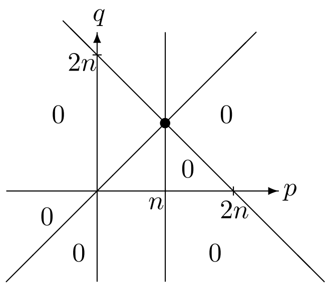
$$\sigma : H_{\mathcal{M}} = \mathrm{Hom}_{DM_{gm}(k)}(.,\mathbb{Z}) \to H. \tag{4.12.a}$$

In particular, we get a cycle class $\sigma_X^n : CH^n(X)_K \to H^{2n}(X,n)$. Let us denote by $Z^n(X,K)$ the group n-codimensional K-cycles in X (simply called *cycles* in what follows) and by $\mathcal{K}^n_{rat}(X)$ (resp. $\mathcal{K}^n_{alg}(X)$) its subgroup formed by cycles rationally (resp. algebraically) equivalent to 0.

Definition 4.13. Using the notations above, we define the group of cycles H-equivalent to 0 as:
$$\mathcal{K}^n_H(X) = \{\alpha \in Z^n(X,K) \mid \sigma_X^n(\alpha) = 0\}.$$

Remark 4.14. The map (4.12.a) induces a morphism of cycle modules $K^M_{*+a} \to \hat{H}^{a,a}$ which corresponds to cohomological symbols $K^M_a(E) \to \bar{H}^a(E,a)$ compatible with corestriction, norm, residues and the action of $K^M_*(E)$.

4.15. We analyze the coniveau spectral sequence (4.0.b) under the assumption (Vanishing) and (Rigidity). The E_1-term is described by the following drawings:



Property (Rigidity) implies that $E_1^{n,n}(X,n) = Z^n(X,K)$. As only one differential goes to $E_r^{n,n}$, we obtain a sequence of epimorphisms:

$$Z^n(X,K) = E_1^{n,n}(X,n) \to E_2^{n,n}(X,n) \to E_3^{n,n}(X,n) \to \dots$$

which become isomorphisms as soon as $r > n$. Thus, if we put

$$\mathcal{K}^n_{(r)}(X) = \mathrm{Ker}(E_1^{n,n}(X,n) \to E_{r+1}^{n,n}(X,n)),$$

we obtain an increasing filtration on $Z^n(X,K)$:

$$\mathcal{K}^n_{(1)}(X) \subset \mathcal{K}^n_{(2)}(X) \subset \dots \subset \mathcal{K}^n_{(n)}(X) \subset Z^n(X,K) \tag{4.15.a}$$

such that $E_r^{n,n}(X,n) = Z^n(X,K)/\mathcal{K}^n_{(r-1)}(X)$.

Note also that $E_n^{n,n} = E_\infty^{n,n}$ is the first step of the coniveau filtration on $H^{2n}(X,n)$ so that we get a monomorphism

$$\epsilon : E_n^{n,n}(X,n) \to H^{2n}(X,n).$$

Note these considerations can be applied to the functor $\mathrm{Hom}_{DM_{gm(k)}}(.,K)$ corresponding to K-rational motivic cohomology. In this case, according to Example 4.9, the $E_r^{n,n} = CH^n(X)_K = H^{2n}_{\mathcal{M}}(X;K(n))$.

Returning to the general case, the natural transformation σ induces a morphism of the coniveau spectral sequences. This induces the following commutative diagram:

$$\begin{tikzcd} & CH^n(X)_K \arrow[r, equal] \arrow[d, two heads, "\tilde\sigma^n_X"] & CH^n(X)_K \arrow[r, "\sim"] \arrow[d, two heads] & H^{2n}_{\mathcal{M}}(X;K(n)) \arrow[d, two heads, "\sigma^n_X"] \\ Z^n(X,K) \arrow[ur, two heads] \arrow[dr, two heads] & & & \\ & E_2^{n,n}(X,n) \arrow[r, two heads] & E_n^{n,n}(X,n) \arrow[r, hook, "\epsilon"] & H^{2n}(X,n) \end{tikzcd} \tag{4.15.b}$$

The following proposition is a generalization of a result of Bloch-Ogus (cf [BO74, (7.4)]).

Proposition 4.16. *Consider the preceding hypothesis and notations. Then the following properties hold:*

(i) *For any scheme X and any integer $n \in \mathbb{N}$, $\mathcal{K}^n_{rat}(X) \subset \mathcal{K}^n_{(1)}(X)$.*
(ii) *For any scheme X and any integer $n \in \mathbb{N}$, $\mathcal{K}^n_{(n)}(X) = \mathcal{K}^n_H(X)$.*

Moreover, the following conditions are equivalent :

(iii) *For any smooth proper scheme X, $\mathcal{K}^1_H(X) = \mathcal{K}^1_{alg}(X)$.*
(iii') *For any smooth proper scheme X and any $n \in \mathbb{N}$, $\mathcal{K}^n_{(1)}(X) = \mathcal{K}^n_{alg}(X)$.*

Note that under the equivalent conditions (iii) and (iii'), the morphism $\tilde\sigma^n_X$ induces, according to (4.11.a), an isomorphism:

$$A^n(X)_K \xrightarrow{\sim} H^n_{\mathrm{Zar}}\big(X;\mathcal{H}^n(n)\big). \tag{4.16.a}$$

Proof. Properties (i) and (ii) are immediate consequences of (4.15.b).

Note that, for $n = 0$, condition (iii') always holds. Note also that (iii) is the particular case $n = 1$ of (iii'), according to assertion (ii). Thus it remains to prove that (iii) implies (iii').

Assume $n > 1$. For the inclusion $\mathcal{K}^n_{alg}(X) \subset \mathcal{K}^n_{(1)}(X)$, we consider $\alpha, \beta \in Z^n(X, K)$ such that α is algebraically equivalent to β. This means there exists a smooth proper connected curve C, points $x_0, x_1 \in C(k)$, and a cycle γ in $Z^n(X \times C, K)$ such that $f_*(g^*(x_0).\gamma) = \alpha$, $f_*(g^*(x_1).\gamma) = \beta$ where $f : X \times C \to X$ and $g : X \times C \to X$ are the canonical projections. Using the functoriality described in paragraph 4.10 applied to the morphism of cycle modules $K^M_* \to \hat{H}^{0,0}$ (Remark 4.14), we get a commutative diagram

$$\begin{tikzcd}
A^1(C; K^M_*)_K \arrow[r, "q^*"] \arrow[d, "(1)"'] & A^1(C \times X; K^M_*)_K \arrow[r, ".\gamma"] \arrow[d] & A^{p+1}(C \times X; K^M_*)_K \arrow[r, "f_*"] \arrow[d] & A^n(X; K^M_*)_K \arrow[d, "(2)"] \\
A^1(C; \hat{H}^{0,0}) \arrow[r, "q^*"] & A^1(C \times X; \hat{H}^{0,0}) \arrow[r, ".\gamma"] & A^{p+1}(C \times X; \hat{H}^{0,0}) \arrow[r, "f_*"] & A^n(X; \hat{H}^{0,0})
\end{tikzcd}$$

Recall the identifications:

$$A^n(X; K^M_*)_n = CH^n(X), \quad A^n(X; \hat{H}^{0,0})_n = A^n(X; \hat{H}^{n,n})_0 = E_2^{n,n}(X, n).$$

According to these ones, the first (resp. n-th) graded piece of the map (1) (resp. (2)) can be identified with the morphism $\tilde{\sigma}^1_X$ (resp. $\tilde{\sigma}^n_X$). In particular, we are reduced to prove that $x_0 - x_1$ belongs to $\mathcal{K}^1_{(1)}(C)$. This finally follows from (iii).

We prove conversely that $\mathcal{K}^n_{(1)}(X) \subset \mathcal{K}^n_{alg}(X)$. Recall $A^n(X; \hat{H}^{n,n})_0$ is the cokernel of the differential (4.4.b)

$$C^{n-1}(X; \hat{H}^{n,n})_0 \xrightarrow{d^{n-1}} C^n(X; \hat{H}^{n,n})_0 = Z^n(X, K).$$

We have to prove that the image of this map consists of the cycles algebraically equivalent to zero. Consider a point $y \in X^{(p-1)}$ with residue field E and an element $\rho \in \bar{H}^{1,1}(E)$. We consider the immersion $Y \xrightarrow{i} X$ of the reduced closure of y in X Using De Jong's theorem, we can consider an alteration $Y' \xrightarrow{f} Y$ such that Y' is smooth over k. Let $\varphi : E \to L$ be the extension of function fields associated with f. According to the basic functoriality of cycle modules 4.10, we obtain a commutative diagram

$$\begin{tikzcd}
\bar{H}^{1,1}(L) \arrow[r, equal] \arrow[d, "N_{L/E}"'] & C^0(Y'; \hat{H}^{1,1})_0 \arrow[r, "d^1_{Y'}"] \arrow[d] & C^1(Y'; \hat{H}^{1,1})_0 \arrow[r, equal] \arrow[d] & Z^1(Y') \arrow[d, "f_*"] \\
\bar{H}^{1,1}(E) \arrow[r, equal] & C^0(Y; \hat{H}^{1,1})_0 \arrow[r, "d^1_Y"] \arrow[d, hook] & C^1(Y; \hat{H}^{1,1})_0 \arrow[r, equal] \arrow[d] & Z^1(Y) \arrow[d, "i_*"] \\
 & C^{p-1}(X; \hat{H}^{p,p})_0 \arrow[r, "d^{p-1}_X"] & C^n(X; \hat{H}^{p,p})_0 \arrow[r, equal] & Z^n(X)
\end{tikzcd}$$

where f_* and i_* are the usual proper pushouts on cycles. Recall from [Ros96, (R2d)] that $N_{L/E} \circ \varphi_* = [L : E].Id$ for the cycle module $\hat{H}^{1,1}$. Thus, $N_{L/E}$ is surjective. As algebraically equivalent cycles are stable by direct images of cycles, we are reduced to the case of the scheme Y', in codimension 1, already obtained above. $\square$

Remark 4.17. In the preceding proof, if we can replace the alteration f by a (proper birational) resolution of singularities, then the theorem is true with integral coefficients. This is the case in characteristic 0 but also when the dimension of X is less or equal than 3 in characteristic p.

4.4. **Mixed Weil cohomologies.** Consider a presheaf of differential graded K-algebras $\mathbb{E}$ over the category of smooth schemes. For any closed pair (X, Z) and any integer n, we put :
$H^n_Z(X, \mathbb{E}) = H^n[\mathrm{Cone}(\mathbb{E}(X) \to \mathbb{E}(X - Z))]$.

Recall from [CD07] that a mixed Weil cohomology theory over k with coefficients in K is a presheaf $\mathbb{E}$ as above satisfying the following properties:

(1) For $X = \mathrm{Spec}(k), \mathbb{A}^1_k, \mathbb{G}_m$,
$$\dim_K H^i(X) = \begin{cases} 1 & \text{if } i = 0 \text{ or } (X = \mathbb{G}_m, i = 1) \\ 0 & \text{otherwise} \end{cases}$$
(2) For any excisive morphism $(Y, T) \to (X, Z)$, the induced morphism $H^*_Z(X, \mathbb{E}) \to H^*_T(Y, \mathbb{E})$ is an isomorphism.
(3) For any smooth schemes X, Y, the exterior cup-product induces an isomorphism
$$\bigoplus_{p+q=n} H^p(X, \mathbb{E}) \otimes_K H^q(X, \mathbb{E}) \to H^n(X \times Y, \mathbb{E}).$$

It is proved in [CD07, 2.7.11] that there is a (covariant) symmetric monoidal triangulated functor
$$R_{\mathbb{E}} : DM_{gm}(k) \to D^b(K)$$
such that
$$H : DM_{gm}(k)^{op} \to K{-}vs,\ \mathcal{M} \mapsto H^0(R_H(\mathcal{M}^\vee))$$
extends the cohomological functor $H^*(., \mathbb{E})$.

The twists on this cohomology theory can be described for any K-vector space V as follows:
$$V(n) = \begin{cases} V \otimes_K \mathrm{Hom}_K(H^1(\mathbb{G}_m, E)^{\otimes n}, K) & \text{if } n \geq 0, \\ V \otimes_K H^1(\mathbb{G}_m, E)^{\otimes, -n} & \text{if } n \leq 0. \end{cases}$$
With these notations, $H(M(X)(-n)[-i]) = H^i(X, \mathbb{E})(n)$. As the functor H is symmetric monoidal, for any smooth projective scheme of dimension n, the morphism $\eta : M(X) \otimes M(X)(-n)[-2n]$ defined in 2.17, induces a perfect pairing, the Poincaré duality pairing,
$$H^i(X, \mathbb{E}) \otimes_K H^{2n-i}(X, \mathbb{E})(n) \to K,\ x \otimes y \mapsto p_*(x.y).$$
As in the preceding section, the unit $1 \in H^0(\mathrm{Spec}(k))$ defines a regulator map
$$\sigma^{q,n} : H^q_{\mathcal{M}}(X; \mathbb{Z}(n)) \to H^q(X, \mathbb{E})(n)$$
compatible with pullbacks, pushouts and products. For any function field L, we deduce a morphism
$$\hat{\sigma}^{q,n} : \bar{H}^q_{\mathcal{M}}(L, \mathbb{Z}(n)) \to \bar{H}^q(L, \mathbb{E})(n)$$
which is compatible with restriction, norm, residues and products. In other words, we get a canonical morphism of cycle modules $\hat{\sigma}^{q,n} : \hat{H}^{q,n}_{\mathcal{M}} \to \hat{\mathbb{E}}^{q,n}$.

Remark 4.18. (1) Regulators are generally understood as "higher cycle classes". In the same way, the preceding morphisms of cycles modules are "higher symbols". Indeed, we obtain the classical (cohomological) symbol map $K^M_n(L) \to \bar{H}^n(L, \mathbb{E})(n)$ in the case $q = n$.
(2) Given a generator of $H^1(\mathbb{G}_m, \mathbb{E})$, we obtain for any integer n, a canonical isomorphism: $H^*(X, \mathbb{E})(n) \simeq H^*(X, \mathbb{E})$. The cycle modules associated with H above thus satisfies the following relation : $\hat{H}^{q,n}_* = \hat{H}^{0,n-q}_{*-q} \simeq \hat{H}^{0,0}_{*-q}$.

Corollary 4.19. *Consider a mixed Weil cohomology $\mathbb{E}$ with the notations above. Let $\mathcal{H}^p(\mathbb{E})$ be the Zariski sheaf associated with $H^p(.,\mathbb{E})$.*

Assume that for any function field L/k and any negative integer i, $\bar{H}^i(L,\mathbb{E}) = 0$. Then, the following conditions are equivalent :

(i) *For any function field L, $\bar{H}^0(L,\mathbb{E}) = K$.*

(ii) *For any integer $p \in \mathbb{N}$ and any projective smooth scheme X, the regulator map $\sigma^{p,p} : H^p_{\mathcal{M}}(.;\mathbb{Z}(p)) \to H^p(.,E)(p)$ induces an isomorphism*

$$A^p(X)_K \to H^p_{\mathrm{Zar}}(X;\mathcal{H}^p(\mathbb{E}))(p).$$

Proof. Remark the assumption implies that for any smooth scheme X and any $i < 0$, $H^i(X,\mathbb{E}) = 0$ – apply the coniveau spectral sequence for X.

$(i) \Rightarrow (ii)$: We apply Proposition 4.16 together with Remark 4.12. Indeed, assumption (Vanishing) and (Rigidity) are among our hypothesis. Remark that (Rigidity) and the Poincaré duality pairing implies that for any smooth projective connected curve $p : C \to \mathrm{Spec}(k)$, the morphism $p_* : H^2(C,\mathbb{E})(1) \to H^0(C,\mathbb{E}) = K$ is an isomorphism. Following classical arguments, this together with the multiplicativity of the cycle class map implies that homological equivalence for $\mathbb{E}$ is between rational and numerical equivalence. From Matsusaka's theorem (cf [Mat57]), these two equivalences coincide for divisors. This implies assumption (iii) of Proposition 4.16.

$(ii) \Rightarrow (i)$: For a d-dimensional smooth projective connected scheme X, we deduce from the coniveau spectral sequence and Poincaré duality that $E_2^{d,d}(X,d) = H^{2d}(X,\mathbb{E})(d) = H^0(X,\mathbb{E})$. Thus property (ii) implies $H^0(X,\mathbb{E}) = K$. If L is the function field of X, we deduce that $\bar{H}^0(L,\mathbb{E}) = K$. Considering any function field E, we easily construct an integral projective scheme X over k with function field E. Applying De Jong's theorem, we find an alteration $\tilde{X} \to X$ such that $\tilde{X}$ is projective smooth and the function field L of $\tilde{X}$ is a finite extension of E and the result now follows from the fact $N_{L/E} : \bar{H}^0(L) \to \bar{H}^0(E)$ is a split epimorphism. □

Remark 4.20. Condition (i) in the previous corollary is only reasonable when the base field k is separably closed (or after an extension to the separable closure of k).

Example 4.21. Assume k is a separably closed field of exponential characteristic p. Condition (i) above is fulfilled by the following mixed Weil cohomology theories : algebraic De Rham cohomology if $p = 0$, rational étale l-adic cohomology if $p \neq l$, rigid cohomology (k is the residue field of a complete valuation ring with field of fraction K). The case of rigid cohomology was in fact our motivation.

Remark 4.22. When k is the field of complex numbers and H is algebraic De Rham cohomology, the filtration on cycles (4.15.a) is usually called the *Bloch-Ogus filtration* – see [Fri95]. It can be compared with other filtrations (see [Nor93], [Fri95]). It is an interesting question whether a similar comparison to that of [Nor93, rem. 5.4] can be obtained in the case of rigid cohomology.

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