Triangulated categories of mixed motives

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Introduction

A. Historical background

A.1. The conjectural theory described by Beilinson. In a landmarking paper, [**Beĭ87**], A. Beilinson stated a series of conjectures which offer a complete renewal of the traditional theory of pure motives invented by A. Grothendieck. Namely, he proposes to extend the notion of pure motives to that of mixed motives with two models in mind: mixed Hodge structures defined by P. Deligne one the one hand, perverse sheaves on the other hand defined in [**BBD82**]. One of the main innovation, considered by Beilinson in analogy with the second model, is to consider a triangulated version of mixed motives in which one could hope to find the more involved theory of abelian mixed motives through the concept of t-structures. This hoped for theory was conjecturally described by Beilinson in [**Beĭ87**, 5.10] under the name of *motivic complexes*.

It was modeled (see *loc. cit.*, paragraph A) on the theory of étale l-torsion (resp. l-adic) sheaves and their derived category as introduced fifty years ago by Grothendieck and M. Artin. The major achievement of Grothendieck and his collaborators in [**SGA4**] was to define a theory of coefficients systems relative to any scheme with a collection of operations, $f_*, f^*, f_!, f^!, \otimes, \mathit{Hom}$, satisfying a set of formulas now called the *Grothendieck six functors formalism* (see section A.5 in this introduction for more details). This formalism, formulated in the language of triangulated categories, ultimately encode a very general duality theory. Note however that the complete duality theory for l-torsion étale sheaves was completed only recently by the work of Gabber [**ILO**].

The theory was also conjectured to be deeply linked with Quillen algebraic K-theory (see [**Beĭ87**, 5.10, §B]). In fact, up to torsion and for a regular scheme S, the ext-groups between two Tate motives over S should coincide with Adams graded parts of Quillen algebraic K-theory.[1]

The ideas of Beilinson were very fecund because, not long after the publication of [**Beĭ87**], one had three candidates for a triangulated category of mixed motives, respectively by M. Hanamura, M. Levine, and V. Voevodsky. In this book, we will focus on Voevodsky's theory.

A.2. Voevodsky's motivic complexes. The first attempt of Voevodsky in defining the category of motivic complexes, in his 1992 Harvard's thesis, introduces the fundamental process of $\mathbf{A}^1$-localization, which amounts to make the affine line contractible in the category of mixed motives, by analogy with the topological case. It also involves the use of the h-topology which was to become fundamental in the area of motives and cohomology. These two ingredients given, Voevodsky defined the triangulated category of (effective) h-motives over any base in [**Voe96**].

However, Voevodsky was aware that his definition will give the correct answer to Beilinson's conjectural construction only with rational coefficients. In [**VSF00**, chap. 5], he introduces another definition of motivic complexes over a perfect field with integral coefficients, still using the $\mathbf{A}^1$-localization process but this time introducing the notion of Nisnevich sheaves with transfers and their derived category (see [**MVW06**] for a detailed exposition). At the time being all the properties foreseen by Beilinson are established for this integral category over a perfect field, except for the construction of the motivic t-structure.[2] It remains to extend this definition to arbitrary bases and to establishes the Grothendieck six functors formalism.

The path in this direction was laid down by Voevodsky in [**Voe10a**] were he uses the theory of relative cycles invented by Suslin and Voevodsky to extend the definition of transfers. This definition was also exploited by Ivorra in [**Ivo07**] to extend the definition of geometric motivic complexes of Voevodsky over any base, avoiding the use of sheaves with transfers. Still it entirely remained to construct Grothendieck six functors formalism for this definition.

A.3. Morel and Voevodsky homotopy theory. Soon after the introduction of Voevodsky's motivic complexes, F. Morel and Voevodsky introduced the more general theory of $\mathbf{A}^1$-homotopy of schemes ([**MV99**]) whose design is to extend the framework of algebraic topology to algebraic geometry and is built around the $\mathbf{A}^1$-localization tool. It is within this theory that was invented another important tool in the motivic homotopy theory, the $\mathbf{P}^1$-stabilization process. From the purely motivic point of view, this amounts to invert the Tate motive $\mathbf{Z}(1)$ for the tensor product. From the homotopical point of view, this operation is much more involved and reveals

[1] See below for the precise statement.

[2] This hoped for t-structure is described in [**Voe92**, Hyp. 0.0.21].

the theory of spectra, objects which incarnate cohomology theories in algebraic topology. These two processes, of $\mathbf{A}^1$-localization and $\mathbf{P}^1$-stabilization, applied to the category of simplicial Nisnevich sheaves, led to the stable $\mathbf{A}^1$-homotopy category of schemes (see [**Jar00**]) a triangulated category with integral coefficients, defined over any base, which generalizes the category of motivic complexes.[3]

Over a perfect field, and with rational coefficients, the relation between homotopy and motives was clarified in an unpublished paper of Morel ([**Mor06**]): the rational stable $\mathbf{A}^1$-homotopy category contains the stable (*i.e.* $\mathbf{P}^1$-stable) version of the category of motivic complexes as an explicit direct factor, called the *+-part* of the stable homotopy category.[4] Then Morel introduces this +-part as a good candidate for the rational version of the triangulated category of motives ([**Mor06**, paragraph at the end of p.2]). We will dubbed the objects of this category the *Morel motives*.

On the other hand, with integral coefficients, O. Röndigs et P.A. Østvær showed that over a field of characteristic 0, the $\mathbf{P}^1$-stable category of motivic complexes coincides with the category of modules over the ring spectrum which represents motivic cohomology (see [**RØ08**]).[5] This ring spectrum was introduced by Voevodsky (see [**Voe98**]) using the theory of relative cycles. It is defined over any base and one is led to consider the category of modules over this ring spectrum as a possible definition of the integral triangulated category of motives.

A.4. Cross functors. The definitive step towards the six functors formalism in motivic homotopy theory was taken up by Voevodsky in a series of lectures were he laid down the theory of *cross functors*. The main theorem of this theory consists in giving a criterion on a system of triangulated categories indexed by schemes, equipped with a basic functoriality, to be able to construct exceptional functors $(f_!, f^!)$ satisfying the properties required by Grothendieck 6 functors formalism. In particular, the system of triangulated categories must satisfy three notable properties: the $\mathbf{A}^1$*-localization property*, the $\mathbf{P}^1$*-stability property* and the *localization property*. Unfortunately, only an introductory part on this theory was released (see [**Del01**]) in which the basic setup is established but which does not contain the proof of the main result.

The writing of this theory was accomplished by J. Ayoub in his thesis (see [**Ayo07a, Ayo07b**]). Ayoub uses the axioms laid down by Voevodsky: he calls a system of triangulated categories satisfying the properties alluded above a *homotopy stable functor*. However, he goes far beyond the original result of Voevodsky: apart the complete theory of cross functors (concerned with $f_!, f^!$), he also studied tensor structures, *constructibility* properties and their stability under the six operations, *t-structures* and *specialization functors* such as the vanishing cycle functor. The main example of a stable homotopy functor is the stable $\mathbf{A}^1$-homotopy category. One readily deduces that the category of Morel motives is also a homotopy stable functor.

However, it is by no means obvious that the category of modules over the motivic homotopy ring spectrum does meet the requirements of a homotopy stable functor. In fact, it can be seen that this is equivalent to Conjecture 17 of Voevodsky in [**Voe02b**].

A.5. Grothendieck 6 functors formalism.

A.5.1. We now give the precise formulation of the *Grothendieck 6 functors formalism*. As presented here, it is extracted from the properties of the derived category of l-torsion étale sheaves obtained in [**SGA4**, tome 3].[6]

A triangulated category $\mathscr{T}$, fibred over the category of schemes, satisfies the *Grothendieck 6 functors formalism* if the following conditions hold:

[3] Heuristically, the essential difference between stable $\mathbf{A}^1$-homotopy and motivic complexes is the presence of transfers in the later case.

[4] See also Theorem 11 in this introduction and its corollary.

[5] See also Theorem 8 in this introduction for an extension of their result.

[6] It also coincides with formulas gathered by Deligne in an unpublished note which he graciously supported us with.

(1) There exists 3 pairs of adjoints functors as follows:

$$\begin{aligned} f^* : \mathscr{T}(X) \rightleftarrows \mathscr{T}(Y) : f_*, & f \text{ any morphism,} \\ f_! : \mathscr{T}(Y) \rightleftarrows \mathscr{T}(X) : f^!, & f \text{ any separated morphism of finite type,} \\ (\otimes, Hom), & \text{ symetric closed monoidal structure on } \mathscr{T}(X). \end{aligned}$$

(2) There exists a structure of a covariant (resp. contravariant) 2-functors on $f \mapsto f_*$, $f \mapsto f_!$ (resp. $f \mapsto f^*$, $f \mapsto f^!$).

(3) There exists a natural transformation

$$\alpha_f : f_! \to f_*$$

which is an isomorphism when f is proper. Moreover, α is a morphism of 2-functors.

(4) For any smooth separated morphism $f : X \to S$ in $\mathscr{S}$ of relative dimension d, there exists a canonical natural isomorphism

$$\mathfrak{p}'_f : f^* \longrightarrow f^!(-d)[-2d]$$

where $?(-d)$ denotes the inverse of the Tate twist iterated d-times. Moreover $\mathfrak{p}'$ is an isomorphism of 2-functors.

(5) For any cartesian square in $\mathscr{S}$:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

such that f is separated of finite type, there exist natural isomorphisms

$$\begin{aligned} g^* f_! &\xrightarrow{\sim} f'_! g'^* , \\ g'_* f'^! &\xrightarrow{\sim} f^! g_* . \end{aligned}$$

(6) For any separated morphism of finite type $f : Y \to X$, there exist natural isomorphisms

$$\begin{aligned} Ex(f_!^*, \otimes) : (f_! K) \otimes_X L &\xrightarrow{\sim} f_!(K \otimes_Y f^* L) , \\ Hom_X(f_!(L), K) &\xrightarrow{\sim} f_* Hom_Y(L, f^!(K)) , \\ f^! Hom_X(L, M) &\xrightarrow{\sim} Hom_Y(f^*(L), f^!(M)) . \end{aligned}$$

(Loc) For any closed immersion $i : Z \to S$ with complementary open immersion j, there exists a distinguished triangle of natural transformations as follows:

$$j_! j^! \xrightarrow{\alpha'_j} 1 \xrightarrow{\alpha_i} i_* i^* \xrightarrow{\partial_i} j_! j^![1]$$

where $\alpha'_?$ (resp. $\alpha_?$) denotes the counit (resp. unit) of the relevant adjunction.

A.5.2. The next part of Grothendieck 6 functors formalism is concerned with duality. This kind of properties appears already in [**Har66**]. It is considered more axiomatically, in the case of étale sheaves, in [**SGA5**, Exp. I].[7] In *loc. cit.*, Grothendieck states the fundamental property of absolute purity and indicates its fundamental link with duality. We state these properties as natural extensions of the properties given in the preceding paragraph; assume $\mathscr{T}$ satisfies these preceding properties:

(7) *Absolute purity.*– For any closed immersion $i : Z \to S$ of regular scheme of (constant) codimension c, there exists a canonical isomorphism:

$$\mathbb{1}_Z(-c)[-2c] \xrightarrow{\sim} i^!(\mathbb{1}_S)$$

where $\mathbb{1}$ denotes the unit object for the tensor product.

[7] The duality properties are stated in the unpublished notes of Deligne as part of the complete formalism.

(8) *Duality.*– Let S be regular scheme and K_S be any invertible object of $\mathscr{T}(S)$. For any separated morphism $f : X \to S$ of finite type, put $K_X = f^!(K_S)$. For any object M of $\mathscr{T}(X)$, put $D_X(M) = \mathit{Hom}(M, K_X)$.

(a) For any X/S as above, K_X is a dualizing object of $\mathscr{T}(X)$. In other words, the canonical map:

$$M \to D_X(D_X(M))$$

is an isomorphism.

(b) For any X/S as above, and any objects M, N of $\mathscr{T}(X)$, we have a canonical isomorphism

$$D_X(M \otimes D_X(N)) \simeq \mathit{Hom}_X(M, N)\,.$$

(c) For any morphism between separated S-schemes of finite type $f : Y \to X$, we have natural isomorphisms

$$\begin{aligned} D_Y(f^*(M)) &\simeq f^!(D_X(M)) \\ f^*(D_X(M)) &\simeq D_Y(f^!(M)) \\ D_X(f_!(N)) &\simeq f_*(D_Y(N)) \\ f_!(D_Y(N)) &\simeq D_X(f_*(N))\ . \end{aligned}$$

A.5.3. The last property we want to exhibit as a natural extension of Grothendieck 6 functors formalism is the compatibility with projective limits of schemes. The basis for the next statement is [**SGA4**, Exp. VI] though it does not appear explicitly. As in the case of the duality property, it should involve some finiteness assumption. Note the formulation below is valid for an arbitrary fibred triangulated monoidal category $\mathscr{T}$.

(9) *Continuity.*– Let $(S_\alpha)_{\alpha \in A}$ be an essentially affine projective system of schemes. Put $S = \varprojlim_{\alpha \in A} S_\alpha$.

Then the canonical functor

$$2\text{-}\varinjlim_{\alpha} \mathscr{T}(S_\alpha) \to \mathscr{T}(S)$$

is an equivalence of monoidal triangulated categories.

B. Voevodsky's motivic complexes

The primary goal of this treatise is to develop the theory of Voevodsky motives, integrally over any base scheme[8], within the framework of sheaves with transfers. Actually, we can define Voevodsky's motives with coefficients in an arbitrary ring Λ and prove all the results stated below in that case but we restrict this presentation to integral coefficients for simplicity.

After refining and completing Suslin-Voevodsky's theory of relative cycles, we introduce the category $\mathscr{S}m^{cor}_{\mathbf{Z},S}$ of integral finite correspondences over smooth S-schemes and the related notion of (Nisnevich) sheaves with transfers over a base scheme S (Def. 10.4.2) as in the usual case of a perfect base field. Following the idea of stable homotopy, we define the triangulated category $\mathrm{DM}(X)$ of *stable motivic complexes* (see Def. 11.1.1) as the $\mathbf{P}^1$-stabilization of the $\mathbf{A}^1$-localization of the derived category of the (Grothendieck) abelian category of sheaves with transfers over S.

One easily gets that the fibred category DM is equipped with the basic functoriality needed by the cross-functor formalism. The main difficulty is the localization property, property (Loc) in Paragraph A.5.1. Unfortunately, though all the functors involved in the formulation of (Loc) are well defined for DM, we can only prove this property when S and Z are smooth over some base scheme (see Prop. 11.4.2). This is not enough to apply Ayoub's results.

However, we are able to construct the 6 operations for DM using the method of Deligne, used in [**SGA4**, XVII], and partially get the Grothendieck 6 functors formalism:

[8]In this introduction, all schemes will be assumed to be noetherian of finite dimension.

Theorem 1 (see Th. 11.4.5). *The triangulated category* DM*, fibred over the category of schemes, satisfies the following part of the properties stated in Paragraph A.5.1:*

- *properties (1), (2), (3),*
- *property (4) when f is an open immersion or f is projective and smooth,*
- *property (5) when g is smooth or f is projective and smooth,*
- *property (6) when f is projective and smooth,*
- *Property (Loc) when S and Z are smooth over some common base scheme.*

One of the application of this theory is that we get a well defined integral motivic cohomology theory for any scheme X:

$$H^{n,m}_{\mathcal{M}}(X,\mathbf{Z}) = \mathrm{Hom}_{\mathrm{DM}(X)}\big(\mathbb{1}_X, \mathbb{1}_X(m)[n]\big)$$

which enjoys the following properties (see section 11.2):

- it admits a ring structure, pullback maps associated with any morphism of schemes compatible with the ring structure,
- it admits push-forward maps with respect to projective morphisms between schemes smooth over some common base, or with respect to some finite morphisms (for example finite flat; see Paragraph 11.2.4),
- it coincides with Voevodsky's motivic cohomology groups when X is smooth over a perfect field (see Example 11.2.3); in particular one gets the following identification with higher Chow groups:

$$H^{n,m}_{\mathcal{M}}(X,\mathbf{Z}) = CH^m(X, 2m-n),$$

- it admits Chern classes and satisfies the projective bundle formula,
- it admits a localization long exact sequence associated with a closed immersion of schemes smooth over some common base.

As in the classical case, any smooth S-scheme X admits a motive $M_S(X)$ over X in $\mathrm{DM}(S)$. Moreover, one defines the Tate motive $\mathbb{1}_S(1)$ as the reduced motive of $\mathbf{P}^1_S$. We defined the category of *constructible motives* $\mathrm{DM}_c(S)$ as the thick triangulated subcategory of DM generated by the objects of the form $M_S(X)(n)$ for a smooth S-scheme X and an integer $n \in \mathbf{Z}$, where $?(n)$ refers to the n-th Tate twist. One gets the following generalization of the classical result obtained by Voevodsky over a perfect field:

Theorem 2 (see Th. 11.1.13). *A motive M in $\mathrm{DM}(S)$ is constructible if and only if it is compact.*[9]

The category $\mathrm{DM}_c(S)$ is equivalent to the category obtained from the bounded homotopy category of the additive category $\mathscr{S}m^{cor}_{\mathbf{Z},S}$ in the following way:

- *take the Verdier quotient modulo the thick triangulated subcategory generated by:*
  - *for any Nisnevich distinguished square* $\begin{array}{ccc} W & \xrightarrow{k} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$ *of smooth S-schemes:*

$$[W] \xrightarrow{g_*-k_*} [U]\oplus[V] \xrightarrow{j^*+f^*} [X]$$

  - *for any smooth S-scheme X, $p : \mathbf{A}^1_X \to X$ the canonical projection:*

$$[\mathbf{A}^1_X] \xrightarrow{p_*} [X],$$

- *invert the Tate twist,*
- *take the pseudo-abelian envelope.*

The triangulated category $\mathrm{DM}_c(X)$ is stable by the operations f^*, f_* when f is smooth projective, and $\otimes$ but we cannot prove the stability for the other operations of DM and a fortiori do not get the duality properties (7) and (8) of the Grothendieck 6 functors formalism.

[9] Recall that M is compact if thefunctor $\mathrm{Hom}(M,-)$ commutes with arbitrary direct sums.

However, we are able to prove the continuity property (9) for the category DM_c:

$$2\text{-}\varinjlim_{\alpha} \mathrm{DM}_c(S_\alpha) \simeq \mathrm{DM}_c(S),$$

where we only require that the transition morphism of (X_α) are affine and dominant (see Theorem 11.1.24) . Note this result allows us to extend the comparison of motivic cohomology with higher Chow groups to arbitrary regular schemes of equal characteristics.

C. Beilinson motives

C.1. Definition and fundamental properties. As anticipated by Morel, the theory of mixed motives with rational coefficients is much simpler and we succeed in establishing a complete formalism for them. Our initial approach differs slightly from that of Morel. We construct, out of the rational stable homotopy category and the ring spectrum associated with rational Quillen K-theory a $\mathbf{Q}$-linear triangulated category $\mathrm{DM}_{\text{Б}}(X)$, which we call the *triangulated category of Beilinson motives* (see Def. 14.2.1). Essentially by construction, in the case where X is regular, we have a natural identification

$$\mathrm{Hom}_{\mathrm{DM}_{\text{Б}}(X)}(\mathbf{Q}_X, \mathbf{Q}_X(p)[q]) \simeq Gr^p_\gamma K_{2p-q}(X)_{\mathbf{Q}}\,,$$

where the right hand side is the graded part of the algebraic K-theory of X with respect to the γ-filtration. These groups were first considered by Beilinson as the rational motivic cohomology groups. We call them the *Beilinson motivic cohomology groups.*

One of the interest of our definition is that the localization property (Loc) can be easily deduced from its validity for the stable homotopy category. Therefore, the cross-functor formalism and more generally all the results of Ayoub can be applied to $\mathrm{DM}_{\text{Б}}$. Using the constructions of this book, we obtain a slightly more general and precise formalism.

THEOREM 3 (see Cor. 14.2.11 and Th. 2.4.50). *All the standard Grothendieck six functors formalism (see Paragraph A.5.1) is verified by the fibred triangulated category* $\mathrm{DM}_{\text{Б}}$*.*

Concerning duality for Beilinson motives, we first deduce from Quillen's localization theorem in algebraic K-theory the absolute purity theorem:

THEOREM 4 (see Th. 14.4.1). *The absolute purity property (see A.5.2(7)) holds for* $\mathrm{DM}_{\text{Б}}$*.*

As said before, this result is not enough to establish duality for Beilinson motives. We first have to use descent theory and resolution of singularities (as first explained by Grothendieck in [**SGA5**, I.3]). Using the existence of trace maps in algebraic K-theory, we prove the following result:

THEOREM 5 (h-descent, see Th. 14.3.3 and Th. 4.4.1). *Consider a finite group G and a pullback square of schemes*

$$\begin{array}{ccc} T & \xrightarrow{h} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array}$$

in which Y is endowed with an action of G over X. Put $U = X - Z$ and assume the following three conditions are satisfied.

(a) The morphism f is proper and surjective.
(b) The induced morphism $f^{-1}(U) \to U$ is finite.
(c) The morphism $f^{-1}(U)/G \to U$ is generically radicial.

Put $a = f \circ i$. Then, for any object M of $\mathrm{DM}_{\text{Б}}(X)$*, we get a canonical distinguished triangle in* $\mathrm{DM}_{\text{Б}}(X)$*:*

$$M \longrightarrow i_*\, i^*(M) \oplus f_*\, f^*(M)^G \longrightarrow a_*\, a^*(M)^G \longrightarrow M[1]$$

where $?^G$ means the invariants under the action of G, and the first (resp. second) map of the triangle is induced by the difference (resp. sum) of the obvious adjunction morphisms.

In fact, we show that this apparently simple result implies a much stronger descent property for the fibred triangulated category DM_{B}, the descent property for the h-topology, thus in particular étale and even flat descent as well as proper descent.

C.2. Constructible Beilinson motives. The next step towards duality for Beilinson motives is the definition of a suitable finiteness condition. As in the case of Voevodsky motives, we define the category of *Beilinson constructible motives*, denoted by $\mathrm{DM}_{\mathrm{B},c}(X)$, as the thick subcategory of $\mathrm{DM}_{\mathrm{B}}(X)$ generated by the motives of the form $M_X(Y)(p) := f_! f^!(\mathbf{Q}_X)(p)$ for $f : Y \to X$ separated smooth of finite type, and $p \in \mathbf{Z}$. This category coincides with the full subcategory of compact objects in $\mathrm{DM}_{\mathrm{B}}(X)$.[10]

The usefulness of this definition comes from the following result, which is the analog of Gabber's finiteness theorem in the l-adic setting. Analogously, its proof relies on absolute purity, (a weak form of) proper descent as well as Gabber's weak uniformization theorem.[11]

THEOREM 6 (finiteness, see Th. 15.2.1). *The subcategory $\mathrm{DM}_{\mathrm{B},c}$ is stable under the six operations of Grothendieck when restricted to excellent schemes.*

The final statement concerning Grothendieck 6 functors formalism in the setting of Beilinson motives is that, when one restricts to constructible Beilinson motives and separated B-schemes of finite type for an excellent scheme B of dimension less than 2, the complete formalism is available:

THEOREM 7 (see Th. 15.2.4 and Prop. 15.1.6). *The fibred category $\mathrm{DM}_{\mathrm{B},c}$ over the category of schemes described above satisfies the complete Grothendieck 6 functors formalism described in section A.5, in particular the* duality property *A.5.2(8) and the* continuity property *A.5.3(9).*

REMARK. Note that the finiteness theorem as well as the duality property are also consequences of [**Ayo07a**], respectively Scholie 2.2.34 and Theorem 2.3.73, applied to DM_{B} when one restricts to quasi-projective schemes over a field or a discrete valuation ring. As ours, the proof of Ayoub uses in an essential way the absolute purity property (Theorem 4 stated above).

C.3. Comparison theorems. In the historical part of this introduction, we saw many approaches for the triangulated category of (rational) motives. We succeed in comparing them all with our definition of Beilinson motives.

Denote by KGL_S the algebraic K-theory spectrum in Morel and Voevodsky's stable homotopy category $\mathrm{SH}(S)$. By virtue of a result of Riou, the γ-filtration on K-theory induces a decomposition of $KGL_{S,\mathbf{Q}}$:

$$KGL_{S,\mathbf{Q}} \simeq \bigoplus_{n\in\mathbf{Z}} H_{\mathrm{B},S}(n)[2n]\,.$$

The ring spectrum $H_{\mathrm{B},S}$ represents Beilinson motivic cohomology. Almost by construction, the category $\mathrm{DM}_{\mathrm{B}}(S)$ is the full subcategory of $\mathrm{SH}_{\mathbf{Q}}(S)$ which consists of objects E such that the unit map $E \to H_{\mathrm{B},S} \otimes E$ is an isomorphism. In fact, our first comparison result relates the theory of Beilinson motives with the approach of Röndigs and Østvær through modules over a ring spectrum:

THEOREM 8 (see Th. 14.2.9). *For any scheme S, there is a canonical equivalence of categories*

$$\mathrm{DM}_{\mathrm{B}}(S) \simeq \mathrm{Ho}(H_{\mathrm{B},S}\text{-}\mathrm{mod})$$

where the left hand side denotes the homotopy category of modules over the ring spectrum $H_{\mathrm{B},S}$.

The next comparison involves the the h-topology: this is the Grothendieck topology on the category of schemes, generated by étale surjective morphisms and proper surjective morphisms. The first published work of Voevodsky on triangulated categories of mixed motives ([**Voe96**]), introduces the $\mathbf{A}^1$-homotopy category of the derived category of h-sheaves. We consider a $\mathbf{Q}$-linear and $\mathbf{P}^1$-stable version of it, which we denote by $\underline{\mathrm{DM}}_{\mathrm{h},\mathbf{Q}}(S)$. By construction, for any S-scheme of finite type X, there is a h-motive $M_S(X)$ in $\underline{\mathrm{DM}}_{\mathrm{h},\mathbf{Q}}(S)$. We define $\mathrm{DM}_{\mathrm{h},\mathbf{Q}}(S)$ as the

[10] Note the striking analogy with perfect complexes.

[11] i.e. that, locally for the h-topology, any excellent scheme is regular, and any closed immersion between excellent schemes is the embedding of a strict normal crossing divisor into a regular scheme.

smallest triangulated full subcategory of $\underline{\mathrm{DM}}_{\mathrm{h},\mathbf{Q}}(S)$ which is stable by (infinite) direct sums, and which contains the objects $M_S(X)(p)$, for X/S *smooth of finite type*, and $p \in \mathbf{Z}$. Using h-descent in $\mathrm{DM}_{Б}$, we get the following comparison result.

THEOREM 9 (see Th. 16.1.2). *If S is excellent, then we have canonical equivalences of categories*

$$\mathrm{DM}_{Б}(S) \simeq \mathrm{DM}_{\mathrm{h},\mathbf{Q}}(S)\,.$$

In fact, we first prove this result for the variant of $\mathrm{DM}_{\mathrm{h},\mathbf{Q}}(S)$ obtained by replacing everywhere the h-topology by the qfh-topology – for the later, coverings are generated by étale covers and finite surjective morphisms. In particular we get an equivalence of categories: $\mathrm{DM}_{\mathrm{h},\mathbf{Q}}(S) \simeq \mathrm{DM}_{\mathrm{qfh},\mathbf{Q}}(S)$. This result allows us to link Beilinson motives with Voevodsky's motivic complexes. Let us denote by $\mathrm{DM}_{\mathbf{Q}}$ the fibred category of stable motivic complexes alluded to in Paragraph B. Using the preceding result in the case of the qfh-topology, we prove:

THEOREM 10 (see Th. 16.1.4). *If S is excellent and geometrically unibranch, then there is a canonical equivalence of categories*

$$\mathrm{DM}_{Б}(S) \simeq \mathrm{DM}_{\mathbf{Q}}(S)\,.$$

In particular, given such a scheme S, we get a description of $\mathrm{DM}_{Б,c}(S)$ as in Theorem 2 cited above. Voevodsky's integral (resp. rational) motivic cohomology is represented in $\mathrm{SH}(S)$ by a ring spectrum $H_{\mathcal{M},S}$ (resp. $H^{\mathbf{Q}}_{\mathcal{M},S}$). The preceding theorem immediately gives an isomorphism of ring spectra:[12]

$$H_{Б,S} \simeq H^{\mathbf{Q}}_{\mathcal{M},S}\,.$$

As Beilinson motivic cohomology ring spectra over different bases are compatible with pullbacks, we easily deduce the following corollary which solves affirmatively conjecture 17 of [**Voe02b**] in some cases, and up to torsion:

COROLLARY. *For any morphism $f : T \to S$ of excellent geometrically unibranch schemes, the canonical map*

$$f^* H^{\mathbf{Q}}_{\mathcal{M},S} \to H^{\mathbf{Q}}_{\mathcal{M},T}$$

is an isomorphism of ring spectra.

The last comparison statement is concerned with the approach of Morel. According to Morel, the category $\mathrm{SH}_{\mathbf{Q}}(S)$ can be decomposed into two factors, one of them being $\mathrm{SH}_{\mathbf{Q}}(S)_+$, that is the part of $\mathrm{SH}_{\mathbf{Q}}(S)$ on which the map $\epsilon : S^0_{\mathbf{Q}} \to S^0_{\mathbf{Q}}$, induced by the permutation of the factors in $\mathbf{G}_m \wedge \mathbf{G}_m$, acts as -1. Let $S^0_{\mathbf{Q}+}$ be the unit object of $\mathrm{SH}_{\mathbf{Q}}(S)_+$.

Using the presentation of Beilinson motives in terms of $H_{Б}$-modules (Theorem 8 cited above) as well as Morel's computation of the motivic sphere spectrum in terms of Milnor-Witt K-theory, we obtain another proof of a result of Morel (see [**Mor06**]):

THEOREM 11 (see Th. 16.2.13). *For any scheme S, the canonical map $S^0_{\mathbf{Q}+} \to H_{Б,S}$ is an isomorphism.*

In fact, we even get the following corollary:

COROLLARY. *For any scheme S, there is a canonical equivalence of categories*

$$\mathrm{SH}_{\mathbf{Q}}(S)_+ \simeq \mathrm{DM}_{Б}(S)\,.$$

---

[12] Note in particular that, when S is regular, we get an isomorphism:

$$H^{p,q}_{\mathcal{M}}(S,\mathbf{Z}) \otimes \mathbf{Q} \simeq Gr^p_\gamma K_{2p-q}(S)_{\mathbf{Q}}$$

which extends the known isomorphism when S has equal characteristics. It is natural with respect to pullbacks and compatible with products.

Recall from Morel theory that, when -1 is a sum of squares in all the residue fields of S, ϵ is equal to $-Id$ on the whole of $\mathrm{SH}_{\mathbf{Q}}(S)$. Thus in that particular case (e.g. S is a scheme over an algebraically closed field), the category of Beilinson motives coincide with the rational stable homotopy category. In general, we can introduce according to Morel the étale variant of $\mathrm{SH}_{\mathbf{Q}}(S)$ denoted by $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})$.[13] As locally for the étale topology, -1 is always a square, and because DM_{B} satisfies étale descent, we get the following final illuminating comparison statement.

COROLLARY. *For any scheme S, there is a canonical equivalence of categories*

$$\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q}) \simeq \mathrm{DM}_{\text{B}}(S)\,.$$

Let us draw a conclusive picture which summarize most of the comparison results we obtained:

COROLLARY. *Given any scheme S, the category $\mathrm{DM}_{\text{B}}(S)$ is a full subcategory of the rational stable homotopy category $\mathrm{SH}_{\mathbf{Q}}(S)$. Given an rational spectrum E over S, the following conditions are equivalent:*

(i) *E is a Beilinson motive,*
(ii) *E is an $H_{\text{B},S}$-module,*
(iii) *E satisfies étale descent,*
(iii') *(S excellent) E satisfies* qfh-*descent,*
(iii") *(S excellent) E satisfies* h-*descent,*
(iv) *(S excellent geometrically unibranch) E admits transfers,*
(v) *the endomorphism $\epsilon \in \mathrm{End}(S^0_{\mathbf{Q}})$ acts by $-Id$ on E* i.e. $\epsilon \otimes 1_E = -1_E$.

REMARK. (see Corollary 14.2.16) Points (iv) and (v) are related to the orientation theory for spectra (not only ring spectra). In fact, $H_{\text{B},S}$ is the universal orientable rational ring spectrum over S.

Let $\mathbf{Q}.Sm_S$ be the $\mathbf{Q}$-linear envelop of the category Sm_S. On obtains (see Example 5.3.43 in conjunction with Par. 5.3.35) that the full subcategory of compact objects of $\mathrm{SH}_{\mathbf{Q}}(S)$ is equivalent to the category obtained from the homotopy category $\mathrm{K}^b(\mathbf{Q}.Sm_S)$ by performing the following operations:

- take the Verdier quotient modulo the thick triangulated subcategory generated by:
  - for any Nisnevich distinguished square $\begin{array}{ccc} W & \xrightarrow{k} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$ of smooth S-schemes:

$$\mathbf{Q}_S(W) \xrightarrow{g_*-k_*} \mathbf{Q}_S(U) \oplus \mathbf{Q}_S(V) \xrightarrow{j^*+f^*} \mathbf{Q}_S(X)$$

  - for any smooth S-scheme X, $p : \mathbf{A}^1_X \to X$ the canonical projection:

$$\mathbf{Q}_S(\mathbf{A}^1_X) \xrightarrow{p_*} \mathbf{Q}_S(X).$$

- invert the Tate twist,
- take the pseudo-abelian envelope.

Let us denote by $\mathrm{D}_{\mathbf{A}^1,c}(S,\mathbf{Q})$ this category. We finally obtain the following concrete description of Beilinson constructible motives:

COROLLARY. *Given any scheme S, the category $\mathrm{DM}_{\text{B},c}(S)$ is equivalent to the full subcategory of $\mathrm{D}_{\mathbf{A}^1,c}(S,\mathbf{Q})$ made by the objects E which satisfies one the following equivalent conditions:*

(i) *(Galois descent) given any smooth S-scheme X and any Galois S-cover $f : Y \to X$ of group G, the canonical map $E \otimes \mathbf{Q}_S(Y)/G \to E \otimes \mathbf{Q}_S(X)$ is an isomorphism,*
(ii) *(Orientability) ϵ acts by $-Id$ on E,*

Recall again the following remarks:

(1) When (-1) is a sum of square in every residue fields of S, conditions (i), (ii) are true for any rational spectrum E over S.

[13] In brief, this is the $\mathbf{P}^1$-stabilization of the $\mathbf{A}^1$-localization of the derived category of sheaves of $\mathbf{Q}$-vector spaces over the *lisse-étale* of S.

(2) When S is excellent and geometrically unibranch, the category $\mathrm{DM}_{\mathrm{B},c}(S)$ is equivalent to the category of rational geometric Voevodsky motives (same definition as in Theorem 2 but replacing $\mathbf{Z}$ by $\mathbf{Q}$).

C.4. Realizations. The last feature of Beilinson motives is that they are easily realizable in various cohomology theories. To get this fact, we use the setting of modules over a strict ring spectrum.[14] Given such a ring spectrum $\mathcal{E}$ in $\mathrm{DM}_{\mathrm{B}}(S)$, one can define, for any S-scheme X, the triangulated category

$$\mathrm{D}(X,\mathcal{E}) = \mathrm{Ho}(\mathcal{E}_X\text{-}\mathrm{mod})\,,$$

where $\mathcal{E}_X = f^*\mathcal{E}$, for $f : X \to S$ the structural map.

We then have realization functors

$$\mathrm{DM}_{\mathrm{B}}(X) \to \mathrm{D}(X,\mathcal{E})\ , \quad M \mapsto \mathcal{E}_X \otimes_X M$$

which commute with the six operations of Grothendieck. Using Ayoub's description of the Betti realization, we obtain:

THEOREM 12. *If $S = \mathrm{Spec}\,(k)$ with k a subfield of $\mathbf{C}$, and if $\mathcal{E}_{Betti}$ represents Betti cohomology in $\mathrm{DM}_{\mathrm{B}}(S)$, then, for any k-scheme of finite type, the full subcategory of compact objects of $\mathrm{D}(X,\mathcal{E}_{Betti})$ is canonically equivalent to the derived category of constructible sheaves of geometric origin $\mathrm{D}^b_c(X(\mathbf{C}),\mathbf{Q})$.*

More generally, if S is the spectrum of some field k, given a mixed Weil cohomology $\mathcal{E}$, with coefficient field (of characteristic zero) $\mathbf{K}$, we get realization functors

$$\mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_c(X,\mathcal{E})\ , \quad M \mapsto \mathcal{E}_X \otimes_X M$$

(where $\mathrm{D}_c(X,\mathcal{E})$ stands for the category of compact objects of $\mathrm{D}(X,\mathcal{E})$), which commute with the six operations of Grothendieck (which preserve compact objects on both sides). Moreover, the category $\mathrm{D}_c(S,\mathcal{E})$ is then canonically equivalent to the bounded derived category of the abelian category of finite dimensional $\mathbf{K}$-vector spaces. As a by-product, we get the following concrete finiteness result: for any k-scheme of finite type X, and for any objects M and N in $\mathrm{D}_c(X,\mathcal{E})$, the $\mathbf{K}$-vector space $\mathrm{Hom}_{\mathrm{D}_c(X,\mathcal{E})}(M,N[n])$ is finite dimensional, and it is trivial for all but a finite number of values of n.

If k is of characteristic zero, this abstract construction gives essentially the usual categories of coefficients (as seen above in the case of Betti cohomology), and in a sequel of this work, we shall prove that one recovers in this way the derived categories of constructible ℓ-adic sheaves (of geometric origin) in any characteristic. But something new happens in positive characteristic:

THEOREM 13. *Let V be a complete discrete valuation ring of mixed characteristic, with field of functions K, and residue field k. Then rigid cohomology is a K-linear mixed Weil cohomology, and thus defines a ring spectrum $\mathcal{E}_{rig}$ in $\mathrm{DM}_{\mathrm{B}}(k)$. We obtain a system of closed symmetric monoidal triangulated categories $\mathrm{D}_{rig}(X) = \mathrm{D}_c(X,\mathcal{E}_{rig})$, for any k-scheme of finite type X, such that*

$$\mathrm{Hom}_{\mathrm{D}_{rig}(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q]) \simeq H^q_{rig}(X)(p)\,,$$

as well as realization functors

$$R_{rig} : \mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_{rig}(X)$$

which preserve the six operations of Grothendieck.

D. Detailed organization

The book is organized in four parts that we now review in more details.

[14] i.e. we say a ring spectrum is *strict* if it is a commutative monoid in the underlying model category.

D.1. Grothendieck six functors formalism (Part 1). The first part is concerned with the formalism described in section A.5 above. It is the foundational part of this work.

We use the language of fibred categories (introduced in [**SGA1**, VI]), complemented by that of 2-functors (or pseudo-functors), in order to describe the 6 functors formalism. We first describe an axioms which allow to derive the core formalism – i.e. the part described in section A.5.1 – from simpler axioms. We do not claim originality in this task: our main contribution is to give a synthesis of the approach of Deligne described in [**SGA4**, XVII] (see also [**Har66**, Appendix]) with that of Voevodsky developed by Ayoub in [**Ayo07a**].

Recall that a (cleaved) fibred category $\mathscr{M}$ over $\mathscr{S}$ can be seen as a family of categories $\mathscr{M}(S)$ for every object S of $\mathscr{S}$ together with a pullback functor $f^* : \mathscr{M}(S) \to \mathscr{M}(T)$ for any morphism $f : T \to S$ of $\mathscr{S}$.[15] Given a suitable class $\mathscr{P}$ of morphisms in $\mathscr{S}$, we set up a systematic study of a particular kind of fibred categories, called $\mathscr{P}$-fibred categories (definition 1.1.10): one where for any f in $\mathscr{P}$, the pullback functor f^* admits a *left* adjoint, generically denoted by $f_\sharp$. The functor $f_\sharp$ has to be thought as a variant of the *exceptional direct image functor*.[16]

In section 1, we study basic properties of $\mathscr{P}$-fibred categories which will be the core of the 6 functors formalism, such as *base change formulas* and *projection formulas* when an additional monoidal structure is involved. These formulas are particular case of a compatibility relation between different kind of functors expressed through a canonical comparison morphism. Such kind of comparison morphisms are generically called *exchange morphisms*. They are very versatile and appears everywhere in the theory (see Paragraphs 1.1.6, 1.1.15, 1.1.24, 1.1.31, 1.1.33, 1.2.5). In fact, they appears fundamentally in Grothendieck 6 functors formalism: in the list of properties A.5.1, they are the isomorphisms of (5), (6) and even (4). In the direction of the full Grothendieck functoriality, we introduce a core axiomatic for $\mathscr{P}$-fibred categories that we consider as minimal: the categories satisfying this axiomatic are called *$\mathscr{P}$-premotivic* (section 1.4). $\mathscr{P}$-premotivic categories will form the basic setting in all this work. They will appear in three different flavours, depending on which particular kind of additional structure we consider on categories: abelian, triangulated and model categories.

In Section 2, we restrict our attention to the triangulated and geometric case, meaning that we consider triangulated $\mathscr{P}$-fibred categories over a suitable category of schemes $\mathscr{S}$. The aim of the section is to develop, and extend, Grothendieck 6 functors formalism in this basic setting. We exhibit many properties of such fibred categories which are indexed in the appendix. Let us concentrate in this introduction on the two main properties which will corresponds respectively to Deligne and Voevodsky's approach on the 6 functors formalism.

The first one, called the *support property* and abbreviated by (Supp), asserts that the adjoint functors of the kind f_*, for f proper, and $j_\sharp$, for j an open immersion, satisfy a gluing property that allows to use the argument of Deligne to construct the exceptional direct image functor $f_!$.[17] Several properties are derived from (Supp) and the basic axioms of $\mathscr{P}$-fibred categories which lead to a partial version of the 6 functors formalism (see Theorem 2.2.14).

The second property, most fundamental in the motivic context, is the *localization property* abbreviated by (Loc), which is in fact part of the 6 functors formalism (see Paragraph A.5.1). It has many interesting consequences and reformulations that are derived in section 2.3.1. Note that (Loc) is also known in the literature as the "gluing formalism". Some of the properties that we prove in *loc.cit.* are already classical (see [**BBD82**]).

The most interesting consequence of (Loc) was discovered by Voevodsky: together with the usual $\mathbf{A}^1$-localization and $\mathbf{P}^1$-stabilization properties of the motivic context, it implies the complete basic 6 functors formalism as stated in Paragraph A.5.1. This was proved by Ayoub in [**Ayo07a**].

[15] These pullback functors are subject to the usual cocycle condition ; see section 1.

[16] This kind of situation frequently happens: analytical case (open immersions), sheaves on the small étale site (étale morphisms), Nisnevich sheaves on the smooth site (smooth morphisms).

[17] In the context of torsion étale sheaves of [**SGA4**, XVII], property (Supp) is a consequence of the proper base change theorem.

In section 2.4, we revisit the proof of Ayoub and give some improvement of his theorems (see Theorem 2.4.50 for the precise statement):

- we remove the quasi-projectivity assumption for the existence of $f_!$, replacing it by the assumption that f is separated of finite type;
- we introduce the *orientation property* which allows to get a simpler more usual form to the purity isomorphism (the one stated in point (4) of A.5.1);
- we give another proof of the main theorem in the oriented case by showing that relative purity is equivalent to some (strong) duality property in the smooth projective case (see Theorem 2.4.42);
- we directly incorporate the monoidal structure whereas Ayoub gives a separate discussion for this.

Apart from these differences, the material of section 2.4 is very similar to that of [**Ayo07a**]. Moreover, in the non oriented case, it should be clear that we rely on the original argument of Ayoub for the proof of Theorem 2.4.42.

Concerning terminology, we have called *motivic triangulated category* (Definition 2.4.45) what Ayoub calls a "monoidal stable homotopy functor".

The remaining of Part 1 is concerned with extensions of Grothendieck 6 functors formalism.

In Section 3, we show how to use the setting of $\mathscr{P}$-fibred model categories as a framework to formulate Deligne's cohomological descent theory.

Unless in trivial cases, object of a derived category are not local.[18] To formulate descent theory in derived categories, the main idea of Deligne was to extend the derived category of a scheme by one relative to a simplicial scheme, usually a hypercover with respect to a Grothendieck topology (see [**SGA4**, Vbis]). The construction consists in first extending the theory of sheaves to the case where the base is a simplicial schemes and then consider the associated derived category.

We generalize this construction to the case of an arbitrary $\mathscr{P}$-fibred category equipped with a model structure.[19] In fact, we show in Section 3.1 how to extend a $\mathscr{P}$-fibred category over a category of schemes to the corresponding category of simplicial schemes and even of arbitrary diagrams of schemes. Most importantly, we show how to extend the fibred model structure to the case of diagrams of schemes (see Prop. 3.1.11).[20] Concretely, this means that we define a derived functor of the kind $\mathbf{L}\varphi^*$ (resp. $\mathbf{R}\varphi_*$) for an arbitrary morphism φ of diagrams of schemes. Let us underline that these derived functors mingles two different kinds of functoriality: the usual pullback f^* (resp. direct image f_*) for a morphism of schemes f together with homotopy colimits (resp. limits) – see the discussion in Paragraph 3.1.12 till Proposition 3.1.16. With that extension in hands, we can easily formulate (cohomological) descent theory for arbitrary Grothendieck topologies on the category of schemes for the homotopy category of a $\mathscr{P}$-fibred model category: see Definition 3.2.5.

The end of Section 3 is devoted to concrete examples of descent in $\mathscr{P}$-fibred model categories, and their relation with properties of the associated homotopy category, assuming it is triangulated, as introduced in Section 2. The first and most simple example corresponds to the case of a Grothendieck topology associated with a cd-structure in the sense of Voevodsky (as the Nisnevich and the cdh-topology. See [**Voe10b**] or Paragraph 2.1.10). In that case, descent can be characterized as the existence of certain distinguished triangles (Mayer-Vietoris for Zariski topology,

[18]The first example of this fact is the circle: any non trivial open subset of S^1 is contractible whereas S^1 itself is not.

[19]Recall that model structures, introduced by Quillen, allow to perform all the usual constructions of derived categories by localizing an arbitrary category with respect to a given class of morphisms called weak equivalences. It contains in particular the usual case of complexes of an arbitrary abelian category with quasi-isomorphisms as weak equivalences. The main construction of the theory of Quillen is that of left (resp. right) derived functors which can be defined by replacing the usual notion of projective (resp. injective) resolution by that of cofibrant (resp. fibrant) resolution.

[20]By restricting the morphisms of diagrams of schemes to a certain class denoted by $\mathscr{P}_{cart}$, we also show how to get a $\mathscr{P}_{cart}$-fibred model category over diagrams of schemes (Rem. 3.1.21) but this is not really needed in the descent theory.

Brown-Gersten for Nisnevich topology): this is Theorem 3.3.2 which is in fact a reformulation of the results of Voevodsky.

We then proceed to the most fundamental case of descent in algebraic geometry, that for proper surjective maps which allows in principle the use of resolution of singularities. In fact, the main result of the whole of Section 3 is a characterization of h-descent which allows to reduce it, for $\mathscr{P}$-fibred homotopy triangulated categories which are rational and motivic, to a simple property easily checked in practice[21]: this is Theorem 3.3.37. Along the way, we proved also the following results interesting in their own:

- several characterization of étale descent (Theorems 3.3.23 and 3.3.32);
- a characterization of qfh-descent (Theorem 3.3.25) as if it was defined by a cd-structure.[22]

In fact, the last point is the heart of the proof of the main Theorem, 3.3.37. Whereas the extension of fibred homotopy categories to diagrams of schemes is not unprecedented (see [**Ayo07b**]), our study of proper and h-descent seems to be completely new. In our opinion, it is one of the most important technical innovation of this book.

In Section 4, we study the extension of Grothendieck 6 functors formalism in *rational* motivic categories, mainly duality and continuity. As already mentioned, the general principle is not knew and follows mainly the path laid by Grothendieck in [**SGA5**].

In the case of an abstract motivic triangulated category – which is for the purpose of descent theory the homotopy category of an underlying fibred model category as seen above – the first task is to introduce a correct property of finiteness inherent to any duality theorem. This is done following Voevodsky, as in the work of Ayoub, by introducing the notion of *constructiblity* in Definition 4.2.1. The name is inspired by the étale case, but the notion of constructibility which we consider here is defined a generation property which really corresponds to what Voevodsky called *geometric motives*: constructible motives in our sense are generated by twists of motives of smooth schemes and are stable by cones, direct factors and finite sums. Let us mention that in good cases, the property of being constructible coincides with that of being compact in a triangulated category, resounding with the theory of perfect complexes (in the context of l-adic sheaves, this corresponds to "constructible of geometric origin").

The main point on constructible motives is the study of their stability under the 6 operations that we get from the axioms of a triangulated motivic category. This is done in Section 4.2. As in the étale case, the crucial point is the stability with respect to the operation f_*, when f is a morphism of finite type between excellent schemes. In Theorem 4.2.24, we give conditions on a motivic triangulated category so that the stability for f_* is guaranteed (then the stability by the other operations follows easily, see 4.2.29). Our proof follows essentially an argument of Gabber. The general principle, going back to [**SGA4**, XIX, 5.1], is to use resolution of singularities to reduce to an absolute purity statement which is among our assumptions.[23]

In Section 4.3, we introduce an important property of motivic triangulated categories, called *continuity*, which allows reasoning that involves projective limits of schemes. In fact, it is shown in Proposition 4.3.4 that this property implies the property (9) of the (extended) Grothendieck 6 functors formalism (see Paragraph A.5.3 above). We also give a criterion for continuity (4.3.6) which will be applied later in concrete cases and draw some interesting consequences.

Finally, Section 4.4 deals with duality in itself for constructible motives, that is property (8) of Paragraph A.5.2. The main theorem 4.4.21 asserts that, under the same condition than Theorem 4.2.24, and if one restricts to schemes that are separated of finite type over an excellent base scheme B of dimension less or equal to 2, then the full duality property holds (see also Corollary 4.4.24). The proof follows the line of proof of the analog Th. 2.3.73 of [**Ayo07a**]. In particular the main point is the fact that constructible motives are generated by some nice motives adapted to the use of resolution of singularities: see Corollary 4.4.3. The main difference with *op. cit.* is

[21]This is the *separation* property defined in 2.1.7. Let us mention here it is a consequence of the existence of well behaved trace maps (see the proof of Theorem 14.3.3).

[22]It is at the origin of the formulation of descent that we gave for DM_{B} in Theorem 5(b) above.

[23]Absolute purity will be proved later for Beilinson motives.

that we use De Jong equivariant resolution of singularities [**dJ97**], so that our assumptions are a little bit more general.

D.2. The constructive part (Part 2). The purpose of this part is to give a method of construction of triangulated categories that satisfies the formalism described in Part 1. We have chosen to mainly use the setting of derived category. Also, we use our notion of $\mathscr{P}$-fibred categories ($\mathscr{P}$-premotivic with a good monoidal structure). Recall this means the pullback functor f^* admits a left adjoint $f_\sharp$ when $f \in \mathscr{P}$. Essentially, $\mathscr{P}$ will be either the class of smooth morphisms of finite type or the class of all morphisms of finite type (eventually separated).

In Section 5.1, starting from a $\mathscr{P}$-premotivic abelian category $\mathscr{A}$, we first show how to prove that the derived $\mathrm{D}(\mathscr{A})$ is also a $\mathscr{P}$-premotivic category. This consists in deriving the structural functors of a $\mathscr{P}$-premotivic category, which is done by building a suitable underlying $\mathscr{P}$-fibred model category in Proposition 5.1.12. Actually, the proof of the axioms of a model category has already appeared in our previous work [**CD09**]. Let us mention the flavor of this model structure: we can describe explicitly cofibrations as well as fibrations, by the use of an adapted Grothendieck topology t. This model structure is linked with cohomological t-descent (as shown later in Proposition 5.2.10). The advantage of our framework is to easily obtain the functoriality of this construction (Paragraph 5.1.23), as well as other homotopical constructions (dg-structure: Rem. 5.1.19, extension to diagrams of schemes: Par. 5.1.20). In paragraph 5.1.c, we also describe in suitable cases the constructible objects of the derived category by a presentation similar to that of Voevodsky's geometric motives over a perfect field.

In Section 5.2 (resp. Section 5.3) we show how to describe the $\mathbf{A}^1$-localization (resp. $\mathbf{P}^1$-stabilization) process in $\mathscr{P}$-premotivic derived categories: to any $\mathscr{P}$-premotivic abelian category $\mathscr{A}$ is associated an $\mathbf{A}^1$-derived category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathbf{P}^1$-stable and $\mathbf{A}^1$-derived category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$) in Definition 5.2.16 (resp. 5.3.22). From the model category obtained in Section 5.1, the construction uses the classical tools of motivic homotopy theory as introduced by Morel and Voevodsky. Again, our framework allows to get the same homotopical constructions as in the simple derived case as well as some nice universal properties. We also get a description of constructible objects under suitable assumptions: Section 5.2.d (resp. 5.3.e). These sections are filled with concrete examples.

In Section 6, we focus on the main (in fact universal) example of motivic derived category, the $\mathbf{A}^1$-derived category of Morel, obtained by the process described above from the abelian premotivic category of abelian sheaves over the smooth Nisnevich site. The main point here is that one gets the localization property for this category by a theorem of Morel and Voevodsky. We give two new contributions on this topic. First we show in Section 6.1 that the $\mathbf{A}^1$-derived category can be embedded in a larger category which naturally contains objects that we can call motives of singular schemes. This is useful to state descent properties and will be essential to study h-motives. Second, we show in Section 6.3 how one can use the $\mathbf{A}^1$-derived category to obtain good properties of another premotivic derived category satisfying suitable assumptions. This will be applied to motivic complexes.

In Section 7, we go back to the case of an arbitrary monoidal $\mathscr{P}$-fibred model category $\mathscr{M}$ and explain how to use the setting of ring spectra and modules over ring spectra in the premotivic context. The main construction associates to a suitable collection of (commutative) ring spectra R in $\mathscr{M}$ a $\mathscr{P}$-fibred monoidal category denoted by $\mathrm{Ho}(R\text{-}\mathrm{mod})$: Proposition 7.2.13. This construction will be used several times:

- in the study of algebraic K-theory (Section 13): the category of modules over K-theory is the fundamental technical tool to get motivic proper descent as well as motivic absolute purity;
- in the study of Beilinson motives when we will relate them with modules over motivic cohomology (Theorem 14.2.9);
- in the study of realizations associated with a mixed Weil cohomology (Section 17).

D.3. Motivic complexes (Part 3). This part is concerned with the constructions described above, in Section B. Our aim is to extend the definition of Voevodsky's integral motivic complexes to any base, then study their functoriality and introduce their non effective, or rather $\mathbf{P}^1$-stable, counter-part.

Our first task, in Section 8, is to revisit Suslin-Voevodsky's theory of *relative cycles* exposed in [**SV00b**]. Indeed, they will be at the heart of the general construction. Our presentation is made to prepare the theory of *finite correspondences*, a particular case of relative cycles. Especially, we want to give a meaning to the following picture representing the composition of finite correspondences α from X to Y and β from Y to Z:

$$\begin{array}{ccccc} \beta \otimes_Y \alpha & \to & \beta & \to & Z. \\ \downarrow & & \downarrow & & \\ \alpha & \longrightarrow & Y & & \\ \downarrow & & & & \\ X & & & & \end{array}$$

(see also (9.1.4.1)). More precisely, we want to interpret this as a diagram of cycles. Thus we are led to consider cycles (with their support) as objects of a category. Concretely, a cycle is considered as a multi-pointed scheme, each point being affected with some multiplicity (an integral or rational number).

This conceptual shift has the advantage of allowing a treatment of cycles analogue to that of algebraic varieties, or rather schemes, promoted by Grothendieck via studying morphisms. Thus, we replace the various groups of relative cycles introduced by Suslin and Voevodsky in *op. cit.* by properties of morphisms of cycles. Here is a list of the principal ones:

- pseudo-dominant (8.1.2), equidimensional (8.1.3 and 8.3.18),
- pre-special (8.1.20),
- special (8.1.28),
- Λ-universal (8.1.48).

The most intriguing one, being *pre-special*, has no counter-part in *op. cit.* Its idea comes from a mistake (fortunately insignificant) in the convention of Suslin and Voevodsky. Indeed, Lemma 3.2.4 of *op. cit.* is false whenever the base S is non reduced and irreducible: then any fat point (x_0, x_1) and any flat S-scheme give a counter-example.[24] The explanation is that the operation of specialization along a fat point does not take into account the geometric multiplicities of the base. On the contrary, when X is flat over an irreducible scheme S, the geometric multiplicity of any irreducible component of X is a multiple of the geometric multiplicity of S. This leads us to the definition of a pre-special morphism of cycles β/α, where a divisibility condition appears in the multiplicities of β with respect to that of α.[25]

The main achievement of Suslin and Voevodsky's theory is the construction of a pullback operation for relative cycles. In our language, it corresponds to a kind of tensor product, more precisely a product of cycles relative to a common base cycle (as for example the cycle $\beta \otimes_Y \alpha$ of the preceding picture). Despite our different presentation, the method to define this operation follows closely the original idea of Suslin and Voevodsky: use the *flatification theorem* of Gruson and Raynaud to reduce to the case of flat base change of cycles. Recall that the key point is to find the correct condition on cycles – or rather morphisms of cycles in our language – so that one obtains a uniquely defined operation independent of the chosen flatification. This is measured by a specialization procedure (Definition 8.1.25) associated with *fat points* (Definition 8.1.22) and leads to the central notion of *special morphisms* of cycles (Definition 8.1.28). An innovation that we introduce in the theory is to give, as soon as possible, local definitions at a point in the style of EGA. This is in particular the case for the property of being special.

Once this notion is in place, one defines for a base cycle α, a special α-cycle β and any morphism $\phi : \alpha' \to \alpha$ the relative product denoted by $\beta \otimes_\alpha \alpha'$, equivalently the base change of

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[24] Explicitly, take $S = Z = \operatorname{Spec}\big(k[t]/(t^2)\big) = \{\eta\}$, $R = \big(k[t]\big)_{(t)}$. The left hand side of the equality of 3.2.4 is $2.\eta$ while the right hand side is η.

[25] To anticipate the remaining of the construction, given a non reduced scheme S, this will allow for the operation of pullback along the immersion $S_{red} \to S$ associated with the reduction of S: it simply corresponds to dividing by the geometric multiplicities of S, as the base change to S_{red} does for flat S-schemes.

β/α along ϕ (Definition 8.1.39). This notion is close to the correspondence homomorphisms of Section 3.2 of *op. cit.* In particular it usually involves denominators. The last important notion, being Λ-*universal*, corresponds to cycles β/α with coefficients in a ring $\Lambda \subset \mathbf{Q}$, which keeps their coefficients in Λ after any base change.

One sees that our language is especially convenient when it is time to consider the stability of certain properties of morphisms of cycles by composition (Cor. 8.2.6) or base change (Cor. 8.1.45). Then the usual statements of intersection theory are proven in Section 8.2, still following or extending Suslin and Voevodsky: commutativity, associativity, projection formulas. This makes our relative product a good extension of the classical notion of exterior product of cycles (over a field).

The focal point of intersection theory is the study of multiplicities. Thus we introduce *Suslin-Voevodsky's multiplicities*, as the ones appearing as a corollary of the existence of the relative cycle $\beta \otimes_\alpha \alpha'$ (Definition 8.1.42). A very important result in the theory, already enlightened by Suslin and Voevodsky, is the fact these multiplicities can be expressed in terms of *Samuel multiplicities*.[26] In fact, we even prove a criterion for the property of being special at a point involving Samuel multiplicities at the branches of the point: see Corollary 8.3.25. Roughly speaking, the multiplicities arising from Samuel's definition at each branches of the point must coincide: then this common value is simply the Suslin-Voevodsky's multiplicity.

Finally, still following the treatment of algebraic geometry by Grothendieck, we introduce in the theory the study of the constructibility of properties of morphisms of cycles (special and Λ-universal). Explicitly, we prove that given a relative cycle β/α, when α is the cycle associated with a scheme S, the locus where β is special (resp. Λ-universal) is an ind-constructible subset of S (Lemma 8.3.4). This allows to prove the good behaviour of these notions with respect to projective limits of schemes (see in particular 8.3.9). This will be the key point when proving the continuity property – (9) of A.5.3 – of the fibred category DM.

The remaining of Part 3, consists in extending the theory of sheaves with transfers introduced by Voevodsky, originally over a perfect field, to the case of an arbitrary base and apply to it the general procedures studied in Part 2 to get the fibred category DM.

In Section 9, we work out the theory of finite correspondences using the formalism of relative cycles. The construction is summarized in Corollary 9.4.1: given a class of morphisms $\mathscr{P}$ contained in the class of separated morphisms of finite type and a ring of coefficients Λ, we produce a monoidal $\mathscr{P}$-fibred category, denoted by $\mathscr{P}^{cor}_\Lambda$, whose fiber over a noetherian scheme S (eventually singular) is the category of $\mathscr{P}$-schemes over S with morphisms the finite correspondences.

In Section 10, we develop the theory of sheaves with transfers along the very same line as the original treatment of Voevodsky. This time, the outcome can be summarized by Corollaries 10.3.11 and 10.3.15: given a class $\mathscr{P}$ of morphisms as above and a suitable Grothendieck topology t, we construct an abelian premotivic category $\mathrm{Sh}_t(\mathscr{P}, \Lambda)$ which is compatible with the topology t (cf Part 2); its fiber over a scheme S is given by t-sheaves of Λ-modules with transfers (in particular presheaves on $\mathscr{P}^{cor}_{\Lambda,S}$).[27] The section is closed with an important comparison result, essentially due to Voevodsky, between Nisnevich sheaves with transfers and sheaves for the qfh-topology (with rational coefficients over geometrically unibranch bases): see Theorem 10.5.14.

Finally, Section 11 is devoted to gather the work done previously and define the stable derived category of motivic complexes DM_Λ, given an arbitrary ring of coefficients Λ. The out-come has already been described in Section B above.

D.4. Beilinson motives (Part 4). This part contains the construction of Beilinson motives as well as the proof of all the properties stated before. It is based on the first and second parts but independent of the third one – except in the comparison statements of Section 16.1.

Section 12 contains a short reminder on the stable homotopy category and the notion of oriented ring spectra.

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[26] When a correct regularity assumption is added, one reduces to the usual Serre's Tor-intersection formula: see 8.3.31 and 8.3.32).

[27] The most notable topologies t that fit in this result are the Nisnevich and the cdh ones. See Section 10.4.

Section 13 is the heart of our construction. It contains a detailed study of the K-theory ring spectrum KGL and the associated notion of KGL-modules in the homotopical sense (based on the formalism introduced in Section 7). Using the works of several authors (most notably: Riou, Nauman, Spitzweck, Østvær), we show how the central results of Quillen on algebraic K-theory give important properties of KGL-modules: absolute purity (Th. 13.6.3) and trace maps (Def. 13.7.4).

In Section 14, we finally introduce the definition of Beilinson motives. Let us describe it in detail now. It is based on the process of Bousfield localization of the stable homotopy category with respect to a cohomology. This operation is fundamental in modern algebraic topology. We apply it in algebraic geometry to the rational stable homotopy category (or, what amount to the same, to the rational stable $\mathbf{A}^1$-derived category of Morel, Section 6) and to the rational K-theory spectrum $KGL_{\mathbf{Q}}$: the Bousfield localization of $\mathrm{D}_{\mathbf{A}^1,\Lambda}(S, \mathbf{Q})$ with respect to $KGL_{\mathbf{Q},S}$ is the category of Beilinson motives $\mathrm{DM}_{\mathrm{B}}(S)$ over S (Definition 14.2.1). Using the preceding study of $KGL_{\mathbf{Q}}$ together with the decomposition of Riou recalled in the beginning of Section C.3, we get the main properties of the premotivic category DM_{B}: the h-descent theorem (14.3.4) and the absolute purity theorem (14.4.1).

Then the theoretical background laid down in Part 1 is applied to DM_{B}, given in particular the complete Grothendieck six functors formalism for constructible Beilinson motives (Section 15). Our work closes on the two main subjects described above on Beilinson motives: the comparison statements (Section 16) and the study of motivic realizations (Section 17).

Notations and conventions

In every section, we will fix a category denoted by $\mathscr{S}$ which will contain our geometric objects. Most of the time, $\mathscr{S}$ will be a category of schemes which are suitable for our needs; the required hypothesis on $\mathscr{S}$ are given at the head of each section. In the text, when no precisions are given, any scheme will be assumed to be an object of $\mathscr{S}$.

When $\mathscr{A}$ is an additive category, we denote by $\mathscr{A}^\natural$ the pseudo-abelian enveloppe of $\mathscr{A}$. We denote by $\mathrm{C}(\mathscr{A})$ the category of complexes of $\mathscr{A}$. We consider $\mathrm{K}(\mathscr{A})$ (resp. $\mathrm{K}^b(\mathscr{A})$) the category of complexes (resp. bounded complexes) of $\mathscr{A}$ modulo the chain homotopy equivalences and when $\mathscr{A}$ is abelian, we let $\mathrm{D}(\mathscr{A})$ be the derived category of $\mathscr{A}$.

If $\mathscr{M}$ is a model category, $\mathrm{Ho}(\mathscr{M})$ will denote its homotopy category.

We will use the notation

$$\alpha : \mathscr{C} \rightleftarrows \mathscr{D} : \beta$$

to mean a pair of functors such that α is left adjoint to β. Similarly, when we speak of an adjoint pair of functors (α, β), α will always be the left adjoint. We will denote by

$$ad(\alpha, \beta) : 1 \to \beta\alpha \text{ (resp. } ad'(\alpha, \beta) : \alpha\beta \to 1)$$

the unit (resp. counit) of the adjunction (α, β). Considering a natural tranformation $\eta : F \to G$ of functors, we usually denote by the same letter η — when the context is clear — the induced natural transformation $AFB \to AGB$ obtained when considering functors A and B composed on the left and right with F and G respectively.

In section 8, we will assume that equidimensional morphims have constant relative dimension.

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Part 1

Fibred categories and the six functors formalism

1. General definitions and axiomatic

1.0. We assume that $\mathscr{S}$ is an arbitrary category.

We shall say that a class $\mathscr{P}$ of morphisms of $\mathscr{S}$ is *admissible* if it is has the following properties.

(Pa) Any isomorphism is in $\mathscr{P}$.

(Pb) The class $\mathscr{P}$ is stable by composition.

(Pc) The class $\mathscr{P}$ is stable by pullbacks: for any morphism $f : X \to Y$ in $\mathscr{P}$ and any morphism $Y' \to Y$, the pullback $X' = Y' \times_Y X$ is representable in $\mathscr{S}$, and the projection $f' : X' \to Y'$ is in $\mathscr{P}$.

The morphisms which are in $\mathscr{P}$ will be called the *$\mathscr{P}$-morphisms.*[28]

In what follows, we assume that an admissible class of morphisms $\mathscr{P}$ is fixed.

1.1. $\mathscr{P}$-fibred categories.

1.1.a. *Definitions.* Let $\mathscr{C}at$ be the 2-category of categories.

1.1.1. Let $\mathscr{M}$ be a fibred category over $\mathscr{S}$, seen as a 2-functor $\mathscr{M} : \mathscr{S}^{op} \to \mathscr{C}at$; see [**SGA1**, Exp. VI]

Given a morphism $f : T \to S$ in $\mathscr{S}$, we shall denote by

$$f^* : \mathscr{M}(S) \to \mathscr{M}(T)$$

the corresponding pullback functor between the corresponding fibers. We shall always assume that $(1_S)^* = 1_{\mathscr{M}(S)}$, and that for any morphisms $W \xrightarrow{g} T \xrightarrow{f} S$ in $\mathscr{S}$, we have structural isomorphisms:

$$g^* f^* \xrightarrow{\sim} (fg)^* \tag{1.1.1.1}$$

which are subject to the usual cocyle condition with respect to composition of morphisms.

Given a morphism $f : T \to S$ in $\mathscr{S}$, if the corresponding inverse image functor f^* has a left adjoint, we shall denote it by

$$f_\sharp : \mathscr{M}(T) \to \mathscr{M}(S)\,.$$

For any morphisms $W \xrightarrow{g} T \xrightarrow{f} S$ in $\mathscr{S}$ such that f^* and g^* have a left adjoint, we have an isomorphism obtained by transposition from the isomorphism (1.1.1.1):

$$(fg)_\sharp \xrightarrow{\sim} f_\sharp g_\sharp\,. \tag{1.1.1.2}$$

Definition 1.1.2. A *pre-$\mathscr{P}$-fibred category $\mathscr{M}$* over $\mathscr{S}$ is a fibred category $\mathscr{M}$ over $\mathscr{S}$ such that, for any morphism $p : T \to S$ in $\mathscr{P}$, the pullback functor $p^* : \mathscr{M}(S) \to \mathscr{M}(T)$ has a left adjoint $p_\sharp : \mathscr{M}(T) \to \mathscr{M}(S)$.

Convention 1.1.3. Usually, we will consider that (1.1.1.1) and (1.1.1.2) are identities. Similarly, we consider that for any object S of $\mathscr{S}$, $(1_S)^* = 1_{\mathscr{M}(S)}$ and $(1_S)_\sharp = 1_{\mathscr{M}(S)}$.[29]

Example 1.1.4. Let S be an object of $\mathscr{S}$. We let $\mathscr{P}/S$ be the full subcategory of the comma category $\mathscr{S}/S$ made of objects over S whose structural morphism is in $\mathscr{P}$. We will usually call the objects of $\mathscr{P}/S$ the *$\mathscr{P}$-objects over S*.

Given a morphism $f : T \to S$ in $\mathscr{S}$ and a $\mathscr{P}$-morphism $\pi : X \to S$, we put $f^*(\pi) = \pi \times_S T$ using the property (Pc) of $\mathscr{P}$ (see 1.0). This defines a functor $f^* : \mathscr{P}/S \to \mathscr{P}/T$.

Given two $\mathscr{P}$-morphisms $f : T \to S$ and $\pi : Y \to T$, we put $f_\sharp(\pi) = f \circ \pi$ using the property (Pb) of $\mathscr{P}$. this defines a functor $f_\sharp : \mathscr{P}/T \to \mathscr{P}/S$. According to the property of pullbacks, $f_\sharp$ is left adjoint to f^*.

We thus get a pre-$\mathscr{P}$-fibred category $\mathscr{P}/? : S \mapsto \mathscr{P}/S$.

[28] In practice, $\mathscr{S}$ will be an adequate subcategory of the category of noetherian schemes and $\mathscr{P}$ will be the class of smooth morphisms (resp. étale morphisms, morphisms of finite type, separated or not necessarily separated) in $\mathscr{S}$.

[29] We can always strictify globally the fibred category structure so that $g^*f^* = (fg)^*$ for any composable morphisms f and g, and so that $(1_S)^* = 1_{\mathscr{M}(S)}$ for any object S of $\mathscr{S}$; moreover, for a morphism h of $\mathscr{S}$ such that a left adjoint of h^* exists, and we can choose the left adjoint functor $h_\sharp$ which we feel as the most convenient for us, depending on the situation we deal with. For instance, if $h = 1_S$, we can choose $h_\sharp$ to be $1_{\mathscr{M}(S)}$, and if $h = fg$, with f^* and g^* having left adjoints, we can choose $h_\sharp$ to be $f_\sharp g_\sharp$ (with the unit and counit naturally induced by composition).

Example 1.1.5. Assume $\mathscr{S}$ is the category of noetherian schemes of finite dimension, and $\mathscr{P} = Sm$. For a scheme S of $\mathscr{S}$, let $\mathscr{H}_\bullet(S)$ be the pointed homotopy category of schemes over S defined by Morel and Voevodsky in [**MV99**]. Then according to *op. cit.*, $\mathscr{H}_\bullet$ is a pre-Sm-fibred category over $\mathscr{S}$.

1.1.6. *Exchange structures I.*– Suppose given a weak $\mathscr{P}$-fibred category $\mathscr{M}$.
Consider a commutative square of $\mathscr{S}$

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow[p]{} & S \end{array}$$

such that p and q are $\mathscr{P}$-morphisms, we get using the identification of convention 1.1.3 a canonical natural transformation

$$Ex(\Delta^*_\sharp) : q_\sharp g^* \xrightarrow{ad(p_\sharp,p^*)} q_\sharp g^* p^* p_\sharp = q_\sharp q^* f^* p_\sharp \xrightarrow{ad'(q_\sharp,q^*)} f^* p_\sharp$$

called the *exchange transformation* between $q_\sharp$ and g^*.

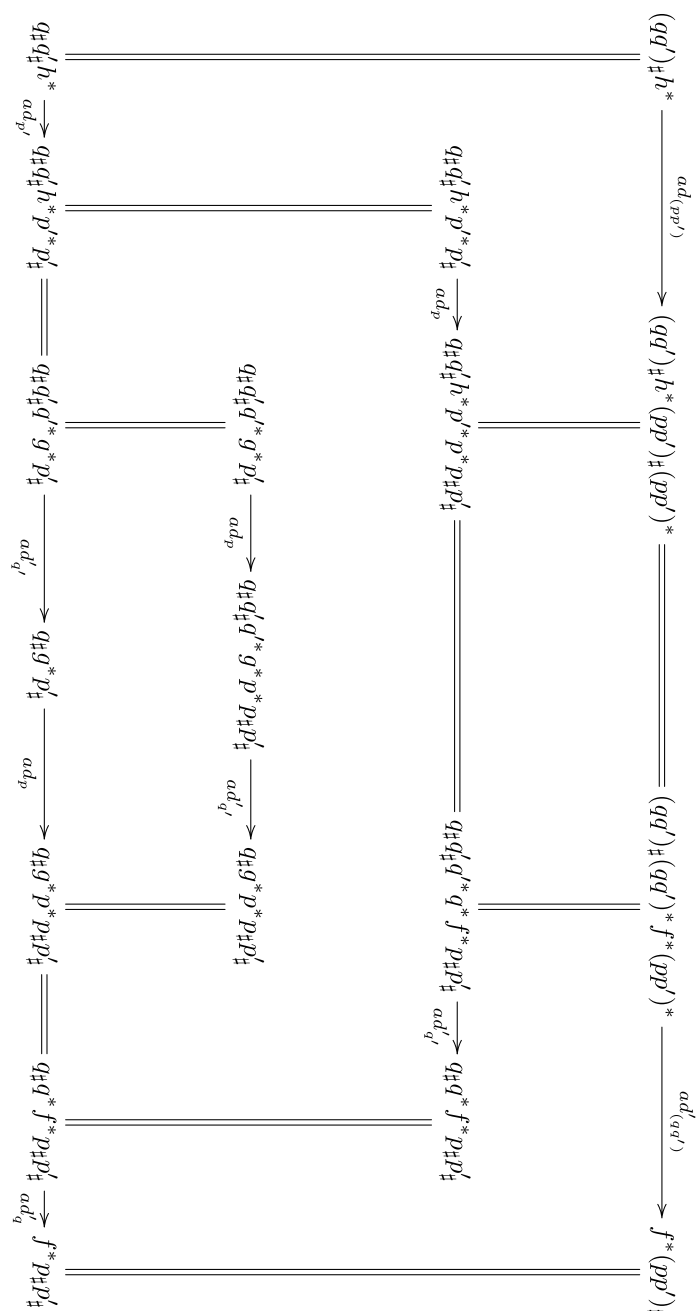
Remark 1.1.7. These exchange transformations satisfy a coherence condition with respect to the relations $(fg)^* = g^* f^*$ and $(fg)_\sharp = f_\sharp g_\sharp$. As an example, consider two commutative squares in $\mathscr{S}$:

$$\begin{array}{ccccc} Z & \xrightarrow{q'} & Y & \xrightarrow{q} & X \\ {\scriptstyle h}\downarrow & \Theta & {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ W & \xrightarrow[p']{} & T & \xrightarrow[p]{} & S \end{array}$$

and let $\Delta \circ \Theta$ be the commutative square made by the exterior maps – it is usually called the horizontal composition of the squares. Then, the following diagram of 2-morphisms is commutative:

$$\begin{array}{ccccc} (qq')_\sharp h^* & & \xrightarrow{Ex(\Delta\circ\Theta)^*_\sharp} & & f^*(pp')_\sharp \\ \| & & & & \| \\ q_\sharp q'_\sharp h^* & \xrightarrow{Ex(\Theta^*_\sharp)} & q_\sharp g^* p'_\sharp & \xrightarrow{Ex(\Delta^*_\sharp)} & f^* p_\sharp p'_\sharp \end{array}$$

To see this, one divides this diagram as follows:



where ad_r (resp. ad'_r) indicates the morphism obtained from the obvious unit morphism (resp. counit morphism) of the adjunction $(r_\sharp, r^*)$ by eventually adding functors on the left or on the right. The reader can check easily that each cell of the above diagram is commutative, proving our claim.

Thus, according to our abuse of notation for natural transformations, Ex behaves as a contravariant functor with respect to the horizontal composition of squares. The same is true for vertical composition of commutative squares.

Remark 1.1.8. In the sequel, we will introduce several exchange transformation between various functor. We speak of an exchange isomorphism when the transformation is an *exchange isomorphism*. When only two kind of functors are involved, say of type a and b, we say that functors of type a and functors of type b commute when the exchange transformation is an isomorphism.

As an example (see also next definition), when the exchange transformation $Ex(\Delta_\sharp^*)$ is an isomorphism, we simply say that f^* and $p_\sharp$ commute – or also that f^* commutes with $p_\sharp$.

1.1.9. Under the assumptions of 1.1.6, we will consider the following property:

($\mathscr{P}$-BC) *$\mathscr{P}$-base change.*– For any cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow[p]{} & S \end{array}$$

such that p is a $\mathscr{P}$-morphism, the exchange transformation

$$Ex(\Delta_\sharp^*) : q_\sharp g^* \to f^* p_\sharp$$

is an isomorphism.[30]

Definition 1.1.10. A *$\mathscr{P}$-fibred category* over $\mathscr{S}$ is a pre-$\mathscr{P}$-fibred category $\mathscr{M}$ over $\mathscr{S}$ which satisfies the property of $\mathscr{P}$-base change.

Example 1.1.11. Consider the notations of example 1.1.4. Then the transitivity property of pullbacks of morphisms in $\mathscr{P}$ amounts to say that $\mathscr{P}/?$ satisfies the $\mathscr{P}$-base change property. Thus, $\mathscr{P}/?$ is in fact a $\mathscr{P}$-fibred category, called *the canonical $\mathscr{P}$-fibred category.*

Definition 1.1.12. A $\mathscr{P}$-fibred category $\mathscr{M}$ over $\mathscr{S}$ is *complete* if, for any morphism $f : T \to S$, the pullback functor $f^* : \mathscr{M}(S) \to \mathscr{M}(T)$ admits a right adjoint $f_* : \mathscr{M}(S) \to \mathscr{M}(T)$.

Remark 1.1.13. In the case where $\mathscr{P}$ is the class of isomorphisms a $\mathscr{P}$-fibred category is what we usually call a bifibred category over $\mathscr{S}$.

Example 1.1.14. The pre-Sm-fibred category $\mathscr{H}_\bullet$ of example 1.1.5 is a complete Sm-fibred category according to [**MV99**].

1.1.15. *Exchange structures II.*– Let $\mathscr{M}$ be a complete $\mathscr{P}$-fibred category. Consider a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow[p]{} & S. \end{array}$$

We obtain an exchange transformation:

$$Ex(\Delta_*^*) : p^* f_* \xrightarrow{ad(g^*,g_*)} g_* g^* p^* f_* = g_* q^* f^* f_* \xrightarrow{ad'(f^*,f_*)} g_* q^*.$$

Assume moreover that p and q are $\mathscr{P}$-morphism. Then we can check that $Ex(\Delta_*^*)$ is the transpose of the exchange $Ex(\Delta_\sharp^*)$. Thus, when Δ is cartesian and p is a $\mathscr{P}$-morphism, $Ex(\Delta_*^*)$ is an isomorphism according to ($\mathscr{P}$-BC).

We can also define an exchange transformation:

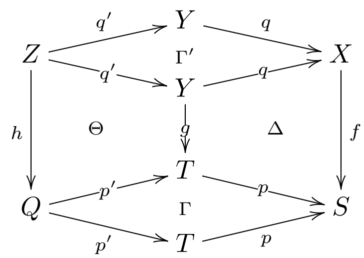
$$Ex(\Delta_{\sharp *}) : p_\sharp g_* \xrightarrow{ad(f^*,f_*)} f_* f^* p_\sharp g_* \xrightarrow{Ex(\Delta_\sharp^*)^{-1}} f_* q_\sharp g^* g_* \xrightarrow{ad'(g^*,g_*)} f_* q_\sharp.$$

Remark 1.1.16. As in remark 1.1.7, we obtain coherence results for these exchange transformations.

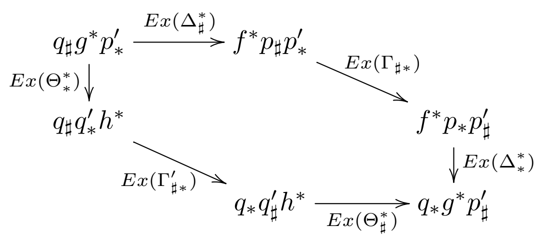
First with respect to the identifications of the kind $f^* g^* = (gf)^*$, $(fg)_* = f_* g_*$, $(fg)_\sharp = f_\sharp g_\sharp$.

[30]In other words, f^* commutes with $p_\sharp$.

Secondly, when several exchange transformations of different kind are involved. As an example, we consider the following commutative diagram in $\mathscr{S}$:



Then the following diagram of natural transformations is commutative:



We leave the verification to the reader (it is analogous to that of Remark 1.1.7 except that it involves also to the compatibility of the unit and counit of an adjunction).

Definition 1.1.17. Let $\mathscr{M}$ be a complete $\mathscr{P}$-fibred category. Consider a commutative square in $\mathscr{S}$

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow[p]{} & S. \end{array}$$

We will say that Δ is *$\mathscr{M}$-transversal* if the exchange transformation

$$Ex(\Delta_*^*) : p^* f_* \to g_* q^*$$

of 1.1.15 is an isomorphism.

Given an admissible class of morphisms Q in $\mathscr{S}$, we say that $\mathscr{M}$ has the *transversality* (resp. *cotransversality*) *property with respect to Q-morphisms*, if, for any cartesian square Δ as above such that f is in Q (resp. p is in Q), Δ is $\mathscr{M}$-transversal.

Remark 1.1.18. Assume $\mathscr{S}$ is a sub-category of the category of schemes. When Q is the class of smooth morphisms (resp. proper morphisms), the cotransversality (resp. transversality) property with respect to Q is usually called the *smooth base change property* (resp. *proper base change property*). See also Definition 2.2.13.

According to Paragraph 1.1.15, we derive the following consequence of our axioms:

Proposition 1.1.19. *Any complete $\mathscr{P}$-fibred category has the cotransversality property with respect to $\mathscr{P}$.*

Let us note for future reference the following corollary:

Corollary 1.1.20. *If $\mathscr{M}$ is a $\mathscr{P}$-fibred category, then, for any monomorphism $j : U \to S$ in $\mathscr{P}$, the functor $j_\sharp$ is fully faithful. If moreover $\mathscr{M}$ is complete, then the functor j_* is fully faithful as well.*

Proof. Because j is a monomorphism, we get a cartesian square in $\mathscr{S}$:

$$\begin{array}{ccc} U & = & U \\ \| & \Delta & \downarrow{\scriptstyle j} \\ U & \xrightarrow[j]{} & S. \end{array}$$

Remark that $Ex(\Delta_\sharp^*) : 1 \to j^* j_\sharp$ is the unit of the adjunction $(j_\sharp, j^*)$. Thus the $\mathscr{P}$-base change property shows that $j_\sharp$ is fully faithful.

Assume $\mathscr{M}$ is complete. We remark similarly that $Ex(\Delta_*^*) : j^* j_* \to 1$ is the counit of the adjunction (j^*, j_*). Thus, the above proposition shows readily that j_* is fully faithful. □

1.1.b. *Monoidal structures.* Let $\mathscr{C}at^\otimes$ be the sub-2-category of $\mathscr{C}at$ made of symmetric monoidal categories whose 1-morphisms are (strong) symmetric monoidal functors and 2-morphisms are symmetric monoidal transformations.

Definition 1.1.21. A *monoidal pre-$\mathscr{P}$-fibred category over $\mathscr{S}$*
is a 2-functor

$$\mathscr{M} : \mathscr{S} \to \mathscr{C}at^\otimes$$

such that $\mathscr{M}$ is a pre-$\mathscr{P}$-fibred category.

In other words, $\mathscr{M}$ is a pre-$\mathscr{P}$-fibred category such that each of its fibers $\mathscr{M}(S)$ is endowed with a structure of a monoidal category, and any pullback morphism f^* is monoidal, with the obvious coherent structures. For an object S of $\mathscr{S}$, we will usually denote by $\otimes_S$ (resp. $\mathbb{1}_S$) the tensor product (resp. unit) of $\mathscr{M}(S)$.
In particular, we then have the following natural isomorphisms:

- for a morphism $f : T \to S$ in $\mathscr{S}$, and objects M, N of $\mathscr{M}(S)$,

$$f^*(M) \otimes_T f^*(N) \xrightarrow{\sim} f^*(M \otimes_S N);$$

- for a morphism $f : T \to S$ in $\mathscr{S}$,

$$f^*(\mathbb{1}_S) \xrightarrow{\sim} \mathbb{1}_T\,.$$

Convention 1.1.22. As in convention 1.1.3, we will generally consider that these structural isomorphisms are identities.

Example 1.1.23. Consider the notations of example 1.1.4.

Using the properties (Pb) and (Pc) of $\mathscr{P}$ (see 1.0), for two S-objects X and Y in $\mathscr{P}/S$, the cartesian product $X \times_S Y$ is an object of $\mathscr{P}/S$. This defines a symmetric monoidal structure on $\mathscr{P}/S$ with unit the trivial S-object S. Moreover, the functor f^* defined in *loc. cit.* is monoidal. Thus, the pre-$\mathscr{P}$-fibred category $\mathscr{P}/?$ is in fact monoidal.

1.1.24. *Monoidal exchange structures I.* Let $\mathscr{M}$ be a monoidal pre-$\mathscr{P}$-fibred category $\mathscr{M}$ over $\mathscr{S}$.

Consider a $\mathscr{P}$-morphism $f : T \to S$, and M (resp. N) an object of $\mathscr{M}(T)$ (resp. $\mathscr{M}(S)$).

We get a morphism in $\mathscr{M}(S)$

$$Ex(f_\sharp^*, \otimes) : f_\sharp(M \otimes_T f^*(N)) \longrightarrow f_\sharp(M) \otimes_S N$$

as the composition

$$f_\sharp(M \otimes_T f^*(N)) \to f_\sharp(f^* f_\sharp(M) \otimes_T f^*(N)) \simeq f_\sharp f^*(f_\sharp(M) \otimes_S N) \to f_\sharp(M) \otimes_S N\,.$$

This map is natural in M and N. It will be called the *exchange transformation* between $f_\sharp$ and $\otimes_T$.

Remark also that the functor $f_\sharp$, as a left adjoint of a symmetric monoidal functor, is colax symmetric monoidal: for any objects M and N of $\mathscr{M}(T)$, there is a canonical morphism

$$f_\sharp(M) \otimes_S f_\sharp(N) \to f_\sharp(M \otimes_T N) \tag{1.1.24.1}$$

natural in M and N, as well as a natural map

$$f_\sharp(\mathbb{1}_T) \to \mathbb{1}_S\,. \tag{1.1.24.2}$$

Remark 1.1.25. As in remark 1.1.7, the preceding exchange transformations satisfy a coherence condition for composable morphisms $W \xrightarrow{g} T \xrightarrow{f} S$. We get in fact a commutative diagram:

$$\begin{array}{ccccc}
(fg)_\sharp\big(M \otimes_S (fg)^*(N)\big) & & \xrightarrow{Ex((fg)^*_\sharp,\otimes)} & & \big((fg)_\sharp(M)\big) \otimes_W N \\
\| & & & & \| \\
f_\sharp g_\sharp\big(M \otimes_S g^* f^*(N)\big) & \xrightarrow{Ex(g^*_\sharp,\otimes)} & f_\sharp\big(g_\sharp(M) \otimes_T f^*(N)\big) & \xrightarrow{Ex(f^*_\sharp,\otimes)} & \big(f_\sharp g_\sharp(M)\big) \otimes_W N
\end{array}$$

As in remark 1.1.16, there is also a coherence relation when different kinds of exchange transformations are involved. Consider a commutative square in $\mathscr{S}$

$$\begin{array}{ccc}
Y & \xrightarrow{q} & X \\
{\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\
T & \xrightarrow[p]{} & S
\end{array}$$

such that p and q are $\mathscr{P}$-morphisms and put $h = f \circ q = p \circ g$. Then the following diagram is commutative:

$$\begin{array}{ccccc}
q_\sharp g^*(M \otimes_T p^* N) & \xrightarrow{Ex(\Delta^*_\sharp)} & f^* p_\sharp(M \otimes_T p^* N) & \xrightarrow{Ex(p^*_\sharp,\otimes)} & f^*(p_\sharp M \otimes_S N) \\
\| & & & & \| \\
q_\sharp(g^* M \otimes_Y q^* f^* N) & \xrightarrow{Ex(q^*_\sharp,\otimes)} & (q_\sharp g^* M) \otimes_X f^* N & \xrightarrow{Ex(\Delta^*_\sharp)} & (f^* p_\sharp M) \otimes_X f^* N
\end{array}$$

We leave the verification to the reader.

1.1.26. Under the assumptions of 1.1.24, we will consider the following property:

($\mathscr{P}$-PF) *$\mathscr{P}$-projection formula.*– For any $\mathscr{P}$-morphism $f : T \to S$ the exchange transformation

$$Ex(f_\sharp, \otimes_T) : f_\sharp(M \otimes_T f^*(N)) \to f_\sharp(M) \otimes_S N$$

is an isomorphism for all M and N.

Definition 1.1.27. A *monoidal* $\mathscr{P}$-fibred category over $\mathscr{S}$ is a monoidal pre-$\mathscr{P}$-fibred category $\mathscr{M} : \mathscr{S}^{op} \to \mathscr{C}at^{\otimes}$ over $\mathscr{S}$ which satisfies the $\mathscr{P}$-projection formula.

Example 1.1.28. Consider the canonical monoidal weak $\mathscr{P}$-fibred category $\mathscr{P}/?$ (see example 1.1.23). The transitivity property of pullbacks implies readily that $\mathscr{P}/?$ satisfies the property ($\mathscr{P}$-PF). Thus, $\mathscr{P}/?$ is in fact a monoidal $\mathscr{P}$-fibred category called *canonical*.

Definition 1.1.29. A monoidal $\mathscr{P}$-fibred category $\mathscr{M}$ over $\mathscr{S}$ is *complete* if it satisfies the following conditions:

(1) $\mathscr{M}$ is complete as a $\mathscr{P}$-fibred category.
(2) For any object S of $\mathscr{S}$, the monoidal category $\mathscr{M}(S)$ is closed (i.e. has an internal Hom).

In this case, we will usually denote by Hom_S the internal Hom in $\mathscr{M}(S)$, so that we have natural bijections

$$\mathrm{Hom}_{\mathscr{M}(S)}(A \otimes_S B, C) \simeq \mathrm{Hom}_{\mathscr{M}(S)}(A, Hom_S(B, C))\,.$$

Example 1.1.30. The $\mathscr{P}$-fibred category $\mathscr{H}_\bullet$ of example 1.1.14 is in fact a complete monoidal $\mathscr{P}$-fibred category. The tensor product is given by the smash product (see [**MV99**]).

1.1.31. *Monoidal exchange structures II.*– Let $\mathscr{M}$ be a complete monoidal $\mathscr{P}$-fibred category. Consider a morphism $f : T \to S$ in $\mathscr{S}$. Then we obtain an exchange transformation:

$$\begin{aligned}
Ex(f^*_*, \otimes_S) : (f_* M) \otimes_S N &\xrightarrow{ad(f^*,f_*)} f_* f^*\big((f_* M) \otimes_S N\big) \\
&= f_*\big((f^* f_* M) \otimes_T f^* N\big) \xrightarrow{ad'(f^*,f_*)} f_*(M \otimes_T f^* N).
\end{aligned}$$

REMARK 1.1.32. As in remark 1.1.25, these exchange transformations are compatible with the identifications $(fg)_* = f_* g_*$ and $(fg)^* = g^* f^*$.

Moreover, there is a coherence relation when composing the exchange transformations of the kind $Ex(f^*_*, \otimes)$ with exchange transformations of the kind $Ex(\Delta^*_*)$ as in *loc. cit.*

Finally, note another kind of coherence relations involving $Ex(f^*_*, \otimes)$, $Ex(\Delta^*_\sharp)$ (resp. $Ex(f^*_\sharp, \otimes)$) and $Ex(\Delta_{\sharp *})$.

We leave the formulation of these coherence relations to the reader, on the model of the preceding ones.

1.1.33. *Monoidal exchange structures III.*– Let $\mathscr{M}$ be a complete monoidal $\mathscr{P}$-fibred category and $f : T \to S$ be a morphism in $\mathscr{S}$.

Because f^* is monoidal, we get by adjunction a canonical isomorphism

$$Hom_S(M, f_* N) \to f_* Hom_T(f^* M, N).$$

Assume that f is a $\mathscr{P}$-morphism. Then from the $\mathscr{P}$-projection formula, we get by adjunction two canonical isomorphisms:

$$f^* Hom_S(M, N) \to Hom_T(f^* M, f^* N),$$
$$Hom_S(f_\sharp M, N) \to f_* Hom_T(M, f^* N)$$

These isomorphisms are generically called *exchange isomorphisms.*

1.1.c. *Geometric sections.*

1.1.34. Consider a weak $\mathscr{P}$-fibred category $\mathscr{M}$.

Let S be a scheme. For any $\mathscr{P}$-morphism $p : X \to S$, we put $M_S(X) := p_\sharp(\mathbb{1}_X)$. According to our conventions, this object is identified with $p_\sharp p^*(\mathbb{1}_S)$. In particular, it defines a covariant functor $M_S : \mathscr{P}/S \to \mathscr{M}(S)$.

Consider a cartesian square in $\mathscr{S}$

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle q}\downarrow & \Delta & \downarrow{\scriptstyle p} \\ T & \xrightarrow[f]{} & S \end{array}$$

such that p is a $\mathscr{P}$-morphism. With the notations of example 1.1.4, $Y = f^*(X)$. Then we get a natural exchange transformation

$$Ex(M_T, f^*) : M_T(f^*(X)) = q_\sharp(\mathbb{1}_Y) = q_\sharp g^*(\mathbb{1}_X) \xrightarrow{Ex(\Delta^*_\sharp)} f^* p_\sharp(\mathbb{1}_X) = f^* M_S(X).$$

In other words, M defines a lax natural transformation $\mathscr{P}/? \to \mathscr{M}$.

Consider $\mathscr{P}$-morphisms $p : X \to S$, $q : Y \to S$. Let $Z = X \times_S Y$ be the cartesian product and consider the cartesian square:

$$\begin{array}{ccc} Z & \xrightarrow{p'} & Y \\ {\scriptstyle q'}\downarrow & \Theta & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & S. \end{array}$$

Using the exchange transformations of the preceding paragraph, we get a canonical morphism

$$Ex(M_S, \otimes_S) : M_S(X \times_S Y) \longrightarrow M_S(X) \otimes_S M_S(Y)$$

as the composition

$$M_S(X \times_S Y) = p_\sharp q'_\sharp p'^*(\mathbb{1}_Y) \xrightarrow{Ex(\Theta^*_\sharp)} p_\sharp p^* q_\sharp(\mathbb{1}_Y) \quad p_\sharp(\mathbb{1}_X \otimes_X p^* q_\sharp(\mathbb{1}_Y))$$
$$\xrightarrow{Ex(p_\sharp, \otimes_X)} p_\sharp(\mathbb{1}_X) \otimes_S q_\sharp(\mathbb{1}_Y) = M_S(X) \otimes_S M_S(Y).$$

In other words, the functor M_S is symmetric colax monoidal.

Remark finally that for any $\mathscr{P}$-morphism $p : T \to S$, and any $\mathscr{P}$-object Y over T, we obtain according to convention an identification $p_\sharp M_T(Y) = M_S(Y)$.

DEFINITION 1.1.35. Given a monoidal pre-$\mathscr{P}$-fibred category $\mathscr{M}$ over $\mathscr{S}$, the lax natural transformation $M : \mathscr{P}/? \to \mathscr{M}$ constructed above will be called the *geometric sections of* $\mathscr{M}$.

The following lemma is obvious from the definitions above:

LEMMA 1.1.36. *let $\mathscr{M}$ be a monoidal $\mathscr{P}$-fibred category. Let $M : \mathscr{P}/? \to \mathscr{M}$ be the geometric sections of $\mathscr{M}$. Then:*

(i) *For any morphism $f : T \to S$ in $\mathscr{S}$, the exchange $Ex(M_T, f^*)$ defined above is an isomorphism.*
(ii) *For any scheme S, the exchange $Ex(M_S, \otimes_S)$ defined above is an isomorphism.*

In other words, M is a cartesian functor and M_S is a (strong) *symmetric monoidal functor.*

1.1.37. In the situation of the lemma we thus obtain the following identifications:

- $f^*M_S(X) \simeq M_T(X \times_S T)$,
- $p_\sharp M_T(Y) \simeq M_S(Y)$,
- $M_S(X \times_S Y) \simeq M_S(X) \otimes_S M_S(Y)$,

whenever it makes sense.

1.1.d. *Twists.*

1.1.38. Let $\mathscr{M}$ be a pre-$\mathscr{P}$-fibred category of $\mathscr{S}$. Recall that a cartesian section of $\mathscr{M}$ (*i.e.* a cartesian functor $A : \mathscr{S} \to \mathscr{M}$) is the data of an object A_S of $\mathscr{M}(S)$ for each object S of $\mathscr{S}$ and of isomorphisms

$$f^*(A_S) \xrightarrow{\sim} A_T$$

for each morphism $f : T \to S$, subject to coherence identities; see [**SGA1**, Exp. VI].

If $\mathscr{M}$ is monoidal, the tensor product of two cartesian sections is defined termwise.

DEFINITION 1.1.39. let $\mathscr{M}$ be a monoidal pre-$\mathscr{P}$-fibred category. A set of *twists* τ for $\mathscr{M}$ is a set of cartesian sections of $\mathscr{M}$ which is stable by tensor product (up to isomorphism), and contains the unit $\mathbb{1}$. For short, when $\mathscr{M}$ is endowed with a set of twists τ, we say also that $\mathscr{M}$ is τ*-twisted*.

1.1.40. Let $\mathscr{M}$ be a monoidal pre-$\mathscr{P}$-fibred category endowed with a set of twists τ.

The tensor product on τ induces a monoid structure that we will denote by + (the unit object of τ will be written 0).

Consider an object $i \in \tau$. For any object S of $\mathscr{S}$, we thus obtain an object $t(i)_S$ in $\mathscr{M}(S)$ associated with i. Given any object M of $\mathscr{M}(S)$, we simply put:

$$M\{i\} = M \otimes_S i_S$$

and call this object the twist of M by i. We also define $M\{0\} = M$.

For any $i, j \in \tau$, and any object M of $\mathscr{M}(S)$, we define $M\{i + j\} = (M\{i\})\{j\}$. Given a morphism $f : T \to S$, an object M of $\mathscr{M}(S)$ and a twist $i \in \tau$, we also obtain $f^*(M\{i\}) = (f^*M)\{i\}$. If f is a $\mathscr{P}$-morphism, for any object M of $\mathscr{M}(T)$, the exchange transformation $Ex(f_\sharp^*, \otimes_T)$ of paragraph 1.1.6 induces a canonical morphism

$$Ex(f_\sharp, \{i\}) : f_\sharp(M\{i\}) \to (f_\sharp M)\{i\}.$$

We will say that $f_\sharp$ *commutes with* τ*-twists* (or simply twists when τ is clear) if for any $i \in \tau$, the natural transformation $Ex(f_\sharp, \{i\})$ is an isomorphism.

DEFINITION 1.1.41. Let $\mathscr{M}$ be a monoidal pre-$\mathscr{P}$-fibred category with a set of twists τ and $M : \mathscr{P}/? \to \mathscr{M}$ be the geometric sections of $\mathscr{M}$.

We say $\mathscr{M}$ is τ*-generated* if for any object S of $\mathscr{S}$, the family of functors

$$\mathrm{Hom}_{\mathscr{M}(S)}(M_S(X)\{i\}, -) : \mathscr{M}(S) \to \mathscr{S}et$$

indexed by a $\mathscr{P}$-object X/S and an element $i \in \tau$ is conservative.

Of course, we do not exclude the case where τ is trivial, but then, we shall simply say that $\mathscr{M}$ is *geometrically generated*.

We shall frequently use the following proposition to characterize complete monoidal $\mathscr{P}$-fibred categories over $\mathscr{S}$:

PROPOSITION 1.1.42. *Let $\mathscr{M} : \mathscr{S} \to \mathscr{C}at^{\otimes}$ be a 2-functor such that:*

(1) For any $\mathscr{P}$-morphism $f : T \to S$, the pullback functor $f^* : \mathscr{M}(S) \to \mathscr{M}(T)$ is monoidal and admits a left adjoint $f_\sharp$ in $\mathscr{C}$.
(2) For any morphism $f : T \to S$, the pullback functor $f^* : \mathscr{M}(S) \to \mathscr{M}(T)$ admits a right adjoint f_* in $\mathscr{C}$.

We consider $\mathscr{M}$ as a monoidal weak $\mathscr{P}$-fibred category and denote by $M : \mathscr{P}/? \to \mathscr{M}$ its associated geometric sections. Suppose given a set of twists τ such that $\mathscr{M}$ is τ-generated. Then, the following assertions are equivalent:

(i) $\mathscr{M}$ satisfies properties ($\mathscr{P}$-BC) and ($\mathscr{P}$-PF)
*(*i.e. *$\mathscr{M}$ is a complete monoidal $\mathscr{P}$-fibred category.)*
(ii) (a) M is a cartesian functor.
(b) For any object S of $\mathscr{S}$, M_S is (strongly) monoidal.
(c) For any $\mathscr{P}$-morphism f, $f_\sharp$ commutes with τ-twists.

PROOF. $(i) \Rightarrow (ii)$: This is obvious (see Lemma 1.1.36).
$(ii) \Rightarrow (i)$: We use the following easy lemma:

LEMMA 1.1.43. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be categories, $F, G : \mathscr{C}_1 \to \mathscr{C}_2$ be two left adjoint functors, and $\eta : F \to G$ be a natural transformation. Let $\mathcal{G}$ be a class of objects of $\mathscr{C}_1$ which is generating in the sense that the family of functors $\mathrm{Hom}_{\mathscr{C}_1}(X, -)$ for X in $\mathcal{G}$ is conservative.*

Then the following conditions are equivalent:

(1) η is an isomorphism.
(2) For all X in $\mathcal{G}$, η_X is an isomorphism.

Given this lemma, to prove ($\mathscr{P}$-BC), we are reduced to check that the exchange transformation $Ex(\Delta_\sharp^*)$ is an isomorphism when evaluated on an object $M_T(U)\{i\}$ for an object U of $\mathscr{P}/T$ and a twist $i \in \tau$. Then it follows from (ii), 1.1.40 and example 1.1.11.[31]
To prove ($\mathscr{P}$-PF), we proceed in two steps first proving the case $M = M_T(U)\{i\}$ and N any object of $\mathscr{M}(S)$ using the same argument as above with the help of 1.1.28. Then, we can prove the general case by another application of the same argument. □

Suppose given a complete monoidal $\mathscr{P}$-fibered category $\mathscr{M}$ with a set of twists τ. Let $f : T \to S$ be a morphism of $\mathscr{S}$. Then the exchange transformation 1.1.31 induces for any $i \in \tau$ an exchange transformation

$$Ex(f_*, \{i\}) : (f_*M)\{i\} \to f_*(M\{i\})\,.$$

DEFINITION 1.1.44. In the situation above, we say that f_* *commutes with τ-twists* (or simply with twists when τ is clear) if, for any $i \in \tau$, the exchange transformation $Ex(f_*, \{i\})$ is an isomorphism.

It will happen frequently that twists are $\otimes$-invertible. Then f_* commutes with twists as its right adjoint does.

1.2. Morphisms of $\mathscr{P}$-fibred categories.

1.2.a. *General case.*

1.2.1. Consider two $\mathscr{P}$-fibred categories $\mathscr{M}$ and $\mathscr{M}'$ over $\mathscr{S}$, as well as a cartesian functor $\varphi^* : \mathscr{M} \to \mathscr{M}'$ between the underlying fibred categories: for any object S of $\mathscr{S}$, we have a functor

$$\varphi_S^* : \mathscr{M}(S) \to \mathscr{M}'(S)\,,$$

[31] The cautious reader will use remark 1.1.7 to check that the corresponding map

$$M_X(U \times_T Y)\{i\} \to M_X(U \times_T Y)\{i\}$$

is the identity.

and for any map $f : T \to S$ in $\mathscr{S}$, we have an isomorphism of functors c_f

(1.2.1.1)
$$\begin{array}{ccc} \mathscr{M}(S) & \xrightarrow{\varphi_S^*} & \mathscr{M}'(S) \\ {\scriptstyle f^*}\downarrow & \swarrow\!\!\!\!/\ c_f & \downarrow{\scriptstyle f^*} \\ \mathscr{M}(T) & \xrightarrow[\varphi_T^*]{} & \mathscr{M}'(T) \end{array} \qquad c_f : f^* \varphi_S^* \xrightarrow{\sim} \varphi_T^* f^*$$

satisfying some cocycle condition with respect to composition in $\mathscr{S}$.

For any $\mathscr{P}$-morphism $p : T \to S$, we construct an exchange morphism

$$Ex(p_\sharp, \varphi^*) : p_\sharp \varphi_T^* \longrightarrow \varphi_S^* p_\sharp$$

as the composition

$$p_\sharp \varphi_T^* \xrightarrow{ad(p_\sharp, p^*)} p_\sharp \varphi_T^* p^* p_\sharp \xrightarrow{c_p^{-1}} p_\sharp p^* \varphi_S^* p_\sharp \xrightarrow{ad'(p_\sharp, p^*)} \varphi_S^* p_\sharp.$$

Definition 1.2.2. Consider the situation above. We say that the cartesian functor

$$\varphi^* : \mathscr{M} \to \mathscr{M}'$$

is a *morphism of $\mathscr{P}$-fibred categories* if, for any $\mathscr{P}$-morphism p, the exchange transformation $Ex(p_\sharp, \varphi^*)$ is an isomorphism.

Example 1.2.3. If $\mathscr{M}$ is a monoidal $\mathscr{P}$-fibred category, then the geometric sections $M : \mathscr{P}/? \to \mathscr{M}$ is a morphism of $\mathscr{P}$-fibred categories (1.1.36).

Definition 1.2.4. Let $\mathscr{M}$ and $\mathscr{M}'$ be two complete $\mathscr{P}$-fibred categories. A *morphism of complete $\mathscr{P}$-fibred categories* is a morphism of $\mathscr{P}$-fibred categories

$$\varphi^* : \mathscr{M} \to \mathscr{M}'$$

such that, for any object S of $\mathscr{S}$, the functor $\varphi_S^* : \mathscr{M}(S) \to \mathscr{M}'(S)$ has a right adjoint

$$\varphi_{*,S} : \mathscr{M}'(S) \to \mathscr{M}(S)\,.$$

When we want to indicate a notation for the right adjoint of a morphism as above, we use the writing

$$\varphi^* : \mathscr{M} \rightleftarrows \mathscr{N} : \varphi_*$$

the left adjoint being in the left hand side.

1.2.5. *Exchange structures III.* Consider a morphism $\varphi^* : \mathscr{M} \to \mathscr{M}'$ of complete $\mathscr{P}$-fibred categories.

Then for any morphism $f : T \to S$ in $\mathscr{S}$, we define exchange transformations

(1.2.5.1) $$Ex(\varphi^*, f_*) : \varphi_S^* f_* \longrightarrow f_* \varphi_T^*,$$

(1.2.5.2) $$Ex(f^*, \varphi_*) : f^* \varphi_{*,S} \longrightarrow \varphi_{*,T} f^*,$$

as the respective compositions

$$\varphi_S^* f_* \xrightarrow{ad(f^*, f_*)} f_* f^* \varphi_S^* f_* \simeq f_* \varphi_T^* f^* f_* \xrightarrow{ad'(f^*, f_*)} f_* \varphi_T^*,$$

$$f^* \varphi_{*,S} \xrightarrow{ad(f^*, f_*)} f^* \varphi_{*,S} f_* f^* \simeq f^* f_* \varphi_{*,T} f^* \xrightarrow{ad'(f^*, f_*)} \varphi_{*,T} f^*.$$

Remark 1.2.6. We warn the reader that $\varphi_* : \mathscr{M}' \to \mathscr{M}$ is not a cartesian functor in general, meaning that the exchange transformation $Ex(f^*, \varphi_*)$ is not necessarily an isomorphism, even when f is a $\mathscr{P}$-morphism.

1.2.b. *Monoidal case.*

DEFINITION 1.2.7. Let $\mathscr{M}$ and $\mathscr{M}'$ be monoidal $\mathscr{P}$-fibred categories.

A *morphisms of monoidal $\mathscr{P}$-fibred categories* is a morphism $\varphi^* : \mathscr{M} \to \mathscr{M}'$ of $\mathscr{P}$-fibred categories such that for any object S of $\mathscr{S}$, the functor $\varphi^*_S : \mathscr{M}(X) \to \mathscr{N}(S)$ has the structure of a (strong) symmetric monoidal functor, and such that the structural isomorphisms (1.2.1.1) are isomorphisms of symmetric monoidal functors.

In the case where $\mathscr{M}$ and $\mathscr{M}'$ are complete monoidal $\mathscr{P}$-fibred categories, we shall say that such a morphism φ^* is a *morphism of complete monoidal $\mathscr{P}$-fibred categories* if φ^* is also a morphism of complete $\mathscr{P}$-fibred categories.

REMARK 1.2.8. If we denote by $M(-,\mathscr{M})$ and $M(-,\mathscr{M}')$ the geometric sections of $\mathscr{M}$ and $\mathscr{M}'$ respectively, we have a natural identification:

$$\varphi^*_S(M_S(X,\mathscr{M})) \simeq M_S(X,\mathscr{M}')\,.$$

1.2.9. *Monoidal exchange structures IV.* Consider a a morphism $\varphi^* : \mathscr{M} \to \mathscr{M}'$ of complete monoidal $\mathscr{P}$-fibred categories. For objects M (resp. N) of $\mathscr{M}(S)$ (resp. $\mathscr{M}'(S)$), we define an exchange transformation

$$Ex(\varphi_*, \otimes, \varphi^*) : (\varphi_{*,S} M) \otimes_S N \to \varphi_{*,S}(M \otimes_T \varphi^*_S N),$$

natural in M and N, as the following composite

$$\begin{aligned}(\varphi_{*,S} M) \otimes_S N &\xrightarrow{ad(\varphi^*,\varphi_*)} \varphi_{*,S}\varphi^*_S((\varphi_{*,S} M) \otimes_S N)\\ &= \varphi_{*,S}((\varphi^*_S \varphi_{*,S} M) \otimes_T \varphi^*_S N) \xrightarrow{ad'(\varphi^*,\varphi_*)} \varphi_{*,S}(M \otimes_T \varphi^*_S N).\end{aligned}$$

As in remark 1.1.32, we get coherence relations between the various exchange transformations associated with a morphism of monoidal $\mathscr{P}$-fibred categories. We left the formulation to the reader.

Note also that, because φ^* is monoidal, we get by adjunction a canonical isomorphism:

$$Hom_{\mathscr{M}(S)}(M, \varphi_{*,S} M') \xrightarrow{\sim} \varphi_{*,S} Hom_{\mathscr{M}'(S)}(\varphi^*_S M, M')\,.$$

1.2.10. Consider two monoidal $\mathscr{P}$-fibred categories $\mathscr{M}$, $\mathscr{M}'$ and a cartesian functor $\varphi^* : \mathscr{M} \to \mathscr{M}'$ such that, for any scheme S, $\varphi^*_S : \mathscr{M}(S) \to \mathscr{M}'(S)$ is monoidal.

Given a cartesian section $K = (K_S)_{S \in \mathscr{S}}$ of $\mathscr{M}$, we obtain for any morphism $f : T \to S$ in $\mathscr{S}$ a canonical map

$$f^* \varphi^*_S(K_S) = \varphi^*_T(f^*(K_S)) \to \varphi^*_T(K_T)$$

which defines a cartesian section of $\mathscr{M}'$, which we denote by $\varphi^*(K)$.

DEFINITION 1.2.11. Let $(\mathscr{M}, \tau)$ and $(\mathscr{M}', \tau')$ be twisted monoidal $\mathscr{P}$-fibred categories. Let $\varphi^* : \mathscr{M} \to \mathscr{M}'$ be a cartesian functor as above (resp. a morphism of monoidal $\mathscr{P}$-fibred categories).

We say that $\varphi^* : (\mathscr{M}, \tau) \to (\mathscr{M}', \tau')$ is *compatible with twists* if for any $i \in \tau$, the cartesian section $\varphi^*(i)$ is in τ' (up to isomorphism in $\mathscr{M}'$).

REMARK 1.2.12. In particular, φ^* induces a map $\tau \to \tau'$ (if we consider the isomorphism classes of objects). Moreover, for any object K of $\mathscr{M}(S)$ and any twist $i \in \tau$, we get an identification:

$$\varphi^*_S(K\{i\}) \simeq (\varphi^*_S K)\{\varphi^*(i)\}.$$

Moreover, the exchange transformation $Ex(\varphi_*, \otimes)$ induces an exchange:

$$Ex(\varphi_*, \{i\}) : \varphi_{*,S}(K)\{i\} \to \varphi_{*,S}\big(K\{\varphi^*(i)\}\big).$$

When this transformation is an isomorphism for any twist $i \in \tau$, we say that φ_* *commutes with twists.*

Note finally that Lemma 1.1.43 allows to prove, as for Proposition 1.1.42, the following useful lemma:

Lemma 1.2.13. *Consider two complete monoidal $\mathscr{P}$-fibred categories $\mathscr{M}$, $\mathscr{M}'$ and denote by $M(-,\mathscr{M})$ and $M(-,\mathscr{M}')$ their respective geometric sections. Let $\varphi^* : \mathscr{M} \to \mathscr{M}'$ be a cartesian functor such that*

(1) For any scheme S, $\varphi^*_S : \mathscr{M}(S) \to \mathscr{M}'(S)$ is monoidal.
(2) For any scheme S, φ^*_S admits a right adjoint $\varphi_{*,S}$.

Assume $\mathscr{M}$ (resp. $\mathscr{M}'$) is τ-generated (resp. τ'-twisted) and that φ^* induces a surjective map from the set of isomorphism classes of τ-twists to the set of isomorphism classes of τ'-twists. Then the following conditions are equivalent:

(i) φ^* is a morphism of complete monoidal $\mathscr{P}$-fibred categories.
(ii) For any object X of $\mathscr{P}/S$, the exchange transformation (cf. 1.2.1)

$$\varphi^* M_S(X,\mathscr{M}) \to M_S(X,\mathscr{M}')$$

is an isomorphism.

1.3. Structures on $\mathscr{P}$-fibred categories.

1.3.a. *Abstract definition.*

1.3.1. We fix a sub-2-category $\mathscr{C}$ of $\mathscr{C}at$ with the following properties[32]:

(1) the 2-functor

$$\mathscr{C}at \to \mathscr{C}at' \ , \quad A \mapsto A^{op}$$

sends $\mathscr{C}$ to $\mathscr{C}'$, where $\mathscr{C}'$ denotes the 2-category whose objects and maps are those of $\mathscr{C}$ and whose 2-morphisms are the 2-morphisms of $\mathscr{C}$, put in the reverse direction.

(2) $\mathscr{C}$ is closed under adjunction: for any functor $u : A \to B$ in $\mathscr{C}$, if a functor $v : B \to A$ is a right adjoint or a left adjoint to u, then v is in $\mathscr{C}$.

(3) the 2-morphisms of $\mathscr{C}$ are closed by transposition: if

$$u : A \rightleftarrows B : v \text{ and } u' : A \rightleftarrows B : v'$$

are two adjunctions in $\mathscr{C}$ (with the left adjoints on the left hand side), a natural transformation $u \to u'$ is in $\mathscr{C}$ if and only if the corresponding natural transformation $v' \to v$ is in $\mathscr{C}$.

We can then define and manipulate $\mathscr{C}$-structured $\mathscr{P}$-fibred categories as follows.

Definition 1.3.2. A *$\mathscr{C}$-structured $\mathscr{P}$-fibred category* (resp. *$\mathscr{C}$-structured complete $\mathscr{P}$-fibred category*) $\mathscr{M}$ over $\mathscr{S}$ is simply a $\mathscr{P}$-fibred category (resp. a complete $\mathscr{P}$-fibred category) whose underlying 2-functor $\mathscr{M} : \mathscr{S}^{op} \to \mathscr{C}at$ factors through $\mathscr{C}$.

If $\mathscr{M}$ and $\mathscr{M}'$ are $\mathscr{C}$-structured fibred categories over $\mathscr{S}$, a cartesian functor $\mathscr{M} \to \mathscr{M}'$ is *$\mathscr{C}$-structured* if the functors $\mathscr{M}(S) \to \mathscr{M}'(S)$ are in $\mathscr{C}$ for any object S of $\mathscr{S}$, and if all the structural 2-morphisms (1.2.1.1) are in $\mathscr{C}$ as well.

Definition 1.3.3. A morphism of $\mathscr{C}$-structured $\mathscr{P}$-fibred categories (resp. $\mathscr{C}$-structured complete $\mathscr{P}$-fibred categories) is a morphism of $\mathscr{P}$-fibred categories (resp. of complete $\mathscr{P}$-fibred categories) which is $\mathscr{C}$-structured as a cartesian functor.

1.3.4. Consider a 2-category $\mathscr{C}$ as in the paragraph 1.3.1. In order to deal with the monoidal case, we will consider also a sub-2-category $\mathscr{C}^{\otimes}$ of $\mathscr{C}$ such that:

(1) The objects of $\mathscr{C}^{\otimes}$ are objects of $\mathscr{C}$ equipped with a symmetric monoidal structure;
(2) the 1-morphisms of $\mathscr{C}^{\otimes}$ are exactly the 1-morphisms of $\mathscr{C}$ which are symmetric monoidal as functors;
(3) the 2-morphisms of $\mathscr{C}^{\otimes}$ are exactly the 2-morphisms of $\mathscr{C}$ which are symmetric monoidal as natural transformations.

Note that $\mathscr{C}^{\otimes}$ satisfies condition (1) of 1.3.1, but it does not satisfies conditions (2) and (3) in general. Instead, we get the following properties:

[32]See the following sections for examples.

(2′) If $u : A \to B$ is a functor in $\mathscr{C}^{\otimes}$, a right (resp. left) adjoint v is a lax[33] (resp. colax) monoidal functor in $\mathscr{C}$.

(3′) Consider adjunctions

$$u : A \rightleftarrows B : v \text{ and } u' : A \rightleftarrows B : v'$$

in $\mathscr{C}$ (with the left adjoints on the left hand side). If $u \to u'$ (resp. $v \to v'$) is a 2-morphism in $\mathscr{C}^{\otimes}$ then $v \to v'$ (resp. $u \to u'$) is a 2-morphism in $\mathscr{C}$ which is a symmetric monoidal transformation of lax (resp. colax) monoidal functors.

We thus adopt the following definition:

Definition 1.3.5. A $(\mathscr{C}, \mathscr{C}^{\otimes})$*-structured monoidal* $\mathscr{P}$*-fibred category* (resp. a $(\mathscr{C}, \mathscr{C}^{\otimes})$*-structured complete monoidal* $\mathscr{P}$*-fibred category*) is simply a monoidal $\mathscr{P}$-fibred category (resp. a complete monoidal $\mathscr{P}$-fibred category) whose underlying 2-functor $\mathscr{M} : \mathscr{S}^{op} \to \mathscr{C}at^{\otimes}$ factors through $\mathscr{C}^{\otimes}$. Morphisms of such objects are defined in the same way.

Note that, with the hypothesis made on $\mathscr{C}$, all the exchange natural transformations defined in the preceding paragraphs lie in $\mathscr{C}$ and satisfy the appropriate coherence property with respect to the monoidal structure.

1.3.b. *The abelian case.*

1.3.6. Let $\mathscr{A}b$ be the sub-2-category of $\mathscr{C}at$ made of the abelian categories, with the additive functors as 1-morphisms, and the natural transformations as 2-morphisms. Obviously, it satisfies properties of 1.3.1. When we will apply one of the definitions 1.3.2, 1.3.3 to the case $\mathscr{C} = \mathscr{A}b$, we will use the simple adjective *abelian* for $\mathscr{A}b$-structured. This allows to speak of *morphisms of abelian* $\mathscr{P}$*-fibred categories.*

Let $\mathscr{A}b^{\otimes}$ be the sub-2-category of $\mathscr{A}b$ made of the abelian monoidal categories, with 1-morphisms the symmetric monoidal additive functors and 2-morphisms the symmetric monoidal natural transformations. It satisfies the hypothesis of paragraph 1.3.4. When we will apply definition 1.3.5 to the case of $(\mathscr{A}b, \mathscr{A}b^{\otimes})$, we will use the simple expression *abelian monoidal* for $(\mathscr{A}b, \mathscr{A}b^{\otimes})$-structured monoidal. This allows to speak of *morphisms of abelian monoidal* $\mathscr{P}$*-fibred categories.*

Lemma 1.3.7. *Consider an abelian* $\mathscr{P}$*-fibred category* $\mathscr{A}$ *such that for any object* S *of* $\mathscr{S}$*,* $\mathscr{A}(S)$ *is a Grothendieck abelian category. Then the following conditions are equivalent:*

(i) $\mathscr{A}$ *is complete.*

(ii) For any morphism $f : T \to S$ *in* $\mathscr{S}$*,* f^* *commutes with sums.*

If in addition, $\mathscr{A}$ *is monoidal, the following conditions are equivalent:*

(i′) $\mathscr{A}$ *is monoidal complete.*

(ii′) (a) For any morphism $f : T \to S$ *in* $\mathscr{S}$*,* f^* *is right exact.*

(b) For any object S *of* $\mathscr{S}$*, the bifunctor* $\otimes_S$ *is right exact.*

In view of this lemma, we adopt the following definition:

Definition 1.3.8. A Grothendieck abelian (resp. Grothendieck abelian monoidal) $\mathscr{P}$-fibred category $\mathscr{A}$ over $\mathscr{S}$ is an abelian $\mathscr{P}$-fibred category which is complete (resp. complete monoidal) and such that for any scheme S, $\mathscr{A}(S)$ is a Grothendieck abelian category.

Remark 1.3.9. Let $\mathscr{A}$ be a Grothendieck abelian monoidal $\mathscr{P}$-fibred category. Conventionally, we will denote by $M_S(-, \mathscr{A})$ its geometric sections. Note that if $\mathscr{A}$ is τ-twisted, then any object of $\mathscr{A}$ is a quotient of a direct sum of objects of shape $M_S(X, \mathscr{A})\{i\}$ for a $\mathscr{P}$-object X/S and a twist $i \in \tau$.

1.3.10. Consider an abelian category $\mathscr{A}$ which admits small sums. Recall the following definition:

An object X of $\mathscr{T}$ is *finitely presented* if the functor $\mathrm{Hom}_{\mathscr{T}}(X, -)$ commutes with small filtering

[33] For any object a, a' in A, F is lax if there exists a structural map $F(a) \otimes F(a') \xrightarrow{(1)} F(a \otimes a')$ satisfying coherence relations (see [**Mac98**, XI. 2]). Colax is defined by reversing the arrow (1).

colimits. A essentially small $\mathcal{G}$ of objects of $\mathscr{A}$ is called generating if for any object A of $\mathscr{A}$ there exists an epimorphism of the form:

$$\bigoplus_{i \in I} G_i \to A$$

where $(G_i)_{i \in I}$ is a family of objects if $\mathcal{G}$.

Definition 1.3.11. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-fibred category over $\mathscr{S}$.

Given a set of twists τ of $\mathscr{A}$, we say $\mathscr{A}$ is *finitely τ-presented* if for any object S of $\mathscr{S}$, for any $\mathscr{P}$-object X/S and any twist $i \in \tau$, the object $M_S(X)\{i\}$ is finitely presented and the class of such objects form an essentially small generating family of $\mathscr{A}(S)$.

Remark 1.3.12.

1.3.c. *The triangulated case.*

1.3.13. Let $\mathscr{T}ri$ be the sub-2-category of $\mathscr{C}at$ made of the triangulated categories, with the triangulated functors as 1-morphisms, and the triangulated natural transformations as 2-morphisms. Then $\mathscr{T}ri$ satisfies the properties of 1.3.1 (property (2) can be found for instance in [**Ayo07a**, Lemma 2.1.23], and we leave property (3) as an exercise for the reader). When we will apply one of the definitions 1.3.2, 1.3.3 to the case $\mathscr{C} = \mathscr{T}ri$, we will use the simple adjective *triangulated* for $\mathscr{T}ri$-structured. This allows to speak of *morphisms of triangulated $\mathscr{P}$-fibred categories.*

Let $\mathscr{T}ri^{\otimes}$ be the sub-2-category of $\mathscr{T}ri$ made of the triangulated monoidal categories, with 1-morphisms the symmetric monoidal triangulated functors and 2-morphisms the symmetric monoidal natural transformations. It satisfies the hypothesis of paragraph 1.3.4. When we will apply definition 1.3.5 to the case of $(\mathscr{T}ri, \mathscr{T}ri^{\otimes})$, we will use the expression *triangulated monoidal* for $(\mathscr{T}ri, \mathscr{T}ri^{\otimes})$-structured monoidal. This allows to speak of *morphisms of triangulated monoidal $\mathscr{P}$-fibred categories.*

Convention 1.3.14. The set of twists of a triangulated monoidal $\mathscr{P}$-fibred category $\mathscr{T}$ will always be of the form $\mathbf{Z} \times \tau$, by which we mean that τ is a set of twists, while $\mathbf{Z} \times \tau$ is the closure of τ by suspension functors $[n]$, $n \in \mathbf{Z}$. In the notation, we shall often make the abuse of only indicating τ. In particular, the expression $\mathscr{T}$ is τ-generated will mean conventionally that $\mathscr{T}$ is $(\mathbf{Z} \times \tau)$-generated in the sense of definition 1.1.41.

1.3.15. Consider a triangulated category $\mathscr{T}$ which admits small sums. Recall the following definitions:
An object X of $\mathscr{T}$ is called *compact* if the functor $\mathrm{Hom}_{\mathscr{T}}(X, -)$ commutes with small sums. A class $\mathcal{G}$ of objects of $\mathscr{T}$ is called generating if the family of functor $\mathrm{Hom}_{\mathscr{T}}(X[n], -)$, $X \in \mathcal{G}$, $n \in \mathbf{Z}$, is conservative.
The triangulated category $\mathscr{T}$ is called *compactly generated* if there exists a generating set $\mathcal{G}$ of compact objects of $\mathscr{T}$. This property of being compact has been generalized by A. Neeman to the property of being *α-small* for some cardinal α (*cf.* [**Nee01**, 4.1.1]) – recall compact=$\aleph_0$-small. Then the property of being compactly generated has been generalized by Neeman to the property of being *well generated*; see [**Kra01**] for a convenient characterization of well generated triangulated categories.

Definition 1.3.16. Let $\mathscr{T}$ be a triangulated $\mathscr{P}$-fibred category over $\mathscr{S}$. We say that $\mathscr{T}$ is *compactly generated* (resp. *well generated*) if for any object S of $\mathscr{S}$, $\mathscr{T}(S)$ admits small sums and is compactly generated (resp. well generated).

Given a set of twists τ for $\mathscr{T}$, we say $\mathscr{T}$ is *compactly τ-generated* if it is compactly generated in the above sense and for any $\mathscr{P}$-object X/S, any twist $i \in \tau$, $M_S(X)\{i\}$ is compact.

1.3.17. For a triangulated category $\mathscr{T}$ which has small sums, given a family $\mathcal{G}$ of objects of $\mathscr{T}$, we denote by $\langle \mathcal{G} \rangle$ the localizing subcategory of $\mathscr{T}$ generated by $\mathcal{G}$, i.e. $\langle \mathcal{G} \rangle$ is the smallest triangulated full subcategory of $\mathscr{T}$ which is stable by small sums and which contains all the objects in $\mathcal{G}$. Recall that, in the case $\mathscr{T}$ is well generated (e.g. if $\mathscr{T}$ compactly generated), then the family $\mathcal{G}$ generates $\mathscr{T}$ (in the sense that the family of functors $\{\mathrm{Hom}_{\mathscr{T}}(X, -)\}_{X \in \mathcal{G}}$ is conservative) if and only if $\mathscr{T} = \langle \mathcal{G} \rangle$. The following lemma is a consequence of [**Nee01**]:

Lemma 1.3.18. *Let $\mathscr{T}$ be a triangulated monoidal $\mathscr{P}$-fibred category over $\mathscr{S}$ with geometric sections M. Assume $\mathscr{T}$ is τ-generated.*

If $\mathscr{T}$ is well generated, then for any object S of $\mathscr{S}$,

$$\mathscr{T}(S) = \langle M_S(X)\{i\}; X/S \text{ a } \mathscr{P}\text{-object}, i \in \tau\rangle$$

Moreover, there exists a regular cardinal α such that all the objects of shape $M_S(X)\{i\}$ are α-compact.

Note finally that the Brown representability theorem of Neeman (*cf.* [**Nee01**]) gives the following lemma (analog of 1.3.7):

Lemma 1.3.19. *Consider a well generated triangulated $\mathscr{P}$-fibred category $\mathscr{T}$. Then the following conditions are equivalent:*

(i) $\mathscr{T}$ is complete.
(ii) For any morphism $f : T \to S$ in $\mathscr{S}$, f^* commutes with sums.

If in addition, $\mathscr{T}$ is monoidal, the following conditions are equivalent:

(i') $\mathscr{T}$ is monoidal complete.
(ii') (a) For any morphism $f : T \to S$ in $\mathscr{S}$, f^* is right exact.
(b) For any object S of $\mathscr{S}$, the bifunctor $\otimes_S$ is right exact.

We finish this section with a proposition which will constitute a useful trick:

Proposition 1.3.20. *Consider an adjunction of triangulated categories*

$$a : \mathscr{T} \rightleftarrows \mathscr{T}' : b.$$

Assume that $\mathscr{T}$ admits a set of compact generators $\mathcal{G}$ such that any object in $a(\mathcal{G})$ in compact in $\mathscr{T}'$. Then b commutes with direct sums. If in addition $\mathscr{T}'$ is well generated then b admits a right adjoint.

Proof. The second assertion follows from the first one according to a corollary of the Brown representability theorem of Neeman (*cf.* [**Nee01**, 8.4.4]).

For the first one, we consider a family $(X_i)_{i\in I}$ of objects of $\mathscr{T}'$ and prove that the canonical morphism

$$\oplus_{i\in I} b(X_i) \to b\left(\oplus_{i\in I} X_i\right)$$

is an isomorphism in $\mathscr{T}$. To prove this, it is sufficient to apply the functor $\mathrm{Hom}_{\mathscr{T}}(G, -)$ for any object G of $\mathcal{G}$. Then the result is obvious from the assumptions. □

We shall often use the following standard argument to produce equivalences of triangulated categories.

Corollary 1.3.21. *Let $a : \mathscr{T} \to \mathscr{T}'$ be a triangulated functor between triangulated categories. Assume that the functor a preserves small sums, and that $\mathscr{T}$ admits a small set of compact generators $\mathcal{G}$, such that $a(\mathcal{G})$ form a family of compact objects in $\mathscr{T}'$. Then a is fully faithful if and only if, for any couple of objects G and G' in $\mathcal{G}$, the map*

$$\mathrm{Hom}_{\mathscr{T}}(G, G'[n]) \to \mathrm{Hom}_{\mathscr{T}'}(a(G), a(G')[n])$$

is bijective for any integer n. If a is fully faithful, then a is an equivalence of categories if and only if $a(\mathcal{G})$ is a generating family in $\mathscr{T}'$.

Proof. Let us prove that this is a sufficient condition. As $\mathscr{T}$ is in particular well generated, by the Brown representability theorem, the functor b admits a right adjoint $b : \mathscr{T}' \to \mathscr{T}$. By virtue of the preceding proposition, the functor b preserves small sums. Let us prove that a is fully faithful. We have to check that, for any object M of $\mathscr{T}$, the map $M \to b(a(M))$ is invertible. As a and b are triangulated and preserve small sums, it is sufficient to check this when M runs over a generating family of objects of $\mathscr{T}$ (e.g. $\mathcal{G}$). As $\mathcal{G}$ is generating, it is sufficient to prove that the map

$$\mathrm{Hom}_{\mathscr{T}}(G, M[n]) \to \mathrm{Hom}_{\mathscr{T}'}(a(G), a(M)[n]) = \mathrm{Hom}_{\mathscr{T}'}(a(G), b(a(M))[n])$$

is bijective for any integer n, which hold then by assumption. The functor a thus identifies $\mathscr{T}$ with the localizing subcategory of $\mathscr{T}'$ generated by $a(\mathcal{G})$; if moreover $a(\mathcal{G})$ is a generating family in $\mathscr{T}'$, then $\mathscr{T}' = \langle a(\mathcal{G})\rangle$, which also proves the last assertion. □

1.3.d. *The model category case.*

1.3.22. We shall use Hovey's book [**Hov99**] for a general reference to the theory of model categories. Note that, following *loc. cit.*, all the model categories we shall consider will have small limits and small colimits.

Let $\mathscr{M}$ be the sub-2-category of $\mathscr{C}at$ made of the model categories, with 1-morphisms the left Quillen functors and 2-morphisms the natural transformations. When we will apply definition 1.3.2 (resp. 1.3.3) to $\mathscr{C} = \mathscr{M}$, we will speak of a *$\mathscr{P}$-fibred model category* for a $\mathscr{M}$-structured $\mathscr{P}$-fibred category $\mathscr{M}$ (resp. morphism of $\mathscr{P}$-fibred model categories). Note that according to the definition of left Quillen functors, $\mathscr{M}$ is then automatically complete.

Given a property (P) of model categories (like being cofibrantly generated, left and/or right proper, combinatorial, stable, etc), we will say that a $\mathscr{P}$-fibred model category $\mathscr{M}$ over $\mathscr{S}$ has the property (P) if, for any object S of $\mathscr{S}$, the model category $\mathscr{M}(S)$ has the property (P).

For the monoidal case, we let $\mathscr{M}^{\otimes}$ be the sub-2-categories of $\mathscr{M}$ made of the symmetric monoidal model categories (see [**Hov99**, Definition 4.2.6]), with 1-morphisms the symmetric monoidal left Quillen functors and 2-morphisms the symmetric monoidal natural transformations, following the conditions of 1.3.4. When we will apply definition 1.3.5 to the case of $(\mathscr{M}, \mathscr{M}^{\otimes})$, we will speak simply of a *monoidal $\mathscr{P}$-fibred model category* (resp. *morphism of monoidal $\mathscr{P}$-fibred model categories*) for a (resp. morphism of) $(\mathscr{M}, \mathscr{M}^{\otimes})$-structured monoidal $\mathscr{P}$-fibred category $\mathscr{M}$. Again, $\mathscr{M}$ is then monoidal complete.

Remark 1.3.23. Let $\mathscr{M}$ be a $\mathscr{P}$-fibred model category over $\mathscr{S}$. Then for any $\mathscr{P}$-morphism $p : X \to Y$, the inverse image functor $p^* : \mathscr{M}(Y) \to \mathscr{M}(X)$ has very strong exactness properties: it preserves small limits and colimits (having both a left and a right adjoint), and it preserves weak equivalences, cofibrations, and fibrations. The only non (completely) trivial assertion here is about the preservation of weak equivalences. For this, one notices first that it preserves trivial cofibrations and trivial fibrations (being both a left Quillen functor and a right Quillen functor). In particular, by virtue of Ken Brown Lemma [**Hov99**, Lemma 1.1.12], it preserves weak equivalences between cofibrant (resp. fibrant) objects. Given a weak equivalence $u : M \to N$ in $\mathscr{M}(Y)$, we can find a commutative square

$$\begin{array}{ccc} M' & \xrightarrow{u'} & N' \\ \downarrow & & \downarrow \\ M & \xrightarrow[u]{} & N \end{array}$$

in which the two vertical maps are trivial fibrations, and where u' is a weak equivalence between cofibrant objects, from which we deduce easily that $p^*(u)$ is a weak equivalence in $\mathscr{M}(X)$.

1.3.24. Consider a $\mathscr{P}$-fibred model category $\mathscr{M}$ over $\mathscr{S}$. By assumption, we get the following pairs of adjoint functors:

(a) For any morphism $f : X \to S$ of $\mathscr{S}$,

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M}(S)) \rightleftarrows \mathrm{Ho}(\mathscr{M}(X)) : \mathbf{R}f_*$$

(b) For any $\mathscr{P}$-morphism $p : T \to S$, the pullback functor

$$\mathbf{L}p_\sharp : \mathrm{Ho}(\mathscr{M}(S)) \rightleftarrows \mathrm{Ho}(\mathscr{M}(T)) : \mathbf{L}p^* = p^* = \mathbf{R}p^*$$

Moreover, the canonical isomorphism of shape $(fg)^* \simeq g^* f^*$ induces a canonical isomorphism $\mathbf{R}(fg)^* \simeq \mathbf{R}g^* \mathbf{R}f^*$. In the situation of the $\mathscr{P}$-base change formula 1.1.9, we obtain also that the base change map

$$\mathbf{L}q_\sharp \mathbf{L}g^* \to \mathbf{L}f^* \mathbf{L}p_\sharp$$

is an isomorphism from the equivalent property of $\mathscr{M}$. Thus, we have defined a complete $\mathscr{P}$-fibred category whose fiber over S is $\mathrm{Ho}(\mathscr{M}(S))$.

Definition 1.3.25. Given a $\mathscr{P}$-fibred model category $\mathscr{M}$ as above, the complete $\mathscr{P}$-fibred category defined above will be denoted by $\mathrm{Ho}(\mathscr{M})$ and called the *homotopy $\mathscr{P}$-fibred category* associated with $\mathscr{M}$.

1.3.26. Assume that $\mathscr{M}$ is a monoidal $\mathscr{P}$-fibred model category over $\mathscr{S}$. Then, for any object S of $\mathscr{S}$, $\mathrm{Ho}(\mathscr{M})(S)$ has the structure of a symmetric closed monoidal category; see [**Hov99**, Theorem 4.3.2]. The (derived) tensor product of $\mathrm{Ho}(\mathscr{M})(S)$ will be denoted by $M \otimes_S^{\mathbf{L}} N$, and the (derived) internal Hom will be written $\mathbf{R}Hom_S(M, N)$, while the unit object will be written $\mathbb{1}_S$.

For any morphism $f : T \to S$ in $\mathscr{S}$, the derived functor $\mathbf{L}f^*$ is symmetric monoidal as follows from the equivalent property of its homologue f^*.

Moreover, for any $\mathscr{P}$-morphism $p : T \to S$ and for any object M in $\mathrm{Ho}(\mathscr{M})(T)$ and any object N in $\mathrm{Ho}(\mathscr{M})(S)$, the exchange map of 1.1.24

$$\mathbf{L}p_\sharp(M \otimes^{\mathbf{L}} p^*(N)) \to \mathbf{L}p_\sharp(M) \otimes^{\mathbf{L}} N$$

is an isomorphism.

Definition 1.3.27. Given a monoidal $\mathscr{P}$-fibred model category $\mathscr{M}$ as above, the complete monoidal $\mathscr{P}$-fibred category defined above will be denoted by $\mathrm{Ho}(\mathscr{M})$ and called the *homotopy monoidal $\mathscr{P}$-fibred category* associated with $\mathscr{M}$.

1.4. Premotivic categories. In the present article, we will focus on a particular type of $\mathscr{P}$-fibred category.

1.4.1. Let $\mathcal{S}$ be a scheme. Assume $\mathscr{S}$ is a full subcategory of the category of $\mathcal{S}$-schemes. In most of this work, we will denote by $\mathscr{S}^{ft}$ the class of morphisms of finite type in $\mathscr{S}$ and by Sm be the class of smooth morphisms of finite type in $\mathscr{S}$. There is an exception to this rule: throughout Part 3, $\mathscr{S}^{ft}$ will be the class of separated morphisms of finite type in $\mathscr{S}$ and Sm will be the class of separated smooth morphisms of finite type in $\mathscr{S}$. However, the axiomatic which we will present in the sequel can be applied identically in each cases so that the reader can freely use the restriction that all morphisms of Sm and $\mathscr{S}^{ft}$ are separated.

In any case, the classes Sm and $\mathscr{S}^{ft}$ are admissible in $\mathscr{S}$ in the sense of 1.0 (this is automatic, for instance, if $\mathscr{S}$ is stable by pullbacks).

Definition 1.4.2. Let $\mathscr{P}$ be an admissible class of morphisms in $\mathscr{S}$.

A *$\mathscr{P}$-premotivic category over $\mathscr{S}$* – or simply *$\mathscr{P}$-premotivic category* when $\mathscr{S}$ is clear – is a complete monoidal $\mathscr{P}$-fibred category over $\mathscr{S}$. A *morphism of $\mathscr{P}$-premotivic categories* is a morphism of complete monoidal $\mathscr{P}$-fibred categories over $\mathscr{S}$.

As a particular case, when $\mathscr{C}$ is the 2-category $\mathscr{T}ri$ of triangulated categories (resp. $\mathscr{A}b$ of abelian categories), a *$\mathscr{P}$-premotivic triangulated (resp. abelian) category over $\mathscr{S}$* is a $(\mathscr{C}, \mathscr{C}^{\otimes})$-structured complete monoidal $\mathscr{P}$-fibred category over $\mathscr{S}$ (def. 1.3.5). Morphisms of $\mathscr{P}$-premotivic triangulated (resp. abelian) categories are defined accordingly.

We will also say: *premotivic* for Sm-premotivic and *generalized premotivic* for $\mathscr{S}^{ft}$-premotivic.

The sections of a $\mathscr{P}$-premotivic category will be called *premotives*.

Example 1.4.3. Let $\mathscr{S}$ be the category of noetherian schemes of finite dimension.

For such a scheme S, recall $\mathscr{H}_\bullet(S)$ is the pointed homotopy category of Morel and Voevodsky; *cf.* examples 1.1.5, 1.1.14, 1.1.30. Then, according to the fact recalled in these examples the 2-functor $\mathscr{H}_\bullet$ is a geometrically generated premotivic category (recall Definition 1.1.41).

For such a scheme S, consider the stable homotopy category $\mathrm{SH}(S)$ of Morel and Voevodsky (see [**Jar00, Ayo07b**]). According to [**Ayo07b**], it defines a triangulated premotivic category denoted by SH. Moreover, it is compactly $(\mathbf{Z} \times \mathbf{Z})$-generated in the sense of definition 1.1.41 where the first factor refers to the suspension and the second one refers to the Tate twist (*i.e.* as a triangulated premotivic category, it is compactly generated by the Tate twists).

1.4.4. Let $\mathscr{T}$ be a $\mathscr{P}$-premotivic triangulated category with geometric sections M and τ be a set of twists for $\mathscr{T}$ (Definition 1.1.39).

Recall from Convention 1.3.14 (resp. and Definition 1.3.16) that $\mathscr{T}$ is said to be τ-generated (resp. compactly τ-generated) if for any scheme S, the family of isomorphism of classes of premotives of the form $M_S(X)\{i\}$ for a $\mathscr{P}$-scheme X over S and a twist $i \in \tau$ is a set of generators (resp. compact generators) for the triangulated category $\mathscr{T}(S)$ (in the respective case, we also assume $\mathscr{T}(S)$ admits small sums).

Let E be a premotive over S and X be a $\mathscr{P}$-scheme over S. For any $(n,i) \in \mathbf{Z} \times \tau$, we define the cohomology of X in degree n and twist i with coefficients in E as:

$$H^{n,i}_{\mathscr{T}}(X,E) = \mathrm{Hom}_{\mathscr{T}(S)}\big(M_S(X), E\{i\}(n)\big).$$

The fact $\mathscr{T}$ is τ-generated amounts to say that any such premotive E is determined by its cohomology.

Example 1.4.5. The premotivic triangulated category SH of the previous example is compactly $\mathbf{Z}$-generated where $\mathbf{Z}$ refers to the Tate twist (in other words it is compactly generated by Tate twists).

Definition 1.4.6. Let $\mathscr{M}$ and $\mathscr{M}'$ be $\mathscr{P}$-premotivic categories.

A morphism of $\mathscr{P}$-premotivic categories (or simply a *premotivic morphism*) is a morphism $\varphi^* : \mathscr{M} \to \mathscr{M}'$ of complete monoidal $\mathscr{P}$-fibred categories. We shall also say that

$$\varphi^* : \mathscr{M} \rightleftarrows \mathscr{M}' : \varphi_*$$

is a *premotivic adjunction*. When moreover $\mathscr{M}$ and $\mathscr{M}'$ are $\mathscr{P}$-premotivic triangulated (resp. abelian) categories, we will ask φ^* is a compatible with the triangulated (resp. additive) structure – as in Definition 1.3.3.

If we assume that $\mathscr{M}$ (resp. $\mathscr{M}'$) is τ-twisted (resp. τ'-twisted), we will say as in Definition 1.2.11 that φ^* is *compatible with twists* if for any $i \in \tau$, $\varphi^*(i)$ belongs up to isomorphism to τ'. We say φ^* is *strictly compatible with twists* if it is compatible with twists and if any element of τ' is isomorphic to the image of an element of τ.

Usually, premotivic categories comes equip with canonical twists (especially the Tate twist) and premotivic morphisms are compatible with twists.

Example 1.4.7. With the hypothesis and notations of 1.4.3, we get a premotivic adjunction

$$\Sigma^\infty : \mathscr{H}_\bullet \rightleftarrows \mathrm{SH} : \Omega^\infty$$

induced by the infinite suspension functor according to [**Jar00**].

1.4.8. Let $\mathscr{T}$ (resp. $\mathscr{A}$) be a triangulated $\mathscr{P}$-premotivic category with geometric sections M and set of twists τ. For any scheme S, we let $\mathscr{T}_{\tau,c}(S)$ be the smallest triangulated thick[34] subcategory of $\mathscr{T}(S)$ which contains premotives of shape $M_S(S)\{i\}$ (resp. $M_S(X,\mathscr{A})\{i\}$) for a $\mathscr{P}$-scheme X/S and a twist $i \in \tau$. This subcategory is stable by the operations f^*, $p_\sharp$ and $\otimes$. In particular, $\mathscr{T}_{\tau,c}$ defines a *not necessarily complete* triangulated (resp. abelian) $\mathscr{P}$-fibred category over $\mathscr{S}$. We also obtain a morphism of triangulated (resp. abelian) monoidal $\mathscr{P}$-fibred categories, fully faithful as a functor,

$$\iota : \mathscr{T}_{\tau,c} \to \mathscr{T}$$

Definition 1.4.9. Consider the notations introduced above. We will call $\mathscr{T}_{\tau,c}$ the *τ-constructible part of $\mathscr{T}$*. For any scheme S, the objects of $\mathscr{T}_{\tau,c}(S)$ will be called *τ-constructible.*

When τ is clear from the context, we will put $\mathscr{T}_c := \mathscr{T}_{\tau,c}$ and use the terminology *constructible.*

Remark 1.4.10. The condition of τ-constructibility is a good categorical notion of finiteness which extends the notion of *geometric motives* as introduced by Voevodsky. In the triangulated motivic case, it will be studied thoroughly in section 4.

Proposition 1.4.11. *Let $\mathscr{T}$ be a τ-twisted $\mathscr{P}$-premotivic triangulated category. Let S be a scheme such that:*

(1) The category $\mathscr{T}(S)$ admits finite sums.

[34] *i.e.* stable by direct factors.

(2) For any $\mathscr{P}$-scheme X over S, and any twist $i \in \tau$, the premotive $M_S(X)\{i\}$ is compact.

Then, a premotive M over S is τ-constructible if and only if it is compact.

PROOF. If $\mathscr{T}$ is any compactly generated triangulated category, then, for any small family C of compact generators, the thick triangulated category of $\mathscr{T}$ generated by C consists exactly of the compact objects of $\mathscr{T}$. □

Thus, when the conditions of this proposition are fulfilled, the category $\mathscr{T}_{\tau,c}(S)$ does not depend on the particular choice of τ. This will often be the case in practice (see 5.1.33, 5.2.39, 5.3.42).

REMARK 1.4.12. The notion of compact objects in a triangulated category was heavily developed by A. Neeman. Its relation with finiteness conditions is particularly emphasized when considering the derived category of complexes of quasi-coherent sheaves over a quasi-compact separated scheme: in this triangulated category, being compact is equivalent to being perfect ([**Nee96**, Cor. 4.3]).

DEFINITION 1.4.13. Consider a τ-generated premotivic category $\mathscr{M}$.

An *enlargement* of $\mathscr{M}$ is the data of a τ'-twisted generalized premotivic category $\underline{\mathscr{M}}$ together with a premotivic adjunction

$$\rho_\sharp : \mathscr{M} \longrightarrow \underline{\mathscr{M}} : \rho^*$$

(where $\underline{\mathscr{M}}$ is considered as a premotivic category in the obvious way), satisfying the following properties:

(a) For any scheme S in $\mathscr{S}$, the functor $\rho_{\sharp,S} : \mathscr{M}(S) \to \underline{\mathscr{M}}(S)$ is fully faithful and its right adjoint $\rho^*_S : \underline{\mathscr{M}}(S) \to \mathscr{M}(S)$ commutes with sums.
(b) $\rho_\sharp$ is strictly compatible with twists.

Again, this notion is defined similarly for a $\mathscr{C}$-structured $\mathscr{P}$-premotivic category.

Note that for any smooth S-scheme X, we get in the context of an enlargement as above the following identifications:

$$\rho_{\sharp,S}(M_S(X)) \simeq \underline{M}_S(X),$$
$$\rho^*_S(\underline{M}_S(X)) \simeq M_S(X)$$

where M (resp. $\underline{M}$) denote the geometric sections of $\mathscr{M}$ (resp. $\underline{\mathscr{M}}$).

Remember also that for any morphism of schemes f and any smooth morphism p, $\rho_\sharp$ commutes with f^* and $p_\sharp$, while ρ^* commutes with f_* and p^*.

2. Triangulated $\mathscr{P}$-fibred categories in algebraic geometry

2.0. In this entire section, we fix a base scheme $\mathcal{S}$ assumed to be noetherian and a full subcategory $\mathscr{S}$ of the category of noetherian $\mathcal{S}$-schemes satisfying the following properties:

(a) $\mathscr{S}$ is closed under finite sums and pullback along morphisms of finite type.
(b) For any scheme S in $\mathscr{S}$, any quasi-projective S-scheme belongs to $\mathscr{S}$.

In sections 2.2 and 2.4, we will add the following assumption on $\mathscr{S}$:

(c) Any separated morphism $f : Y \to X$ in $\mathscr{S}$, admits a compactification in $\mathscr{S}$ in the sense of [**SGA4**, 3.2.5], i.e. admits a factorization of the form

$$Y \xrightarrow{j} \bar{Y} \xrightarrow{p} X$$

where j is an open immersion, p is proper, and $\bar{Y}$ belongs to $\mathscr{S}$. Furthermore, if f is quasi-projective, then p can be chosen to be projective.
(d) Chow's lemma holds in $\mathscr{S}$ (i.e., for any proper morphism $Y \to X$ in $\mathscr{S}$, there exists a projective birational morphism $p : Y_0 \to Y$ in $\mathscr{S}$ such that fp is projective as well).

A category $\mathscr{S}$ satisfying all these properties will be called *adequate* for future references.[35]

We also fix an admissible class $\mathscr{P}$ of morphisms in $\mathscr{S}$ and a complete triangulated $\mathscr{P}$-fibred category $\mathscr{T}$. We will add the following assumptions:

(d) In section 2.2 and 2.3, $\mathscr{P}$ contains the open immersions.
(e) In section 2.4, $\mathscr{P}$ contains the smooth morphisms of $\mathscr{S}$.

In the case $\mathscr{T}$ is monoidal, we denote by

$$M : \mathscr{P}/? \to \mathscr{T}$$

its geometric sections.

According to the convention of 1.4.2, we will speak of the *premotivic case* when $\mathscr{P}$ is the class of smooth morphisms of finite type[36] in $\mathscr{S}$ and $\mathscr{T}$ is a premotivic triangulated category.

2.1. Elementary properties.

DEFINITION 2.1.1. We say that $\mathscr{T}$ is additive, if for any finite family $(S_i)_{i\in\mathscr{I}}$ of schemes in $\mathscr{S}$, the canonical map

$$\mathscr{T}\left(\coprod_i S_i\right) \to \prod_i \mathscr{T}(S_i)$$

is an equivalence.

Recall this property implies in particular that $\mathscr{T}(\varnothing) = 0$.

LEMMA 2.1.2. *Let S be a scheme, $p : \mathbf{A}^1_S \to S$ be the canonical projection. The following conditions are equivalent:*

(i) The functor $p^* : \mathscr{T}(S) \to \mathscr{T}(\mathbf{A}^1_S)$ is fully faithful.
(ii) The counit adjunction morphism $1 \to p_*p^*$ is an isomorphism.

In the premotivic case, these conditions are equivalent to the following ones:

(iii) The unit adjunction morphism $p_\sharp p^* \to 1$ is an isomorphism.
(iv) The morphism $M_S(\mathbf{A}^1_S) \xrightarrow{p_*} \mathbb{1}_S$ induced by p is an isomorphism.
(iv') For any smooth S-scheme X, the morphism $M_S(\mathbf{A}^1_X) \xrightarrow{(1_X\times p)_*} M_S(X)$ is an isomorphism.

The only thing to recall is that in the premotivic case, $p_\sharp p^*(M) = M_S(\mathbf{A}^1_S)\otimes M$ and $p_*p^*(M) = \mathit{Hom}_S(M_S(\mathbf{A}^1_S), M)$.

DEFINITION 2.1.3. The equivalent conditions of the previous lemma will be called the *homotopy property* for $\mathscr{T}$, denoted by (Htp).

2.1.4. Recall that a *sieve* R of a scheme X is a class of morphisms in $\mathscr{S}/X$ which is stable by composition on the right by any morphism of schemes (see [**SGA4**, I.4]).

Given such a sieve R, we will say that $\mathscr{T}$ *is R-separated* if the class of functors f^* for $f \in R$ is conservative. Given two sieves R, R' of X, the following properties are immediate:

(a) If $R \subset R'$ then $\mathscr{T}$ is R-separated implies $\mathscr{T}$ is R'-separated.
(b) If $\mathscr{T}$ is R-separated and is R'-separated then $\mathscr{T}$ is $(R \cup R')$-separated.

A family of morphisms $(f_i : X_i \to X)_{i\in I}$ of schemes defines a sieve $R = \langle f_i, i \in I\rangle$ such that f is in R if and only if there exists $i \in I$ such that f can be factored through f_i. Obviously,

(c) $\mathscr{T}$ is R-separated if and only if the family of functors $(f_i^*)_{i\in I}$ is conservative.

[35]For instance, the scheme $\mathcal{S}$ can be the spectrum of a prime field or of a Dedekind domain. The category $\mathscr{S}$ might be the category of all noetherian $\mathcal{S}$-schemes of finite dimension or simply the category of quasi-projective $\mathcal{S}$-schemes. In all these cases, property (c) is ensured by Nagata's theorem (see [**Con07**]) and property (d) by Chow's lemma (see [**EGA2**, 5.6.1]).

[36]or smooth separated morphisms of finite type when applying this section in Part 3

Recall that a topology on $\mathscr{S}$ is the data for any scheme X of a set of sieves of X satisfying certain stability conditions (*cf.* [**SGA4**, II, 1.1]), called t-covering sieves. A pre-topology t_0 on $\mathscr{S}$ is the data for any scheme X of a set of families of morphisms of shape $(f_i : X_i \to X)_{i \in I}$ satisfying certain stability conditions (*cf.* [**SGA4**, II, 1.3]), called t_0-covers. A pre-topology t_0 generated a unique topology t.

Definition 2.1.5. Let t be a Grothendieck topology on $\mathscr{S}$. We say that $\mathscr{T}$ is *t-separated* if the following property holds:

(t-sep) For any t-covering sieve R, $\mathscr{T}$ is R-separated in the sense defined above.

Obviously, given two topologies t and t' on $\mathscr{S}$ such that t' is finer than t, if $\mathscr{T}$ is t-separated then it is t'-separated.

If the topology t on $\mathscr{S}$ is generated by a pre-topology t_0 then $\mathscr{T}$ is t-separated if and only if for any t_0-covers $(f_i)_{i \in I}$, the family of functors $(f_i^*)_{i \in I}$ is conservative – use [**SGA4**, 1.4] and 2.1.4(a)+(c).

2.1.6. Recall that a morphism of schemes $f : T \to S$ is radicial if it is injective and for any point t of T, the residual extension induced by f at t is radicial (*cf.* [**EGA1**, 3.5.4, 3.5.8])[37] The following definition is inspired by [**Ayo07a**, Def. 2.1.160].

Definition 2.1.7. We say that $\mathscr{T}$ is *separated* (resp. *semi-separated*) if $\mathscr{T}$ is separated for the topology generated by surjective families of morphisms of finite type (resp. finite radicial morphisms) in $\mathscr{S}$. We also denote by (Sep) (resp. (sSep)) this property.

Remark 2.1.8. If $\mathscr{T}$ is additive, property (Sep) (resp. (sSep)) is equivalent to ask that for any surjective morphism of finite type (resp. finite surjective radicial morphism) $f : T \to S$ in $\mathscr{S}$, the functor f^* is conservative.

Proposition 2.1.9. *Assume $\mathscr{T}$ is semi-separated and satisfies the transversality property with respect to finite surjective radicial morphisms.*

Then for any finite surjective radicial morphism $f : Y \to X$, the functor

$$f^* : \mathscr{T}(X) \to \mathscr{T}(Y)$$

is an equivalence of categories.

Proof. We first consider the case when $f = i$ is in addition a closed immersion. In this case, we can consider the pullback square below.

$$\begin{array}{ccc} Y & = & Y \\ \| & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow[i]{} & Z \end{array}$$

Using the transversality property with respect to i, we see that the counit $i^* i_* \to 1$ is an isomorphism. It thus remains to prove that the unit map $1 \to i_* i^*$ is an isomorphism. As i^* is conservative by semi-separability, it is sufficient to check that

$$i^* \to i^* i_* i^*(M)$$

is an isomorphism. But this is a section of the map $i^* i_* i^*(M) \to i^*(M)$, which is already known to be an isomorphism.

Consider now the general case of a finite radicial extension f. We introduce the pullback square

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{p} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[f]{} & X \end{array}$$

[37] It is equivalent to ask that f is universally injective. When f is surjective, this is equivalent to ask that f is a universal homeomorphism.

Consider the diagonal immersion $i : Y \to Y \times_X Y$. Because Y is noetherian and p is separable, i is finite (*cf.* [**EGA2**, 6.1.5]) thus a closed immersion. As p is a universal homeomorphism, the same is true for its section i. The preceding case thus implies that i^* is an equivalence of categories. Moreover, as $pi = qi = 1_Y$, we see that p^* and q^* are both quasi-inverses to i^*, which implies that they are isomorphic equivalences of categories. More precisely, we get canonical isomorphisms of functors

$$i^* \simeq p_* \simeq q_* \quad \text{and} \quad i_* \simeq p^* \simeq q^*.$$

We check that the unit map $1 \to f_* f^*$ is an isomorphism. Indeed, by semi-separability, it is sufficient to prove this after applying the functor f^*, and we get, using the transversality property for f:

$$f^* \simeq i^* p^* f^* \simeq q_* p^* f^* \simeq f^* f_* f^*.$$

We then check that the counit map $f^* f_* \to 1$ is an isomorphism as well. In fact, using again the transversality property for f, we have isomorphisms

$$f^* f_*(M) \simeq q_* p^*(M) \simeq i^* i_*(M) \simeq M.$$

□

2.1.10. Recall from [**Voe10b**] that a cd-structure on $\mathscr{S}$ is a collection P of commutative squares of schemes

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & Q & \downarrow f \\ A & \xrightarrow[e]{} & X \end{array}$$

which is closed under isomorphisms. We will say that a square Q in P is P-distinguished.

Voevodsky associates to P a topology t_P, the smallest topology such that:

- for any P-distinguished square Q as above, the sieve generated by $\{f : A \to X, e : Y \to X\}$ is t_P-covering on X.
- the empty sieve covers the empty scheme.

Example 2.1.11. A *Nisnevich distinguished square* is a square Q as above such that Q is cartesian, f is étale, e is an open embedding with reduced complement Z and the induced map $f^{-1}(Z) \to Z$ is an isomorphism. The corresponding cd-structure is called the *upper cd-structure* (see section 2 of [**Voe10c**]). Because we work with noetherian schemes, the corresponding topology is the *Nisnevich topology* (see proposition 2.16 of *loc.cit.*).

A *proper* cdh-*distinguished square* is a square Q as above such that Q is cartesian, f is proper, e is a closed embedding with open complement U and the induced map $f^{-1}(U) \to U$ is an isomorphism. The corresponding cd-structure is called the *lower cd-structure*. The topology associated with the lower cd-structure is called the *proper cdh-topology*.

The topology generated by the lower and upper cd-structures is by definition (according to the preceding remark on Nisnevich topology) the *cdh-topology*.

All these three examples are complete cd-structures in the sense of [**Voe10b**, 2.3].

Lemma 2.1.12. *Let P be a complete cd-structure (see* [**Voe10b**, def 2.3]*) on $\mathscr{S}$ and t_P be the associated topology. The following conditions are equivalent:*

(i) $\mathscr{T}$ is t_P-separated.
(ii) For any distinguished square Q for P of the above form, the pair of functors (e^*, f^*) is conservative.

Proof. This follows from the definition of a complete cd-structure and 2.1.4(a). □

Remark 2.1.13. If we assume that $\mathscr{S}$ is stable by arbitrary pullback then any cd-structure P on $\mathscr{S}$ such that P-distinguished squares are stable by pullback is complete (see [**Voe10b**, 2.4]).

2.2. Exceptional functors, following Deligne.

2.2.a. *The support axiom.*

2.2.1. Consider an open immersion $j : U \to S$. Applying 1.1.15 to the cartesian square

$$\begin{array}{ccc} U & = & U \\ \| & & \downarrow{\scriptstyle j} \\ U & \xrightarrow[j]{} & S \end{array}$$

we get a canonical natural transformation

$$\gamma_j : j_\sharp = j_\sharp 1_* \xrightarrow{Ex(\Delta_{\sharp *})} j_* 1_\sharp = j_*.$$

Recall that the functors $j_\sharp$ and j_* are fully faithful (see Corollary 1.1.20).

Note that according to remark 1.1.7, this natural transformation is compatible with the identifications of the kind $(jk)_\sharp = j_\sharp k_\sharp$ and $(jk)_* = j_* k_*$.

LEMMA 2.2.2. *Let S be a scheme, U and V be subschemes such that $S = U \sqcup V$. We let $h : U \to S$ (resp. $k : V \to S$) be the canonical open immersions.*

Assume that the functor $(h^*, k^*) : \mathscr{T}(S) \to \mathscr{T}(U) \times \mathscr{T}(V)$ is conservative and that $\mathscr{T}(\varnothing) = 0$. Then the natural transformation γ_h (resp. γ_k) is an isomorphism. Moreover, the functor (h^*, k^*) is then an equivalence of categories.

PROOF. As $h_\sharp$ and h_* are fully faithful, we have $h^* h_\sharp \simeq h^* h_*$. By $\mathscr{P}$-base change, we also get $k^* h_\sharp \simeq k^* h_* \simeq 0$. It remains to prove the last assertion. The functor $R = (h^*, k^*)$ has a left adjoint L defined by L=$h_\sharp \oplus k_\sharp$:

$$L(M, N) = h_\sharp(M) \oplus k_\sharp(N)\,.$$

The natural transformation $LR \to 1$ is an isomorphism: to see this, is it sufficient to evaluate at h^* and k^*, which gives an isomorphism in $\mathscr{T}(U)$ and $\mathscr{T}(V)$ respectively. The natural transformation $1 \to RL$ is also an isomorphism because $h_\sharp$ and $k_\sharp$ are fully faithful. □

REMARK 2.2.3. Assume $\mathscr{T}$ is Zariski separated (definition 2.1.5). Then, as a corollary of this lemma, $\mathscr{T}$ is additive (definition 2.1.1) if and only if $\mathscr{T}(\varnothing) = 0$.

2.2.4. *Exchange structures V.*– Assume $\mathscr{T}$ is additive. We consider a commutative square of schemes

$$\begin{array}{ccc} V & \xrightarrow{k} & T \\ {\scriptstyle q}\downarrow & \Delta & \downarrow{\scriptstyle p} \\ U & \xrightarrow[j]{} & S \end{array} \tag{2.2.4.1}$$

such that j, k are an open immersions and p, q are a proper morphisms.

This diagram can be factored into the following commutative diagram:

$$\begin{array}{ccccc} V & & \xrightarrow{\qquad k \qquad} & & \\ & \searrow^{l} & & & \\ & & U \times_S T & \xrightarrow{j'} & T \\ & {\scriptstyle q} & \downarrow{\scriptstyle p'} & \Theta & \downarrow{\scriptstyle p} \\ & & U & \xrightarrow{j} & S. \end{array}$$

Then l is an open and closed immersion so that the previous lemma implies the canonical morphism $\gamma_l : l_\sharp \to l_*$ is an isomorphism. As a consequence, we get a natural exchange transformation

$$Ex(\Delta_{\sharp *}) : j_\sharp q_* = j_\sharp p'_* l_* \xrightarrow{Ex(\Theta_{\sharp *})} p_* j'_\sharp l_* \xrightarrow{\gamma_l^{-1}} p_* j'_\sharp l_\sharp = p_* k_\sharp$$

using the exchange of 1.1.15. Note that, with the notations introduced in 2.2.1, the following diagram is commutative.

(2.2.4.2)
$$\begin{array}{ccc} j_\sharp q_* & \xrightarrow{Ex(\Delta_{\sharp *})} & p_* k_\sharp \\ {\scriptstyle \gamma_j q_*}\downarrow & & \downarrow{\scriptstyle p_* \gamma_k} \\ j_* q_* & \xrightarrow{\sim} (jq)_* = (pk)_* \xleftarrow{\sim} & p_* k_* \end{array}$$

Indeed one sees first that it is sufficient to treat the case where Δ is cartesian. Then, as $j_\sharp$ is a fully faithful left adjoint to j^* it is sufficient to check that (2.2.4.2) commutes after having applied j^*. Using the cotransversality property with respect to open immersions, one sees then that this consists to verify the commutativity of (2.2.4.2) when j is the identity, in which case it is trivial.

Definition 2.2.5. Let $p : T \to S$ be a proper morphism in $\mathscr{S}$.

We say that the triangulated $\mathscr{P}$-fibred category $\mathscr{T}$ satisfies the *support property* with respect to p, denoted by (Supp_p), if it is additive and for any commutative square of shape (2.2.4.1) the exchange transformation $Ex(\Delta_{\sharp *}) : j_\sharp q_* \to p_* k_\sharp$ defined above is an isomorphism.

We say that $\mathscr{T}$ satisfies the *support property*, also denoted by (Supp), if it satisfies (Supp_p) for all proper morphism p in $\mathscr{S}$.

By definition, it is sufficient to check the last property of property (Supp) in the case where Δ is cartesian.

2.2.b. *Exceptional direct image.*

2.2.6. We denote by $\mathscr{S}^{sep}$ (resp. $\mathscr{S}^{open}$, $\mathscr{S}^{prop}$) the sub-category of the category $\mathscr{S}$ with the same objects but morphisms are separated morphisms of finite type (resp. open immersions, proper morphisms). We denote by

$$\mathscr{T}_* : \mathscr{S} \to \mathscr{T}ri^{\otimes}$$
$$\text{resp. } \mathscr{T}_\sharp : \mathscr{S}^{open} \to \mathscr{T}ri^{\otimes}$$

the 2-functor defined respectively by morphisms of type f_* and $j_\sharp$ (f any morphism of schemes). The proposition below is essentially based on a result of Deligne [**SGA4**, XVII, 3.3.2]:

Proposition 2.2.7. *Assume $\mathscr{T}$ is a monoidal $\mathscr{P}$-fibred category and satisfies property (Supp). Then there exists a unique 2-functor*

$$\mathscr{T}_! : \mathscr{S}^{sep} \to \mathscr{T}ri^{\otimes}$$

with the property that

$$\mathscr{T}_!|_{\mathscr{S}^{prop}} = \mathscr{T}_*|_{\mathscr{S}^{prop}}, \quad \mathscr{T}_!|_{\mathscr{S}^{open}} = \mathscr{T}_\sharp$$

and for any commutative square Δ of shape (2.2.4.1)*, the composition of the structural isomorphisms*

$$j_\sharp q_* = j_! q_! \simeq (jq)_! = (pk)_! \simeq p_! k_! = p_* k_\sharp$$

is equal to the exchange transformation $Ex(\Delta_{\sharp *})$.

2.2.8. Under the assumptions of the proposition, for any separated morphism of finite type $f : Y \to X$, we will denote by $f_! : \mathscr{T}(Y) \to \mathscr{T}(X)$ the functor $\mathscr{T}_!(f)$. The functor $f_!$ is called the *direct image functor with compact support* or the *left exceptional functor* associated with f.

Proof. We recall the principle of the proof of Deligne. Let $f : Y \to X$ be a separated morphism of finite type in $\mathscr{S}$.

Let $\mathscr{C}_f$ be the category of compactifications of f in $\mathscr{S}$, i.e. of factorizations of f of the form

(2.2.8.1)
$$Y \xrightarrow{j} \bar{Y} \xrightarrow{p} X$$

where j is an open immersion, p is proper, and $\bar{Y}$ belongs to $\mathscr{S}$. Morphisms of $\mathscr{C}_f$ are given by commutative diagrams of the form

$$\begin{array}{ccccc} & & \bar{Y}' & & \\ & \overset{j'}{\nearrow} & \downarrow \pi & \overset{p'}{\searrow} & \\ Y & & & & X. \\ & \underset{j}{\searrow} & \bar{Y} & \underset{p}{\nearrow} & \end{array} \tag{2.2.8.2}$$

in $\mathscr{S}$. To any compactification of f of shape (2.2.8.1), we associate the functor $p_* j_\sharp$.
To any morphism of compactifications (2.2.8.2), we associate a natural isomorphism

$$p'_* j'_\sharp = p_* \pi_* j'_\sharp \xrightarrow{Ex(\Delta_{\sharp *})^{-1}} p_* j_\sharp 1_* = p_* j_\sharp.$$

where Δ stands for the commutative square made by removing π in the diagram (2.2.8.2), and $Ex(\Delta_{\sharp *})$ is the corresponding natural transformation (see 2.2.4). The compatibility of $Ex(\Delta_{\sharp *})$ with composition of morphisms of schemes shows that we have defined a functor

$$\Gamma_f : \mathscr{C}_f^{op} \to \mathit{Hom}(\mathscr{T}(Y), \mathscr{T}(X))$$

which sends all the maps of $\mathscr{C}_f$ to isomorphisms (by the support property).

The category $\mathscr{C}_f$ is non-empty by the assumption 2.0(c) on $\mathscr{S}$, and it is in fact left filtering; see [**SGA4**, XVII, 3.2.6(ii)]. This defines a canonical functor $f_! : \mathscr{T}(Y) \to \mathscr{T}(X)$, independent of any choice compactification of f, defined in the category of functors $\mathit{Hom}(\mathscr{T}(Y), \mathscr{T}(X))$ by the formula

$$f_! = \varinjlim_{\mathscr{C}_f^{op}} \Gamma_f \, .$$

If $f = p$ is proper, then the compactification

$$Y \xrightarrow{=} Y \xrightarrow{p} X$$

is an initial object of $\mathscr{C}_f$, which gives a canonical identification $p_! = p_*$. Similarly, if $f = j$ is an open immersion, then the compactification

$$Y \xrightarrow{j} X \xrightarrow{=} X$$

is a terminal object of $\mathscr{C}_j$, so that we get a canonical identification $j_! = j_\sharp$.

This construction is compatible with composition of morphisms. Let $g : Z \to Y$ and $f : Y \to X$ be two separated morphisms of finite type in $\mathscr{S}$. For any a couple of compactifications

$$Z \xrightarrow{k} \bar{Z} \xrightarrow{q} Y \text{ and } Y \xrightarrow{j} \bar{Y} \xrightarrow{p} X$$

of f and g respectively, we can choose a compactification

$$\bar{Z} \xrightarrow{h} T \xrightarrow{r} Y$$

of jq, and we get a canonical isomorphism

$$f_! \, g_! \simeq p_* \, j_\sharp \, q_* \, k_\sharp \simeq p_* \, r_* \, h_\sharp \, k_\sharp \simeq (pr)_* \, (hk)_\sharp \simeq (fg)_! \, .$$

The independence of these isomorphisms with respect to the choices of compactification follows from [**SGA4**, XVII, 3.2.6(iii)]. The cocycle conditions (i.e. the associativity) also follows formally from [**SGA4**, XVII, 3.2.6]. The uniqueness statement is obvious. □

2.2.9. This construction is functorial in the following sense.
Define a 2-*functor with support on* $\mathscr{T}$ to be a triple $(\mathscr{D}, a, b)$, where:

(i) $\mathscr{D} : \mathscr{S}^{sep} \to \mathscr{T}ri$ is a 2-functor (we shall write the structural coherence isomorphisms as $c_{g,f} : \mathscr{D}(gf) \xrightarrow{\sim} \mathscr{D}(g)\mathscr{D}(f)$ for composable arrows f and g in $\mathscr{S}^{sep}$);
(ii) $a : \mathscr{T}_*|_{\mathscr{S}^{prop}} \to \mathscr{D}|_{\mathscr{S}^{prop}}$ and $b : \mathscr{T}_\sharp \to \mathscr{D}|_{\mathscr{S}^{open}}$ are morphisms of 2-functors which agree on objects, i.e. such that for any scheme S in $\mathscr{S}$, we have

$$\psi_S = a_S = b_S : \mathscr{T}(S) \to \mathscr{D}(S) \, ;$$

(iii) for any commutative square of shape (2.2.4.1) in which j and k are open immersions, while p and q are proper morphisms, the diagram below commutes.

$$\begin{array}{ccccc}
\psi_S\, j_\sharp q_* & & \xrightarrow{\psi_S Ex(\Delta_{\sharp *})} & & \psi_S\, p_* k_\sharp \\
\downarrow{\scriptstyle b\, q_*} & & & & \downarrow{\scriptstyle a k_\sharp} \\
\mathscr{D}(j)\psi_U q_* & & & & \mathscr{D}(p)\psi_T k_\sharp \\
\downarrow{\scriptstyle \mathscr{D}(j)a} & & & & \downarrow{\scriptstyle \mathscr{D}(p)b} \\
\mathscr{D}(j)\mathscr{D}(q)\psi_V & \xrightarrow{c_{j,q}^{-1}} & \mathscr{D}(jq) = \mathscr{D}(pk)\psi_V & \xleftarrow{c_{p,k}^{-1}} & \mathscr{D}(p)\mathscr{D}(k)\psi_V
\end{array}$$

Morphisms of 2-functors with support on $\mathscr{T}$

$$(\mathscr{D}, a, b) \to (\mathscr{D}', a', b')$$

are defined in the obvious way: these are morphisms of 2-functors $\mathscr{D} \to \mathscr{D}'$ which preserve all the structure on the nose.

Using the arguments of the proof of 2.2.7, one checks easily that the category of 2-functors with support has an initial object, which is nothing else but the 2-functor $\mathscr{T}_!$ together with the identities of $\mathscr{T}_*|_{\mathscr{S}^{prop}}$ and of $\mathscr{T}_\sharp$ respectively. In particular, for any 2-functor $\mathscr{D} : \mathscr{S}^{sep} \to \mathscr{T}ri$, a morphism of 2-functors $\mathscr{T}_! \to \mathscr{D}$ is completely determined by its restrictions to $\mathscr{S}^{prop}$ and $\mathscr{S}^{open}$, and by its compatibility with the exchange isomorphisms of type $Ex(\Delta_{\sharp *})$ in the sense described in condition (iii) above.

Proposition 2.2.10. *Assume that $\mathscr{T}$ satisfies the support property and consider the notations of Proposition 2.2.7. For any separated morphism of finite type f in $\mathscr{S}$, there exists a canonical natural transformation*

$$\alpha_f : f_! \to f_* \,.$$

The collection of maps α_f defines a morphism of 2-functors

$$\alpha : \mathscr{T}_! \to \mathscr{T}_*|_{\mathscr{S}^{sep}} \ , \quad f \mapsto (\alpha_f : f_! \to f_*)$$

whose restrictions to $\mathscr{S}^{prop}$ and $\mathscr{S}^{open}$ are respectively the identity and the morphism of 2-functors $\gamma : \mathscr{T}_\sharp \to \mathscr{T}_*|_{\mathscr{S}^{open}}$ defined in 2.2.1.

Proof. The identities $f_* = f_*$ for f proper (resp. projective) and the exchange natural transformations of type $Ex(\Delta_{\sharp *})$ turns $\mathscr{T}_*|_{\mathscr{S}^{sep}}$ into a 2-functor with support (resp. restricted support) on $\mathscr{T}$ (property (iii) of 2.2.9 is expressed by the commutative square (2.2.4.2)). □

Proposition 2.2.11. *Let $\mathscr{T}'$ be another triangulated complete $\mathscr{P}$-fibred category over $\mathscr{S}$. Assume that $\mathscr{T}$ and $\mathscr{T}'$ both have the support property, and consider given a triangulated morphism of $\mathscr{P}$-fibred categories $\varphi^* : \mathscr{T} \to \mathscr{T}'$ (recall definition 1.2.2).*

Then, there is a canonical family of natural transformations

$$Ex(\varphi^*, f_!) : \varphi^*_X\, f_! \to f_!\, \varphi^*_Y$$

for each separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, which is functorial with respect to composition in $\mathscr{S}$ (i.e. defines a morphism of 2-functors) and such that, the following conditions are verified:

(a) if f is proper, then, under the identification $f_! = f_*$, the map $Ex(\varphi^*, f_!)$ is the exchange transformation $Ex(\varphi^*, f_*) : \varphi^*_X\, f_* \to f_*\, \varphi^*_Y$ defined in 1.2.5;

(b) if f is an open immersion, then, under the identification $f_! = f_\sharp$, the map $Ex(\varphi^*, f_!)$ is the inverse of the exchange isomorphism $Ex(f_\sharp, \varphi^*) : f_\sharp\, \varphi^*_Y \to \varphi^*_X\, f_\sharp$ defined in 1.2.1.

Proof. The exchange maps of type $Ex(\varphi^*, f_*)$ define a morphism of 2-functors

$$a : \mathscr{T}_*|_{\mathscr{S}^{prop}} \to \mathscr{T}'_*|_{\mathscr{S}^{prop}} = \mathscr{T}_!'|_{\mathscr{S}^{prop}}$$

while the inverse of the exchange isomorphisms of type $Ex(f_\sharp, \varphi^*)$ define a morphism of 2-functors

$$b : \mathscr{T}_\sharp \to \mathscr{T}'_\sharp = \mathscr{T}_!'|_{\mathscr{S}^{open}} \,,$$

in such a way that the triple $(\mathscr{T}_!', a, b)$ is a 2-functor with support on $\mathscr{T}$. □

COROLLARY 2.2.12. *Suppose $\mathscr{T}$ satisfies the support property and consider the notations of proposition 2.2.7.*

(1) *For any cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

such that f is separated of finite type, there exists a canonical natural transformation

$$Ex(\Delta_!^*) : g^* f_! \to f'_! {g'}^*$$

compatible with horizontal and vertical compositions of squares, and satisfying the following identifications in $\mathscr{T}(X')$

(a) *f proper:*

$$\begin{array}{ccc} g^* f_! & \xrightarrow{Ex(\Delta_!^*)} & f'_! {g'}^* \\ \| & & \| \\ g^* f_* & \xrightarrow{Ex(\Delta_*^*)} & f'_* {g'}^*, \end{array}$$

(b) *f open immersion:*

$$\begin{array}{ccc} g^* f_! & \xrightarrow{Ex(\Delta_!^*)} & f_! {g'}^* \\ \| & & \| \\ g^* f_\sharp & \xrightarrow{Ex(\Delta_\sharp^*)^{-1}} & f'_\sharp {g'}^*. \end{array}$$

Moreover, when g is a $\mathscr{P}$-morphism, $Ex(\Delta_!^*)$ is an isomorphism.

(2) *For any cartesian square Δ as in (1), assuming f is separated of finite type and g is a $\mathscr{P}$-morphism, there exists a canonical natural transformation*

$$Ex(\Delta_{\sharp\,!}) : g_\sharp f'_! \to f_! g'_\sharp$$

compatible with horizontal and vertical compositions of squares, and satisfying the following identifications in $\mathscr{T}(X')$

(a) *f proper:*

$$\begin{array}{ccc} g_\sharp f'_! & \xrightarrow{Ex(\Delta_{\sharp !})} & f_! g'_\sharp \\ \| & & \| \\ g_\sharp f'* & \xrightarrow{Ex(\Delta_{\sharp *})} & f_* g'_\sharp, \end{array}$$

(b) *f open immersion:*

$$\begin{array}{ccc} g_\sharp f'_! & \xrightarrow{Ex(\Delta_{\sharp !})} & f_! g'_\sharp \\ \| & & \| \\ g_\sharp f'_\sharp & = & f_\sharp g'_\sharp. \end{array}$$

(3) *If furthermore $\mathscr{T}$ is monoidal then for any separated morphism of finite type $f : Y \to X$, there is a natural transformation*

$$Ex(f_!^*, \otimes) : (f_! K) \otimes L \to f_!(K \otimes f^* L)$$

which is compatible with respect to composition in $\mathscr{S}$, and such that, in each of the following cases, we have the following identifications:

(a) *f proper:*

$$\begin{array}{ccc} (f_! K) \otimes L & \xrightarrow{Ex(f_!^*,\otimes)} & f_!(K \otimes f^* L) \\ \| & & \| \\ (f_* K) \otimes L & \xrightarrow{Ex(f_*^*,\otimes)} & f_*(K \otimes f^* L), \end{array}$$

(b) *f open immersion:*

$$\begin{array}{ccc} (f_! K) \otimes L & \xrightarrow{Ex(f_!^*,\otimes)} & f_!(K \otimes f^* L) \\ \| & & \| \\ (f_\sharp K) \otimes L & \xrightarrow{Ex(f_\sharp^*,\otimes)^{-1}} & f_\sharp(K \otimes f^* L). \end{array}$$

As in the previous analogous cases, the natural transformations $Ex(\Delta_!^*)$, $Ex(\Delta_{\sharp,!})$ and $Ex(f_!^*, \otimes)$ will be called *exchange transformations*.

PROOF. To prove (1), consider a fixed map $g : X' \to X$ in $\mathscr{S}$. We consider the triangulated $\mathscr{P}/X$-fibred categories $\mathscr{T}'$ and $\mathscr{T}''$ over $\mathscr{S}/X$ defined by $\mathscr{T}'(Y) = \mathscr{T}(Y)$ and $\mathscr{T}''(Y) = \mathscr{T}(Y')$ for any X-scheme Y (in $\mathscr{S}$), with $g' : Y' = Y \times_X X' \to Y$ the map obtained from $Y \to X$ by pullback along g. The collection of functors

$${g'}^* : \mathscr{T}(Y) \to \mathscr{T}(Y')$$

define an exact morphism of triangulated $\mathscr{P}/X$-fibred categories over $\mathscr{S}/X$ (by the $\mathscr{P}$-base change formula):

$$\varphi^* : \mathscr{T}' \to \mathscr{T}'' .$$

Applying the preceding proposition to the latter gives (1). The fact that we get an isomorphism whenever g is a $\mathscr{P}$-morphism follows from the $\mathscr{P}$-base change formula and from paragraph 1.1.15.

For point (2), we consider the notations above assuming that g is a $\mathscr{P}$-morphism. The collection of functors

$$g'_\sharp : \mathscr{T}(Y') \to \mathscr{T}(Y)$$

associated with an X-scheme Y, $g' : Y' = Y \times_X X' \to Y$ obtained from g as above, define an exact morphism of triangulated $\mathscr{P}/X$-fibred categories over $\mathscr{S}/X$ (applying again the $\mathscr{P}$-base change formula):

$$\varphi^* : \mathscr{T}'' \to \mathscr{T}' .$$

Applying the preceding proposition to the latter gives (2).

The proof of (3) is similar: fix a scheme X in $\mathscr{S}$, as well as an object L in $\mathscr{T}(X)$. Let $\mathscr{T}'$ be the restriction of $\mathscr{T}$ to $\mathscr{S}/X$ as above. We can consider L as a cartesian section of $\mathscr{T}'$, and by the $\mathscr{P}$-projection formula, we then have an exact morphism of triangulated $\mathscr{P}/X$-fibred categories over $\mathscr{S}/X$:

$$L \otimes (-) : \mathscr{T}' \to \mathscr{T}' .$$

Here again, we can apply the preceding proposition and conclude. □

2.2.c. *Further properties.* We will be particularly interested in the following properties of the triangulated $\mathscr{P}$-fibred category $\mathscr{T}$.

DEFINITION 2.2.13. Let $f : Y \to X$ be a morphism in $\mathscr{S}$. We introduced the following properties for $\mathscr{T}$, assuming in the third case that $\mathscr{T}$ is monoidal:

(Adj$_f$) The functor f_* admits a right adjoint. Under this assumption, we denote by $f^!$ the right adjoint of f_*.

(BC$_f$) Any cartesian square of $\mathscr{S}$ of the form

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

is $\mathscr{T}$-transversal (Def. 1.1.17) – *i.e.* the exchange transformation

$$Ex(\Delta^*_*) : g^* f_* \to f'_* g'^*$$

associated with Δ is an isomorphism.

(PF$_f$) For any object premotive M over Y, and N over X, the exchange transformation (see paragraph 1.1.31)

$$Ex(f^*_*, \otimes_X) : (f_* M) \otimes_X N \to f_*(M \otimes_Y f^* N)$$

is an isomorphism.

We denote by (Adj) (resp. (BC), (PF)) the property (Adj$_f$) (resp. (BC$_f$), (PF$_f$)) for any *proper* morphism f in $\mathscr{S}$ and call it the *adjoint property* (resp. *proper base change property*, *projection formula*).

We can summarize the construction and properties introduced in this section as follows:

THEOREM 2.2.14. *Assume $\mathscr{T}$ satisfies the properties (Supp) and (Adj).*

Then for any separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, there exists an essentially unique pair of adjoint functors

$$f_! : \mathscr{T}(Y) \rightleftarrows \mathscr{T}(X) : f^!$$

called the exceptional functors*, such that:*

(1) *There exists a structure of a covariant (resp. contravariant) 2-functor on* $f \mapsto f_!$ *(resp.* $f \mapsto f^!$*).*
(2) *There exists a natural transformation* $\alpha_f : f_! \to f_*$ *compatible with composition in* f *which is an isomorphism when* f *is proper.*
(3) *For any open immersion* j, $j_! = j_\sharp$ *and* $j^! = j^*$.
(4) *For any cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

in which f *is separated and of finite type, there exists natural transformations*

$$Ex(\Delta^*_!) : g^* f_! \to f'_! g'^*,$$
$$Ex(\Delta^!_*) : g'_* f'^! \to f^! g_*$$

which are isomorphisms in the following three cases:
- f *is an open immersion.*
- g *is a* $\mathscr{P}$*-morphism.*
- $\mathscr{T}$ *satisfies the proper base change property (BC).*

Assume that $\mathscr{T}$ *is in addition monoidal. Then the following property holds:*

(5) *For any separated morphism of finite type* $f : Y \to X$ *in* $\mathscr{S}$*, there exists natural transformations*

$$Ex(f^*_!, \otimes) : (f_! K) \otimes_X L \longrightarrow f_!(K \otimes_Y f^* L)\,,$$
$$Hom_X(f_!(L), K) \longrightarrow f_* Hom_Y(L, f^!(K))\,,$$
$$f^! Hom_X(L, M) \longrightarrow Hom_Y(f^*(L), f^!(M))\,.$$

which are isomorphisms in the following cases:
- f *is an open immersion.*
- $\mathscr{T}$ *satisfies the projection formula (PF).*

Indeed the existence of $f_!$ follows from Proposition 2.2.7 while that of $f^!$ follows directly from assumption (Adj). Assertions (1) and (3) follows from the construction, (2) is Proposition 2.2.10, (4) (resp. (5)) follows from Corollary 2.2.12 and the definition of (BC) (resp. (PF)). Note also that the second and third isomorphisms in (5) are obtained by transposition from $Ex(f_!, \otimes)$.

2.2.15. While the properties (BC_f) and (PF_f) are only reasonable in practice for proper morphisms, this is not the case for the property (Adj_f). Recall that an exact functor between well generated triangulated categories admits a right adjoint if and only if it commutes with small sums: this is an immediate consequence of the *Brown representability theorem* proved by Neeman (*cf.* [**Nee01**, 8.4.4]).

Proposition 2.2.16. *Assume that* $\mathscr{T}$ *is a compactly* τ*-generated triangulated premotivic category over* $\mathscr{S}$.

Then, for any morphism of schemes $f : T \to S$*, the functor* $f_* : \mathscr{T}(T) \to \mathscr{T}(S)$ *admits a right adjoint.*

Proof. This follows directly from Proposition 1.3.20. □

2.3. The localization property.

2.3.a. *Definition.*

2.3.1. Consider a closed immersion $i : Z \to S$ in $\mathscr{S}$. Let $U = S - Z$ be the complement open subscheme of S and $j : U \to S$ the canonical immersion. We will use the following consequence of the triangulated $\mathscr{P}$-fibred structure on $\mathscr{T}$:

(a) The unit $1 \to j^* j_\sharp$ is an isomorphism.
(b) The counit $j^* j_* \to 1$ is an isomorphism.

(c) $i^* j_\sharp = 0$.
(d) $j^* i_* = 0$.
(e) The composite map $j_\sharp j^* \xrightarrow{ad'(j_\sharp, j^*)} 1 \xrightarrow{ad(i^*, i_*)} i_* i^*$ is zero.

In fact, the first four relations all follow from the base change property ($\mathscr{P}$-BC). Relation (e) is a consequence of (d) once we have noticed that the following square is commutative

$$\begin{array}{ccc} j_\sharp j^* & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ j_\sharp j^* i_* i^* & \rightarrow & i_* i^*. \end{array}$$

For the closed immersion i and the triangulated category $\mathscr{T}$, we introduce the property (Loc_i) made of the following assumptions:

(a) The pair of functors (j^*, i^*) is conservative.
(b) The counit $i^* i_* \xrightarrow{ad'(i^*, i_*)} 1$ is an isomorphism.

Definition 2.3.2. We say that $\mathscr{T}$ satisfies the *localization property*, denoted by (Loc), if:

(1) $\mathscr{T}(\varnothing) = 0$.
(2) For any closed immersion i in $\mathscr{S}$, (Loc_i) is satisfied.

The main consequence of the localization axiom is that it leads to the situation of the six gluing functor (*cf.* [**BBD82**, prop. 1.4.5]):

Proposition 2.3.3. *Let $i : Z \to S$ be a closed immersion with complementary open immersion $j : U \to S$ such that (Loc_i) is satisfied.*

(1) The functor i_* admits a right adjoint $i^!$.
(2) For any K in $\mathscr{T}(S)$, there exists a unique map $\partial_{i,K} : i_* i^* K \to j_\sharp j^* K[1]$ such that the triangle
$$j_\sharp j^* K \xrightarrow{ad'(j_\sharp, j^*)} K \xrightarrow{ad(i^*, i_*)} i_* i^* K \xrightarrow{\partial_{i,K}} j_\sharp j^* K[1]$$
is distinguished. The map $\partial_{i,K}$ is functorial in K.
(3) For any K in $\mathscr{T}(S)$, there exists a unique map $\partial'_{i,K} : j_* j^* K \to i_* i^! K[1]$ such that the triangle
$$i_* i^! K \xrightarrow{ad'(i_*, i^!)} K \xrightarrow{ad(j^*, j_*)} j_* j^* K \xrightarrow{\partial'_{i,K}} i_* i^! K[1]$$
is distinguished. The map $\partial'_{i,K}$ is functorial in K.

Under the property (Loc_i), the canonical triangles appearing in (2) and (3) above are called the *localization triangles* associated with i.

Proof. We first consider point (2). For the existence, we consider a distinguished triangle

$$j_\sharp j^* K \xrightarrow{ad'(j_\sharp, j^*)} K \xrightarrow{\pi} C \xrightarrow{+1}$$

Applying 2.3.1(e), we obtain a factorization

$$\begin{array}{ccc} K & \xrightarrow{ad(i^*, i_*)} & i_* i^* K \\ & {\scriptstyle\pi}\searrow \quad C \quad \nearrow{\scriptstyle w} & \end{array}$$

We prove w is an isomorphism. According to the above triangle, $j^* C = 0$. From 2.3.1(d), $j^* i_* i^* K = 0$ so that $j^* w$ is an isomorphism. Applying i^* to the above distinguished triangle, we obtain from 2.3.1(c) that $i^* \pi$ is an isomorphism. Thus, applying i^* to the above commutative diagram together with (Loc_i) (b), we obtain that $i^* w$ is an isomorphism which concludes.

Consider a map $K \xrightarrow{u} L$ in $\mathscr{T}(S)$ and suppose we have chosen maps a and b in the diagram:

$$\begin{array}{ccccccc} j_\sharp j^* K & \xrightarrow{ad'(j_\sharp, j^*)} & K & \xrightarrow{ad(i^*, i_*)} & i_* i^* K & \xrightarrow{a} & j_\sharp j^* K[1] \\ {\scriptstyle u}\downarrow & & {\scriptstyle u}\downarrow & & & & \downarrow{\scriptstyle u} \\ j_\sharp j^* L & \xrightarrow{ad'(j_\sharp, j^*)} & L & \xrightarrow{ad(i^*, i_*)} & i_* i^* L & \xrightarrow{b} & j_\sharp j^* L[1] \end{array}$$

such that the horizontal lines are distinguished triangles. We can find a map $h : i_*i^*K \to i_*i^*L$ completing the previous diagram into a morphism of triangles. Then the map $w = h - i_*i^*(u)$ satisfy the relation $w \circ ad(i^*, i_*) = 0$. Thus it can be lifted to a map in $\mathrm{Hom}(j_\sharp j^*K[1], i_*i^*L)$. But this is zero by adjunction and the relation 2.3.1(d). This proves both the naturality of $\partial_{i,K}$ and its uniqueness.

For point (1) and (3), for any object K of $\mathscr{T}(S)$, we consider a distinguished triangle

$$D \to K \xrightarrow{ad(j^*,j_*)} j_*j^*K \xrightarrow{+1}$$

According to 2.3.1(b), $j^*D = 0$. Thus according to the triangle of point (2) applied to D, we obtain $D = i_*i^*D$. Arguing as for point (2), we thus obtain that D is unique and depends functorialy on K so that, if we put $i^!K = i^*D$, point (1) and (3) follows. □

REMARK 2.3.4. Consider the hypothesis and notations of the previous proposition.

(1) By transposition from 2.3.1(d), we deduce that $i^!j_* = 0$.
(2) Assume that i is a $\mathscr{P}$-morphism. Then the $\mathscr{P}$-base change formula implies that $i^*j_* = 0$. Dually, we get that $i^!j_\sharp = 0$. By adjunction, we thus obtain $\partial_{i,K} = 0$ and $\partial'_{i,K} = 0$ for any object K so that both localization triangles are split. In that case, we get that $\mathscr{T}(S) = \mathscr{T}(Z) \times \mathscr{T}(U)$.[38]

The preceding proposition admits the following reciprocal statement:

LEMMA 2.3.5. *Consider a closed immersion $i : Z \to S$ in $\mathscr{S}$ with complementary open immersion $j : U \to S$. Then the following properties are equivalent:*

(i) $\mathscr{T}$ satisfies (Loc_i).
(ii) (a) The functor i_* is conservative.
(b) For any object K of $\mathscr{T}(S)$, there exists a map $i_*i^*(K) \to j_\sharp j^*(K)[1]$ which fits into a distinguished triangle

$$j_\sharp j^*(K) \xrightarrow{ad'(j_\sharp,j^*)} K \xrightarrow{ad(i^*,i_*)} i_*i^*(K) \to j_\sharp j^*(K)[1]$$

PROOF. The fact (i) implies (ii) follows from Proposition 2.3.3. Conversely, (ii)(b) implies that the pair (i^*, j^*) is conservative and it remains to prove (Loc_i) (b). Let K be an object of $\mathscr{T}(S)$. Consider the distinguished triangle given by (ii)(b):

$$j_\sharp j^*(K) \xrightarrow{ad'(j_\sharp,j^*)} K \xrightarrow{ad(i^*,i_*)} i_*i^*(K) \to j_\sharp j^*(K)[1].$$

If we apply i_* on the left to this triangle, we get using 2.3.1(d) that the morphism

$$i_*(K) \xrightarrow{ad(i^*,i_*).i_*} i_*i^*i_*(K)$$

is an isomorphism. Hence, by the zig-zag equation, the morphism

$$i_*i^*i_*(K) \xrightarrow{i_*.ad'(i^*,i_*)} i_*(K)$$

is an isomorphism. Property (ii)(a) thus implies that $i^*i_*(K) \simeq K$. □

2.3.b. *First consequences of localization.* The following statement is straightforward.

PROPOSITION 2.3.6. *Assume $\mathscr{T}$ satisfies the localization property and consider a scheme S in $\mathscr{S}$.*

(1) Let S_{red} be the reduced scheme associated with S. The canonical immersion $S_{red} \xrightarrow{\nu} S$ induces an equivalence of categories:

$$\nu^* : \mathscr{T}(S) \to \mathscr{T}(S_{red}).$$

(2) For any any partition
(3) partition $(S_i \xrightarrow{\nu_i} S)_{i \in I}$ of S by locally closed subsets, the family of functors $(\nu_i^*)_{i\in I}$ is conservative (S_i is considered with its canonical structure of a reduced subscheme of S).

LEMMA 2.3.7. *If $\mathscr{T}$ satisfies the localization property (Loc) then it is additive.*

[38] This remark explains why the localization property is too strong for generalized premotivic categories.

Proof. Note that, by assumption, $\mathscr{T}(\varnothing) = 0$. Then the assertion follows directly from Lemma 2.2.2. □

Proposition 2.3.8. *If $\mathscr{T}$ satisfies the localization property then it satisfies the* cdh*-separation property.*

Proof. Consider a cartesian square of schemes

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & Q & \downarrow p \\ A & \xrightarrow[e]{} & X. \end{array}$$

According to Lemma 2.1.12, we have only to check that the pair of functors (e^*, p^*) is conservative when Q is a Nisnevich (or respectively a proper cdh) distinguished square. Let $\nu : A' \to X$ be the complementary closed (resp. open) immersion to e, where A' has the induced reduced subscheme (resp. induced subscheme) structure. Consider the cartesian square

$$\begin{array}{ccc} Y & \longleftarrow & B' \\ p\downarrow & & \downarrow q \\ X & \xleftarrow[\nu]{} & A' \end{array}$$

By assumption on Q, q is an isomorphism. According to (Loc) (ii), (e^*, ν^*) is conservative. This concludes. □

The following proposition can be found in a slightly less precise and general form in [**Ayo07a**, 2.1.162].[39]

Proposition 2.3.9. *Assume $\mathscr{T}$ satisfies the localization property.*
Then the following conditions are equivalent:

(i) $\mathscr{T}$ is separated.
(ii) For a morphism $f : T \to S$ in $\mathscr{S}$, $f^* : \mathscr{T}(S) \to \mathscr{T}(T)$ is conservative whenever f is:
  - *(a) a finite étale cover;*
  - *(b) finite, faithfully flat and radicial.*

Proof. Only $(ii) \Rightarrow (i)$ requires a proof. Consider a surjective morphism of finite type $f : T \to S$ in $\mathscr{S}$. According to [**EGA4**, 17.16.4], there exists a partition $(S_i)_{i\in I}$ of S by (affine) subschemes and a family of maps of the form

$$S_i'' \xrightarrow{g_i} S_i' \xrightarrow{h_i} S_i$$

such that g_i (resp. h_i) satisfies assumption (a) (resp. (b)) above and such that for any $i \in I$, $f \times_S S_i''$ admits a section. Thus, Proposition 2.3.6 concludes. □

2.3.c. *Localization and exchange properties.*

2.3.10. Consider a morphism of complete triangulated $\mathscr{P}$-fibred categories over $\mathscr{S}$:

$$\varphi^* : \mathscr{T} \to \mathscr{T}'.$$

Recall that for any morphism $f : Y \to X$, there is an exchange transformation (1.2.5.1):

$$Ex(\varphi^*, f_*) : \varphi_X^* f_* \longrightarrow f_* \varphi_Y^*.$$

If $\mathscr{T}$ and $\mathscr{T}'$ satisfies the support axiom and f is separated of finite type, we have constructed (Proposition 2.2.11) another exchange transformation:

$$Ex(\varphi^*, f_!) : \varphi_X^* f_! \longrightarrow f_! \varphi_Y^*.$$

Proposition 2.3.11. *Consider a morphism $\varphi^* : \mathscr{T} \to \mathscr{T}'$ as above.*

(1) Let $i : Z \to X$ be a closed immersion such that $\mathscr{T}$ and $\mathscr{T}'$ satisfy property (Loc$_i$). Then the exchange $Ex(\varphi^*, i_*) : \varphi_X^* i_* \to i_* \varphi_Z^*$ is an isomorphism.

[39] A warning: the proof in *loc. cit.* seems to require that the schemes are excellent.

(2) Assume $\mathscr{T}$ and $\mathscr{T}'$ satisfy property (Loc).
Then the following conditions are equivalent:
(i) For any integer $n > 0$ and any scheme X in $\mathscr{S}$, the exchange $Ex(\varphi^*, p_{n*})$ is an isomorphism where $p_n : \mathbf{P}^n_X \to X$ is the canonical projection.
(ii) For any proper morphism $f : Y \to X$, the exchange $Ex(\varphi^*, f_*)$ is an isomorphism.
(3) Assume $\mathscr{T}$ and $\mathscr{T}'$ satisfy properties (Loc) and (Supp).
Then conditions (i) and (ii) above are equivalent to the following one:
(iii) For any separated morphism $f : Y \to X$ of finite type, the exchange $Ex(\varphi^*, f_!)$ is an isomorphism.

REMARK 2.3.12. We will simply say that φ^* commutes with $f_!$ when assertion (iii) is fulfilled. For an important case where this happens, see Proposition 2.4.53.

PROOF. Assertion (1) follows easily from the conservativity of (i^*, j^*) where j is the complementary open immersion and the relations of paragraph 2.3.1. Assertion (3) is an easy consequence of the definition of $f_!$ and the exchange $Ex(\varphi^*, f_!)$.

Concerning assertion (2), we have to prove that (i) implies (ii). We fix a morphism $f : Y \to X$ and prove that the exchange $Ex(\varphi^*, f_*) : \varphi^*_Y f_* \to f_* \varphi^*_X$ is an isomorphism.

We first treat the case where f is projective. According to Proposition 2.3.8, $\mathscr{T}'$ satisfies the Zariski separation property. Using the ($\mathscr{P}$-BC) property, we see that the problem is local in X so that we can assume X is affine. Then X admits an ample line bundle and there exists an integer $n > 0$ such that f can be factored ([**EGA2**, (5.5.4)(ii)]) into a closed immersion $i : Y \to \mathbf{P}^n_X$ and the projection $p_n : \mathbf{P}^n_X \to X$. Thus, assertion (1) and assumption (i) allows to conclude.

To treat the general case, we argue by noetherian induction on Y, assuming that for any proper closed subscheme T of Y, the result is known for the restriction of f to T. In fact, the case $T = \varnothing$ is obvious because $\mathscr{T}(\varnothing) = 0$.

According to Chow's lemma [**EGA2**, 5.6.2], there exists a morphism $p : Y_0 \to Y$ such that:

(a) p and $f \circ p$ are projective morphisms.
(b) There exists a dense open subscheme V_0 of Y over which p is an isomorphism.

Let T be the complement of V in Y equipped with its reduced subscheme structure. Let j and i be the respective immersion of T and V in Y. According to point (3) of Proposition 2.3.3, it is sufficient to prove that the following natural transformations are isomorphisms:

$$\varphi^*_Y f_* i_* \to f_* \varphi^*_X i_*. \tag{2.3.12.1}$$

$$\varphi^*_Y f_* j_* \to f_* \varphi^*_X j_*. \tag{2.3.12.2}$$

Concerning the first one, we consider the following commutative diagram:

$$\begin{array}{ccccc} \varphi^*_Y f_* i_* & \xrightarrow{Ex(\varphi^*, f_*)} & f_* \varphi^*_X i_* & \xrightarrow{Ex(\varphi^*, i_*)} & f_* i_* \varphi^*_X \\ \| & & & & \| \\ \varphi^*_Y (fi)_* & & \xrightarrow{Ex(\varphi^*, (fi)_*)} & & (fi)_* \varphi^*_X. \end{array}$$

Thus the result follows from assertion (1) and the induction hypothesis.

Concerning the natural transformation (2.3.12.2), we consider the pullback square

$$\begin{array}{ccc} V_0 & \xrightarrow{l} & Y_0 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{j} & Y. \end{array}$$

Assumption (b) above says that q is an isomorphism which implies the relation: $j_* = p_* l_* q^*$. In particular, it is sufficient to prove that the natural transformation $\varphi^*_Y f_* p_* \to f_* \varphi^*_X p_*$ is an isomorphism. This follows from the commutativity of the following diagram

$$\begin{array}{ccccc} \varphi^*_Y f_* p_* & \xrightarrow{Ex(\varphi^*, f_*)} & f_* \varphi^*_X p_* & \xrightarrow{Ex(\varphi^*, p_*)} & f_* p_* \varphi^*_X \\ \| & & & & \| \\ \varphi^*_Y (fp)_* & & \xrightarrow{Ex(\varphi^*, (fp)_*)} & & (fp)_* \varphi^*_X, \end{array}$$

according to the projective case treated above and assumption (b). The proof is complete. □

COROLLARY 2.3.13. *In the next statements, we assume $\mathscr{T}$ is monoidal when it is needed.*

(1) Let $i : Z \to X$ be a closed immersion such that $\mathscr{T}$ satisfies property (Loc$_i$). Then $\mathscr{T}$ satisfies property (Supp$_i$) (resp. (BC$_i$), (PF$_i$)).

(2) Assume $\mathscr{T}$ satisfies the localization property. Then the following properties of $\mathscr{T}$ are equivalent:

(i) For any integer $n > 0$ and any scheme X in $\mathscr{S}$, $p_n : \mathbf{P}^n_X \to X$ being the canonical projection, $\mathscr{T}$ satisfies (Supp$_{p_n}$) (resp. (BC$_{p_n}$), (PF$_{p_n}$)).

(ii) $\mathscr{T}$ satisfies (Supp) (resp. (BC), (PF)).

(3) Assume $\mathscr{T}$ is well generated and satisfies the localization property. Then the following properties of $\mathscr{T}$ are equivalent:

(i') For any integer $n > 0$ and any scheme X in $\mathscr{S}$, $p_n : \mathbf{P}^n_X \to X$ being the canonical projection, $\mathscr{T}$ satisfies (Adj$_{p_n}$).

(ii') $\mathscr{T}$ satisfies (Adj).

PROOF. As in the proof of Corollary 2.2.12, each respective case of assertions (1) and (2) follows from the previous proposition applied to a particular type of morphisms $\varphi^* : \mathscr{T}' \to \mathscr{T}''$ of complete $\mathscr{P}$-fibred triangulated categories over a subcategory $\mathscr{S}'$ of $\mathscr{S}$.

For property (Supp), we proceed as follows. We fix an open immersion $j : U \to X$ and let $\mathscr{S}' = \mathscr{S}/X$. For any Y/X, we let $j_Y = Y \times_X U \to Y$ be the pullback of j. We put $\mathscr{T}'(Y) = \mathscr{T}(Y \times_X U)$ and $\mathscr{T}''(Y) = \mathscr{T}(Y)$ and let φ^*_Y be the functor:

$$j_{Y\sharp} : \mathscr{T}(Y \times_X U) \to \mathscr{T}(Y).$$

For the property (BC) (resp. (PF)), we refer the reader to the proof of assertion (1) (resp. (2)) in Corollary 2.2.12.

Finally we consider assertion (3). It is sufficient to prove that (*i'*) implies (*ii'*).
According to the Brown representability theorem [**Nee01**, 8.4.4], the property (Adj$_f$) for a proper morphism f is equivalent to ask that f_* preserves small sum.
Consider an arbitrary set I. For any scheme S, we put $\mathscr{T}^I(S) = \mathscr{T}(S)^I$, that is the category of families of object of $\mathscr{T}(S)$ indexed by I. Then $\mathscr{T}^I$ is obviously a complete triangulated $\mathscr{P}$-fibred category over $\mathscr{S}$ (limits and colimits are computed termwise). For any scheme S, we consider the functor:

$$\varphi^*_S : \mathscr{T}^I(S) \to \mathscr{T}(S), (M_i)_{i\in I} \mapsto \sum_{i\in I} M_i.$$

Then $\varphi^* : \mathscr{T}^I \to \mathscr{T}$ is obviously a morphism of complete $\mathscr{P}$-fibred categories. Thus, given condition (*i'*), the preceding proposition applied to φ^* shows that for any proper morphism f, f_* commutes with sums indexed by I. As this is true for any I, we obtain (*ii'*). □

2.3.d. *Localization and monoidal structure.*

2.3.14. Assume $\mathscr{T}$ is monoidal and let M denote its geometric sections. Fix a closed immersion $i : Z \to S$ in $\mathscr{S}$ with complementary open immersion $j : U \to S$. We fix an object $M_S(S/S-Z)$ of $\mathscr{T}(S)$ and a distinguished triangle

$$M_S(S-Z) \xrightarrow{j_*} \mathbb{1}_S \xrightarrow{p_i} M_S(S/S-Z) \xrightarrow{d_i} M_S(S-Z)[1]. \tag{2.3.14.1}$$

Remark that according to 2.3.1(c), the map $i^*(p_i) : \mathbb{1}_Z \to i^*M_S(S/S-Z)$ is an isomorphism. Given any object K in $\mathscr{T}(S)$, we thus obtain an isomorphism

$$i^*(M_S(S/S-Z) \otimes_S K) = i^*(M_S(S/S-Z)) \otimes_Z i^*(K) \xrightarrow{(i^*p_i)^{-1}} \mathbb{1}_Z \otimes_Z i^*(K) = i^*(K)$$

which is natural in K. It induces by adjunction a map

$$\psi_{i,K} : M_S(S/S-Z) \otimes_S K \to i_*i^*(K) \tag{2.3.14.2}$$

which is natural in K.
For any $\mathscr{P}$-scheme X/S, we put $M_S(X/X - X_Z) = M_S(S/S - Z) \otimes_S M_S(X)$ so that we get from (2.3.14.1) a canonical distinguished triangle:

$$M_S(X - X_Z) \xrightarrow{j_{X*}} M_S(X) \to M_S(X/X - X_Z) \to M_S(X - X_Z)[1].$$

The map (2.3.14.2) for $K = M_S(X)$ gives a canonical map

$$\psi_{i,X} : M_S(X/X - X_Z) \to i_*(M_Z(X_Z)). \tag{2.3.14.3}$$

PROPOSITION 2.3.15. *Consider the previous hypothesis and notations. Then the following conditions are equivalent:*

(i) $\mathscr{T}$ *satisfies the property (Loc$_i$).*

(ii) *(a) The functor i_* is conservative.*
(b) The morphism $\psi_{i,S} : M_S(S/S - Z) \to i_*(\mathbb{1}_Z)$ is an isomorphism.
(c) For any object K of $\mathscr{T}(S)$, the exchange transformation

$$Ex(i_*^*, \otimes) : (i_*\mathbb{1}_Z) \otimes_S K \to i_* i^* K$$

is an isomorphism.

(iii) *(a) The functor i_* is conservative.*
(b) The morphism $\psi_{i,S} : M_S(S/S - Z) \to i_*(\mathbb{1}_Z)$ is an isomorphism.
(c) For any objects K and L of $\mathscr{T}(S)$, the exchange transformation

$$Ex(i_*^*, \otimes) : (i_*K) \otimes_S L \to i_*(K \otimes_Z i^* L)$$

is an isomorphism.

Assume in addition that $\mathscr{T}$ is well generated and τ-generated as a triangulated $\mathscr{P}$-fibred category. Then the above conditions are equivalent to the following one:

(iv) *(a) The functor i_* is conservative, commutes with direct sums and with τ-twists.*
(b) The morphism $\psi_{i,X} : M_S(X/X - X_Z) \to i_*(M_Z(X_Z))$ is an isomorphism for any $\mathscr{P}$-scheme X/S.

In particular, (Loc$_i$) implies that for any object K of $\mathscr{T}(S)$, the localization triangle of 2.3.3

$$j_\sharp j^*(K) \to K \to i_* i^*(K) \xrightarrow{\partial_K} j_\sharp j^*(K)[1]$$

is canonically isomorphic (through exchange transformations) to the triangle (2.3.14.1) tensored with K.

PROOF. $(i) \Rightarrow (iii)$: According to (Loc$_i$) (a), we need only to check that the maps in (iii)(b) and (iii)(c) are isomorphisms after applying i^* and j^*. This follows easily from (Loc$_i$) (b).
$(iii) \Rightarrow (ii)$: Obvious
$(ii) \Rightarrow (i)$: According to (ii)(b), the distinguished triangle (2.3.14.1) is isomorphic to a triangle of the form

$$j_\sharp j^*(\mathbb{1}_S) \xrightarrow{ad'(j_\sharp, j^*)} \mathbb{1}_S \xrightarrow{ad(i^*, i_*)} i_* i^*(\mathbb{1}_S) \to j_\sharp j^*(\mathbb{1}_S).$$

According to (ii)(c), this latter triangle tensored with K is isomorphic through exchange transformations to a triangle of the form

$$j_\sharp j^*(K) \xrightarrow{ad'(j_\sharp, j^*)} K \xrightarrow{ad(i^*, i_*)} i_* i^*(K) \to j_\sharp j^*(K).$$

Thus Lemma 2.3.5 allows to conclude.

To end the proof, we remark by using the equations for the adjunction (i^*, i_*) that for any object M of $\mathscr{T}(S)$, the following diagram is commutative:

$$\begin{array}{ccccc}
 & & i_* i^*(\mathbb{1}_S) \otimes K & = & i_*(\mathbb{1}_Z) \otimes K \\
 & \nearrow^{\psi_i \otimes 1_K} & & & \downarrow{\scriptstyle Ex(i_*^*, \otimes)} \\
M_S(S/S - Z) \otimes K & & & & \\
 & \searrow_{\psi_{i,K}} & & & \\
 & & i_* i^*(K) & = & i_*(\mathbb{1}_Z \otimes i^* i^*(K)).
\end{array}$$

Note that (i) implies that i_* is conservative and commutes with direct sums (see 2.3.3) and (ii)(c) implies it commutes with twists. According to the above diagram, (ii)(b) implies (iv)(b).
We prove that reciprocally that (iv) implies (ii). Because (ii)(b) (resp. (ii)(a)) is a particular case of (iv)(b) (resp. (iv)(a)), we have only to prove (ii)(b). In view of the previous diagram, we are reduced to prove that for any object K of $\mathscr{T}(S)$, the map $\psi_{i,K}$ is an isomorphism. Consider the full subcategory $\mathscr{U}$ of $\mathscr{T}(S)$ made of the objects K such that $\psi_{i,K}$ is an isomorphism. Then $\mathscr{U}$ is triangulated. Using (iv)(a), $\mathscr{U}$ is stable by small sums and τ-twists. By assumption, it contains the objects of the form $M_S(X)$ for a $\mathscr{P}$-scheme X/S. Thus, because $\mathscr{T}$ is well generated by assumption, Lemma 1.3.18 concludes. □

Lemma 2.3.16. *Consider a closed immersion $i : Z \to S$. We assume the following conditions are satisfied in addition to that of 2.0:*

- *$\mathscr{T}$ is well generated, τ-generated, and satisfies the Zariski separation property.*
- *For any $\mathscr{P}$-scheme X_0/Z and any point x_0 of X_0, there exists an open neighbourhood U_0 of x_0 in X_0 and a $\mathscr{P}$-scheme U/S such that $U_0 = U \times_S Z$.*[40]

Then the functor i_* is conservative.

Proof. Consider an object K of $\mathscr{T}(Z)$ such that $i_*(K) = 0$. We prove that $K = 0$. Because $\mathscr{T}$ is τ-generated, it is sufficient to prove that for a $\mathscr{P}$-morphism $p_0 : X_0 \to Z$ and a twist $(n, m) \in \mathbf{Z} \times \tau$,

$$\mathrm{Hom}_{\mathscr{T}(Z)}(M_Z(X_0)\{m\}[n], K) = 0.$$

Because $M_Z(X_0) = p_{0\sharp}(\mathbb{1}_{X_0})$, this equivalent to prove that

$$\mathrm{Hom}_{\mathscr{T}(X_0)}(\mathbb{1}_{X_0}\{m\}[n], p_0^*(K)) = 0.$$

Using the Zariski separation property on $\mathscr{T}$, this latter assumption is local in X_0. Thus, according to the assumption on the class $\mathscr{P}$, we can assume there exists a $\mathscr{P}$-scheme X/S such that $X_0 = X \times_S Z$. Thus $M_Z(X_0)\{m\}[n] = i^*(M_S(X)\{m\}[n])$ and the initial assumption on K allows to conclude. □

Note for future applications the following interesting corollaries:

Corollary 2.3.17. *Assume $\mathscr{T}$ is a premotivic triangulated category which is compactly τ-generated for a group of twists τ (*i.e.* any twists in τ admits a tensor inverse) and which satisfies the Zariski separation property.*

Then, for any closed immersion i, the functor i_* is conservative, commutes with sums and with twists.

This is a consequence of lemmas 2.3.16 and 2.2.16. In fact, under these conditions, i_* commutes with arbitrary τ-twists because it is true for its (left) adjoint i^*.

Corollary 2.3.18. *Assume $\mathscr{T}$ satisfies the assumptions of the preceding corollary. Then the following conditions on a closed immersion i are equivalent:*

(i) $\mathscr{T}$ satisfies the property (Loc$_i$).
(ii) For any scheme S in $\mathscr{S}$ and any smooth S-scheme X, the map (2.3.14.3)

$$\psi_{i,X} : M_S(X/X - X_Z) \to i_*M_Z(X_Z)$$

is an isomorphism.

We finish this section with the following useful result:

Proposition 2.3.19. *Assume $\mathscr{T}$ is τ-generated and consider a τ'-generated triangulated $\mathscr{P}$-fibred category $\mathscr{T}'$ and a morphism*

$$\varphi^* : (\mathscr{T}, \tau) \rightleftarrows (\mathscr{T}', \tau') : \varphi_* .$$

We assume the following properties:

[40] This property is trivial when $\mathscr{P}$ is the class of open immersions or the class of morphisms of finite type in $\mathscr{S}$. It is also true when $\mathscr{P}$ is the class of étale morphism or $\mathscr{P} = Sm$ (*cf.* [**EGA4**, 18.1.1]).

(a) *the morphism φ^* is strictly compatible with twists;*
(b) *$\mathscr{T}'$ is well generated.*

We consider a closed immersion $i : Z \to S$ and further assume the following properties:

(c) *$\mathscr{T}$ satisfies the property (Loc$_i$).*
(d) *The exchange transformation $Ex(\varphi^*, i_*) : \varphi^* i_* \to i_* \varphi^*$ is an isomorphism.*
(e) *The functor $i_* : \mathscr{T}'(Z) \to \mathscr{T}'(S)$ commutes with τ'-twists.*[41]

Then $\mathscr{T}'$ satisfies the property (Loc$_i$).

Proof. Note that, under the above assumptions, φ_* is conservative (in fact, for any $\mathscr{P}$-scheme X/S and any twists $i \in \tau'$, the premotive $M_S(X)\{i\}$ is in the essential image of φ^*). Thus, if $i_* : \mathscr{T}(Z) \to \mathscr{T}(S)$ is conservative (resp. commute with sums), then $i_* : \mathscr{T}'(S) \to \mathscr{T}'(S)$ is conservative (resp. commute with sums) using the isomorphism $\varphi_* i_* \simeq i_* \varphi_*$.
Let M (resp. M') be the geometric sections of $\mathscr{T}$ (resp. $\mathscr{T}'$). As in 2.3.14, we fix a distinguished triangle

$$M_S(S-Z) \xrightarrow{j_*} \mathbb{1}_S \xrightarrow{p_i} M_S(S/S-Z) \xrightarrow{d_i} M_S(S-Z)[1].$$

and we put $M'_S(S/S-Z) = \varphi^* M_S(S/S-Z)$. According to *loc. cit.*, we thus get for any $\mathscr{P}$-scheme X/S canonical maps

$$\psi_{i,X} : M_S(X/X - X_Z) \to i_* M_Z(X_Z),$$
$$\psi'_{i,X} : M'_S(X/X - X_Z) \to i_* M'_Z(X_Z).$$

By construction, the following diagram is commutative:

$$\begin{array}{ccccc}
\varphi^* M_S(X/X - X_Z) & \xrightarrow{\varphi^*\psi_{i,X}} & \varphi^* i_* M_Z(X_Z) & \xrightarrow{Ex(\varphi^*, i_*)} & i_* \varphi^* M_Z(X_Z) \\
\| & & & & \| \\
M'_S(X/X - X_Z) & & \xrightarrow{\psi'_{i,X}} & & M'_Z(X_Z)
\end{array}$$

Thus, Proposition 2.3.15 allows to conclude. □

2.4. Purity and the theorem of Ayoub. Recall we assume $\mathscr{P} = Sm$ in this section.

2.4.a. *The stability property.* The following section is directly inspired by the work of Ayoub in [**Ayo07a**, §1.5].[42] We claim no originality except for a closer look on the needed axioms.

Definition 2.4.1. A *pointed smooth S-scheme* will be a couple (f, s) of morphisms of $\mathscr{S}$ such that $f : X \to S$ is a smooth separated morphism of finite type and $s : S \to X$ is a section of f.

We associate with a pointed smooth scheme (f, s) the following endofunctor of $\mathscr{T}(S)$

$$\mathcal{T}h(f, s) := f_\sharp s_*$$

called the associated *Thom transformation.*

If $\mathscr{T}$ satisfies (Adj$_s$) (recall: s_* admits a right adjoint denoted by $s^!$), we put

$$\mathcal{T}h'(f, s) := s^! f^*$$

and call it the associated *adjoint Thom transformation.*

Remark 2.4.2. Note that because f is separated, s is a closed immersion.

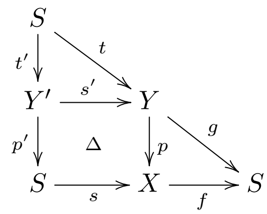
Example 2.4.3. (1) Let $p : E \to X$ be a vector bundle and s_0 be its zero section. Following [**Ayo07a**], we put $\mathcal{T}h(E) := \mathcal{T}h(p, s_0)$ and call it simply the Thom transformation associated with E/X.
(2) Consider a pointed smooth S-scheme (f, s) such that f is étale. Then s is an open and closed immersion. Thus, if $\mathscr{T}$ is additive, $s_* = s_\sharp$ according to Lemma 2.2.2. In particular, $\mathcal{T}h(f, s) = Id_S$.

[41] This will be satisfied if any τ'-twists is invertible because the left adjoint of i_* commutes with τ'-twists.
[42] See also [**Del01**, §5].

Definition 2.4.4. We will say that $\mathscr{T}$ satisfies the *stability property*, denoted by (Stab), if for any point smooth scheme (f, s), the Thom transformation $\mathcal{T}h(f, s)$ is an equivalence of categories.

2.4.5. Consider a commutative diagram in $\mathscr{S}$ of the form

(2.4.5.1)



such that Δ is a cartesian square, (f, s), (g, t) are smooth pointed schemes and g is a smooth separated morphism of finite type. Then we get a canonical exchange morphism:

$$\mathcal{T}h(g, t) = f_\sharp p_\sharp s'_* t'_* \xrightarrow{Ex(\Delta_{\sharp *})} f_\sharp s_* p'_\sharp t'_* = \mathcal{T}h(f, s)\mathcal{T}h(p', t'). \tag{2.4.5.2}$$

This is an isomorphism as soon as $Ex(\Delta_{\sharp *})$ is an isomorphism. The following lemma gives a sufficient condition for this to happen.

Lemma 2.4.6. *Consider the above notations. If $\mathscr{T}$ satisfies (Loc_s) then the natural transformations $Ex(\Delta_{\sharp *})$ is an isomorphism for any square Δ as above.*

This lemma follows easily from the definition of (Loc_s), the relations of paragraph 2.3.1 and the $\mathscr{P}$-base change formula ($\mathscr{P}$-BC). It motivates the next definition:

Definition 2.4.7. We say that $\mathscr{T}$ satisfies the *weak localization property* (wLoc) if it satisfies (Loc_s) for any closed immersion s which admits a smooth retraction.

Proposition 2.4.8. *Assume that $\mathscr{T}$ satisfies the Nisnevich separation property. Then the following conditions are equivalent:*

(i) $\mathscr{T}$ satisfies (wLoc).
(ii) For any scheme S and any closed immersion $i : Z \to X$ between smooth S-schemes, $\mathscr{T}$ satisfies (Loc_i).

Proof. Of course, (ii) implies (i). We prove the reciprocal statement. The Nisnevich separation property says that for any Nisnevich cover $f : X' \to X$, the functor f^* is conservative. We deduce from that point the properties (Loc_i) (a) and (Loc_i) (b) are local in X with respect to the Nisnevich topology – for (b), one also uses the smooth projection formula. Thus, we can conclude as locally for the Nisnevich topology, i admits a smooth retraction (see for example [**Dég07**, 4.5.11]). □

Applying the second point of Example 2.4.3, we easily deduce from that construction the following kind of excision property:

Lemma 2.4.9. *Assume that $\mathscr{T}$ satisfies (wLoc).*

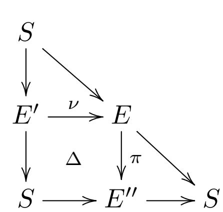
Then, given any diagram (2.4.5.1) *satisfying the assumption as above and such that p is étale, the natural transformation* (2.4.5.2) *gives an isomorphism:*

$$\mathcal{T}h(g, t) \xrightarrow{\sim} \mathcal{T}h(f, s).$$

2.4.10. To any short exact sequence of vector bundles over a scheme S

$$0 \to E' \xrightarrow{\nu} E \xrightarrow{\pi} E'' \to 0, \tag{σ}$$

we can associate a commutative diagram



where the non labeled map are either the canonical projections or the zero sections of the relevant vector bundles, and Δ is cartesian. Using the notation of Example 2.4.3, the exchange transformation (2.4.5.2) associated with this diagram has the following form:

$$\mathcal{T}h(\sigma) : \mathcal{T}h(E) \longrightarrow \mathcal{T}h(E'') \circ \mathcal{T}h(E').$$

Recall from the above that this natural transformation is an isomorphism as soon as $\mathscr{T}$ satisfies (wLoc).

Proposition 2.4.11. *Assume $\mathscr{T}$ satisfies (wLoc) and (*Zar*-sep). Then the following conditions are equivalent:*

(i) The complete triangulated Sm-fibred category $\mathscr{T}$ satisfies the stability property.
(ii) For any scheme S, the Thom transformation $\mathcal{T}h(\mathbf{A}^1_S)$ is an equivalence of categories.

Proof. We have to prove that (ii) implies (i). Note that according to the above paragraph, we already now that for any scheme S and any integer $n \geq 0$, $\mathcal{T}h(\mathbf{A}^n_S) \simeq \mathcal{T}h(\mathbf{A}^1_S)^{\circ,n}$ is an equivalence.

We consider a smooth pointed scheme $(f : X \to S, s)$ and we prove that $\mathcal{T}h(f,s)$ is an equivalence.

Recall that (Loc_s) implies (Adj) s (first point of Proposition 2.3.3). In particular, $\mathcal{T}h(f,s)$ admits a right adjoint $\mathcal{T}h'(f,s)$ and we have to prove that the adjunction morphisms are isomorphisms.

Consider an open immersion $j : U \to S$ and let (f_0, s_0) be the restriction of the smooth S-point (f,s) over U. Property (Loc_s) implies (BC_s) (Corollary 2.3.13). Thus, using also property ($\mathscr{P}$-BC), we obtain a canonical isomorphism:

$$j^*\mathcal{T}h(f,s) \xrightarrow{\sim} \mathcal{T}h(f_0,s_0)j^*.$$

Recall also that (Loc_s) implies (Supp_s) (again Corollary 2.3.13). Thus we get a canonical isomorphism:

$$j_\sharp\mathcal{T}h(f_0,s_0) \xrightarrow{\sim} \mathcal{T}h(f,s)j_\sharp$$

which gives by adjunction an isomorphism:

$$\mathcal{T}h'(f_0,s_0)j^* \xrightarrow{\sim} j^*\mathcal{T}h'(f,s).$$

Thus, (Zar-sep) shows that the property for $\mathcal{T}h(f,s)$ to be an equivalence is Zariski local in S.

Consider a point $a \in S$, $x = s(a)$. As X is smooth over S, there exists an open subscheme $U \subset X$, an integer $n \geq 0$ and an étale S-morphism $\pi : U \to \mathbf{A}^n_S$ which fits into the following cartesian square:

$$\begin{array}{ccc} S_0 & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\nu} & \mathbf{A}^n_S \end{array}$$

where ν is the zero section (*cf.* [**EGA4**, 17.12.2]). Note that the scheme $S_0 = s^{-1}(U)$ is an open neighbourhood of a in S. Let us put $X_0 = f^{-1}(S_0)$ and $U_0 = U \cap X_0$. Then we get the following commutative diagram:

$$\begin{array}{ccccc} & & X_0 & & \\ & \overset{s_0}{\nearrow} & \uparrow & \overset{f_0}{\searrow} & \\ S_0 & \xrightarrow{s'_0} & U_0 & \xrightarrow{f'_0} & S_0 \\ & \underset{\nu_0}{\searrow} & \downarrow{\scriptstyle \pi_0} & \nearrow & \\ & & \mathbf{A}^n_{S_0} & & \end{array}$$

where π_0 is the restriction of π above S_0 and ν_0 is again the zero section. According to Lemma 2.4.9, we get isomorphisms

$$\mathcal{T}h(f_0,s_0) \simeq \mathcal{T}h(f'_0,s'_0) \simeq \mathcal{T}h(\mathbf{A}^n_S).$$

Thus, according to the beginning of the proof, $\mathcal{T}h(f_0,s_0)$ is an equivalence. This concludes because S_0 is an open neighbourhood of a in S. $\square$

Definition 2.4.12. Assume that $\mathscr{T}$ is monoidal.

(1) For any smooth pointed scheme $(f : X \to S, s)$, we put $M_S\big(X/X - s(S)\big) := f_\sharp s_*(\mathbb{1}_S)$.
(2) For any vector bundle E/S with projection f and zero section s, we define the *Thom premotive* associated with E over S as $MTh_S(E) = f_\sharp s_*(\mathbb{1}_S)$.

2.4.13. We assume $\mathscr{T}$ is monoidal and satisfies properties (wLoc) and (Zar-sep).

In each case of the previous definition, if we apply $f_\sharp$ to the distinguished triangle obtained from point (2) of Proposition 2.3.3 applied to s, we get the following canonical distinguished triangles:

$$M_S\big(X - s(S)\big) \to M_S(X) \to M_S\big(X/X - s(S)\big) \xrightarrow{+1}$$
$$M_S(E^\times) \to M_S(E) \to MTh_S(E) \xrightarrow{+1}$$

where the first map is induced by the obvious open immersion.

Moreover, property (Loc_s) implies (PF_s) (see Corollary 2.3.13). Thus for any premotive K over S, the following composite map is an isomorphism:

(2.4.13.1)
$$\begin{aligned}\mathcal{T}h(f,s).K = f_\sharp s_*(K) = f_\sharp s_*(\mathbb{1}_S \otimes_S s^* f^*(K)) &\xrightarrow{Ex(s_*^*,\otimes)^{-1}} f_\sharp(s_*(\mathbb{1}_S) \otimes_X f^*(K)) \\ &\xrightarrow{Ex(f_\sharp^*,\otimes)} (f_\sharp s_*(\mathbb{1}_S)) \otimes_S K = M_S(X/X - s(S)) \otimes_S K\end{aligned}$$

Similarly, in the case of a vector bundle E/S, we get a canonical isomorphism:

$$\mathcal{T}h(E).K \xrightarrow{\sim} MTh_S(E) \otimes_S K.$$

From these isomorphisms, we deduce easily the following corollary of the previous proposition:

Corollary 2.4.14. *Consider the above notations and assumptions. Then the following properties are equivalent:*

(i) *$\mathscr{T}$ satisfies the stability property.*
(ii) *For any smooth pointed scheme $(X \to S, s)$, the premotive $M_S(X/X - s(S))$ is $\otimes$-invertible.*
(iii) *For any vector bundle E/S the Thom premotive $MTh_S(E)$ is $\otimes$-invertible.*
(iv) *For any scheme S, the premotive $MTh_S(\mathbf{A}^1_S)$ is $\otimes$-invertible.*

Remark 2.4.15. Assume that $\mathscr{T}$ satisfies the assumptions and the equivalent conditions of the previous corollary. Then, under the notations of Paragraph 2.4.10, we associate with the exact sequence (σ) a canonical isomorphism

(2.4.15.1)
$$Th_S(\sigma) : MTh_S(E) \to MTh_S(E'') \otimes_S MTh_S(E').$$

Recall that Deligne introduced in [**Del87**, 4.12] the Picard category $\underline{K}(S)$ of *virtual vector bundle* over a scheme S.

Then, it follows from the above isomorphism and the universal properties of $\underline{K}(S)$ (see [**Del87**, 4.3]) that the functor MTh_S can be extended uniquely to a symmetric monoidal functor:

$$MTh_S : \underline{K}(S) \to \mathscr{T}(S).$$

The reader is referred to [**Ayo07a**, th. 1.5.18] for a detailed argument.

2.4.16. Assume $\mathscr{T}$ is monoidal. For any scheme S, the canonical projection $p : \mathbf{P}^1_S \to S$ is a split epimorphism. A splitting is given by the inclusion of the infinite point $\nu : S \to \mathbf{P}^1_S$. The induced map $p_* : M_S(\mathbf{P}^1_S) \to \mathbb{1}_S$ is a split epimorphism. Thus it admits a kernel K in the triangulated category $\mathscr{T}(S)$.

Definition 2.4.17. Under the above assumption and notations, we define the *Tate premotive* over S as the object $\mathbb{1}_S(1) = K[-2]$ of $\mathscr{T}(S)$.

The monoid generated by the cartesian section $(\mathbb{1}_S)_S$ defines a canonical $\mathbf{N}$-twist on $\mathscr{T}$ called the *Tate twist*. The n-th Tate twist of an object K is denoted by $K(n)$.

2.4.18. Consider again the assumption of Paragraph 2.4.13.

According to Lemma 2.4.9, we get a canonical isomorphism

$$MTh_S(\mathbf{A}^1_S) = M_S(\mathbf{A}^1_S/\mathbf{A}^1_S - \{0\}) \to M_S(\mathbf{P}^1_S/\mathbf{P}^1_S - \{0\}).$$

On the other hand, $\mathbb{1}_S(1)[2]$ is by definition the cokernel of the monomorphism $\nu_* : \mathbb{1}_S \to M_S(\mathbf{P}^1_S)$. Thus we get a canonical morphism:

$$\mathbb{1}_S(1)[2] \to M_S(\mathbf{P}^1_S/\mathbf{P}^1_S - \{0\}) \xrightarrow{\sim} MTh_S(\mathbf{A}^1_S). \tag{2.4.18.1}$$

From this definition and Corollary 2.4.14 the following result is obvious:

Corollary 2.4.19. *Consider the above assumption and notations. Then the following conditions are equivalent:*

(i) $\mathscr{T}$ satisfies the homotopy property.
(ii) For any scheme S, the arrow (2.4.18.1) *is an isomorphism.*

When these equivalent assertions are satisfied, the following conditions are equivalent:

(iii) $\mathscr{T}$ satisfies the stability property.
(iv) For any scheme S, the Tate premotive $\mathbb{1}_S(1)$ is $\otimes$-invertible.

2.4.b. *The purity property.*

2.4.20. Let $f : X \to S$ be a smooth proper morphism in $\mathscr{S}$. We consider the following cartesian square:

$$\begin{array}{ccc} X \times_S X & \xrightarrow{f''} & X \\ {\scriptstyle f'}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & S \end{array} \tag{2.4.20.1}$$

where f' (resp. f'') is the projection on the first (resp. second) factor. Let $\delta : X \to X \times_S X$ be the diagonal embedding. Note that (f', δ) is a smooth pointed scheme which depends only on f. We put:

$$\Sigma_f := \mathcal{T}h(f', \delta) = f'_\sharp \delta_*.$$

We then define a canonical morphism:

$$\mathfrak{p}_f : f_\sharp = f_\sharp f''_* \delta_* \xrightarrow{Ex(\Delta_{\sharp *})} f_* f'_\sharp \delta_* = f_* \circ \Sigma_f$$

using the exchange transformation introduced in paragraph 1.1.15.

Definition 2.4.21. We say that f is *$\mathscr{T}$-pure*, or simply *pure* when $\mathscr{T}$ is clear, when the following conditions are satisfied:

(1) The natural transformation Σ_f is an equivalence.
(2) The morphism $\mathfrak{p}_f : f_\sharp \to f_* \circ \Sigma_f$ is an isomorphism.

Then $\mathfrak{p}_f$ is called the *purity isomorphism* associated with f. We say also that f is *universally $\mathscr{T}$-pure* if f is pure after any base change along a morphism of $\mathscr{S}$.

We introduce the following properties on $\mathscr{T}$:

- $\mathscr{T}$ satisfies the *purity property* (Pur) if any proper smooth morphism is pure.
- $\mathscr{T}$ satisfies the *weak purity property* (wPur) if for any scheme S and any integer $n > 0$, the canonical projection $p_n : \mathbf{P}^n_S \to S$ is pure.

Remark 2.4.22. Consider the above notations and assume f is pure.

Then f_* admits a right adjoint $f^!$ and we deduce by transposition from $\mathfrak{p}_f$ a canonical isomorphism:

$$\mathfrak{p}'_f : f^* \to \Sigma_f^{-1} \circ f^!.$$

Recall also that, when δ_* admits a right adjoint $\delta^!$, Σ_f admits as a right adjoint the transformation $\Omega_f := \delta^! f^*$. In particular, $\Omega_f = \Sigma_f^{-1}$.

The following lemma shows the importance of the purity property.

LEMMA 2.4.23. *Assume that $\mathscr{T}$ satisfies (wLoc). Let $f : Y \to X$ be a proper smooth morphism. If f is universally pure then the following conditions hold:*

(1) $\mathscr{T}$ satisfies (Supp_f) and (BC_f).

(2) For any cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{f}} & Y \\ {\scriptstyle h}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

such that g is smooth, the exchange transformation:

$$Ex(\Delta_{\sharp *}) : g_\sharp \tilde{f}_* \to f_* h_\sharp$$

is an isomorphism.

(3) If moreover $\mathscr{T}$ is monoidal then $\mathscr{T}$ satisfies (PF_f).

PROOF. We first prove condition (2). By assumption, the natural transformation $\Sigma_{\tilde{f}}$ is an equivalence. for f and $\tilde{f}$: by assumption the natural transformations $\Sigma_f = f'_\sharp \delta_*$ and $\Sigma_{\tilde{f}} = \tilde{f}'\tilde{\delta}_*$) are equivalences. Thus, it is sufficient to prove that the natural transformation

$$g_\sharp \tilde{f}_* \Sigma_{\tilde{f}} \xrightarrow{Ex(\Delta_{\sharp *})} f_* h_\sharp \Sigma_{\tilde{f}}$$

is an isomorphism.

For matter of notations, let us also introduce the following cartesian squares:

$$\begin{array}{ccccc} Z & \xrightarrow{\tilde{\delta}} & Z \times_Y Z & \xrightarrow{\tilde{f}'} & Z \\ {\scriptstyle h}\downarrow & \Gamma & {\scriptstyle k}\downarrow & \Theta & \downarrow{\scriptstyle h} \\ X & \xrightarrow[\delta]{} & X \times_S X & \xrightarrow[f']{} & X \end{array}$$

using the notations of 2.4.20. Thus, by definition: $\Sigma_f = f'_\sharp \delta_*$, $\Sigma_{\tilde{f}} = \tilde{f}'\tilde{\delta}_*$. Then we consider the following diagram of exchange transformations:

$$\begin{array}{ccccccc} g_\sharp \tilde{f}_\sharp & & \xrightarrow{\mathfrak{p}_{\tilde{f}}} & & & & g_\sharp \tilde{f}_* \tilde{f}'_\sharp \tilde{\delta}_* \\ \| & & & & & & \downarrow{\scriptstyle Ex(\Delta_{\sharp *})} \\ f_\sharp h_\sharp & \xrightarrow{\mathfrak{p}_f} & f_* f'_\sharp \delta_* h_\sharp & \xleftarrow{Ex(\Gamma_{\sharp *})} & f_* f'_\sharp k_\sharp \tilde{\delta}_* & = & f_* h_\sharp \tilde{f}'_\sharp \tilde{\delta}_* \end{array}$$

Note that it only involves exchange transformations of type $Ex(?_{\sharp *})$: it is commutative by compatibility of these exchange transformations with composition. By assumption, the transformations $\mathfrak{p}_f$ and $\mathfrak{p}_{\tilde{f}}$ are isomorphisms. Moreover the property (Loc_δ) is satisfied and it implies (Supp_δ) according to Corollary 2.3.13. Thus $Ex(\Gamma_{\sharp *})$ is an isomorphism and this concludes the proof of (2).

For condition (1), we note that (2) already implies (Supp_f). Thus we have only to prove (BC_f). We consider a square of shape Δ as in the statement of the lemma without assuming that g is smooth. We have to prove that

$$Ex(\Delta^*_*) : g^* f_* \to \tilde{f}_* h^*$$

is an isomorphism. We proceed as for condition (2). It is sufficient to prove that $Ex(\Delta^*_*)$ is an isomorphism after composition on the right with Σ_f. Then we consider the following commutative diagram of exchange transformations:

$$\begin{array}{ccccccc} g^* f_\sharp & & & \xrightarrow{\mathfrak{p}_f} & & & g^* f_* f'_\sharp \delta_* \\ {\scriptstyle Ex(\Delta^*_\sharp)}\downarrow & & & & & & \downarrow{\scriptstyle Ex(\Delta^*_*)} \\ \tilde{f}_\sharp h^* & \xrightarrow{\mathfrak{p}_{\tilde{f}}} & \tilde{f}_* \tilde{f}'_\sharp \tilde{\delta}_* h^* & \xleftarrow{Ex(\Gamma^*_*)} & \tilde{f}_* \tilde{f}'_\sharp k^* \delta_* & \xleftarrow{Ex(\Theta^*_\sharp)} & \tilde{f}_* h^* f'_\sharp \delta_* \end{array}$$

According to ($\mathscr{P}$-BC), $Ex(\Delta_\sharp^*)$ and $Ex(\Theta_\sharp^*)$ are isomorphisms. By assumption, $\mathfrak{p}_f$ and $\mathfrak{p}_{\tilde{f}}$ are isomorphisms. Moreover, property (Loc_δ) is satisfied and this implies $Ex(\Gamma_*^*)$ is an isomorphism according to Corollary 2.3.13. Condition (1) is proved.

It remains to prove (3). We consider again the notations of the cartesian diagram (2.4.20.1). For any premotives K over X and L over S, we consider the following commutative diagram of exchange transformations (see Remark 1.1.32):

$$\begin{array}{ccc}
f_\sharp\big(K\otimes f^*(L)\big) & \xrightarrow{\ \mathfrak{p}_f\ } & f_*f'_\sharp\delta_*\big(K\otimes\delta^*f'^*f^*(L)\big) \\
 & & \uparrow{\scriptstyle Ex(\delta_*^*,\otimes)} \\
 & & f_*f'_\sharp\big(\delta_*(K)\otimes f'^*f^*(L)\big) \\
\downarrow{\scriptstyle Ex(f_\sharp^*,\otimes)} & & \downarrow{\scriptstyle Ex(f_\sharp'^*,\otimes)} \\
 & & f_*\big(f'_\sharp\delta_*(K)\otimes f^*(L)\big) \\
 & & \uparrow{\scriptstyle Ex(f_*^*,\otimes)} \\
f_\sharp(K)\otimes L & \xrightarrow{\ \mathfrak{p}_f\ } & f_*f'_\sharp\delta_*(K)\otimes L.
\end{array}$$

By definition, the exchanges $Ex(f_\sharp^*,\otimes)$ and $Ex(f_\sharp'^*,\otimes)$ are isomorphisms. By assumption, the arrows labelled $\mathfrak{p}_f$ are isomorphisms. Moreover, the property (Loc_δ) is satisfied: Corollary 2.3.13 implies that $Ex(\delta_*^*,\otimes)$ is an isomorphism. We deduce from this that the arrow $Ex(f_*^*,\otimes)$ is an isomorphism. This concludes the proof of (3) as the functor $\Sigma_f = f'_\sharp\delta_*$ is an equivalence according to the hypothesis on f. $\square$

2.4.24. Assume that $\mathscr{T}$ satisfies the support property (Supp). Then we can extend Definition 2.4.21 to the case of a smooth separated morphism of finite type $f : X \to S$. We still consider the cartesian square (2.4.20.1) and the diagonal embedding $\delta : X \to X \times_S X$. Again, (f', δ) is a smooth pointed scheme so that we can put

$$\Sigma_f := \mathcal{T}h(f',\delta) = f'_\sharp\delta_*$$

and we define a canonical morphism:

$$\mathfrak{p}_f : f_\sharp = f_\sharp f''_!\delta_! \xrightarrow{Ex(\Delta_{\sharp\,!})} f_!f'_\sharp\delta_! = f_!\circ\Sigma_f. \tag{2.4.24.1}$$

using the exchange transformation of point (2) in Corollary 2.2.12.

Definition 2.4.25. Using the notations above, we say that f is *$\mathscr{T}$-pure*, or simply *pure* when $\mathscr{T}$ is clear, when the following conditions are satisfied:

(1) The natural transformation Σ_f is an equivalence.
(2) The morphism $\mathfrak{p}_f : f_\sharp \to f_! \circ \Sigma_f$ is an isomorphism.

We can easily deduce from the construction of the exchange transformation $Ex(\Delta_{\sharp\,!})$ that, when $\mathscr{T}$ satisfies properties (Stab) and (Pur), any smooth separated morphism of finite type f is pure. The following theorem is a consequence of the formalism developed previously.

Theorem 2.4.26. *Assume that $\mathscr{T}$ satisfies the localization and weak purity properties. Then the following conditions hold:*

(1) $\mathscr{T}$ satisfies the stability property.
(2) $\mathscr{T}$ satisfies the support and base change properties.
If moreover $\mathscr{T}$ is monoidal, it satisfies the projection formula.
(3) Any smooth separated morphism of finite type is pure.
(4) For any projective morphism f, the property (Adj_f) holds.
If moreover $\mathscr{T}$ is well generated, then the adjoint property holds in general.

Proof. We start by proving condition (1). As (Loc) implies (Zar-sep), we can apply Proposition 2.4.11 and we have only to prove that for any scheme S, $\mathcal{T}h(\mathbf{A}_S^1)$ is an equivalence. Let $s : S \to \mathbf{A}_S^1$ be the zero section and $j : \mathbf{A}_S^1 \to \mathbf{P}_S^1$ be the canonical open immersion. Put $t = j \circ s$.

According to Lemma 2.4.9, j induces an isomorphism $\mathcal{T}h(\mathbf{A}^1_S) \simeq \mathcal{T}h(p_1, s)$. Consider now the following cartesian squares:

$$\begin{array}{ccccc}
S & \xrightarrow{s} & \mathbf{P}^1_S & \xrightarrow{p_1} & S \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} & \Delta & \downarrow{\scriptstyle s} \\
\mathbf{P}^1_S & \xrightarrow[\delta]{} & \mathbf{P}^1_S \times_S \mathbf{P}^1_S & \xrightarrow[p'_1]{} & \mathbf{P}^1_S
\end{array}$$

where p'_1 (resp. δ) is the projection on the first factor (resp. diagonal embedding). The property (Loc$_s$) implies that $s^*s_* = 1$ and that the exchange transformation $Ex(\Delta_{\sharp *})$ is an isomorphism according to Corollary 2.3.13. Thus we get an isomorphism of functors:

$$\mathcal{T}h(p_1, s) = p_{1\sharp}s_* = s^*s_*p_{1\sharp}s_* \xrightarrow{Ex(\Delta_{\sharp *})^{-1}} s^*p'_{1\sharp}s'_*s_* = s^*p'_{1\sharp}\delta_*s_* = s^*\Sigma_{p_1}s_*$$

and this proves (1) because p_1 is pure.

Condition (2) follows simply from Corollary 2.3.13. In fact, for any scheme S, the weak purity assumption on $\mathscr{T}$ implies that $p_n : \mathbf{P}^n_S \to S$ is universally pure. Thus, Lemma 2.4.23 implies properties (Supp$_{p_n}$) and (BC$_{p_n}$) so that we can apply Corollary 2.3.13 to get (Supp) and (BC). The same argument applies to the property (PF) in the monoidal case.

For condition (3), we consider a smooth separated morphism of finite type $g : Y \to S$ and we prove it is pure. According to (1), Σ_g is an equivalence. Thus, by definition of $\mathfrak{p}_g$, it is sufficient to prove that for any cartesian square:

$$\begin{array}{ccc}
Z & \xrightarrow{\tilde{f}} & Y \\
{\scriptstyle h}\downarrow & \Delta & \downarrow{\scriptstyle g} \\
X & \xrightarrow[f]{} & S
\end{array}$$

with f separated of finite type, the exchange transformation

$$Ex(\Delta_{\sharp\,!}) : g_\sharp \tilde{f}_! \to f_! h_\sharp$$

is an isomorphism.
To do this, we apply Proposition 2.3.11, as in the case of Corollary 2.3.13. We consider the obvious complete $\mathscr{S}m$-fibred triangulated categories $\mathscr{T}'$ and $\mathscr{T}''$ over $\mathscr{S}/S$ which to an S-scheme Y associates:

- $\mathscr{T}'(Y) = \mathscr{T}(Y \times_S X)$.
- $\mathscr{T}''(Y) = \mathscr{T}(Y)$.

We consider the morphism $\varphi^* : \mathscr{T}' \to \mathscr{T}''$ such that for any S-scheme Y, $\varphi^*_Y = (Y \times_S p)_\sharp$. As for any scheme S, $p_n : \mathbf{P}^n_S \to S$ is universally pure, Lemma 2.4.23 shows that φ^* satisfies condition (i) of Proposition 2.3.11. According to that Proposition, (i) is equivalent to condition (iii), and (iii) is precisely what we want.

It remains only to prove condition (4). According to property (Pur), any smooth proper morphism f satisfies (Adj$_f$). According to (Loc) and Proposition 2.3.3 any closed immersion i satisfies (Adj$_i$). It follows easily that any projective morphism f satisfies (Adj$_f$). When $\mathscr{T}$ is well generated, we simply apply point (4) of Corollary 2.3.13. □

Remark 2.4.27. In particular, in the assumption of the previous theorem, if $\mathscr{T}$ satisfies properties (Loc), (wPur) and (Adj)[43], we can apply Theorem 2.2.14 to $\mathscr{T}$ so that we get a complete formalism of operations $(f^*, f_*, f_!, f^!)$ satisfying all the desired formulas.

Thus the preceding theorem gives another look at the main result of [**Ayo07a**, 1.4.2]. In fact, the proof given here is simpler as the assumptions of our theorem are stronger. However, we do not use the homotopy property in our theorem.

[43]Note that under the assumptions of the previous theorem, we know that for any proper smooth morphism f, f_* admits a right adjoint. The same is true for a proper morphism which can be factorized as a closed immersion followed by a smooth proper morphism according to (Loc).

We end up this section with the theorem of Ayoub [**Ayo07a**, 1.4.2], which can be stated in a simpler form according to the preceding theorem:

THEOREM 2.4.28 (Ayoub). *Assume $\mathscr{T}$ satisfies the localization, homotopy and stability properties.*

Then $\mathscr{T}$ is weakly pure.

In fact, this theorem is proved explicitly in *op. cit.*, Theorem 1.7.9.

REMARK 2.4.29. Recall that Ayoub proves more than just this theorem: indeed he constructs the whole formalism of the 6 functors for quasi-projective morphisms for his *monoidal homotopy stable functors* – see again [**Ayo07a**]. The work we have done here is to isolate the crucial properties of purity and weak purity. Also, using the construction of Deligne, we have showed how to avoid the assumption of quasi-projectiveness made by Ayoub. Finally, the interest of Theorem 2.4.26 is to give a possible approach to the *6 functors formalism* without requiring the homotopy property ; this is a question which has been indirectly adressed by many mathematicians (Bloch, Esnault, Barbieri-Viale, ...)

2.4.c. *Duality, purity and orientation.*

2.4.30. This section is concerned with the relation between purity and duality. We will assume that $\mathscr{T}$ is premotivic.

Recall that an object M of a monoidal category $\mathscr{M}$ is called *strongly dualizable* if there exists an object M' such that $(M' \otimes -)$ is both right and left adjoint to $(M \otimes -)$. Then, M' is called the *strong dual* of M.

In case $\mathscr{M}$ is closed monoidal, we will say that a morphism of the form

$$\mu : M \otimes M' \to \mathbb{1}$$

is a *perfect pairing* if the natural transformation

$$(M \otimes -) \to Hom(M', -)$$

obtained from μ by adjunction is an isomorphism. Then M is strongly dualizable with dual M'.

PROPOSITION 2.4.31. *Let $f : X \to S$ be a smooth proper morphism.*

If f is pure then the premotive $M_S(X)$ is strongly dualizable in $\mathscr{T}(S)$ with dual:

$$f_*(\mathbb{1}_X) \simeq f_\sharp\big(\Omega_f(\mathbb{1}_X)\big)$$

where Ω_f denotes the inverse of Σ_f.

PROOF. By assumption, Σ_f is an automorphism of the category $\mathscr{T}(X)$. Moreover, the identification (2.4.13.1) can be rewritten as $\Sigma_f(M) = \Sigma_f(\mathbb{1}_X) \otimes_X M$ for any premotive M over X. The fact Σ_f is an equivalence means that $\Sigma_f(\mathbb{1}_X)$ is a $\otimes$-invertible object, whose inverse is $T := \Omega_f(\mathbb{1}_S)$. In particular, we get: $\Omega_f(M) = T \otimes M$.

According to the Sm-projection formula, the functor $M_S(X) \otimes .$ is isomorphic to $f_\sharp f^*$. Thus, its right adjoint is $f_* f^*$. As f is pure by assumption, this last functor is isomorphic to $f_\sharp \Omega_f f^*$. Using the observation at the beginning of the proof and the Sm-projection formula again, we obtain:

$$f_\sharp \Omega_f f^*(N) = f_\sharp(T \otimes f^*(N)) = f_\sharp(T) \otimes N.$$

Moreover, the right adjoint of $f_\sharp \Omega_f f^*$ is $f_* \Sigma_f f^*$. Using again the purity isomorphism for f, this last functor can be identified with $f_\sharp f^*$ and this concludes. □

2.4.32. Assume again that the premotivic triangulated category $\mathscr{T}$ satisfies properties (wLoc) and (Nis-sep).

Let S be a scheme. A *smooth closed S-pair* will be pair (X, Z) of smooth S-schemes such that Z is closed subscheme of X. We consider the canonical projection $p : X \to S$ and the immersion $i : Z \to X$ associated with (X, Z). Note that according to Proposition 2.4.8, $\mathscr{T}$ satisfies property (Loc$_i$). Then we define the premotive of (X, Z) as follows:

$$M_S(X/X - Z) := p_\sharp i_*(\mathbb{1}_Z). \tag{2.4.32.1}$$

According to property (Loc$_i$), we thus get a canonical distinguished triangle:

$$M_S(X-Z) \xrightarrow{j_*} M_S(X) \to M_S(X/X-Z) \xrightarrow{+1} \tag{2.4.32.2}$$

Note that given any smooth morphism $p : S \to S_0$, we get obviously:

$$p_\sharp M_S(X/X-Z) = M_{S_0}(X/X-Z). \tag{2.4.32.3}$$

Moreover, given any morphism $f : T \to S$, we get an exchange isomorphism:

$$f^* M_S(X/X-Z) \xrightarrow{\sim} M_T(X_T/X_T - Z_T). \tag{2.4.32.4}$$

A morphism of smooth closed S-pairs $(Y,T) \to (X,Z)$ will be a couple (f,g) which fits into a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X, \end{array}$$

with i, k the canonical immersions, and such that $T = f^{-1}(Z)$ as a set. We can associate with (f,g) a morphism of premotives:

$$M_S(Y/Y-T) = q_\sharp k_* g^*(\mathbb{1}_Z) \xrightarrow{Ex(\Delta_*^*)^{-1}} q_\sharp f^* i_*(\mathbb{1}_Z) \xrightarrow{Ex_\sharp^*} 1^* p_\sharp i_*(\mathbb{1}_Z) = M_S(X/X-Z).$$

Indeed, the exchange map $Ex(\Delta_*^*)$ is an isomorphism according to (Loc$_i$) and Corollary 2.3.13.

It is easy to check that the triangle (2.4.32.2) is functorial with respect to morphisms of closed S-pairs. Before proving the next theorem, we state the following lemma.

Lemma 2.4.33. *Consider the assumptions and notations above.*

Let $(f,g) : (Y,T) \to (X,Z)$ be a morphism of smooth closed S-pairs such that f is étale and g is an isomorphism. Then the induced map $M_S(Y/Y-T) \to M_S(X/X-Z)$ is an isomorphism.

Proof. According to the identification 2.4.32.3, it is sufficient to treat the case where $X = Z$. Let $U = X - Z$ and $j : U \to X$ be the obvious immersion. Then (f,j) is a Nisnevich cover of X. According to (Nis-sep), it is sufficient to prove that the pullback of $M_X(Y/Y-T) \to M_X(X/X-Z)$ along f and j is an isomorphism. This is obvious using 2.4.32.4. □

2.4.34. We consider again the assumption of the paragraph preceding the above lemma.

Fix a smooth closed S-pair (X,Z). Let $B_Z X$ (resp. $B_Z(\mathbf{A}^1_X)$) be the blow-up of X (resp. $\mathbf{A}^1_X$) with center in Z (resp. $\{0\} \times Z$). We define the deformation space associated with (X,Z) as the S-scheme $D_Z X = B_Z(\mathbf{A}^1_X) - B_Z X$. Note also $D_Z Z = \mathbf{A}^1_Z$ is a closed subscheme of $D_Z X$; the couple $(D_Z X, \mathbf{A}^1_Z)$ is a smooth closed S-pair.
Let $N_Z X$ be the normal bundle of Z in X. The scheme $D_Z X$ is fibred over $\mathbf{A}^1$. Moreover, the 0-fiber of $(D_Z X, \mathbf{A}^1)$ is the closed pair $(N_Z X, Z)$ corresponding to the zero section and the 1-fiber is the closed pair (X,Z). In particular, we get the following morphisms of closed pairs:

$$(X,Z) \xrightarrow{d_1} (D_Z X, \mathbf{A}^1_Z) \xleftarrow{d_0} (N_Z X, Z) \tag{2.4.34.1}$$

We are now ready to state the purity theorem for smooth closed pairs in our abstract formalism. Though our assumptions are more general, this theorem follows exactly from the method of Morel and Voevodsky used to prove this result in the homotopy category $\mathscr{H}$ (see [**MV99**, §3, 2.24]):

Theorem 2.4.35. *Consider the above assumptions and notations and suppose that $\mathscr{T}$ satisfies the homotopy property. Then the morphisms*

$$M_S(X/X-Z) \xrightarrow{d_{1*}} M_S(D_Z X/D_Z X - \mathbf{A}^1_Z) \xleftarrow{d_{0*}} M_S(N_Z X/N_Z^\times X) =: MTh_S(N_Z X).$$

are isomorphisms.

Proof. By noetherian induction and the preceding lemma, the statement is local in X for the Nisnevich topology. Thus, because (X,Z) is a smooth closed S-pair, we can assume that there exists an étale map $\pi : X \to \mathbf{A}_S^{n+c}$ such that $\pi^{-1}(\mathbf{A}_S^c) = Z$ – *cf.* [**EGA4**, 17.12.2]. Consider the pullback square

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{A}^n \times Z & \xrightarrow{1\times\pi|Z} & \mathbf{A}^n \times \mathbf{A}_S^c. \end{array}$$

There is an obvious closed immersion $Z \to X'$ and its image is contained in $q^{-1}(Z)$. As q is étale, Z is a direct factor of $q^{-1}(Z)$. Put $W = q^{-1}(Z) - Z$ and $\Omega = X' - W$. Thus Ω is an open subscheme of X', and the reader can check that p and q induces morphisms of smooth closed S-pairs

$$(X, Z) \leftarrow (\Omega, Z) \to (\mathbf{A}_Z^n, Z).$$

Applying again the preceding lemma, these morphisms induces isomorphisms on the associated premotives. Thus we are reduced to the case of the closed S-pair $(\mathbf{A}_Z^n, Z)$. A direct computation shows that $D_Z(\mathbf{A}_Z^n) \simeq \mathbf{A}^1 \times \mathbf{A}_Z^n$. Under this isomorphism d_0 (resp. d_1) corresponds to the 0-section (resp. 1-section) of $\mathbf{A}^1 \times \mathbf{A}_Z^n$ corresponding to the first factor. Thus, we conclude using the homotopy property. □

2.4.36. The interest of the previous theorem is to simplify the purity isomorphism. Let us restate the assumptions on the triangulated premotivic category $\mathscr{T}$:

- $\mathscr{T}$ satisfies properties (Nis-sep), (wLoc) and (Htp).

Then applying the above theorem, we get for any smooth closed S-pair (X, Z) a canonical isomorphism

$$\mathfrak{p}_{X,Z} : M_S(X/X - Z) \to MTh_S(N_Z X) \tag{2.4.36.1}$$

Corollary 2.4.37. *Consider the assumptions and notations above.*

(1) *For any smooth pointed S-scheme (f, s) and any premotive K over S, we get a canonical isomorphism*

$$\mathcal{T}h(f, s).K \simeq M_S(X/X - s(S)) \otimes_S K \xrightarrow{\mathfrak{p}_{X,S}} MTh_S(N_s) \otimes_S K.$$

where the first isomorphism is given by the map (2.4.13.1) *and N_s is the normal bundle of s.*

(2) *For any smooth separated morphism of finite type $f : X \to S$ with tangent bundle*[44] *T_f, and any premotive K over X, we get a canonical isomorphism:*

$$\mathfrak{p}_{XX,X} : \Sigma_f(K) \xrightarrow{\sim} MTh_X(T_f) \otimes_X K$$

– here, (XX, X) stands for the closed pair corresponding to the diagonal embedding of X/S.

In the assumption of point (2), we thus get a canonical map:

$$f_\sharp(K) \xrightarrow{\mathfrak{p}_f} f_!(\Sigma_f K) \xrightarrow{\sim} f_!\big(MTh_X(T_f) \otimes_X K\big) \tag{2.4.37.1}$$

that we will still denote by $\mathfrak{p}_f$ and call the *purity isomorphism* associated with f.

Definition 2.4.38. Assume the triangulated premotivic category $\mathscr{T}$ satisfies (wLoc). As usual, $M(1)$ denotes the Tate twist of a premotive M.

An *orientation* $\mathfrak{t}$ of $\mathscr{T}$ will be the data for each smooth scheme X and each vector bundle E/X of rank n of an isomorphism

$$\mathfrak{t}_E : MTh_X(E) \to \mathbb{1}_X(n)[2n],$$

called the *Thom isomorphism*, satisfying the following coherence properties:

[44] We define T_f as the normal bundle of the diagonal immersion $\delta : X \to X \times_S X$.

(a) Given a scheme X and an isomorphism of vector bundles $\varphi : E \to F$ of ranks n over X, the following diagram is commutative:

$$\begin{array}{ccc} MTh_X(E) & \xrightarrow{\varphi_*} & MTh_X(F). \\ & \searrow^{\mathfrak{t}_E} \quad \swarrow^{\mathfrak{t}_F} & \\ & \mathbb{1}_X(n)[2n] & \end{array}$$

(b) For any morphism $f : Y \to X$ of schemes, and any vector bundle E/X of rank n with pullback F over Y, the following diagram commutes:

$$\begin{array}{ccc} f^*(MTh_X(E)) & \xrightarrow{f^*\mathfrak{t}_E} & f^*(\mathbb{1}_X(n)[2n]) \\ \sim\downarrow & & \downarrow\sim \\ MTh_Y(F) & \xrightarrow{\mathfrak{t}_F} & \mathbb{1}_Y(n)[2n] \end{array}$$

where the vertical maps are the canonical isomorphisms.

(c) For any scheme X and any exact sequence (σ) of vector bundles over X

$$0 \to E' \xrightarrow{\nu} E \xrightarrow{\pi} E'' \to 0,$$

if n (resp. m) denotes the rank of the vector bundle E' (resp. E''), the following diagram commutes:

$$\begin{array}{ccc} MTh_X(E) & \xrightarrow{Th_X(\sigma)} & MTh_X(E') \otimes MTh_X(E'') \\ \mathfrak{t}_E\downarrow & & \downarrow \mathfrak{t}_{E'}\otimes\mathfrak{t}_{E''} \\ \mathbb{1}_X(n+m)[2n+2m] & \longrightarrow & \mathbb{1}_X(n)[2n] \otimes \mathbb{1}_X(m)[2m] \end{array}$$

where the map $Th_X(\sigma)$ is the isomorphism (2.4.15.1) associated with (σ) and the bottom vertical one is the obvious identification.

We will also say that $\mathscr{T}$ is *oriented* when the choice of one particular orientation is not essential.

Note that the Thom isomorphism can be viewed as a cohomology class in

$$H^{2n,n}_{\mathscr{T}}(Th_X(E)) := \operatorname{Hom}_{\mathscr{T}(X)}\big(MTh_X(E), \mathbb{1}_S(n)[2n]\big)$$

which in classical homotopy theory is called the *Thom class*.

2.4.39. Suppose the triangulated premotivic category $\mathscr{T}$ satisfies the following properties:

- $\mathscr{T}$ satisfies properties (Nis-sep), (wLoc), (Htp).
- $\mathscr{T}$ admits an orientation $\mathfrak{t}$.

Consider a smooth closed S-pair (X, Z) of codimension n. Let p (resp. q) be the structural morphism of X/S (resp. Z/S) and $i : Z \to X$ the associated immersion. Then we associate with (X, Z) the following form of the purity isomorphism:

(2.4.39.1) $$\mathfrak{p}^{\mathfrak{t}}_{X,Z} : M_S(X/X - Z) \xrightarrow{\mathfrak{p}_{X,Z}} MTh_S(N_Z X) \xrightarrow{q_\sharp(\mathfrak{t}_{N_Z X})} M_S(Z)(n)[2n]$$

where $\mathfrak{p}_{X,Z}$ is the isomorphism (2.4.36.1). For future reference, note that we deduce from this the so-called Gysin morphism:

(2.4.39.2) $$i^* : M_S(X) \xrightarrow{\pi} M_S(X/X - Z) \xrightarrow{\mathfrak{p}^{\mathfrak{t}}_{X,Z}} M_S(Z)(n)[2n]$$

where π is the following map:

$$M_S(X) = p_\sharp(\mathbb{1}_X) \xrightarrow{ad(i^*,i_*)} p_\sharp i_* i^*(\mathbb{1}_X) = M_S(X/X - Z).$$

As a particular case, we get using the notation of Corollary 2.4.37, point (2), an isomorphism:

$$\mathfrak{p}^{\mathfrak{t}}_{XX,X} : \Sigma_f(K) \xrightarrow{\mathfrak{p}_{XX,X}} MTh_X(T_f) \otimes K \xrightarrow{\mathfrak{t}_{T_f}} K(d)[2d]$$

In particular, when $\mathscr{T}$ satisfies property (Supp), the purity comparison map associated with f can be rewritten as:

$$\mathfrak{p}_f^{\mathfrak{t}} : f_\sharp \xrightarrow{\mathfrak{p}_f} f_! \circ \Sigma_f \xrightarrow{\mathfrak{p}^{\mathfrak{t}}_{X X,X}} f_!(d)[2d] \tag{2.4.39.3}$$

EXAMPLE 2.4.40. Assume as in the above definition that $\mathscr{T}$ is premotivic and satisfies properties (wLoc) and (Nis-sep).

We suppose the following two additional conditions are fulfilled:

(a') There exists a morphism of triangulated premotivic categories:

$$\varphi^* : \mathrm{SH} \leftrightarrows \mathscr{T} : \varphi_*$$

where SH is the stable homotopy category of Morel and Voevodsky – see Example 1.4.3.

(b') For any scheme X, let $\mathrm{Pic}(X)$ be the Picard group of X. We assume there exists an application

$$c_1 : \mathrm{Pic}(X) \to H^{2,1}_{\mathscr{T}}(X) := \mathrm{Hom}_{\mathscr{T}(X)}(M(X), \mathbb{1}_X(1)[2])$$

which is natural with respect to contravariant functoriality – we do not require c_1 is a morphism of abelian groups.

Then one can apply the results of [**Dég08**] to $\mathscr{T}(X)$ for any scheme X. All the references which follows will be within *loc. cit.*: according to section 2.3.2, the triangulated category $\mathscr{T}(X)$ satisfies the axioms of Paragraph 2.1.[45] Then the existence of the Thom isomorphism follows from Proposition 4.3 and, more explicitly, from Paragraph 4.4. Property (a) and (b) of the above definition are easy – explicitly, this is a consequence of 4.10 – and Property (c) follows from Lemma 4.30.

To sum up, the assumptions (a') and (b') guarantees the existence of a canonical orientation of $\mathscr{T}$ in the sense of the above definition. Moreover, the purity isomorphism (2.4.39.1) as well as the Gysin morphism (2.4.39.2) associated in the preceding paragraph for this particular orientation coincide with the one defined in [**Dég08**] (see in particular the uniqueness statement of [**Dég08**, Prop. 4.3]).

Note moreover that assuming $\mathscr{T}$ satisfies all the properties above except (b'), the data of an orientation of $\mathscr{T}$ is equivalent to the data of a map c_1 as in (b'). Indeed, if $\mathfrak{t}$ is an orientation of $\mathscr{T}$, given any line bundle L/X with zero section s, we put $c_1(L) = \rho(\mathfrak{t}_L)$ where ρ is the following composite map:

$$H^{2,1}_{\mathscr{T}}(Th_X(L)) \to H^{2,1}_{\mathscr{T}}(L) \xrightarrow{s^*} H^{2,1}_{\mathscr{T}}(X)$$

where the first map is induced by the canonical projection $M_X(L) \to MTh_X(L)$. Then c_1 depends only on the isomorphism classes of L/X – property (a) of the above definition – and it is compatible with pullbacks – property (c) of the above definition.

2.4.41. We now assume the following conditions on the triangulated premotivic category $\mathscr{T}$:

- $\mathscr{T}$ satisfies properties (Nis-sep), (wLoc), (Htp) and (Stab).
- $\mathscr{T}$ admits an orientation $\mathfrak{t}$.

Let $f : X \to S$ be a smooth proper morphism of dimension d. Note we do not need that $\mathscr{T}$ satisfies property (Supp) to rewrite the purity comparison map as follows:

$$\mathfrak{p}_f^{\mathfrak{t}} : f_\sharp \to f_*(d)[2d] \tag{2.4.41.1}$$

(see Paragraph 2.4.39).

Note also that using the Gysin morphism (2.4.39.2) associated with the diagonal immersion $\delta : X \to X \times_S X$, we get the following morphism:

$$\mu_f^{\mathfrak{t}} : M_S(X) \otimes M_S(X)(-d)[-2d] = M_S(X \times_S X)(-d)[-2d] \xrightarrow{\delta^*} M_S(X) \xrightarrow{f_*} \mathbb{1}_S. \tag{2.4.41.2}$$

[45]Note in particular that for any smooth closed S-pair, we obtain a canonical isomorphism in $\mathscr{T}(S)$ of the form:

$$\varphi^*(\Sigma^\infty X/X - Z) \simeq M_S(X/X - Z)$$

where one the left hand side $X/X - Z$ stands for the homotopy cofiber of the open immersion $(X - Z) \to X$ while the left hand side is defined by Equality (2.4.32.1).

THEOREM 2.4.42. *Consider the assumptions and notations above. Then the following conditions are equivalent:*

(i) f *is pure:* $\mathfrak{p}_f$ *is an isomorphism.*
(i') *The natural transformation* $\mathfrak{p}_f.f^*$ *is an isomorphism.*
(ii) *The premotive* $M_S(X)$ *is strongly dualizable and* $\mu^{\mathfrak{t}}_f$ *is a perfect pairing.*

PROOF. In this proof, we put $\tau(K) = K(d)[2d]$. As $\mathscr{T}$ satisfies property (Stab), f_* commutes with Tate twist (def. 1.1.44). This means we the following exchange transformation is an isomorphism:

$$Ex_\tau : \tau f_* \to f_*\tau. \tag{2.4.42.1}$$

We first prove that (i) is equivalent to (i'). One implication is obvious so that we have only to prove that (i') implies (i). Guided by a method of Ayoub (see [**Ayo07a**, 1.7.14, 1.7.15], we will construct a right inverse ϕ_1 and a left inverse ϕ_2 to the morphism $\mathfrak{p}^{\mathfrak{t}}_f$ as the following composite maps:

$$\phi_1 : f_*\tau \xrightarrow{ad(f^*,f_*)} f_*f^*f_*\tau \xrightarrow{Ex_\tau^{-1}} f_*f^*\tau f_* = f_*\tau f^* f_* \xrightarrow{(\mathfrak{p}^{\mathfrak{t}}_f.f^*f_*)^{-1}} f_\sharp f^* f_* \xrightarrow{ad'(f^*,f_*)} f_\sharp$$
$$\phi_2 : f_*\tau \xrightarrow{\beta_f} f_*\tau f^* f_\sharp \xrightarrow{(\mathfrak{p}^{\mathfrak{t}}_f.f^*f_\sharp)^{-1}} f_\sharp f^* f_\sharp \xrightarrow{ad'(f_\sharp,f^*)} f_\sharp.$$

Let us check that $\mathfrak{p}^{\mathfrak{t}}_f \circ \phi_1 = 1$. To prove this relation, we prove that the following diagram is commutative:

$$\begin{array}{ccccccccccc}
f_*\tau & \xrightarrow{ad(f^*,f_*)} & f_*f^*f_*\tau & \xrightarrow{Ex_\tau^{-1}} & f_*\tau f^*f_* & \xrightarrow{(\mathfrak{p}^{\mathfrak{t}}_f f^*f_*)^{-1}} & f_\sharp f^*f_* & \xrightarrow{ad'(f^*,f_*)} & f_\sharp & \xrightarrow{\mathfrak{p}^{\mathfrak{t}}_f} & f_*\tau \\
\| & & \| & & \| & & \| & & (1) & & \| \\
 & & & & f_*\tau f^*f_* & \xrightarrow{(\mathfrak{p}^{\mathfrak{t}}_f f^*f_*)^{-1}} & f_\sharp f^*f_* & \xrightarrow{\mathfrak{p}^{\mathfrak{t}}_f f^*f_*} & f_*\tau f f_* & \xrightarrow{ad'(f^*,f_*)} & f_*\tau \\
 & & & & \| & & & (2) & & & \| \\
 & & f_*f^*f_*\tau & \xrightarrow{Ex_\tau^{-1}} & f_*\tau f^*f_* & & & \xrightarrow{ad'(f^*,f_*)} & & & f_*\tau \\
 & & \| & & & & (3) & & & & \| \\
f_*\tau & \xrightarrow{ad(f^*,f_*)} & f_*f^*f_*\tau & & & & \xrightarrow{ad'(f^*,f_*)} & & & & f_*\tau.
\end{array}$$

The commutativity of (1) and (2) is obvious and the commutativity of (3) follows from Formula (2.4.42.1) defining Ex_τ. Then the result follows from the usual formula between the unit and counit of an adjunction. The relation $\phi_2 \circ \mathfrak{p}^{\mathfrak{t}}_f = 1$ is proved using the same kind of computations.

It remains to prove that (i) and (i') are equivalent to (ii). We already know from Proposition 2.4.31 that (i) implies the premotive $M_S(X)$ is strongly dualizable. Saying that $\mu^{\mathfrak{t}}_f$ is a perfect pairing amounts to prove that the natural transformation obtained by adjunction

$$d^{\mathfrak{t}}_f : (M_S(X) \otimes -) \to \mathit{Hom}(M_S(X), -(d)[2d])$$

is an isomorphism. On the other hand, as we have already seen previously, the smooth projection formula implies an identification of functors:

$$\begin{aligned} f_\sharp f^* &\simeq (M_S(X) \otimes -), \\ f_* f^* &\simeq \mathit{Hom}(M_S(X), -). \end{aligned} \tag{2.4.42.2}$$

Thus, to finish the proof, it will be enough to show that the map

$$f_\sharp f^* \xrightarrow{\mathfrak{p}^{\mathfrak{t}}_f f^*} f_*\tau f^* = f_* f^* \tau.$$

is equal to $d^{\mathfrak{t}}_f$ through the identifications (2.4.42.2).

Let us consider the following cartesian square

$$\begin{array}{ccc} X\times_S X & \xrightarrow{f''} & X \\ {\scriptstyle f'}\big\downarrow & \Delta & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & S \end{array}$$

and put $g = f\circ f''$. According to the definition of $\mu^{\mathfrak{t}}_f$, and notably Formula (2.4.39.2) for the Gysin map δ^*, the natural transformation of functors $(\mu^{\mathfrak{t}}_f \otimes -)$ can be described as the following compositum:

$$f_\sharp f^* f_\sharp f^* \xrightarrow{Ex(\Delta^*_\sharp)} f_\sharp f'_\sharp f''^* f^* = g_\sharp g^* \xrightarrow{ad(\delta^*,\delta_*)} g_\sharp \delta_*\delta^* g^*$$
$$= f_\sharp f'_\sharp \delta_* f^* \xrightarrow{\mathfrak{p}^{\mathfrak{t}}_{X X,X}} f_\sharp \tau f^* = f_\sharp f^*\tau \xrightarrow{ad'(f_\sharp,f^*)} \tau.$$

Note in particular that the base change map $Ex(\Delta^*_\sharp)$ corresponds to the first identification in Formula (2.4.41.2). Thus we have to prove the preceding composite map is equal to the following one, obtained by adjunction from $\mathfrak{p}^{\mathfrak{t}}_f$:

$$f_\sharp f^* f_\sharp f^* = f_\sharp f^* f_\sharp f''_* \delta_* f^* \xrightarrow{Ex(\Delta_{\sharp *})} f_\sharp f^* f_* f'_\sharp \delta_* f^*$$
$$\xrightarrow{\mathfrak{p}^{\mathfrak{t}}_{X X,X}} f_\sharp f^* f_* \tau f^* = f_\sharp f^* f_* f^* \tau \xrightarrow{ad'(f^*,f_*)} f_\sharp f^*\tau \xrightarrow{ad'(f_\sharp,f^*)} \tau$$

This amounts to prove, after some easy cancellation, the commutativity of the following diagram:

$$\begin{array}{ccccc} f^* f_\sharp & = & f^* f_\sharp f''_*\delta_* & \xrightarrow{Ex(\Delta_{\sharp *})} & f^* f_* f''_\sharp \delta_* \\ {\scriptstyle Ex(\Delta^*_\sharp)}\big\downarrow & & & & \big\downarrow{\scriptstyle ad'(f^*,f_*)} \\ f'_\sharp f''^* & \xrightarrow{ad(\delta^*,\delta_*)} & f'_\sharp \delta_*\delta^* f''^* & = & f'_\sharp\delta_*. \end{array}$$

According to the definition of the exchange transformation $Ex(\Delta_{\sharp *})$ (cf Paragraph 1.1.14), we can divide this diagram into the following pieces:

$$\begin{array}{ccccccccc} f^* f_\sharp & = & f^* f_\sharp f''_*\delta_* & \xrightarrow{ad(f^*,f_*)} & f^* f_* f^* f_\sharp f''_*\delta_* & \xrightarrow{Ex(\Delta^*_\sharp)} & f^* f_* f'_\sharp f''^* f''_*\delta_* & \xrightarrow{ad'(f''^*,f''_*)} & f^* f_* f''_\sharp\delta_* \\ {\scriptstyle Ex(\Delta^*_\sharp)}\big\downarrow & & {\scriptstyle Ex(\Delta^*_\sharp)}\big\downarrow & & \nearrow{\scriptstyle ad(f^*,f_*)} & & \big\downarrow{\scriptstyle ad'(f^*,f_*)} & & \big\downarrow{\scriptstyle ad'(f^*,f_*)} \\ f'_\sharp f''^* & = & f'_\sharp f''^* f''_*\delta_* & & = & & f'_\sharp f''^* f''_*\delta_* & \xrightarrow{ad'(f''^*,f''_*)} & f'_\sharp\delta_* \\ \| & & & & (*) & & & & \| \\ f'_\sharp f''^* & & & & \xrightarrow{ad(\delta^*,\delta_*)} & & & & f'_\sharp\delta_*. \end{array}$$

Every part of this diagram is obviously commutative except for part $(*)$. As $f''\delta = 1$, the axioms of a 2-functors (for f^* and f_* say) implies that the unit map

$$\alpha : f'_\sharp f''^* \to f'_\sharp f''^* (f''\delta)_* (f''\delta)^*$$

is the canonical identification that we get using $1_* = 1$ and $1^* = 1$. We can consider the following diagram:

$$\begin{array}{ccccccc}
f'_\sharp f''^* & \multicolumn{3}{c}{\overset{\alpha}{=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=}} & f'_\sharp f''^*(f''\delta)_*(f''\delta)^* & =\!=\!=\!= & f'_\sharp f''^* f''_* \delta_* \\
\| & & & & \| & & \\
f'_\sharp f''^* & \xrightarrow{ad(f''^*, f''_*)} & f'_\sharp f''^* f''_\sharp f''^* & \xrightarrow{ad(\delta^*, \delta_*)} & f'_\sharp f''^*(f''\delta)_*(f''\delta)^* & & \downarrow{\scriptstyle ad'(f''^*, f''_*)} \\
\| & & {\scriptstyle ad'(f''^*, f''_*)}\downarrow & & \downarrow{\scriptstyle ad'(f''^*, f''_*)} & & \\
f'_\sharp f''^* & =\!=\!=\!= & f'_\sharp f''^* & \xrightarrow{ad(\delta^*, \delta_*)} & f'_\sharp \delta_* \delta^* f''^* & =\!=\!=\!= & f'_\sharp \delta_*
\end{array}$$

for which each part is obviously commutative. This concludes. □

As a corollary, together with the results of [**Dég08**], we get the following theorem:

Corollary 2.4.43. *Let us assume the following conditions on the triangulated premotivic category $\mathscr{T}$:*

*(a) $\mathscr{T}$ satisfies properties (*Nis*-sep), (wLoc), (Htp) and (Stab).*
(b) $\mathscr{T}$ admits an orientation $\mathfrak{t}$.
(c) There exists a morphism of triangulated premotivic categories:

$$\varphi^* : \mathrm{SH} \rightleftarrows \mathscr{T} : \varphi_* \ .$$

Then any smooth projective morphism is $\mathscr{T}$-pure. In particular, $\mathscr{T}$ is weakly pure.

Proof. According to Example 2.4.40, one can apply the results of [**Dég08**] to the triangulated category $\mathscr{T}(X)$. Then it follows from [**Dég08**, 5.23] that condition (ii) of the above theorem is satisfied. □

Remark 2.4.44. This theorem is to be compared with the result of Ayoub recalled in Theorem 2.4.28. On the one hand, if $\mathscr{T}$ satisfies the localization property, we get another proof of this result under the additional assumption that $\mathscr{T}$ is oriented. On the other hand, the above theorem does not require the assumption that $\mathscr{T}$ satisfies (Loc); this is important as we can only prove (wLoc) for the category DM_Λ introduced in Definition 11.1.1.

2.4.d. *Motivic categories.* This section summarizes the main constructions of this part and draws a conclusive theorem.

Definition 2.4.45. A *motivic triangulated category over $\mathscr{S}$* is a premotivic triangulated category over $\mathscr{S}$ which satisfies the homotopy, stability, localization and adjoint property.

Remark 2.4.46. Without the adjoint property, this definition corresponds to what Ayoub called a *monoidal stable homotopy* 2*-functor* (cf [**Ayo07a**, def. 2.3.1]). We think our shorter terminology fits well in the spirit of the current theory of mixed motives.

Remark 2.4.47. Assume $\mathscr{T}$ is a premotivic triangulated category such that:

(1) $\mathscr{T}$ is well generated.
(2) $\mathscr{T}$ satisfies the homotopy and stability properties.
(3) $\mathscr{T}$ satisfies the localization property.

Then $\mathscr{T}$ is a motivic triangulated category in the above sense. Indeed, property (Adj) is proved under the above assumptions in point (4) of Theorem 2.4.26. Note also that if $\mathscr{T}$ is compactly τ-generated, we simply obtain property (Adj) from Lemma 2.2.16.[46]

[46]In our examples, (1) will always be satisfied, (2) will be obtained by construction and (3) will be the hard point.

EXAMPLE 2.4.48. According to the previous remark, the premotivic category SH of example 1.4.3 is a motivic category. In fact, property (1) is proved in [**Ayo07a**, 4.5.67], property (2) follows by definition and property (3) is proved in [**Ayo07a**, 4.5.44].

2.4.49. In the next theorem, we summarize what is now called the *Grothendieck 6 functors formalism*. In fact, this is a consequence of the axioms in the above definition, as a result of the work done in previous sections. More precisely:

- We apply Theorem 2.4.26 using the theorem of Ayoub recalled in 2.4.28, and use the generalized theorem of Morel and Voevodsky, Theorem 2.4.35, to get the form (2.4.37.1) of the purity isomorphism.
- In the case where $\mathscr{T}$ is oriented, we use the form (2.4.41.1) of the purity isomorphism. Recall that, when $\mathscr{T}$ satisfies assumption (c) of Corollary 2.4.43, then we have given a different proof of the Theorem of Ayoub and the theorem below follows from 2.4.26 and 2.4.43.

THEOREM 2.4.50. *Let $\mathscr{T}$ be a motivic triangulated category.*

Then, for any separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, there exists a pair of adjoint functors, the exceptional functors*,*

$$f_! : \mathscr{T}(Y) \rightleftarrows \mathscr{T}(X) : f^!$$

such that:

1. *There exists a structure of a covariant (resp. contravariant) 2-functor on $f \mapsto f_!$ (resp. $f \mapsto f^!$).*
2. *There exists a natural transformation $\alpha_f : f_! \to f_*$ which is an isomorphism when f is proper. Moreover, α is a morphism of 2-functors.*
3. *For any smooth separated morphism of finite type $f : X \to S$ in $\mathscr{S}$ with tangent bundle T_f, there are canonical natural isomorphisms*

$$\mathfrak{p}_f : f_\sharp \longrightarrow f_!\big(MTh_X(T_f) \otimes_X .\big)$$
$$\mathfrak{p}'_f : f^* \longrightarrow MTh_X(-T_f) \otimes_X f^!$$

which are dual to each other – the Thom premotive $MTh_X(T_f)$ is $\otimes$-invertible with inverse $MTh_X(-T_f)$.

If $\mathscr{T}$ admits an orientation $\mathfrak{t}$ and f has dimension d then there are canonical natural isomorphisms

$$\mathfrak{p}^{\mathfrak{t}}_f : f_\sharp \longrightarrow f_!(d)[2d]$$
$$\mathfrak{p}'^{\mathfrak{t}}_f : f^* \longrightarrow f^!(-d)[-2d]$$

which are dual to each other.

4. *For any cartesian square:*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

such that f is separated of finite type, there exist natural isomorphisms

$$g^* f_! \xrightarrow{\sim} f'_! g'^* \,,$$
$$g'_* f'^! \xrightarrow{\sim} f^! g_* \,.$$

5. *For any separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, there exist natural isomorphisms*

$$Ex(f_!^*, \otimes) : (f_! K) \otimes_X L \xrightarrow{\sim} f_!(K \otimes_Y f^* L)\,,$$
$$Hom_X(f_!(L), K) \xrightarrow{\sim} f_* Hom_Y(L, f^!(K))\,,$$
$$f^! Hom_X(L, M) \xrightarrow{\sim} Hom_Y(f^*(L), f^!(M))\,.$$

Remark 2.4.51. It is important to precise that in the case where the morphisms in $\mathscr{S}$ are assumed to be quasi-projective, this theorem is proved by Ayoub in [**Ayo07a**] if we except the case where $\mathscr{T}$ is oriented in point (3).[47]

With regards to this theorem, our contribution is to extend the result of Ayoub to the non quasi-projective case and to consider the oriented case – which is crucial in the theory of motives. Recall also we have given another proof of this result in the case where the motivic category $\mathscr{T}$ satisfies in addition the assumptions of Corollary 2.4.43 – which will always be the case for the different categories of motives introduced here.

Remark 2.4.52. The purity isomorphism is compatible with composition. Given smooth separated morphisms of finite type

$$Y \xrightarrow{g} X \xrightarrow{f} S$$

we obtain (*cf.* [**EGA4**, 17.2.3]) an exact sequence of vector bundles over Y

$$(\sigma) \qquad 0 \to g^{-1}T_f \to T_{fg} \to T_g \to 0.$$

which according to Remark 2.4.15 induces an isomorphism:

$$\epsilon_\sigma : MTh_Y(T_{fg}) \xrightarrow{MTh_Y(\sigma)} MTh_Y(T_g) \otimes_Y MTh_Y(g^{-1}T_f) \xrightarrow{\sim} g^* MTh_X(T_f) \otimes_Y MTh_Y(T_g).$$

One can check the following diagram is commutative:

$$\begin{array}{ccc}
(fg)_\sharp(K) & = & f_\sharp g_\sharp(K) \\
\Big\downarrow \mathfrak{p}_{fg} & & \Big\downarrow \mathfrak{p}_f \circ \mathfrak{p}_g \\
 & & f_!\Big(MTh_X(T_f) \otimes_X g_!\big(MTh_Y(T_g) \otimes_Y K\big)\Big) \\
 & & \Big\downarrow Ex(g_!^*, \otimes)^{-1} \\
 & & f_!g_!\big(g^*MTh_Y(T_f) \otimes_Y MTh_Y(T_g) \otimes_Y K\big) \\
 & & \Big\downarrow \epsilon_\sigma^{-1} \\
(fg)_!(MTh(T_{fg}) \otimes K) & = & f_!g_!(MTh(T_{fg}) \otimes K).
\end{array}$$

This is not an easy check.[48] In fact, this is one of the key technical point in the proof of the main Theorem of Ayoub ([**Ayo07a**, 1.4.2]). We refer the reader to [**Ayo07a**, 1.5] for details.

Note also that given the commutativity of the above diagram, if $\mathscr{T}$ admits an orientation $\mathfrak{t}$, it readily follows from axiom (c) of Definition 2.4.38 that the following diagram is commutative:

$$\begin{array}{ccc}
(fg)_\sharp(K) & = & f_\sharp g_\sharp(K) \\
\Big\downarrow \mathfrak{p}^{\mathfrak{t}}_{fg} & & \Big\downarrow \mathfrak{p}^{\mathfrak{t}}_f \circ \mathfrak{p}^{\mathfrak{t}}_g \\
(fg)_!(K)(n+m)[2n+2m] & = & f_!g_!(K)(n+m)[2n+2m]
\end{array}$$

where n (resp. m) is the relative dimension of f (resp. g).

Morphisms of triangulated motivic categories are compatible with Grothendieck 6 operations in the following sense:

Proposition 2.4.53. *Let $\mathscr{T}$ and $\mathscr{T}'$ be motivic triangulated categories and*

$$\varphi^* : \mathscr{T} \rightleftarrows \mathscr{T}' : \varphi_*$$

be an adjunction of premotivic categories.

---

[47] This theorem was first announced by Voevodsky but only notes covering the basic setting were to be found by the time Ayoub wrote the proof.

[48] The main point is to check that the isomorphism of Theorem 2.4.35 is compatible with composition (of closed immersions). On that particular point, see [**Dég08**, Th. 4.32, Cor. 4.33].

Then φ^* (resp. φ_*) commutes with the operations f^* (resp. f_*), for any morphism of schemes f, as well as with the operation $p_!$ (resp. $p^!$), for any separated morphism of finite type p.

Moreover, φ^* is monoidal and for any premotive $M \in \mathscr{T}(S)$, $N \in \mathscr{T}'(S)$, the canonical map

$$Hom(M, \varphi_*(N)) \to \varphi_* Hom(\varphi^*(M), N)$$

is an isomorphism.

PROOF. The only thing to prove is that φ^* commutes with $p_!$ as the other statements follows either from the definitions or by adjunction. This follows from Proposition 2.3.11, the purity property in $\mathscr{T}$ and $\mathscr{T}'$ (property (3) in the above theorem) and the fact φ^* commutes with $p_\sharp$ when p is smooth by assumption. □

REMARK 2.4.54. With additional assumptions on $\mathscr{T}$ and $\mathscr{T}'$, and over a field, we will see that φ^* commutes with all of the six operations (see Theorem 4.4.25).

3. Descent in $\mathscr{P}$-fibred model categories

3.0. In this section, $\mathscr{S}$ is an abstract category and $\mathscr{P}$ an admissible class of morphisms in $\mathscr{S}$.

In section 3.3 however, we will consider as in 2.0 a noetherian base scheme $\mathcal{S}$ and we will assume that $\mathscr{S}$ is an adequate category of $\mathcal{S}$-schemes satisfying the following condition on $\mathscr{S}$:

(a) Any scheme in $\mathscr{S}$ is finite dimensional.

Moreover, in sections 3.3.c and 3.3.d, we will even assume:

(a′) Any scheme in $\mathscr{S}$ is quasi-excellent and finite dimensional.

We fix an admissible class $\mathscr{P}$ of morphisms in $\mathscr{S}$ which contains the class of étale morphisms in $\mathscr{S}$ and a stable combinatorial $\mathscr{P}$-fibred model category $\mathscr{M}$ over $\mathscr{S}$.

In section 3.3.d, we will assume furthermore that:

(b) The stable model $\mathscr{P}$-fibred category $\mathscr{M}$ is $\mathbf{Q}$-linear (see 3.2.14).

3.1. Extension of $\mathscr{P}$-fibred categories to diagrams.

3.1.a. *The general case.*

3.1.1. Assume given a $\mathscr{P}$-fibered category $\mathscr{M}$ over $\mathscr{S}$. Then $\mathscr{M}$ can be extended to $\mathscr{S}$-diagrams (i.e. functors from a small category to $\mathscr{S}$) as follows. Let I be a small category, and $\mathscr{X}$ a functor from I to $\mathscr{S}$. For an object i of I, we will denote by $\mathscr{X}_i$ the fiber of $\mathscr{X}$ at i (i.e. the evaluation of $\mathscr{X}$ at i), and, for a map $u : i \to j$ in I, we will still denote by $u : \mathscr{X}_i \to \mathscr{X}_j$ the morphism induced by u. We define the category $\mathscr{M}(\mathscr{X}, I)$ as follows.

An object of $\mathscr{M}(\mathscr{X}, I)$ is a couple (M, a), where M is the data of an object M_i in $\mathscr{M}(\mathscr{X}_i)$ for any object i of I, and a is the data of a morphism $a_u : u^*(M_j) \to M_i$ for any morphism $u : i \to j$ in I, such that, for any object i of I, the map a_{1_i} is the identity of M_i (we will always assume that 1_i^* is the identity functor), and, for any composable morphisms $u : i \to j$ and $v : j \to k$ in I, the following diagram commutes.

$$\begin{array}{ccc} u^*v^*(M_k) & \xrightarrow{\simeq} & (vu)^*(M_k) \\ {\scriptstyle u^*(a_v)}\downarrow & & \downarrow{\scriptstyle a_{vu}} \\ u^*(M_j) & \xrightarrow[a_u]{} & M_i \end{array}$$

A morphism $p : (M, a) \to (N, b)$ is a collection of morphisms

$$p_i : M_i \to N_i$$

in $\mathscr{M}(\mathscr{X}_i)$, for each object i in I, such that, for any morphism $u : i \to j$ in I, the following diagram commutes.

$$\begin{array}{ccc} u^*(M_j) & \xrightarrow{u^*(p_j)} & u^*(N_j) \\ {\scriptstyle a_u}\downarrow & & \downarrow{\scriptstyle b_u} \\ M_i & \xrightarrow[p_i]{} & N_i \end{array}$$

In the case where $\mathscr{M}$ is a monoidal $\mathscr{P}$-fibred category, the category $\mathscr{M}(\mathscr{X}, I)$ is naturally endowed with a symmetric monoidal structure. Given two objects (M, a) and (N, b) of $\mathscr{M}(\mathscr{X}, I)$, their tensor product

$$(M, a) \otimes (N, b) = (M \otimes N, a \otimes b)$$

is defined as follows. For any object i of I,

$$(M \otimes N)_i = M_i \otimes N_i \,,$$

and for any map $u : i \to j$ in I, the map $(a \otimes b)_u$ is the composition of the isomorphism $u^*(M_j \otimes N_j) \simeq u^*(M_j) \otimes u^*(N_j)$ with the morphism

$$a_u \otimes b_u : u^*(M_j) \otimes u^*(N_j) \to M_i \otimes N_i \,.$$

Note finally that if $\mathscr{M}$ is a complete monoidal $\mathscr{P}$-fibred category, then $\mathscr{M}(\mathscr{X}, I)$ admits an internal Hom.

3.1.2. *Evaluation functors.* Assume now that for any S, $\mathscr{M}(S)$ admits small sums.

For each object i of I, we have a functor

$$\begin{aligned} i^* : \mathscr{M}(\mathscr{X}, I) &\to \mathscr{M}(\mathscr{X}_i) \\ (M, a) &\longmapsto M_i \end{aligned} \tag{3.1.2.1}$$

called the *evaluation functor* associated with i. This functor i^* has a left adjoint

$$i_\sharp : \mathscr{M}(\mathscr{X}_i) \to \mathscr{M}(\mathscr{X}, I) \tag{3.1.2.2}$$

defined as follows. If M is an object of $\mathscr{M}(\mathscr{X}_i)$, then $i_\sharp(M)$ is the data (M', a') such that for any object j of I,

$$\left(i_\sharp(M)\right)_j = M'_j = \coprod_{u \in \mathrm{Hom}_I(j,i)} u^*(M) \,, \tag{3.1.2.3}$$

and, for any morphism $v : k \to j$ in I, the map a'_v is the canonical map induced by the collection of maps

$$v^* u^*(M) \simeq (uv)^*(M) \to \coprod_{w \in \mathrm{Hom}_I(k,i)} w^*(M) \tag{3.1.2.4}$$

for $u \in \mathrm{Hom}_I(j, i)$.

If we assume that $\mathscr{M}$ is a complete $\mathscr{P}$-fibred category and that $\mathscr{M}(S)$ admits small products for any S, then i^* has a right adjoint

$$i_* : \mathscr{M}(\mathscr{X}_i) \to \mathscr{M}(\mathscr{X}, I) \tag{3.1.2.5}$$

given, for any object M of $\mathscr{M}(\mathscr{X}_i)$ by the formula

$$\left(i_*(M)\right)_j = \prod_{u \in \mathrm{Hom}_I(i,j)} u_*(M), \tag{3.1.2.6}$$

with transition map given by the dual formula of 3.1.2.4.

3.1.3. *Functoriality.* Assume that $\mathscr{M}$ if a $\mathscr{P}$-fibred category suth that for any object S of $\mathscr{S}$, $\mathscr{M}(S)$ has small colimits.

Remember that, if $\mathscr{X}$ and $\mathscr{Y}$ are $\mathscr{S}$-diagrams, indexed respectively by small categories I and J, a morphism of $\mathscr{S}$-diagrams $\varphi : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ is a couple $\varphi = (\alpha, f)$, where $f : I \to J$ is a functor, and $\alpha : \mathscr{X} \to f^*(\mathscr{Y})$ is a natural transformation (where $f^*(\mathscr{Y}) = \mathscr{Y} \circ f$). In particular, for any object i of I, we have a morphism

$$\alpha_i : \mathscr{X}_i \to \mathscr{Y}_{f(i)}$$

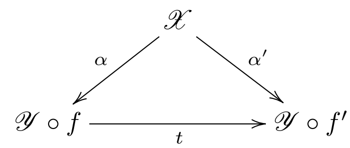
in $\mathscr{S}$. This turns $\mathscr{S}$-diagrams into a strict 2-category: the identity of $(\mathscr{X}, I)$ is the couple $(1_{\mathscr{X}}, 1_I)$, and, if $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ and $\psi = (\beta, g) : (\mathscr{Y}, J) \to (\mathscr{Z}, K)$ are two composable morphisms, the morphism $\psi \circ \varphi : (\mathscr{X}, I) \to (\mathscr{Z}, K)$ is the couple (gf, γ), where for each object i of I, the map

$$\gamma_i : \mathscr{X}_i \to \mathscr{Z}_{g(f(i))}$$

is the composition

$$\mathscr{X}_i \xrightarrow{\alpha_i} \mathscr{Y}_{f(i)} \xrightarrow{\beta_{f(i)}} \mathscr{Z}_{g(f(i))}\,.$$

There is also a notion of natural transformation between morphisms of $\mathscr{S}$-diagrams: if $\varphi = (\alpha, f)$ and $\varphi' = (\alpha', f')$ are two morphisms from $(\mathscr{X}, I)$ to $(\mathscr{Y}, J)$, a natural transformation t from φ to φ' is a natural transformation $t : f \to f'$ such that the following diagram of functors commutes.



This makes the category of $\mathscr{S}$-diagrams a (strict) 2-category.

To a morphism of diagrams $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, J)$, we associate a functor

$$\varphi^* : \mathscr{M}(\mathscr{Y}, J) \to \mathscr{M}(\mathscr{X}, I)$$

as follows. For an object (M, a) of $\mathscr{M}(\mathscr{Y})$, $\varphi^*(M, a) = (\varphi^*(M), \varphi^*(a))$ is the object of $\mathscr{M}(\mathscr{X})$ defined by $\varphi^*(M)_i = \alpha_i^*(M_{f(i)})$ for i in I, and by the formula

$$\varphi^*(a)_u = \alpha_i^*(a_{f(u)}) : \alpha_i^*\, f(u)^*(M_{f(j)}) = u^*\, \alpha_j^*(M_{f(j)}) \to \alpha_i^*(M_{f(i)})$$

for $u : i \to j$ in I.

We will say that a morphism $\varphi : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ is a *$\mathscr{P}$-morphism* if, for any object i in I, the morphism $\alpha_i : \mathscr{X}_i \to \mathscr{Y}_{f(i)}$ is a $\mathscr{P}$-morphism. For such a morphism φ, the functor φ^* has a left adjoint which we denote by

$$\varphi_\sharp : \mathscr{M}(\mathscr{X}, I) \to \mathscr{M}(\mathscr{Y}, J)\,.$$

For instance, given a $\mathscr{S}$-diagram $\mathscr{X}$ indexed by a small category I, each object i of I defines a $\mathscr{P}$-morphism of diagrams $i : \mathscr{X}_i \to (\mathscr{X}, I)$ (where $\mathscr{X}_i$ is indexed by the terminal category), so that the corresponding the functor $i_\sharp$ corresponds precisely to (3.1.2.2).

Assume that $\mathscr{M}$ is a complete $\mathscr{P}$-fibred category such that $\mathscr{M}(S)$ has small limits for any object S of $\mathscr{S}$. Then the functor φ^* has a right adjoint which we denote by

$$\varphi_* : \mathscr{M}(\mathscr{X}, I) \to \mathscr{M}(\mathscr{Y}, J)\,.$$

In the case where φ is the morphism $i : \mathscr{X}_i \to (\mathscr{X}, I)$ defined by an object i of I, i_* corresponds precisely to (3.1.2.5).

REMARK 3.1.4. This construction can be applied in particular to any Grothendieck abelian (monoidal) $\mathscr{P}$-fibred category (*cf.* definition 1.3.8). The triangulated case cannot be treated in general without assuming a thorough structure – this is the purpose of the next section.

3.1.b. *The model category case.*

3.1.5. Let $\mathscr{M}$ be a $\mathscr{P}$-fibred model category over $\mathscr{S}$ (*cf.* 1.3.22). Given a $\mathscr{S}$-diagram $\mathscr{X}$ indexed by a small category I, we will say that a morphism of $\mathscr{M}(\mathscr{X}, I)$ is a *termwise weak equivalence* (resp. a *termwise fibration*, resp. a *termwise cofibration*) if, for any object i of I, its image by the functor i^* is a weak equivalence (resp. a fibration, resp. a cofibration) in $\mathscr{M}(\mathscr{X}_i)$.

PROPOSITION 3.1.6. *If $\mathscr{M}$ is a cofibrantly generated $\mathscr{P}$-fibred model category over $\mathscr{S}$, then, for any $\mathscr{S}$-diagram $\mathscr{X}$ indexed by a small category I, the category $\mathscr{M}(\mathscr{X}, I)$ is a cofibrantly generated model category whose weak equivalences (resp. fibrations) are the termwise weak equivalences (resp. the termwise fibrations). This model category structure on $\mathscr{M}(\mathscr{X}, I)$ will be called the* projective model structure.

Moreover, any cofibration of $\mathscr{M}(\mathscr{X}, I)$ is a termwise cofibration, and the family of functors

$$i^* : \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I) \to \mathrm{Ho}(\mathscr{M})(\mathscr{X}_i)\ , \quad i \in \mathrm{Ob}(I)\,,$$

is conservative.

If $\mathscr{M}$ is left proper (resp. right proper, resp. combinatorial, resp. stable), then so is the projective model category structure on $\mathscr{M}(\mathscr{X})$.

Proof. Let $\mathscr{X}^\delta$ be the $\mathscr{S}$-diagram indexed by the set of objects of I (seen as a discrete category), whose fiber at i is $\mathscr{X}_i$. Let $\varphi : (\mathscr{X}^\delta, Ob\, I) \to (\mathscr{X}, I)$ be the inclusion (i.e. the map which is the identity on objects and which is the identity on each fiber). As φ is clearly a $\mathscr{P}$-morphism, we have an adjunction

$$\varphi_\sharp : \mathscr{M}(\mathscr{X}^\delta, Ob\, I) \simeq \prod_i \mathscr{M}(\mathscr{X}_i) \rightleftarrows \mathscr{M}(\mathscr{X}, I) : \varphi^* \,.$$

The functor $\varphi_\sharp$ can be made explicit: it sends a family of objects $(M_i)_i$ (with M_i in $\mathscr{M}(\mathscr{X}_i)$) to the sum of the $i_\sharp(M_i)$'s indexed by the set of objects of I. Note also that this proposition is trivially verified whenever $\mathscr{X}^\delta = \mathscr{X}$. Using the explicit formula for $i_\sharp$ given in 3.1.2, it is then straightforward to check that the adjunction $(\varphi_\sharp, \varphi^*)$ satisfies the assumptions of [**Cra95**, Theorem 3.3], which proves the existence of the projective model structure on $\mathscr{M}(\mathscr{X}, I)$. Furthermore, the generating cofibrations (resp. trivial cofibrations of $\mathscr{M}(\mathscr{X}, I)$) can be described as follows. For each object i of I, let A_i (resp. B_i) be a generating set of cofibrations (resp. of trivial cofibrations in $\mathscr{M}(\mathscr{X}_i)$. The class of termwise trivial fibrations (resp. of termwise fibrations) of $\mathscr{M}(\mathscr{X}, I)$ is the class of maps which have the right lifting property with respect to the set $A = \cup_{i \in I}\, i_\sharp(A_i)$ (resp. to the set $B = \cup_{i \in I}\, i_\sharp(B_i)$). Hence, the set A (resp. B) generates the class of cofibrations (resp. of trivial cofibrations). In particular, as any element of A is a termwise cofibration (which follows immediately from the explicit formula for $i_\sharp$ given in 3.1.2), and as termwise cofibrations are stable by pushouts, transfinite compositions and retracts, any cofibration is a termwise cofibration (by the small object argument).

As any fibration (resp. cofibration) of $\mathscr{M}(\mathscr{X}, I)$ is a termwise fibration (resp. a termwise cofibration), it is clear that, whenever the model categories $\mathscr{M}(\mathscr{X}_i)$ are right (resp. left) proper, the model category $\mathscr{M}(\mathscr{X}, I)$ has the same property.

The functor φ^* preserves fibrations and cofibrations, while it also preserves and detects weak equivalences (by definition). This implies that the induced functor

$$\varphi^* : \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I) \to \mathrm{Ho}(\mathscr{M})(\mathscr{X}^\delta, Ob\, I) \simeq \prod_i \mathrm{Ho}(\mathscr{M})(\mathscr{X}_i)$$

is conservative (using the facts that the set of maps from a cofibrant object to a fibrant object in the homotopy category of a model category is the set of homotopy classes of maps, and that a morphism of a model category is a weak equivalence if and only if it induces an isomorphism in the homotopy category). As φ^* commutes to limits and colimits, this implies that it commutes to homotopy limits and to homotopy colimits (up to weak equivalences). Using the conservativity property, this implies that a commutative square of $\mathscr{M}(\mathscr{X}, I)$ is a homotopy pushout (resp. a homotopy pullback) if and only if it is so in $\mathscr{M}(\mathscr{X}^\delta, Ob\, I)$. Remember that stable model categories are characterized as those in which a commutative square is a homotopy pullback square if and only if it is a homotopy pushout square. As a consequence, if all the model categories $\mathscr{M}(\mathscr{X}_i)$ are stable, as $\mathscr{M}(\mathscr{X}^\delta, Ob\, I)$ is then obviously stable as well, the model category $\mathscr{M}(\mathscr{X}, I)$ has the same property.

It remains to prove that, if $\mathscr{M}(X, I)$ is a combinatorial model category for any object X of $\mathscr{S}$, then $\mathscr{M}(\mathscr{X}, I)$ is combinatorial as well. For each object i in I, let G_i be a set of accessible generators of $\mathscr{M}(\mathscr{X}_i)$. Note that, for any object i of I, the functor $i_\sharp$ has a left adjoint i^* which commutes to colimits (having itself a right adjoint i_*). It is then easy to check that the set of objects of shape $i_\sharp(M)$, for M in G_i and i in I, is a small set of accessible generators of $\mathscr{M}(\mathscr{X}, I)$. This implies that $\mathscr{M}(\mathscr{X}, I)$ is accessible and ends the proof. □

Proposition 3.1.7. *Let $\mathscr{M}$ be a combinatorial $\mathscr{P}$-fibred model category over $\mathscr{S}$. Then, for any $\mathscr{S}$-diagram $\mathscr{X}$ indexed by a small category I, the category $\mathscr{M}(\mathscr{X}, I)$ is a combinatorial model category whose weak equivalences (resp. cofibrations) are the termwise weak equivalences (resp. the termwise cofibrations). This model category structure on $\mathscr{M}(\mathscr{X}, I)$ will be called the* injective model structure[49]. *Moreover, any fibration of the injective model structure on $\mathscr{M}(\mathscr{X},I)$ is a termwise fibration.*

If $\mathscr{M}$ is left proper (resp. right proper, resp. stable), then so is the injective model category structure on $\mathscr{M}(\mathscr{X},I)$.

Proof. See [**Bar10**, Theorem 2.28] for the existence of such a model structure (if, for any object X in $\mathscr{S}$, all the cofibrations of $\mathscr{M}(X)$ are monomorphisms, this can also be done following mutatis mutandis the proof of [**Ayo07a**, Proposition 4.5.9]). Any trivial cofibration of the projective model structure being a termwise trivial cofibration, any fibration of the injective model structure is a fibration of the projective model structure, hence a termwise fibration.

The assertions about properness follows from their analogs for the projective model structure and from [**Cis06**, Corollary 1.5.21] (or can be proved directly; see [**Bar10**, Proposition 2.31]). Similarly, the assertion on stability follows from their analogs for the projective model structure. □

3.1.8. From now on, we assume that a combinatorial $\mathscr{P}$-fibred model category $\mathscr{M}$ over $\mathscr{S}$ is given. Then, for any $\mathscr{S}$-diagram $(\mathscr{X},I)$, we have two model category structures on $\mathscr{M}(\mathscr{X},I)$, and the identity defines a left Quillen equivalence from the projective model structure to the injective model structure. This fact will be used for the understanding of the functorialities coming from morphisms of diagrams of S-schemes.

3.1.9. The category of $\mathscr{S}$-diagrams admits small sums. If $\{(\mathscr{Y}_j,I_j)\}_{j\in J}$ is a small family of $\mathscr{S}$-diagrams, then their sum is the $\mathscr{S}$-diagram $(\mathscr{X},I)$, where

$$I=\coprod_{j\in J} I_j\,,$$

and $\mathscr{X}$ is the functor from I to $\mathscr{S}$ defined by

$$\mathscr{X}_i=\mathscr{Y}_j \quad \text{whenever } i\in I_j.$$

Proposition 3.1.10. *For any small family of $\mathscr{S}$-diagrams $\{(\mathscr{Y}_j,I_j)\}_{j\in J}$, the canonical functor*

$$\mathrm{Ho}(\mathscr{M})\Big(\coprod_{j\in J}\mathscr{Y}_j\Big)\to\prod_{j\in J}\mathrm{Ho}(\mathscr{M})(\mathscr{Y}_j)$$

is an equivalence of categories.

Proof. The functor

$$\mathscr{M}\Big(\coprod_{j\in J}\mathscr{Y}_j\Big)\to\prod_{j\in J}\mathscr{M}(\mathscr{Y}_j)$$

is an equivalence of categories. It thus remains an equivalence after localization. To conclude, it is sufficient to see that the homotopy category of a product of model categories is the product of their homotopy categories, which follows rather easily from the explicit description of the homotopy category of a model category; see e.g. [**Hov99**, Theorem 1.2.10]. □

Proposition 3.1.11. *Let $\varphi=(\alpha,f):(\mathscr{X},I)\to(\mathscr{Y},J)$ be a morphism of $\mathscr{S}$-diagrams.*

(i) The adjunction $\varphi^*:\mathscr{M}(\mathscr{Y},J)\rightleftarrows\mathscr{M}(\mathscr{X},I):\varphi_*$ is a Quillen adjunction with respect to the injective model structures. In particular, it induces a derived adjunction

$$\mathbf{L}\varphi^*:\mathrm{Ho}(\mathscr{M})(\mathscr{Y},J)\rightleftarrows\mathrm{Ho}(\mathscr{M})(\mathscr{X},I):\mathbf{R}\varphi_*\,.$$

(ii) If φ is a $\mathscr{P}$-morphism, then the adjunction $\varphi_\sharp:\mathscr{M}(\mathscr{X},I)\rightleftarrows\mathscr{M}(\mathscr{Y},J):\varphi^*$ is a Quillen adjunction with respect to the projective model structures, and the functor φ^* preserves weak equivalences. In particular, we get a derived adjunction

$$\mathbf{L}\varphi_\sharp:\mathrm{Ho}(\mathscr{M})(\mathscr{X},I)\rightleftarrows\mathrm{Ho}(\mathscr{M})(\mathscr{Y},J):\mathbf{L}\varphi^*=\varphi^*=\mathbf{R}\varphi^*\,.$$

[49]Quite unfortunately, this corresponds to the 'semi-projective' model structure introduced in [**Ayo07a**, Def. 4.5.8].

Proof. The functor φ^* obviously preserves termwise cofibrations and termwise trivial cofibrations (we reduce to the case of a morphism of $\mathscr{S}$ using the explicit description of φ^* given in 3.1.3), which proves the first assertion. Similarly, the second assertion follows from the fact that, under the assumption that φ is a $\mathscr{P}$-morphism, the functor φ^* preserves termwise weak equivalences (see Remark 1.3.23), as well as termwise fibrations. □

3.1.12. The computation of the (derived) functors $\mathbf{R}\varphi_*$ (and $\mathbf{L}\varphi_\sharp$ whenever it makes sense) given by Proposition 3.1.11 has to do with homotopy limits (and homotopy colimits). It is easier to first understand this in the non derived version as follows.

Consider first the trivial case of a constant $\mathscr{S}$-diagram: let X be an object of $\mathscr{S}$, and I a small category. Then, seeing X as the constant functor $I \to \mathscr{S}$ with value X, we have a projection map $p_I : (X, I) \to X$. From the very definition, the category $\mathscr{M}(X, I)$ is simply the category of presheaves on I with values in $\mathscr{M}(X)$, so that the inverse image functor

$$p_I^* : \mathscr{M}(X) \to \mathscr{M}(X, I) = \mathscr{M}(X)^{I^{op}} \tag{3.1.12.1}$$

is the 'constant diagram functor', while its right adjoint

$$\varprojlim_{I^{op}} = p_{I,*} : \mathscr{M}(X, I) \to \mathscr{M}(X) \tag{3.1.12.2}$$

is the limit functor, and its left adjoint,

$$\varinjlim_{I^{op}} = p_{I,\sharp} : \mathscr{M}(X, I) \to \mathscr{M}(X) \tag{3.1.12.3}$$

is the colimit functor.

Let S be an object of $\mathscr{S}$. A *$\mathscr{S}$-diagram over S* is the data of a $\mathscr{S}$-diagram $(\mathscr{X}, I)$, together with a morphism of $\mathscr{S}$-diagrams $p : (\mathscr{X}, I) \to S$ (i.e. its a $\mathscr{S}/S$-diagram). Such a map p factors as

$$(\mathscr{X}, I) \xrightarrow{\pi} (S, I) \xrightarrow{p_I} S\,, \tag{3.1.12.4}$$

where $\pi = (p, 1_I)$. Then one checks easily that, for any object M of $\mathscr{M}(\mathscr{X}, I)$, and for any object i of I, one has

$$\pi_*(M)_i \simeq p_{i,*}(M_i)\,, \tag{3.1.12.5}$$

where $p_i : \mathscr{X}_i \to S$ is the structural map, from which we deduce the formula

$$p_*(M) \simeq \varprojlim_{i \in I^{op}} \pi_*(M)_i \simeq \varprojlim_{i \in I^{op}} p_{i,*}(M_i)\,, \tag{3.1.12.6}$$

Remark that, if I is a small category with a terminal object ω, then any $\mathscr{S}$-diagram $\mathscr{X}$ indexed by I is a $\mathscr{S}$-diagram over $\mathscr{X}_\omega$, and we deduce from the computations above that, if $p : (\mathscr{X}, I) \to \mathscr{X}_\omega$ denotes the canonical map, then, for any object M of $\mathscr{M}(\mathscr{X}, I)$,

$$p_*(M) \simeq M_\omega\,. \tag{3.1.12.7}$$

Consider now a morphism of $\mathscr{S}$-diagrams $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, J)$. For each object j, we can form the following pullback square of categories.

$$\begin{array}{ccc} I/j & \xrightarrow{u_j} & I \\ {\scriptstyle f/j}\downarrow & & \downarrow{\scriptstyle f} \\ J/j & \xrightarrow[v_j]{} & J \end{array} \tag{3.1.12.8}$$

in which J/j is the category of objects of J over j (which has a terminal object, namely $(j, 1_j)$, and v_j is the canonical projection; the category I/j is thus the category of pairs (i, a), where i is an object of I, and $a : f(i) \to j$ a morphism in J. From this, we can form the following pullback of $\mathscr{S}$-diagrams

$$\begin{array}{ccc} (\mathscr{X}/j, I/j) & \xrightarrow{\mu_j} & (\mathscr{X}, I) \\ {\scriptstyle \varphi/j}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (\mathscr{Y}/j, J/j) & \xrightarrow[\nu_j]{} & (\mathscr{Y}, J) \end{array} \tag{3.1.12.9}$$

in which $\mathscr{X}/j = \mathscr{X} \circ u_j$, $\mathscr{Y}/j = \mathscr{Y} \circ v_j$, and the maps μ_j and ν_j are the one induced by u_j and v_j respectively. For an object M of $\mathscr{M}(\mathscr{X}, I)$ (resp. an object N of $\mathscr{M}(\mathscr{Y}, J)$), we define M/j (resp. N/j) as the object of $\mathscr{M}(\mathscr{X}/j, I/j)$ (resp. of $\mathscr{M}(\mathscr{Y}/j, J/j)$) obtained as $M/j = \mu_j^*(M)$ (resp. $N/j = \nu_j^*(N)$). With these conventions, for any object M of $\mathscr{M}(\mathscr{X}, I)$ and any object j of the indexing category J, one gets the formula

$$\varphi_*(M)_j \simeq (\varphi/j)_*(M/j)_{(j,1_j)} \simeq \varprojlim_{(i,a)\in I/j^{op}} \alpha_{i,*}(M_i)\,. \tag{3.1.12.10}$$

This implies that the natural map

$$\varphi_*(M)/j = \nu_j^*\,\varphi_*(M) \to (\varphi/j)_*\,\mu_j^*(M) = (\varphi/j)_*(M/j) \tag{3.1.12.11}$$

is an isomorphism: to prove this, it is sufficient to obtain an isomorphism from (3.1.12.11) after evaluating by any object $(j', a : j' \to j)$ of J/j, which follows readily from (3.1.12.10) and from the obvious fact that $(I/j)/(j', a)$ is canonically isomorphic to I/j'.

In order to deduce from the computations above their derived versions, we need two lemmata.

LEMMA 3.1.13. *Let $\mathscr{X}$ be a $\mathscr{S}$-diagram indexed by a small category I, and i an object of I. Then the evaluation functor*

$$i^* : \mathscr{M}(\mathscr{X}, I) \to \mathscr{M}(\mathscr{X}_i)$$

is a right Quillen functor with respect to the injective model structure, and it preserves weak equivalences.

PROOF. Proving that the functor i^* is a right Quillen functor is equivalent to proving that its left adjoint (3.1.2.2) is a left Quillen functor with respect to the injective model structure, which follows immediately from its computation (3.1.2.3), as, in any model category, cofibrations and trivial cofibrations are stable by small sums. The last assertion is obvious from the very definition of the weak equivalences in $\mathscr{M}(\mathscr{X}, I)$. □

LEMMA 3.1.14. *For any pullback square of $\mathscr{S}$-diagrams of shape* (3.1.12.9)*, the functors*

$$\mu_j^* : \mathscr{M}(\mathscr{X}, I) \to \mathscr{M}(\mathscr{X}/j, I/j)\,, \quad M \mapsto M/j$$
$$\nu_j^* : \mathscr{M}(\mathscr{Y}, I) \to \mathscr{M}(\mathscr{Y}/j, J/j)\;, \quad N \mapsto N/j$$

are right Quillen functors with respect to the injective model structure, and they preserve weak equivalences.

PROOF. It is sufficient to prove this for the functor μ_j^* (as ν_j^* is simply the special case where $I = J$ and f is the identity). The fact that μ_j^* preserves weak equivalences is obvious, so that it remains to prove that it is a right Quillen functor. We thus have to prove that left adjoint of μ_j^*,

$$\mu_{j,\sharp} : \mathscr{M}(\mathscr{X}/j, I/j) \to \mathscr{M}(\mathscr{X}, I)\,,$$

is a left Quillen functor. In other words, we have to prove that, for any object i of I, the functor

$$i^*\mu_{j,\sharp} : \mathscr{M}(\mathscr{X}, I) \to \mathscr{M}(\mathscr{X})$$

is a left Quillen functor. For any object M of $\mathscr{M}(\mathscr{X}, I)$, we have a natural isomorphism

$$i^*\mu_{j,\sharp}(M) \simeq \coprod_{a\in \mathrm{Hom}_J(f(i),j)} (i,a)_\sharp(M_i)\,.$$

But we know that the functors $(i,a)_\sharp$ are left Quillen functors, so that the stability of cofibrations and trivial cofibrations by small sums and this description of the functor $i^*\mu_{j,\sharp}$ achieves the proof. □

Proposition 3.1.15. *Let S be an object of $\mathscr{S}$, and $p : (\mathscr{X}, I) \to S$ a $\mathscr{S}$-diagram over S, and consider the canonical factorization* (3.1.12.4). *For any object M of* $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$*, there are canonical isomorphisms and* $\mathrm{Ho}(\mathscr{M})(S)$*:*

$$\mathbf{R}\pi_*(M)_i \simeq \mathbf{R}p_{i,*}(M_i) \quad \textit{and} \quad \mathbf{R}p_*(M) \simeq \mathbf{R}\varprojlim_{i\in I^{op}} \mathbf{R}p_{i,*}(M_i)\,.$$

In particular, if furthermore the category I has a terminal object ω, then

$$\mathbf{R}p_*(M) \simeq \mathbf{R}p_{\omega,*}(M_\omega)\,.$$

Proof. This follows immediately from (3.1.12.5) and (3.1.12.6) and from the fact that deriving (right) Quillen functors is compatible with composition. □

Proposition 3.1.16. *We consider the pullback square of $\mathscr{S}$-diagrams* (3.1.12.9) *(as well as the notations thereof). For any object M of* $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$*, and any object j of J, we have natural isomorphisms*

$$\mathbf{R}\varphi_*(M)_j \simeq \mathbf{R}\varprojlim_{(i,a)\in I/j^{op}} \mathbf{R}\alpha_{i,*}(M_i) \quad \textit{and} \quad \mathbf{R}\varphi_*(M)/j \simeq \mathbf{R}(\varphi/j)_*(M/j)$$

in $\mathrm{Ho}(\mathscr{M})(\mathscr{Y}_j)$ *and in* $\mathrm{Ho}(\mathscr{M})(\mathscr{Y}/j, J/j)$ *respectively.*

Proof. Using again the fact that deriving right Quillen functors is compatible with composition, by virtue of Lemma 3.1.13 and Lemma 3.1.14, this is a direct translation of (3.1.12.10) and (3.1.12.11). □

Proposition 3.1.17. *Let $u : T \to S$ be a $\mathscr{P}$-morphism of $\mathscr{S}$, and $p : (\mathscr{X}, I) \to S$ a $\mathscr{S}$-diagram over S. Consider the pullback square of $\mathscr{S}$-diagrams*

$$\begin{array}{ccc} (\mathscr{Y}, I) & \xrightarrow{\varphi} & (\mathscr{X}, I) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ T & \xrightarrow[u]{} & S \end{array}$$

(i.e. $\mathscr{Y}_i = T \times_S \mathscr{X}_i$ for any object i of I). Then, for any object M of $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$*, the canonical map*

$$\mathbf{L}u^*\,\mathbf{R}p_*(M) \to \mathbf{R}q_*\,\mathbf{L}v^*(M)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(T)$.

Proof. By Remark 1.3.23, the functor ν^* is both a left and a right Quillen functor which preserves weak equivalences, so that the functor $\mathbf{L}\nu^* = \nu^* = \mathbf{R}\nu^*$ preserves homotopy limits. Hence, by Proposition 3.1.15, one reduces to the case where I is the terminal category, i.e. to the transposition of the isomorphism given by the $\mathscr{P}$-base change formula ($\mathscr{P}$-BC) for the homotopy $\mathscr{P}$-fibred category $\mathrm{Ho}(\mathscr{M})$ (see 1.1.19). □

3.1.18. A morphism of $\mathscr{S}$-diagrams $\nu = (\alpha, f) : (\mathscr{Y}', J') \to (\mathscr{Y}, J)$, is *cartesian* if, for any arrow $i \to j$ in J', the induced commutative square

$$\begin{array}{ccc} \mathscr{Y}'_i & \longrightarrow & \mathscr{Y}'_j \\ {\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \alpha_j} \\ \mathscr{Y}_{f(i)} & \longrightarrow & \mathscr{Y}_{f(j)} \end{array}$$

is cartesian.

A morphism of $\mathscr{S}$-diagrams $\nu = (\alpha, f) : (\mathscr{Y}', J') \to (\mathscr{Y}, J)$ is *reduced* if $J = J'$ and $f = 1_J$.

PROPOSITION 3.1.19. *Let $\nu : (\mathscr{Y}', J) \to (\mathscr{Y}, J)$ be a reduced cartesian $\mathscr{P}$-morphism of $\mathscr{S}$-diagrams, and $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ a morphism of $\mathscr{S}$-diagrams. Consider the pullback square of $\mathscr{S}$-diagrams*

$$\begin{array}{ccc} (\mathscr{X}', I) & \xrightarrow{\mu} & (\mathscr{X}, I) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ (\mathscr{Y}', J) & \xrightarrow[\nu]{} & (\mathscr{Y}, J) \end{array}$$

(i.e. $\mathscr{X}'_i = \mathscr{Y}'_{f(i)} \times_{\mathscr{Y}_{f(i)}} \mathscr{X}_i$ for any object i of I). Then, for any object M of $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$, the canonical map

$$\mathbf{L}\nu^* \, \mathbf{R}\varphi_*(M) \to \mathbf{R}\psi_* \, \mathbf{L}\mu^*(M)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(\mathscr{Y}', J)$.

PROOF. By virtue of Proposition 3.1.6, it is sufficient to prove that the map

$$j^*\mathbf{L}\nu^* \, \mathbf{R}\varphi_*(M) \to j^*\mathbf{R}\psi_* \, \mathbf{L}\mu^*(M)$$

is an isomorphism for any object j of J. Let $p : (\mathscr{X}/j, I/j) \to \mathscr{Y}_j$ and $q : (\mathscr{X}'/j, J, j) \to \mathscr{Y}'_j$ be the canonical maps. As ν is cartesian, we have a pullback square of $\mathscr{S}$-diagrams

$$\begin{array}{ccc} (\mathscr{X}'/j, I/j) & \xrightarrow{\mu/j} & (\mathscr{X}/j, I/j) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathscr{Y}'_j & \xrightarrow[\nu_j]{} & \mathscr{Y}_j \end{array}$$

But ν_j being a $\mathscr{P}$-morphism, by virtue of Proposition 3.1.17, we thus have an isomorphism

$$\mathbf{L}\nu_j^* \, \mathbf{R}p_*(M/j) \simeq \mathbf{R}q_* \, \mathbf{L}(\mu/j)^*(M/j) = \mathbf{R}q_*(\mathbf{L}\mu^*(M)/j)\,.$$

Applying Proposition 3.1.16 and the last assertion of Proposition 3.1.15 twice, we also have canonical isomorphisms

$$j^*\mathbf{R}\varphi_*(M) \simeq \mathbf{R}p_*(M/j) \quad \text{and} \quad j^*\mathbf{R}\psi_* \, \mathbf{L}\mu^*(M) \simeq \mathbf{R}q_*(\mathbf{L}\mu^*(M)/j)\,.$$

The obvious identity $j^*\mathbf{L}\nu^* = \mathbf{L}\nu_j^* j^*$ achieves the proof. □

COROLLARY 3.1.20. *Under the assumptions of Proposition 3.1.19, for any object N of the category $\mathrm{Ho}(\mathscr{M})(\mathscr{Y}', j)$, the canonical map*

$$\mathbf{L}\mu_\sharp \, \mathbf{L}\psi^*(N) \to \mathbf{L}\varphi^* \, \mathbf{L}\nu_\sharp(N)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$.

REMARK 3.1.21. The class of cartesian $\mathscr{P}$-morphisms form an admissible class of morphisms in the category of $\mathscr{S}$-diagrams, which we denote by $\mathscr{P}_{cart}$. Proposition 3.1.11 and the preceding corollary thus asserts that $\mathrm{Ho}(\mathscr{M})$ is a $\mathscr{P}_{cart}$-fibred category over the category of $\mathscr{S}$-diagrams.

3.1.22. We shall deal sometimes with diagrams of $\mathscr{S}$-diagrams. Let I be a small category, and $\mathscr{F}$ a functor from I to the category of $\mathscr{S}$-diagrams. For each object i of I, we have a $\mathscr{S}$-diagram $(\mathscr{F}(i), J_i)$, and, for each map $u : i \to i'$, we have a functor $f_u : J_i \to J_{i'}$ as well as a natural transformation $\alpha_u : \mathscr{F}(i) \to \mathscr{F}(i') \circ f_u$, subject to coherence identities. In particular, the correspondance $i \mapsto J_i$ defines a functor from I to the category of small categories. Let $I_{\mathscr{F}}$ be the cofibred category over I associated to it; see [**SGA1**, Exp. VI]. Explicitely, $I_{\mathscr{F}}$ is described as follows. The objects are the couples (i, x), where i is an object of I, and x is an object of J_i. A morphism $(i, x) \to (i', x')$ is a couple (u, v), where $u : i \to i'$ is a morphism of I, and $v : f_u(x) \to x'$ is a morphism of $J_{i'}$. The identity of (i, x) is the couple $(1_i, 1_x)$, and, for two morphisms $(u, v) : (i, x) \to (i', x')$ and $(u', v') : (i', x') \to (i'', x'')$, their composition $(u'', v'') : (i, x) \to (i'', x'')$ is defined by $u'' = u' \circ u$, while v'' is the composition of the map

$$f_{u''}(x) = f_{u'}(f_u(x)) \xrightarrow{f_{u'}(v)} f_{u'}(x') \xrightarrow{v'} x''\,.$$

The functor $p : I_{\mathscr{F}} \to I$ is simply the projection $(i,x) \mapsto i$. For each object i of I, we get a canonical pullback square of categories

(3.1.22.1)
$$\begin{array}{ccc} J_i & \xrightarrow{\ell_i} & I_{\mathscr{F}} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ e & \xrightarrow[i]{} & I \end{array}$$

in which i is the functor from the terminal category e which corresponds to the object i, and ℓ_i is the functor defined by $\ell_i(x) = (i,x)$.

The functor $\mathscr{F}$ defines a $\mathscr{S}$-diagram $(\int \mathscr{F}, I_{\mathscr{F}})$: for an object (i,x) of $I_{\mathscr{F}}$, $(\int \mathscr{F})_{(i,x)} = \mathscr{F}(i)_x$, and for a morphism $(u,v) : (i,x) \to (i',x')$, the map

$$(u,v) : (\textstyle\int \mathscr{F})_{(i,x)} = \mathscr{F}(i)_x \to (\textstyle\int \mathscr{F})_{(i',x')} = \mathscr{F}(i')_{x'}$$

is simply the morphism induced by α_u and v. For each object i of I, there is a natural morphism of $\mathscr{S}$-diagrams

(3.1.22.2)
$$\lambda_i : (\mathscr{F}(i), J_i) \to (\textstyle\int \mathscr{F}, I_{\mathscr{F}})\,,$$

given by $\lambda_i = (1_{\mathscr{F}(i)}, \ell_i)$

Proposition 3.1.23. *Let X be an object of $\mathscr{S}$, and $f : \mathscr{F} \to X$ a morphism of functors (where X is considered as the constant functor from I to $\mathscr{S}$-diagrams with value the functor from e to $\mathscr{S}$ defined by X). Then, for each object i of I, we have a canonical pullback square of $\mathscr{S}$-diagrams*

$$\begin{array}{ccc} (\mathscr{F}(i), J_i) & \xrightarrow{\lambda_i} & (\int \mathscr{F}, I_{\mathscr{F}}) \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow[i]{} & (X, I) \end{array}$$

in which φ and φ_i are the obvious morphisms induced by f (where, this time, (X,I) is seen as the constant functor from I to $\mathscr{S}$ with value X).

Moreover, for any object M of $\mathrm{Ho}(\mathscr{M})(\int \mathscr{F}, I_{\mathscr{F}})$, the natural map

$$i^* \,\mathbf{R}\varphi_*(M) = \mathbf{R}\varphi_*(M)_i \to \mathbf{R}\varphi_{i,*}\, \lambda_i^*(M)$$

is an isomorphism. In particular, if we also write by abuse of notation f for the induced map of $\mathscr{S}$-diagrams from $(\int \mathscr{F}, I_{\mathscr{F}})$ to X, we have a natural isomorphism

$$\mathbf{R}f_*(M) \simeq \mathbf{R}\varprojlim_{i \in I^{op}} \mathbf{R}\varphi_{i,*}\, \lambda_i^*(M)\,.$$

Proof. This pullback square is the one induced by (3.1.22.1). We shall prove first that the map

$$i^* \,\mathbf{R}\varphi_*(M) = \mathbf{R}\varphi_*(M)_i \to \mathbf{R}\varphi_{i,*}\, \lambda_i^*(M)$$

is an isomorphism in the particular case where I has a terminal object ω and $i = \omega$. By virtue of Propositions 3.1.15 and 3.1.16, we have isomorphisms

(3.1.23.1)
$$\omega^* \,\mathbf{R}\varphi_*(M) \simeq \mathbf{R}\varprojlim_{i \in I^{op}} \mathbf{R}\varphi_*(M)_i \simeq \mathbf{R}\varprojlim_{(i,x) \in I^{op}_{\mathscr{F}}} \mathbf{R}\varphi_{i,x,*}\,(M_{(i,x)})\,,$$

where $\varphi_{i,x} : \mathscr{F}(i)_x \to X$ denotes the map induced by f. We are thus reduced to prove that the canonical map

(3.1.23.2)
$$\mathbf{R}\varprojlim_{(i,x) \in I^{op}_{\mathscr{F}}} \mathbf{R}\varphi_{i,x,*}\,(M_{(i,x)}) \to \mathbf{R}\varprojlim_{x \in J^{op}_{\omega}} \mathbf{R}\varphi_{\omega,x,*}\,(M_{(\omega,x)}) \simeq \mathbf{R}\varphi_{\omega,*}\, \lambda_\omega^*(M)$$

is an isomorphim. As $I_{\mathscr{F}}$ is cofibred over I, and as ω is a terminal object of I, the inclusion functor $\ell_\omega : J_\omega \to I_{\mathscr{F}}$ has a left adjoint, whence is coaspherical in any weak basic localizer (i.e. is homotopy cofinal); see [**Mal05**, 1.1.9, 1.1.16 and 1.1.25]. As any model category defines a

Grothendieck derivator ([**Cis03**, Thm. 6.11]), it follows from [**Cis03**, Cor. 1.15] that the map (3.1.23.2) is an isomorphism.

To prove the general case, we proceed as follows. Let $\mathscr{F}/i$ be the functor obtained by composing $\mathscr{F}$ with the canonical functor $v_i : I/i \to I$. Then, keeping track of the conventions adopted in 3.1.12, we check easily that $(I/i)_{\mathscr{F}/i} = (I_{\mathscr{F}})/i$ and that $\int(\mathscr{F}/i) = (\int\mathscr{F})/i$. Moreover, the pullback square (3.1.22.1) is the composition of the following pullback squares of categories.

$$\begin{array}{ccccc} J_i & \xrightarrow{a_i} & I_{\mathscr{F}}/i & \xrightarrow{u_i} & I_{\mathscr{F}} \\ {\scriptstyle q}\downarrow & & {\scriptstyle p/i}\downarrow & & \downarrow{\scriptstyle p} \\ e & \xrightarrow[(i,1_i)]{} & I/i & \xrightarrow[v_i]{} & I \end{array}$$

The pullback square of the proposition is thus the composition of the following pullback squares.

$$\begin{array}{ccccc} (\mathscr{F}(i), J_i) & \xrightarrow{\alpha_i} & (\int\mathscr{F}/i, I_{\mathscr{F}}/i) & \xrightarrow{\mu_i} & (\int\mathscr{F}, I_{\mathscr{F}}) \\ {\scriptstyle \varphi_i}\downarrow & & {\scriptstyle \varphi/i}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow[(i,1_i)]{} & (X, I/i) & \xrightarrow[v_i]{} & (X, I) \end{array}$$

The natural transformations

$$(i,1_i)^*\,\mathbf{R}(\varphi/i)_* \to \mathbf{R}\varphi_{i,*}\,\alpha_i^* \quad \text{and} \quad v_i^*\,\mathbf{R}\varphi_* \to \mathbf{R}(\varphi/i)_*\,\mu_i^*$$

are both isomorphisms: the first one comes from the fact that $(i,1_i)$ is a terminal object of I/i, and the second one from Proposition 3.1.16. We thus get:

$$\begin{aligned} i^*\,\mathbf{R}\varphi_*(M) &\simeq (i,1_i)^*\,v_i^*\,\mathbf{R}\varphi_*(M) \\ &\simeq (i,1_i)^*\,\mathbf{R}(\varphi/i)_*\,\mu_i^*(M) \\ &\simeq \mathbf{R}\varphi_{i,*}\,\alpha_i^*\,\mu_i^*(M) \\ &\simeq \mathbf{R}\varphi_{i,*}\,\lambda_i^*(M)\,. \end{aligned}$$

The last assertion of the proposition is then a straightforward application of Proposition 3.1.15. □

PROPOSITION 3.1.24. *If $\mathscr{M}$ is a monoidal $\mathscr{P}$-fibred combinatorial model category over $\mathscr{S}$, then, for any $\mathscr{S}$-diagram $\mathscr{X}$ indexed by a small category I, the injective model structure turns $\mathscr{M}(\mathscr{X}, I)$ into a symmetric monoidal model category. In particular, the categories $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$ are canonically endowed with a closed symmetric monoidal structure, in such a way that, for any morphism of $\mathscr{S}$-diagrams $\varphi : (\mathscr{X}, I) \to (\mathscr{Y}, J)$, the functor $\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})(\mathscr{Y}, J) \to \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$ is symmetric monoidal.*

PROOF. This is obvious from the definition of a symmetric monoidal model category, as the tensor product of $\mathscr{M}(\mathscr{X}, I)$ is defined termwise, as well as the cofibrations and the trivial cofibrations. □

PROPOSITION 3.1.25. *Assume that $\mathscr{M}$ is a monoidal $\mathscr{P}$-fibred combinatorial model category over $\mathscr{S}$, and consider a reduced cartesian $\mathscr{P}$-morphism $\varphi = (\alpha, f) : (\mathscr{X}, I) \to (\mathscr{Y}, I)$. Then, for any object M in $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$ and any object N in $\mathrm{Ho}(\mathscr{M})(\mathscr{Y}, I)$, the canonical map*

$$\mathbf{L}\varphi_\sharp(M \otimes^{\mathbf{L}} \varphi^*(N)) \to \mathbf{L}\varphi_\sharp(M) \otimes^{\mathbf{L}} N$$

is an isomorphism.

PROOF. Let i be an object of I. It is sufficient to prove that the map

$$i^*\mathbf{L}\varphi_\sharp(M \otimes^{\mathbf{L}} \varphi^*(N)) \to i^*\mathbf{L}\varphi_\sharp(M) \otimes^{\mathbf{L}} N$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(\mathscr{X}_i)$. Using Corollary 3.1.20, we see that this map can be identified with the map

$$\mathbf{L}\varphi_{i,\sharp}(M_i \otimes^{\mathbf{L}} \varphi_i^*(N_i)) \to \mathbf{L}\varphi_{i,\sharp}(M_i) \otimes^{\mathbf{L}} N_i\,,$$

which is an isomorphism according to the $\mathscr{P}$-projection formula for the homotopy $\mathscr{P}$-fibred category $\mathrm{Ho}(\mathscr{M})$. □

3.1.26. Let $(\mathscr{X}, I)$ be a $\mathscr{S}$-diagram. An object M of $\mathscr{M}(\mathscr{X}, I)$ is *homotopy cartesian* if, for any map $u : i \to j$ in I, the structural map $u^*(M_j) \to M_i$ induces an isomorphism

$$\mathbf{L}u^*(M_i) \simeq M_j$$

in $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$ (i.e. if there exists a weak equivalence $M'_j \to M_j$ with M'_j cofibrant in $\mathscr{M}(\mathscr{X}_j)$ such that the map $u^*(M'_j) \to M_i$ is a weak equivalence in $\mathscr{M}(\mathscr{X}_i)$).

We denote by $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)_{hcart}$ the full subcategory of $\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$ spanned by homotopy cartesian sections.

Definition 3.1.27. A cofibrantly generated model category $\mathscr{V}$ is *tractable* if there exist sets I and J of cofibrations between cofibrant objects which generate the class of cofibrations and the class of trivial cofibrations respectively.

Remark 3.1.28. If $\mathscr{M}$ is a combinatorial and tractable $\mathscr{P}$-fibred model category over $\mathscr{S}$, then so are the projective and the injective model structures on $\mathscr{M}(\mathscr{X}, I)$; see [**Bar10**, Thm. 2.28 and 2.30].

Proposition 3.1.29. *If $\mathscr{M}$ is tractable, then the inclusion functor*

$$\mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)_{hcart} \to \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I)$$

admits a right adjoint.

Proof. This follows from the fact that the cofibrant homotopy cartesian sections are the cofibrant objects of a right Bousfield localization of the injective model structure on $\mathscr{M}(\mathscr{X}, I)$; see [**Bar10**, Theorem 5.25]. □

Definition 3.1.30. Let $\mathscr{M}$ and $\mathscr{M}'$ two $\mathscr{P}$-fibred model categories over $\mathscr{S}$. A *Quillen morphism* γ from $\mathscr{M}$ to $\mathscr{M}'$ is a morphism of $\mathscr{P}$-fibred categories $\gamma : \mathscr{M} \to \mathscr{M}'$ such that $\gamma^* : \mathscr{M}(X) \to \mathscr{M}'(X)$ is a left Quillen functor for any object X of $\mathscr{S}$.

Remark 3.1.31. If $\gamma : \mathscr{M} \to \mathscr{M}'$ is a Quillen morphism between $\mathscr{P}$-fibred combinatorial model categories, then, for any $\mathscr{S}$-diagram $(\mathscr{X}, I)$, we get a Quillen adjunction

$$\gamma^* : \mathscr{M}(\mathscr{X}, I) \rightleftarrows \mathscr{M}'(\mathscr{X}, I) : \gamma_*$$

(with the injective model structures as well as with the projective model structures).

Proposition 3.1.32. *For any Quillen morphism $\gamma : \mathscr{M} \to \mathscr{M}'$, the derived adjunctions*

$$\mathbf{L}\gamma^* : \mathrm{Ho}(\mathscr{M})(X) \rightleftarrows \mathrm{Ho}(\mathscr{M}')(X) : \mathbf{R}\gamma_*$$

define a morphism of $\mathscr{P}$-fibred categories $\mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{M}')$ over $\mathscr{S}$. If moreover $\mathscr{M}$ and $\mathscr{M}'$ are combinatorial, then the morphism $\mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{M}')$ extends to a morphism of $\mathscr{P}_{cart}$-fibred categories over the category of $\mathscr{S}$-diagrams.

Proof. This follows immediately from [**Hov99**, Theorem 1.4.3]. □

3.2. Hypercovers, descent, and derived global sections.

3.2.1. Let $\mathscr{S}$ be an essentially small category, and $\mathscr{P}$ an admissible class of morphisms in $\mathscr{S}$. We assume that a Grothendieck topology t on $\mathscr{S}$ is given. We shall write $\mathscr{S}^{\amalg}$ for the full subcategory of the category of $\mathscr{S}$-diagrams whose objects are the small families $X = \{X_i\}_{i \in I}$ of objects of $\mathscr{S}$ (seen as functors from a discrete category to $\mathscr{S}$). The category $\mathscr{S}^{\amalg}$ is equivalent to the full subcategory of the category of presheaves of sets on $\mathscr{S}$ spanned by sums of representable presheaves. In particular, small sums are representable in $\mathscr{S}^{\amalg}$ (but note that the functor from $\mathscr{S}$ to $\mathscr{S}^{\amalg}$ does not preserve sums). Finally, we remark that the topology t extends naturally to a Grothendieck topology on $\mathscr{S}^{\amalg}$ such that the topology t on $\mathscr{S}$ is the topology induced from the inclusion $\mathscr{S} \subset \mathscr{S}^{\amalg}$. The covering maps for this topology on $\mathscr{S}^{\amalg}$ will be called *t-covers* (note that the inclusion $\mathscr{S} \subset \mathscr{S}^{\amalg}$ is continuous and induces an equivalence between the topos of t-sheaves on $\mathscr{S}$ and the topos of t-sheaves on $\mathscr{S}^{\amalg}$).

Let Δ be the category of non-empty finite ordinals. Remember that a simplicial object of $\mathscr{S}^{\amalg}$ is a presheaf on Δ with values in $\mathscr{S}^{\amalg}$. For a simplicial set K and an object X of $\mathscr{S}^{\amalg}$, we denote by $K \times X$ the simplicial object of $\mathscr{S}^{\amalg}$ defined by

$$(K \times X)_n = \coprod_{x \in K_n} X \quad , \qquad n \geq 0 \, .$$

We write Δ^n for the standard combinatorial simplex of dimension n, and $i_n : \partial\Delta^n \to \Delta^n$ for its boundary inclusion.

A morphism $p : \mathscr{X} \to \mathscr{Y}$ between simplicial objects of $\mathscr{S}^{\amalg}$ is a *t-hypercover* if, locally for the t-topology, it has the right lifting property with respect to boundary inclusions of standard simplices, which, in a more precise way, means that, for any integer $n \geq 0$, any object U of $\mathscr{S}^{\amalg}$, and any commutative square

$$\begin{array}{ccc} \partial\Delta^n \times U & \xrightarrow{x} & \mathscr{X} \\ {\scriptstyle i_n \times 1}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n \times U & \xrightarrow[y]{} & \mathscr{Y} \end{array} \qquad ,$$

there exists a t-covering $q : V \to U$, and a morphism of simplicial objects $z : \Delta^n \times V \to \mathscr{X}$, such that the diagram bellow commutes.

$$\begin{array}{ccc} \partial\Delta^n \times V & \xrightarrow{x(1\times q)} & \mathscr{X} \\ {\scriptstyle i_n \times 1}\downarrow & \nearrow^{z} & \downarrow{\scriptstyle p} \\ \Delta^n \times V & \xrightarrow[y(1\times q)]{} & \mathscr{Y} \end{array}$$

A *t-hypercover* of an object X of $\mathscr{S}^{\amalg}$ is a a t-hypercover $p : \mathscr{X} \to X$ (where X is considered as a constant simplicial object).

Remark 3.2.2. This definition of t-hypercover is equivalent to the one given in [**SGA4**, Exp. V, 7.3.1.4].

3.2.3. Let $\mathscr{X}$ be a simplicial object of $\mathscr{S}^{\amalg}$. It is in particular a functor from the category Δ^{op} to the category of $\mathscr{S}$-diagrams, so that the constructions and considerations of 3.1.22 apply to $\mathscr{X}$. In particular, there is a $\mathscr{S}$-diagram $\tilde{\mathscr{X}}$ associated to $\mathscr{X}$, namely $\tilde{\mathscr{X}} = (\int \mathscr{X}, (\Delta^{op})_{\mathscr{X}})$. More explicitely, for each integer $n \geq 0$, there is a family $\{\mathscr{X}_{n,x}\}_{x \in K_n}$ of objects of $\mathscr{S}$, such that

$$\mathscr{X}_n = \coprod_{x \in K_n} \mathscr{X}_{n,x} \, . \tag{3.2.3.1}$$

In fact, the sets K_n form a simplicial set K, and the category $(\Delta^{op})_{\mathscr{X}}$ can be identified over Δ^{op} to the category $(\Delta/K)^{op}$, where Δ/K is the fibred category over Δ whose fiber over n is the set K_n (seen as a discrete category), i.e. the category of simplices of K. We shall call K the *underlying simplicial set of $\mathscr{X}$*, while the decomposition (3.2.3.1) will be called the *local presentation of $\mathscr{X}$*. The construction $\mathscr{X} \mapsto \tilde{\mathscr{X}}$ is functorial. If $p : \mathscr{X} \to \mathscr{Y}$ is a morphism of simplicial objects of $\mathscr{S}^{\amalg}$, we shall still denote by $p : \tilde{\mathscr{X}} \to \tilde{\mathscr{Y}}$ the induced morphism of $\mathscr{S}$-diagrams. In particular, for a morphism of $p : \mathscr{X} \to X$, where X is an object of $\mathscr{S}^{\amalg}$, $p : \tilde{\mathscr{X}} \to X$ denotes the corresponding morphism of $\mathscr{S}$-diagrams.

Let $\mathscr{M}$ be a $\mathscr{P}$-fibred combinatorial model category over $\mathscr{S}$. Given a simplicial object $\mathscr{X}$ of $\mathscr{S}^{\amalg}$, we define the category $\mathrm{Ho}(\mathscr{M})(\mathscr{X})$ by the formula:

$$\mathrm{Ho}(\mathscr{M})(\mathscr{X}) = \mathrm{Ho}(\mathscr{M})(\textstyle\int \mathscr{X}, (\Delta^{op})_{\mathscr{X}}) \, . \tag{3.2.3.2}$$

Given an object X of $\mathscr{S}^{\amalg}$ and a morphism $p : \mathscr{X} \to X$, we have a derived adjunction

$$\mathbf{L}p^* : \mathrm{Ho}(\mathscr{M})(X) \rightleftarrows \mathrm{Ho}(\mathscr{M})(\mathscr{X}) : \mathbf{R}p_* \, . \tag{3.2.3.3}$$

PROPOSITION 3.2.4. *Consider an object X of $\mathscr{S}$, a simplicial object $\mathscr{X}$ of $\mathscr{S}^{\amalg}$, as well as a morphism $p : \mathscr{X} \to X$. Denote by K the underlying simplicial set of $\mathscr{X}$, and for each integer $n \geq 0$ and each simplex $x \in K_n$, write $p_{n,x} : \mathscr{X}_{n,x} \to X$ for the morphism of $\mathscr{S}^{\amalg}$ induced by the local presentation of $\mathscr{X}$* (3.2.3.1)*. Then, for any object M of* $\mathrm{Ho}(\mathscr{M})(X)$*, there are canonical isomorphisms*

$$\mathbf{R}p_*\mathbf{R}p^*(M) \simeq \mathbf{R}\varprojlim_{n\in\Delta} \mathbf{R}p_{n,*}\mathbf{L}p_n^*(M) \simeq \mathbf{R}\varprojlim_{n\in\Delta} \Big(\prod_{x\in K_n} \mathbf{R}p_{n,x,*}\mathbf{L}p_{n,x}^*(M) \Big) .$$

PROOF. The first isomorphism is a direct application of the last assertion of Proposition 3.1.23 for $\mathscr{F} = \mathscr{X}$, while the second one follows from the first one by Proposition 3.1.10. □

DEFINITION 3.2.5. Given an object Y of $\mathscr{S}^{\amalg}$, an object M of $\mathrm{Ho}(\mathscr{M})(Y)$ will be said to satisfy *t-descent* if it has the following property: for any morphism $f : X \to Y$ and any t-hypercover $p : \mathscr{X} \to X$, the map

$$\mathbf{R}f_*\,\mathbf{L}f^*(M) \to \mathbf{R}f_*\,\mathbf{R}p_*\,\mathbf{L}p^*\,\mathbf{L}f^*(M)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(Y)$.

We shall say that $\mathscr{M}$ (or by abuse, that $\mathrm{Ho}(\mathscr{M})$) satisfies *t-descent* if, for any object Y of $\mathscr{S}^{\amalg}$, any object of $\mathrm{Ho}(\mathscr{M})(Y)$ satisfies t-descent.

PROPOSITION 3.2.6. *If $Y = \{Y_i\}_{i\in I}$ is a small family of objects of $\mathscr{S}$ (seen as an object of $\mathscr{S}^{\amalg}$), then an object M of* $\mathrm{Ho}(\mathscr{M})(Y)$ *satisfies t-descent if and only if, for any $i \in I$, any morphism $f : X \to Y_i$ of $\mathscr{S}$, and any t-hypercover $p : \mathscr{X} \to X$, the map*

$$\mathbf{R}f_*\,\mathbf{L}f^*(M_i) \to \mathbf{R}f_*\,\mathbf{R}p_*\,\mathbf{L}p^*\,\mathbf{L}f^*(M_i)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(Y_i)$.

PROOF. This follows from the definition and from Proposition 3.1.10. □

COROLLARY 3.2.7. *The $\mathscr{P}$-fibred model category $\mathscr{M}$ satisfies t-descent if and only if, for any object X of $\mathscr{S}$, and any t-hypercover $p : \mathscr{X} \to X$, the functor*

$$\mathbf{L}p^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\mathscr{X})$$

is fully faithful.

PROPOSITION 3.2.8. *If $\mathscr{M}$ satisfies t-descent, then, for any t-cover $f : Y \to X$, the functor*

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Y)$$

is conservative.

PROOF. Let $f : Y \to X$ be a t-cover, and $u : M \to M'$ a morphism of $\mathrm{Ho}(\mathscr{M})(X)$ whose image by $\mathbf{L}f^*$ is an isomorphism. We can consider the Čech t-hypercover associated to f, that is the simplicial object $\mathscr{Y}$ over X defined by

$$\mathscr{Y}_n = \underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n+1 \text{ times}} .$$

Let $p : \mathscr{Y} \to X$ be the canonical map. For each $n \geq 0$, the map $p_n : \mathscr{Y}_n \to X$ factor through f, from wich we deduce that the functor

$$\mathbf{L}p_n^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\mathscr{Y}_n)$$

sends u to an isomorphism. This implies that the functor

$$\mathbf{L}p^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\mathscr{Y})$$

sends u to an isomorphism as well. But, as $\mathscr{Y}$ is a t-hypercover of X, the functor $\mathbf{L}p^*$ is fully faithful, from which we deduce that u is an isomorphism by the Yoneda Lemma. □

3.2.9. Let $\mathscr{V}$ be a complete and cocomplete category. For an object X of $\mathscr{S}$, define $\mathrm{PSh}\,(\mathscr{S}/X, \mathscr{V})$ as the category of presheaves on $\mathscr{S}/X$ with values in $\mathscr{V}$. Then $\mathrm{PSh}\,(C/-, \mathscr{V})$ is a $\mathscr{P}$-fibred category (where, by abuse of notations, $\mathscr{S}$ denotes also the class of all maps in $\mathscr{S}$): this is a special case of the constructions explained in 3.1.2 applied to $\mathscr{V}$, seen as a fibred category over the terminal category. To be more explicit, for each object X of $\mathscr{S}^{\amalg}$, we have a $\mathscr{V}$-enriched Yoneda embedding

$$\mathscr{S}^{\amalg}/X \times \mathscr{V} \to \mathrm{PSh}\,(\mathscr{S}/X, \mathscr{V}) \quad , \qquad (U, M\} \mapsto U \otimes M\,, \tag{3.2.9.1}$$

where, if $U = \{U_i\}_{i\in I}$ is a small family of objects of $\mathscr{S}/X$, $U \otimes M$ is the presheaf

$$V \mapsto \coprod_{i\in I} \coprod_{a\in \mathrm{Hom}_{\mathscr{S}/S}(V,U_i)} M\,. \tag{3.2.9.2}$$

For a morphism $f : X \to Y$ in $\mathscr{S}$, the functor

$$f^* : \mathrm{PSh}\,(\mathscr{S}/Y, \mathscr{V}) \to \mathrm{PSh}\,(\mathscr{S}/X, \mathscr{V})$$

is the functor defined by composition with the corresponding functor $\mathscr{S}/X \to \mathscr{S}/Y$. The functor f^* has always a left adjoint

$$f_\sharp : \mathrm{PSh}\,(\mathscr{S}/X, \mathscr{V}) \to \mathrm{PSh}\,(\mathscr{S}/Y, \mathscr{V})\ ,$$

which is the unique colimit preserving functor defined by

$$f_\sharp(U \otimes M) = U \otimes M\,,$$

where, on the left hand side U is considered as an object over X, while, on the right hand side, U is considered as an object over Y by composition with f. Similary, if all the pullbacks by f are representable in $\mathscr{S}$ (e.g. if f is a $\mathscr{P}$-morphism), the functor f^* can be described as the colimit preserving functor defined by the formula

$$f^*(U \otimes M) = (X \times_Y U) \otimes M\,.$$

If $\mathscr{V}$ is a cofibrantly generated model category, then, for each object X of $\mathscr{S}$, the category $\mathrm{PSh}\,(\mathscr{S}/X, \mathscr{V})$ is naturally endowed with the *projective model category structure*, i.e. with the cofibrantly generated model category structure whose weak equivalences and fibrations are defined termwise (this is Proposition 3.1.6 applied to $\mathscr{V}$, seen as a fibred category over the terminal category). The cofibrations of the projective model category structure on $\mathrm{PSh}\,(\mathscr{S}/X, \mathscr{V})$ will be called the *projective cofibrations.* If moreover $\mathscr{V}$ is combinatorial (resp. left proper, resp. right proper, resp. stable), so is $\mathrm{PSh}\,(\mathscr{S}/X, \mathscr{V})$. In particular, if $\mathscr{V}$ is a combinatorial model category, then $\mathrm{PSh}\,(\mathscr{S}/-, \mathscr{V})$ is a $\mathscr{P}$-fibred combinatorial model category over $\mathscr{S}$.

According to Definition 3.2.5, it thus makes sense to speak of t-descent in $\mathrm{PSh}\,(\mathscr{S}/-, \mathscr{V})$.

If $U = \{U_i\}_{i\in I}$ is a small family of objects of $\mathscr{S}$ over X, and if F is a presheaf over $\mathscr{S}/X$, we define

$$F(U) = \prod_{i\in I} F(U_i)\,. \tag{3.2.9.3}$$

the functor $F \mapsto F(U)$ is a right adjoint to the functor $E \mapsto U \otimes E$.

We remark that a termwise fibrant presheaf F on $\mathscr{S}/X$ satisfies t-descent if and only if, for any object Y of $\mathscr{S}^{\amalg}$, and any t-hypercover $\mathscr{Y} \to Y$ over X, the map

$$F(Y) \to \mathbf{R}\varprojlim_{n\in\Delta} F(\mathscr{Y}_n)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{V})$.

Proposition 3.2.10. *If $\mathscr{V}$ is combinatorial and left proper, then the category of presheaves $\mathrm{PSh}\,(\mathscr{S}/X, \mathscr{V})$ admits a combinatorial model category structure whose cofibrations are the projective cofibrations, and whose fibrant objects are the termwise fibrant objects which satisfy t-descent. This model category structure will be called the* t-local model category structure*, and the corresponding homotopy category will be denoted by* $\mathrm{Ho}_t(\mathrm{PSh}\,(\mathscr{S}/X, \mathscr{V}))$.

Moreover, any termwise weak equivalence is a weak equivalence for the t-local model structure, and the induced functor

$$a^* : \mathrm{Ho}(\mathrm{PSh}\,(\mathscr{S}/X,\mathscr{V})) \to \mathrm{Ho}_t(\mathrm{PSh}\,(\mathscr{S}/X,\mathscr{V}))$$

admits a fully faithful right adjoint

$$a_* : \mathrm{Ho}_t(\mathrm{PSh}\,(\mathscr{S}/X,\mathscr{V})) \to \mathrm{Ho}(\mathrm{PSh}\,(\mathscr{S}/X,\mathscr{V}))$$

whose essential image consists precisely of the full subcategory of $\mathrm{Ho}(\mathrm{PSh}\,(\mathscr{S}/X,\mathscr{V}))$ *spanned by the presheaves which satisfy t-descent.*

Proof. Let H be the class of maps of shape

$$\operatorname*{hocolim}_{n\in\Delta^{op}} \mathscr{Y}_n \otimes E \to Y \otimes E\,, \tag{3.2.10.1}$$

where Y is an object of $\mathscr{S}^{\amalg}$ over X, $\mathscr{Y} \to Y$ is a t-hypercover, and E is a cofibrant replacement of an object which is either a source or a target of a generating cofibration of $\mathscr{V}$. Define the t-local model category structure as the left Bousfield localization of $Pr(\mathscr{S}/X,\mathscr{V})$ by H; see [**Bar10**, Theorem 4.7]. We shall call *t-local weak equivalences* the weak equivalences of the t-local model category structure. For each object Y over X, the functor $Y \otimes (-)$ is a left Quillen functor from $\mathscr{V}$ to $Pr(\mathscr{S}/X,\mathscr{V})$. We thus get a total left derived functor

$$Y \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{V}) \to \mathrm{Ho}_t(\mathrm{PSh}\,(\mathscr{S}/X,\mathscr{V}))$$

whose right adjoint is the evaluation at Y. For any object E of $\mathscr{V}$ and any t-local fibrant presheaf F on $\mathscr{S}/X$ with values in $\mathscr{V}$, we thus have natural bijections

$$\mathrm{Hom}(E, F(Y)) \simeq \mathrm{Hom}(Y \otimes^{\mathbf{L}} E, F)\,, \tag{3.2.10.2}$$

and, for any simplicial object $\mathscr{Y}$ of $\mathscr{S}/X$, identifications

$$\mathrm{Hom}(E,\, \mathbf{R}\varprojlim_{n\in\Delta} F(\mathscr{Y}_n)) \simeq \mathrm{Hom}(\,\mathbf{L}\varinjlim_{n\in\Delta} \mathscr{Y}_n \otimes^{\mathbf{L}} E, F)\,, \tag{3.2.10.3}$$

One sees easily that, for any t-hypercover $\mathscr{Y} \to Y$ and any cofibrant object E of $\mathscr{V}$, the map

$$\mathbf{L}\varinjlim_{n\in\Delta} \mathscr{Y}_n \otimes^{\mathbf{L}} E \to Y \otimes^{\mathbf{L}} E \tag{3.2.10.4}$$

is an isomorphism in the t-local homotopy category $\mathrm{Ho}_t(\mathrm{PSh}\,(\mathscr{S}/X,\mathscr{V}))$: by the small object argument, the smallest full subcategory of $\mathrm{Ho}(\mathrm{PSh}\,(\mathscr{S}/X,\mathscr{V}))$ which is stable by homotopy colimits and which contains the source and the targets of the generating cofibrations is $\mathrm{Ho}_t(\mathrm{PSh}\,(\mathscr{S}/X,\mathscr{V}))$ itself, and the class of objects E of $\mathscr{V}$ such that the map (3.2.10.4) is an isomorphism in $\mathrm{Ho}(\mathscr{V})$ is sable by homotopy colimits. Similarly, we see that, for any object E, the functor $(-) \otimes^{\mathbf{L}} E$ preserves sums. As a consequence, we get from (3.2.10.2) and (3.2.10.3) that the fibrant objects of the t-local model category structure are precisely the termwise fibrant objects F of the projective model structure which satisfy t-descent. The last part of the proposition follows from the general yoga of left Bousfield localizations. □

3.2.11. Let $\mathscr{M}$ be a $\mathscr{P}$-fibred combinatorial model category over $\mathscr{S}$, and S an object of $\mathscr{S}$. Denote by

$$\mathscr{S} : \mathscr{S}/S \to \mathscr{S}$$

the canonical forgetful functor. Then there is a canonical morphism of $\mathscr{S}$-diagrams

$$\sigma : (\mathscr{S}, \mathscr{S}/S) \to (S, \mathscr{S}/S) \tag{3.2.11.1}$$

(where $(S,\mathscr{S}/S)$ stands for the constant diagram with value S). This defines a functor

$$\mathbf{R}\sigma_* : \mathrm{Ho}(\mathscr{M})(\mathscr{S},\mathscr{S}/S) \to \mathrm{Ho}(\mathscr{M})(S,\mathscr{S}/S) = \mathrm{Ho}(\mathrm{PSh}\,(\mathscr{S}/S,\mathscr{M}(S)))\,. \tag{3.2.11.2}$$

For an object M of $\mathrm{Ho}(\mathscr{M})(S)$, one defines the presheaf of *geometric derived global sections of M over S* by the formula

$$\mathbf{R}\Gamma_{geom}(-,M) = \mathbf{R}\sigma_*\,\mathbf{L}\sigma^*(M)\,. \tag{3.2.11.3}$$

This is a presheaf on $\mathscr{S}/S$ with values in $\mathscr{M}(S)$ whose evaluation on a morphism $f : X \to S$ is, by virtue of Propositions 3.1.15 and 3.1.16,

$$\mathbf{R}\Gamma_{geom}(X, M) \simeq \mathbf{R}f_*\,\mathbf{L}f^*(M)\,. \tag{3.2.11.4}$$

PROPOSITION 3.2.12. *For an object M of* $\mathrm{Ho}(\mathscr{M})(S)$*, the following conditions are equivalent.*

(a) The object M satisfies t-descent.
(b) The presheaf $\mathbf{R}\Gamma_{geom}(-, M)$ satisfies t-descent.

PROOF. For any morphism $f : X \to S$ and any t-hypercover $p : \mathscr{X} \to X$ over S, we have, by Proposition 3.2.4 and formula (3.2.11.4), an isomorphism

$$\mathbf{R}f_*\mathbf{R}p_*\,\mathbf{L}p^*\,\mathbf{L}f^*(M) \simeq \mathbf{R}\varprojlim_{n\in\Delta}\mathbf{R}\Gamma_{geom}(\mathscr{X}_n, M)\,.$$

From there, we see easily that conditions (a) and (b) are equivalent. □

3.2.13. The preceding proposition allows to reduce descent problems in a fibred model category to descent problems in a category of presheaves with values in a model category. On can even go further and reduce the problem to category of presheaves with values in an 'elementary model category' as follows.

Consider a model category $\mathscr{V}$. Then one can associate to $\mathscr{V}$ its corresponding *prederivator* $\mathbf{Ho}(\mathscr{V})$, that is the strict 2-functor from the 2-category of small categories to the 2-category of categories, defined by

$$\mathbf{Ho}(\mathscr{V})(I) = \mathrm{Ho}(\mathscr{V}^{I^{op}}) = \mathrm{Ho}(\mathrm{PSh}\,(I, \mathscr{V})) \tag{3.2.13.1}$$

for any small category I. More explicitly: for any functor $u : I \to J$, one gets a functor

$$u^* : \mathbf{Ho}(\mathscr{V})(J) \to \mathbf{Ho}(\mathscr{V})(I)$$

(induced by the composition with u), and for any morphism of functors

$$I \underset{v}{\overset{u}{\rightrightarrows}} J \quad (\Downarrow \alpha)\quad ,$$

one has a morphism of functors

$$\mathbf{Ho}(\mathscr{V})(I) \underset{v^*}{\overset{u^*}{\leftleftarrows}} \mathbf{Ho}(\mathscr{V})(J) \quad (\alpha^* \Uparrow)\quad .$$

Moreover, the prederivator $\mathbf{Ho}(\mathscr{V})$ is then a Grothendieck derivator; see [**Cis03**, Thm. 6.11]. This means in particular that, for any functor between small categories $u : I \to J$, the functor u^* has a left adjoint

$$\mathbf{L}u_\sharp : \mathbf{Ho}(\mathscr{V})(I) \to \mathbf{Ho}(\mathscr{V})(J) \tag{3.2.13.2}$$

as well as a right adjoint

$$\mathbf{R}u_* : \mathbf{Ho}(\mathscr{V})(I) \to \mathbf{Ho}(\mathscr{V})(J) \tag{3.2.13.3}$$

(in the case where $J = e$ is the terminal category, then $\mathbf{L}u_\sharp$ is the homotopy colimit functor, while $\mathbf{R}u_*$ is the homotopy limit functor).

If $\mathscr{V}$ and $\mathscr{V}'$ are two model categories, a *morphism of derivators*

$$\Phi : \mathbf{Ho}(\mathscr{V}) \to \mathbf{Ho}(\mathscr{V}')$$

is simply a morphism of 2-functors, that is the data of functors

$$\Phi_I : \mathbf{Ho}(\mathscr{V})(I) \to \mathbf{Ho}(\mathscr{V}')(I)$$

together with coherent isomorphisms

$$u^*(\Phi_J(F)) \simeq \Phi_I(u^*(F))$$

for any functor $u : I \to J$ and any presheaf F on J with values in $\mathscr{V}$ (see [**Cis03**, p. 210] for a precise definition).

Such a morphism Φ is said to be *continuousmorphism!continuous* if, for any functor $u : I \to J$, and any object F of $\mathbf{Ho}(\mathscr{V})(I)$, the canonical map

$$\Phi_J \, \mathbf{R}u_*(F) \to \mathbf{R}u_* \, \Phi_I(F) \tag{3.2.13.4}$$

is an isomorphism. One can check that a morphism of derivators Φ is continuous if and only if it commutes with homotopy limits (i.e. if and only if the maps (3.2.13.4) are isomorphisms in the case where $J = e$ is the terminal category); see [**Cis08**, Prop. 2.6]. For instance, the total right derived functor of any right Quillen functor defines a continuous morphism of derivators; see [**Cis03**, Prop. 6.12].

Dually a morphism Φ of derivators is *cocontinuous* if, for any functor $u : I \to J$, and any object F of $\mathbf{Ho}(\mathscr{V})(I)$, the canonical map

$$\mathbf{L}u_! \, \Phi_I(F) \to \Phi_J \, \mathbf{L}u_!(F) \tag{3.2.13.5}$$

is an isomorphism.

3.2.14. We shall say that a stable model category $\mathscr{V}$ is $\mathbf{Q}$-*linear* if all the objects of the triangulated category $\mathrm{Ho}(\mathscr{V})$ are uniquely divisible.

THEOREM 3.2.15. *Let $\mathscr{V}$ be a model category (resp. a stable model category, resp. a $\mathbf{Q}$-linear stable model category), and denote by $\mathcal{S}$ the model category of simplicial sets (resp. the stable model category of S^1-spectra, resp. the $\mathbf{Q}$-linear stable model category of complexes of $\mathbf{Q}$-vector spaces). Denote by $\mathbb{1}$ the unit object of the closed symmetric monoidal category $\mathrm{Ho}(\mathcal{S})$.*

Then, for each object E of $\mathrm{Ho}(\mathscr{V})$, there exists a unique continuous morphism of derivators

$$\mathbf{R}\mathrm{Hom}(E, -) : \mathbf{Ho}(\mathscr{V}) \to \mathbf{Ho}(\mathcal{S})$$

such that, for any object F of $\mathrm{Ho}(\mathscr{V})$, there is a functorial bijection

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(\mathbb{1}, \mathbf{R}\mathrm{Hom}(E, F)) \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{V})}(E, F)) \, .$$

PROOF. Note that the stable $\mathbf{Q}$-linear case follows from the stable case and from the fact that the derivator of complexes of $\mathbf{Q}$-vector spaces is (equivalent to) the full subderivator of the derivator of S^1-spectra spanned by uniquely divisible objects.

It thus remains to prove the theorem in the case where $\mathscr{V}$ be a model category (resp. a stable model category) and $\mathcal{S}$ is the model category of simplicial sets (resp. the stable model category of S^1-spectra). The existence of $\mathbf{R}\mathrm{Hom}(E, -)$ follows then from [**Cis03**, Prop. 6.13] (resp. [**CT11**, Lemma A.6]).

For the unicity, as we don't really need it here, we shall only sketch the proof (the case of simplicial sets is done in [**Cis03**, Rem. 6.14]). One uses the universal property of the derivator $\mathbf{Ho}(\mathcal{S})$: by virtue of [**Cis08**, Cor. 3.26] (resp. of [**CT11**, Thm. A.5]), for any model category (resp. stable model category) $\mathscr{V}'$ there is a canonical equivalence of categories between the category of cocontinous morphisms from $\mathbf{Ho}(\mathcal{S})$ to $\mathbf{Ho}(\mathscr{V}')$ and the homotopy category $\mathrm{Ho}(\mathscr{V})$. As a consequence, the derivator $\mathbf{Ho}(\mathcal{S})$ admits a unique closed symmetric monoidal structure, and any derivator (resp. triangulated derivator) is naturally and uniquely enriched in $\mathbf{Ho}(\mathcal{S})$; see [**Cis08**, Thm. 5.22]. More concretely, this universal property gives, for any object E in $\mathrm{Ho}(\mathscr{V}')$, a unique cocontinuous morphism of derivators

$$\mathbf{Ho}(\mathcal{S}) \to \mathbf{Ho}(\mathscr{V}') \quad , \qquad K \mapsto K \otimes E$$

such that $\mathbb{1} \otimes E = E$. For a fixed K in $\mathbf{Ho}(\mathcal{S})(I)$, this defines a cocontinuous morphism of derivators

$$\mathbf{Ho}(\mathscr{V}') \to \mathbf{Ho}(\mathscr{V}'^{I^{op}}) \quad , \qquad E \mapsto K \otimes E$$

which has a right adjoint

$$\mathbf{Ho}(\mathscr{V}'^{I^{op}}) \to \mathbf{Ho}(\mathscr{V}') \quad , \qquad F \mapsto F^K \, .$$

Let

$$\mathbf{R}\mathrm{Hom}(E, -) : \mathbf{Ho}(\mathscr{V}) \to \mathbf{Ho}(\mathcal{S})$$

be a continuous morphism such that, for any object F of $\mathscr{V}$, there is a functorial bijection

$$i_F : \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(\mathbb{1}, \mathbf{R}\mathrm{Hom}(E,F)) \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{V})}(E,F))\,.$$

Then, for any object K of $\mathbf{Ho}(\mathcal{S})(I)$, and any object F of $\mathbf{Ho}(\mathscr{V})(I)$ a canonical isomorphism

$$\mathbf{R}\mathrm{Hom}(E,F^K) \simeq \mathbf{R}\mathrm{Hom}(E,F)^K$$

which is completely determined by being the identity for $K = \mathbb{1}$ (this requires the full universal property of $\mathbf{Ho}(\mathcal{S})$ given by by [**Cis08**, Thm. 3.24] (resp. by the dual version of [**CT11**, Thm. A.5])). We thus get from the functorial bijections i_F the natural bijections:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Ho}(\mathcal{S})(I)}(K, \mathbf{R}\mathrm{Hom}(E,F)) &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(\mathbb{1}, \mathbf{R}\mathrm{Hom}(E,F)^K)\\ &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(\mathbb{1}, \mathbf{R}\mathrm{Hom}(E,F^K))\\ &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{V})}(E,F^K)\\ &\simeq \mathrm{Hom}_{\mathbf{Ho}(\mathscr{V})(I)}(K\otimes E,F)\,.\end{aligned}$$

In other words, $\mathbf{R}\mathrm{Hom}(E,-)$ has to be a right adjoint to $(-)\otimes E$. □

REMARK 3.2.16. The preceding theorem mostly holds for abstract derivators. The only problem is for the existence of the morphism $\mathbf{R}\mathrm{Hom}(E,-)$ (the unicity is always clear). However, this problem disapears for derivators which have a Quillen model (as we have seen above), as well as for triangulated derivators (see [**CT11**, Lemma A.6]). Hence Theorem 3.2.15 holds in fact for any triangulated Grothendieck derivator.

In the case when $\mathscr{V}$ is a combinatorial model category (which, in practice, will essentially always be the case), the enrichment over simplicial sets (resp, in the stable case, over spectra) can be constructed via Quillen functors by Dugger's presentation theorems [**Dug01**] (resp. [**Dug06**]).

COROLLARY 3.2.17. *Let $\mathscr{M}$ be a $\mathscr{P}$-fibred combinatorial model category (resp. a stable $\mathscr{P}$-fibred combinatorial model category, resp. a $\mathbf{Q}$-linear stable $\mathscr{P}$-fibred combinatorial model category) over $\mathscr{S}$, and $\mathcal{S}$ the model category of simplicial sets (resp. the stable model category of S^1-spectra, resp. the $\mathbf{Q}$-linear stable model category of complexes of $\mathbf{Q}$-vector spaces).*

Consider an object S of $\mathscr{S}$, a morphism $f : X \to S$, and a morphism of $\mathscr{S}$-diagrams $p : (\mathscr{X}, I) \to X$ over S. Then, for an object M of $\mathrm{Ho}(\mathscr{M})(S)$, the following conditions are equivalent.

(a) *The map*

$$\mathbf{R}f_*\,\mathbf{L}f^*(M) \to \mathbf{R}f_*\,\mathbf{R}p_*\,\mathbf{L}p^*\,\mathbf{L}f^*(M)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(S)$.

(b) *The map*

$$\mathbf{R}\Gamma_{geom}(X,M) \to \mathbf{R}\varprojlim_{i\in I^{op}} \mathbf{R}\Gamma_{geom}(\mathscr{X}_i, M)$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(S)$.

(c) *For any object E of $\mathrm{Ho}(\mathscr{M})(S)$, the map*

$$\mathbf{R}\mathrm{Hom}(E, \mathbf{R}\Gamma_{geom}(X,M)) \to \mathbf{R}\varprojlim_{i\in I^{op}} \mathbf{R}\mathrm{Hom}(E, \mathbf{R}\Gamma_{geom}(\mathscr{X}_i, M))$$

is an isomorphism in $\mathrm{Ho}(\mathcal{S})$.

PROOF. The equivalence between (a) and (b) follows from Propositions 3.1.15 and 3.1.16, which give the formula

$$\mathbf{R}f_*\,\mathbf{R}p_*\,\mathbf{L}p^*\,\mathbf{L}f^*(M) \simeq \mathbf{R}\varprojlim_{i\in I^{op}} \mathbf{R}\Gamma_{geom}(\mathscr{X}_i, M)\,.$$

The identification

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{S})}(\mathbb{1}, \mathbf{R}\mathrm{Hom}(E,F)) \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(E,F)$$

and the Yoneda Lemma show that a map in $\mathrm{Ho}(\mathscr{M})(S)$ is an isomorphism if and only its image by $\mathbf{R}\mathrm{Hom}(E,-)$ is an isomorphism for any object E of $\mathrm{Ho}(\mathscr{M})(S)$. Moreover, as $\mathbf{R}\mathrm{Hom}(E,-)$

is continuous, for any small category I and any presheaf F on I with values in $\mathscr{M}(S)$, there is a canonical isomorphism

$$\mathbf{R}\mathrm{Hom}(E,\ \mathbf{R}\varprojlim_{i\in I^{op}} F_i)) \simeq \mathbf{R}\varprojlim_{i\in I^{op}} \mathbf{R}\mathrm{Hom}(E, F_i))\,.$$

This proves the equivalence between contitions (b) and (c). □

Corollary 3.2.18. *Under the assumptions of Corollary 3.2.17, given an object S of $\mathscr{S}$, an object M of* $\mathrm{Ho}(\mathscr{M})(S)$ *satisfies t-descent if and only if, for any object E of* $\mathrm{Ho}(\mathscr{M})(S)$ *the presheaf of simplicial sets (resp. of S^1-spectra, resp. of complexes of* $\mathbf{Q}$*-vector spaces)*

$$\mathbf{R}\mathrm{Hom}(E, \mathbf{R}\Gamma_{geom}(-, M))$$

satisfies t-descent over $\mathscr{S}/S$.

Proof. This follows from the preceding corollary, using the formula given by Proposition 3.2.4. □

Remark 3.2.19. We need the category $\mathscr{S}$ to be small in some sense to apply the two preceding corollaries because we need to make sense of the projective model category structure of Proposition 3.2.10. However, we can use these corollaries even if the site $\mathscr{S}$ is not small as well: we can either use the theory of universes, or apply these corollaries to all the adequate small subsites of $\mathscr{S}$. As a consequence, we shall feel free to use Corollaries 3.2.17 and 3.2.18 for non necessarily small sites $\mathscr{S}$, leaving to the reader the task to avoid set-theoretic difficulties according to her/his taste.

Definition 3.2.20. For an S^1-spectrum E and an integer n, we define its nth cohomology group $H^n(E)$ by the formula

$$H^n(E) = \pi_{-n}(E)\,,$$

where π_i stands for the ith stable homotopy group functor.

Let $\mathscr{M}$ be a monoidal $\mathscr{P}$-fibred stable combinatorial model category over $\mathscr{S}$. Given an object S of $\mathscr{S}$ as well as an object M of $\mathrm{Ho}(\mathscr{M})(S)$, we define the presheaf of *absolute derived global sections of M over S* by the formula

$$\mathbf{R}\Gamma(-, M) = \mathbf{R}\mathrm{Hom}(\mathbb{1}_S, \mathbf{R}\Gamma_{geom}(-, M))\,.$$

For a map $X \to S$ of $\mathscr{S}$, we thus have the *absolute cohomology of X with coefficients in M*, $\mathbf{R}\Gamma(X, M)$, as well as the *cohomology groups of X with coefficients in M*:

$$H^n(X, M) = H^n(\mathbf{R}\Gamma(X, M))\,.$$

We have canonical isomorphisms of abelian groups

$$\begin{aligned} H^n(X, M) &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(\mathbb{1}_S, \mathbf{R}f_*\,\mathbf{L}f^*(M)) \\ &\simeq \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbb{1}_X, \mathbf{L}f^*(M))\,. \end{aligned}$$

Note that, if moreover $\mathscr{M}$ is $\mathbf{Q}$-linear, the presheaf $\mathbf{R}\Gamma(-, M)$ can be considered as a presheaf of complexes of $\mathbf{Q}$-vector spaces on $\mathscr{S}/S$.

3.3. Descent over schemes. The aim of this section is to give natural sufficient conditions for $\mathscr{M}$ to satisfy descent with respect to various Grothendieck topologies[50].

[50] In fact, using remark 3.2.16, all of this section (results and proofs) holds for an abstract algebraic prederivator in the sense of Ayoub [**Ayo07a**, Def. 2.4.13] without any changes (note that the results of 3.1.b are in fact a proof that (stable) combinatorial fibred model categories over $\mathscr{S}$ give rise to algebraic prederivators). The only interest of considering a fibred model category over $\mathscr{S}$ is that it allows to formulate things in a little more naive way.

3.3.a. *Localization and Nisnevich descent.*

3.3.1. Recall from example 2.1.11 that a *Nisnevich distinguished square* is a pullback square of schemes

$$\begin{array}{ccc} V & \xrightarrow{l} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[j]{} & X \end{array} \tag{3.3.1.1}$$

in which f is étale, j is an open immersion with reduced complement Z and the induced morphism $f^{-1}(Z) \to Z$ is an isomorphism.

For any scheme X in $\mathscr{S}$, we denote by X_{Nis} the small Nisnevich site of X.

THEOREM 3.3.2 (Morel-Voevodsky). *Let $\mathscr{V}$ be a (combinatorial) model category and T a scheme in $\mathscr{S}$. For a presheaf F on T_{Nis} with values in $\mathscr{V}$, the following conditions are equivalent.*

(i) *$F(\varnothing)$ is a terminal object in* $\mathrm{Ho}(\mathscr{V})$*, and for any Nisnevich distinguished square* (3.3.1.1) *in T_{Nis}, the square*

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(V) \end{array}$$

is a homotopy pullback square in $\mathscr{V}$.

(ii) *The presheaf F satisfies Nisnevich descent on T_{Nis}.*

PROOF. By virtue of corollaries 3.2.17 and 3.2.18, it is sufficient to prove this in the case where $\mathscr{V}$ is the usual model category of simplicial sets, in which case this is precisely Morel and Voevodsky's theorem; see [**MV99, Voe10b, Voe10c**]. □

3.3.3. Consider a Nisnevich distinguished square (3.3.1.1) and put $a = jg = fl$. According to our general assumption 3.0, the maps a, j and f are $\mathscr{P}$-morphisms. For any object M of $\mathscr{M}(X)$, we obtain a commutative square in $\mathscr{M}$ (which is well defined as an object in the homotopy of commutative squares in $\mathscr{M}(X)$):

$$\begin{array}{ccc} \mathbf{L}a_\sharp a^* M & \longrightarrow & \mathbf{L}f_\sharp f^*(M) \\ \downarrow & & \downarrow \\ \mathbf{L}j_\sharp j^*(M) & \longrightarrow & M. \end{array} \tag{3.3.3.1}$$

We also obtain another commutative square in $\mathscr{M}$ by appyling the functor $\mathbf{R}Hom_X(-, \mathbb{1}_X)$:

$$\begin{array}{ccc} M & \longrightarrow & \mathbf{R}f_*\, f^*(M) \\ \downarrow & & \downarrow \\ \mathbf{R}j_*\, j^*(M) & \longrightarrow & \mathbf{R}a_*\, a^*(M). \end{array} \tag{3.3.3.2}$$

PROPOSITION 3.3.4. *If the category* $\mathrm{Ho}(\mathscr{M})$ *has the localization property, then for any Nisnevich distinguished square* (3.3.1.1) *and any object M of* $\mathrm{Ho}(\mathscr{M})(X)$*, the squares* (3.3.3.1) *and* (3.3.3.2) *are homotopy cartesians.*

PROOF. Let $i : Z \to X$ be the complement of the open immersion j (Z being endowed with the reduced structure) and $p : f^{-1}(Z) \to Z$ the map induced by f.

We have only to prove that one of the squares (3.3.3.1), (3.3.3.2) are cartesian. We choose the square (3.3.3.1).

Because the pair of functor $(\mathbf{L}i^*, j^*)$ is conservative on $\mathrm{Ho}(\mathscr{M})(X)$, we have only to check that the pullback of (3.3.3.1) along j^* or $\mathbf{L}i^*$ is homotopy cartesian. But, using the $\mathscr{P}$-base change property, we see that the image of (3.3.3.1) by j^* is (canonically isomorphic to) the commutative square

$$\begin{array}{ccc} \mathbf{L}g_\sharp a^*(M) & = & \mathbf{L}g_\sharp a^*(M) \\ \downarrow & & \downarrow \\ j^*(M) & = & j^*(M) \end{array}$$

which is obviously homotopy cartesian.

Using again the $\mathscr{P}$-base change property, we obtain that the image of (3.3.3.1) by $\mathbf{L}i^*$ is isomorphic in $\mathrm{Ho}(\mathscr{M})$ to the square

$$\begin{array}{ccc} 0 & \longrightarrow & p_\sharp p^* \mathbf{L}i^*(M) \\ \| & & \downarrow \\ 0 & \longrightarrow & \mathbf{L}i^*(M) \end{array}$$

which is again obviously homotopy cartesian because p is an isomorphism (note for this last reason, $p_\sharp = \mathbf{L}p_\sharp$). □

Corollary 3.3.5. *If* $\mathrm{Ho}(\mathscr{M})$ *has the localization property then it satisfies Nisnevich descent.*

Proof. This corollary thus follows immediately from Corollary 3.2.17, Theorem 3.3.2 and Proposition 3.3.4. □

Remark 3.3.6. Note that using Theorem 3.3.2, if we assume only that $\mathrm{Ho}(\mathscr{M})$ satisfies Nisnevich descent, then the squares (3.3.3.1) and (3.3.3.2) are homotopy cartesians for any Nisnevich distinguished square (3.3.1.1).

Assume that $\mathscr{M}$ is monoidal with geometric sections M. Let S be a base scheme and consider a Nisnevich distinguished square (3.3.1.1) of smooth S-schemes. Then the fact that the square (3.3.3.1) is homotopy cartesian implies there exists a *canonical* distinguished triangle:

$$M_S(V) \xrightarrow{g_*+l_*} M_S(U) \oplus M_S(Y) \xrightarrow{f_*+j_*} M_S(X) \longrightarrow M_S(V)[1]$$

It is called the *Mayer-Vietoris triangle* associated with the square (3.3.1.1).

3.3.b. *Proper base change isomorphism and descent by blow-ups.*

3.3.7. Recall from example 2.1.11 that a cdh-*distinguished square* is a pullback square of schemes

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array} \qquad (3.3.7.1)$$

in which f is proper surjective, i a closed immersion and the induced map $f^{-1}(X - Z) \to X - Z$ is an isomorphism.

Recall from Example 2.1.11 the cdh-*topology* is the Grothendieck topology on the category of schemes generated by Nisnevich coverings and by coverings of shape $\{Z \to X, Y \to X\}$ for any cdh-distinguished square (3.3.7.1).

Theorem 3.3.8 (Voevodsky). *Let* $\mathscr{V}$ *be a (combinatorial) model category. For a presheaf* F *on* $\mathscr{S}$ *with values in* $\mathscr{V}$*, the following conditions are equivalent.*

(i) *The presheaf* F *satisfies* cdh-*descent on* $\mathscr{S}$*.*

(ii) The presheaf F satisfies Nisnevich descent and, for any cdh-*distinguished square* (3.3.7.1) *of* $\mathscr{S}$*, the square*

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(Z) & \longrightarrow & F(T) \end{array}$$

is a homotopy pullback square in $\mathscr{V}$*.*

PROOF. It is sufficient to prove this in the case where $\mathscr{V}$ is the usual model category of simplicial sets; see corollaries 3.2.17 and 3.2.18. As the distinguished cdh-squares define a bounded regular and reduced *cd*-structure on $\mathscr{S}$, the equivalence between (i) and (ii) follows from Voevodsky's theorems on descent with respect to topologies defined by *cd*-structures [**Voe10b, Voe10c**]. □

3.3.9. Consider a cdh-distinguished square (3.3.7.1) and put $a = ig = fk$. For any object M of $\mathscr{M}(X)$, we obtain a commutative square in $\mathscr{M}$ (which is well defined as an object in the homotopy of commutative squares in $\mathscr{M}(X)$):

(3.3.9.1)
$$\begin{array}{ccc} M & \longrightarrow & \mathbf{R}f_* \, \mathbf{L}f^*(M) \\ \downarrow & & \downarrow \\ \mathbf{R}i_* \, \mathbf{L}i^*(M) & \longrightarrow & \mathbf{R}a_* \, \mathbf{L}a^*(M) \end{array}$$

PROPOSITION 3.3.10. *Assume* $\operatorname{Ho}(\mathscr{M})$ *satisfies the localization property and the transversality property with respect to proper morphisms. Then the following conditions hold:*

(i) For any cdh-*distinguished square* (3.3.7.1)*, and any object M of* $\operatorname{Ho}(\mathscr{M})(X)$ *the commutative square* (3.3.9.1) *is homotopy cartesian.*
(ii) The $\mathscr{P}$-fibred model category $\operatorname{Ho}(\mathscr{M})$ *satisfies* cdh-*descent.*

PROOF. We first prove (i). Consider a cdh-distinguished square (3.3.7.1) and let $j : U \to X$ be the complement open immersion of i. As the pair of functor $(\mathbf{L}i^*, j^*)$ is conservative on $\operatorname{Ho}(\mathscr{M})(X)$, we have only to check that the image of (3.3.9.1) under $\mathbf{L}i^*$ and j^* is homotopy cartesian.

Using projective transversality, we see that the image of (3.3.9.1) by the functor $\mathbf{L}i^*$ is (isomorphic to) the homotopy pullback square

$$\begin{array}{ccc} \mathbf{L}i^*(M) & \longrightarrow & \mathbf{R}g_* \, \mathbf{L}g^* \, \mathbf{L}i^*(M) \\ \| & & \| \\ \mathbf{L}i^*(M) & \longrightarrow & \mathbf{R}g_* \, \mathbf{L}g^* \, \mathbf{L}i^*(M) \end{array}.$$

Let $h : f^{-1}(U) \to U$ be the pullback of f over U. As j is an open immersion, it is by assumption a $\mathscr{P}$-morphism and the $\mathscr{P}$-base change formula implies that the image of (3.3.9.1) by j^* is (isomorphic to) the commutative square

$$\begin{array}{ccc} \mathbf{L}j^*(M) & \longrightarrow & \mathbf{R}h_*\mathbf{L}h^*\mathbf{L}j^*(M) \\ \downarrow & & \downarrow \\ 0 & = & 0 \end{array}$$

which is obviously homotopy cartesian because h is an isomorphism.

We then prove (ii). We already know that $\mathscr{M}$ satisfies Nisnevich descent (Corollary 3.3.5). Thus, by virtue of the equivalence between conditions (i) and (ii) of Theorem 3.3.8, the computation above, together with corollaries 3.2.17 and 3.2.18 imply that $\mathscr{M}$ satisfies cdh-descent. □

3.3.11. To any cdh-distinguished square (3.3.7.1), one associates a diagram of schemes $\mathscr{Y}$ over X as follows. Let $\ulcorner$ be the category freely generated by the oriented graph

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \\ c & & \end{array} \tag{3.3.11.1}$$

Then $\mathscr{Y}$ is the functor from $\ulcorner$ to $\mathscr{S}/X$ defined by the following diagram.

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array} \tag{3.3.11.2}$$

We then have a canonical map $\varphi : \mathscr{Y} \to X$, and the second assertion of Theorem 3.3.10 can be reformulated by saying that the adjunction map

$$M \to \mathbf{R}\varphi_* \, \mathbf{L}\varphi^*(M)$$

is an isomorphism for any object M of $\mathrm{Ho}(\mathscr{M})(X)$: indeed, by virtue of Proposition 3.1.15, $\mathbf{R}\varphi_* \, \mathbf{L}\varphi^*(M)$ is the homotopy limit of the diagram

$$\begin{array}{ccc} & & \mathbf{R}f_* \, \mathbf{L}f^*(M) \\ & & \downarrow \\ \mathbf{R}i_* \, \mathbf{L}i^*(M) & \longrightarrow & \mathbf{R}a_* \, \mathbf{L}a^*(M) \end{array}$$

in $\mathrm{Ho}(\mathscr{M})(X)$. In other words, if $\mathscr{M}$ has the properties of localization and of projective transversality, then the functor

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\mathscr{Y}, \ulcorner)$$

is fully faithful.

3.3.c. *Proper descent with rational coefficients I: Galois excision.* From now on, we assume that any scheme in $\mathscr{S}$ is quasi-excellent[51] (in fact, we shall only use the fact that the normalization of a quasi-excellent schemes gives rise to a finite surjective morphism, so that, in fact, universally japanese schemes would be enough). We fix a scheme S in $\mathscr{S}$, and we shall work with S-schemes in $\mathscr{S}$ (assuming these form an essentially small category).

3.3.12. The h-*topology* (resp. the qfh-*topology)* is the Grothendieck topology on the category of schemes associated to the pretopology whose coverings are the universal topological epimorphisms (resp. the quasi-finite universal topological epimorphisms). This topology has been introduced and studied by Voevodsky in [**Voe96**].

The h-topology is finer than the cdh-topology and, of course, finer than the qfh-topology. The qfh-topology is in turn finer than the étale topology. An interesting feature of the h-topology (resp. of the qfh-topology) is that any proper (resp. finite) surjective map is an h-cover. In fact, the h-topology (resp. the qfh-topology) can be described as the topology generated by the Nisnevich coverings and by the proper (resp. finite) surjective maps; see Lemma 3.3.28 (resp. Lemma 3.3.27) below for a precise statement.

3.3.13. Consider a morphism of schemes $f : Y \to X$. Consider the group of automorphisms $G = \mathrm{Aut}_Y(X)$ of the X-scheme Y.

Assuming X is connected, we say according to [**SGA1**, exp. V] that f is a *Galois cover* if it is finite étale (thus surjective) and G operates transitively and faithfully on any (or simply one) of the geometric fibers of Y/X. Then G is called the *Galois group* of Y/X.[52]

[51] See 4.1.1 below for a reminder on quasi-excellent schemes.

[52] The map f induces a one to one correspondence between the generic points of Y and that of X. For any generic point $y \in Y$, $x = f(y)$, the residual extension κ_y/κ_x is a Galois extension with Galois group G.

When X is not connected, we will still say that f is a *Galois cover* if it is so over any connected component of X. Then G will be called the *Galois group* of X. If $(X_i)_{i\in I}$ is the family connected components of X, then G is the product of the Galois groups G_i of $f \times_X X_i$ for each $i \in I$. The group G_i is equal to the Galois group of any residual extension over a generic point of X_i.

The following definition is an extension of the definition 5.5 of [**SV00b**]:

Definition 3.3.14. A *pseudo-Galois cover* is a finite surjective morphism of schemes $f : Y \to X$ which can be factored as

$$Y \xrightarrow{f'} X' \xrightarrow{p} X$$

where f' is a Galois cover and p is radicial[53] (such a p is automatically finite and surjective).

Note that the group G defined by the Galois cover f' is independent of the choice of the factorization. In fact, if $\bar{X}$ denotes the semi-localization of X at its generic points, considering the cartesian squares

$$\begin{array}{ccccc} \bar{Y} & \longrightarrow & \bar{X}' & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{f'} & X' & \xrightarrow{p} & X \end{array}$$

then $G = \mathrm{Aut}_{\bar{X}}(\bar{Y})$ – for any point $y \in \bar{Y}$, $x' = f'(y)$, $x = f(y)$, $\kappa_{x'}/\kappa_x$ is the maximal radicial sub-extension of the normal extension κ_y/κ_x. It will be called the *Galois group* of Y/X.

Remark also that Y is a G-torsor over X locally for the qfh-topology (i.e. it is a Galois object of group G in the qfh-topos of X): this comes from the fact that finite radicial epimorphisms are isomorphisms locally for the qfh-topology (any universal homeomorphism has this property by [**Voe96**, prop. 3.2.5]).

Let $f : Y \to X$ be a finite morphism, and G a finite group acting on Y over X. Note that, as Y is affine on X, the scheme theoretic quotient Y/G exists; see [**SGA1**, Exp. V, Cor. 1.8]. Such scheme-theoretic quotients are stable by flat pullbacks; see [**SGA1**, Exp. V, Prop. 1.9].

Definition 3.3.15. Let G be finite group. A qfh-*distinguished square of group* G is a pullback square of S-schemes of shape

(3.3.15.1)
$$\begin{array}{ccc} T & \xrightarrow{h} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array}$$

in which Y is endowed with an action of G over X, and satisfying the following three conditions.

(a) The morphism f is finite and surjective.
(b) The induced morphism $f^{-1}(X - Z) \to f^{-1}(X - Z)/G$ is flat.
(c) The morphism $f^{-1}(X - Z)/G \to X - Z$ is radicial.

Immediate examples of qfh-distinguished squares of trivial group are the following. The scheme Y might be the normalization of X, and Z is a nowhere dense closed subscheme out of which f is an isomorphism; or Y is dense open subscheme of X which is the disjoint union of its irreducible components; or Y is a closed subscheme of X inducing an isomorphism $Y_{red} \simeq X_{red}$.

A qfh-distinguished square of group G (3.3.15.1) will be said to be *pseudo-Galois* if Z is nowhere dense in X and if the map $f^{-1}(X - Z) \to X - Z$ is a pseudo-Galois cover of group G.

The main examples of pseudo-Galois qfh-distinguished squares will come from the following situation.

Proposition 3.3.16. *Consider an irreducible normal scheme X, and a finite extension L of its field of functions $k(X)$. Let K be the inseparable closure of $k(X)$ in L, and assume that L/K is a Galois extension of group G. Denote by Y the normalization of X in L. Then the action of*

[53] See 2.1.6 for a reminder on radicial morphisms.

G on $k(Y) = L$ extends naturally to an action on Y over X. Furthermore, there exists a closed subscheme Z of X, such that the pullback square

$$\begin{array}{ccc} T & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array}$$

is a pseudo-Galois qfh-*distinguished square of group G.*

Proof. The action of G on L extends naturally to an action on Y over X by functoriality. Furthermore, Y/G is the normalization of X in K, so that $Y/G \to X$ is finite radicial and surjective (see [**Voe96**, Lemma 3.1.7] or [**Bou98**, V, §2, n° 3, lem. 4]). By construction, Y is generically a Galois cover over Y/G, which implies the result (see [**EGA4**, Cor. 18.2.4]). □

3.3.17. For a given S-scheme T, we shall denote by $L(T)$ the corresponding representable qfh-sheaf of sets (remember that the qfh-topology is not subcanonical, so that $L(T)$ has to be distinguished from T itself). Beware that, in general, there is no reason that, given a finite group G acting on T, the scheme-theoretic quotient $L(T/G)$ (whenever defined) and the qfh-sheaf-theoretic quotient $L(T)/G$ would coincide.

Lemma 3.3.18. *Let $f : Y \to X$ be a separated morphism, G a finite group acting on Y over X, and Z a closed subscheme of X such that f is finite and surjective over $X - Z$, and such that the quotient map $f^{-1}(X - Z) \to f^{-1}(X - Z)/G$ is flat, while the map $f^{-1}(X - Z)/G \to X - Z$ is radicial. For $g \in G$, write $g : Y \to Y$ for the corresponding automorphism of Y, and define Y_g as the image of the diagonal $Y \to Y \times_X Y$ composed with the automorphism $1_Y \times_X g : Y \times_X Y \to Y \times_X Y$. Then, if $T = Z \times_X Y$, we get a* qfh-*cover of $Y \times_X Y$ by closed subschemes:*

$$Y \times_X Y = (T \times_Z T) \cup \bigcup_{g \in G} Y_g \, .$$

Proof. Note that, as f is separated, the diagonal $Y \to Y \times_X Y$ is a closed embedding, so that the Y_g's are closed subschemes of $Y \times_X Y$. As the map $Y \times_{Y/G} Y \to Y \times_X Y$ is a universal homeomorphism, we may assume that $Y/G = X$. It is sufficient to prove that, if y and y' are two geometric points of Y whose images coincide in X and do not belong to Z, there exists an element g of G such that $y' = gy$ (which means that the pair (y, y') belongs to Y_g). For this purpose, we may assume, without loss of generality, that $Z = \varnothing$. Then, by assumption, Y is flat over X, from which we get the identification $(Y \times_X Y)/G \simeq Y \times_X (Y/G) \simeq Y$ (where the action of G on $Y \times_X Y$ is trivial on the first factor and is induced by the action on Y on the second factor). This achieves the proof. □

Proposition 3.3.19. *For any* qfh-*distinguished square of group G* (3.3.15.1)*, the commutative square*

$$\begin{array}{ccc} L(T)/G & \longrightarrow & L(Y)/G \\ \downarrow & & \downarrow \\ L(Z) & \longrightarrow & L(X) \end{array}$$

is a pullback and a pushout in the category of qfh-*sheaves. Moreover, if X is normal and if Z is nowhere dense in X, then the canonical map $L(Y)/G \to L(Y/G) \simeq L(X)$ is an isomorphism of* qfh-*sheaves (which implies that $L(T)/G \to L(Z)$ is an isomorphism as well).*

Proof. Note that this commutative square is a pullback because it was so before taking the quotients by G (as colimits are universal in any topos). As f is a qfh-cover, it is sufficient to prove that

$$\begin{array}{ccc} L(T) \times_{L(Z)} L(T)/G & \longrightarrow & L(Y) \times_{L(X)} L(Y)/G \\ \downarrow & & \downarrow \\ L(T) & \longrightarrow & L(Y) \end{array}$$

is a pushout square. This latter square fits into the following commutative diagram

$$\begin{array}{ccc} L(T) & \longrightarrow & L(Y) \\ \downarrow & & \downarrow \\ L(T) \times_{L(Z)} L(T)/G & \longrightarrow & L(Y) \times_{L(X)} L(Y)/G \\ \downarrow & & \downarrow \\ L(T) & \longrightarrow & L(Y) \end{array}$$

in which the two vertical composed maps are identities (the vertical maps of the upper commutative square are obtained from the diagonals by taking the quotients under the natural action of G on the right component). It is thus sufficient to prove that the upper square is a pushout. As the lower square is a pullback, the upper one shares the same property; moreover, all the maps in the upper commutative square are monomorphisms of qfh-sheaves, so that it is sufficient to prove that the map $(L(T) \times_{L(Z)} L(T)/G) \amalg L(Y) \to L(Y) \times_{L(X)} L(Y)/G$ is an epimorphism of qfh-sheaves. According to Lemma 3.3.18, this follows from the commutativity of the diagram

$$\begin{array}{ccc} L(T \times_Z T) \amalg \left(\coprod_{g \in G} L(Y_g)\right) & \longrightarrow & L(Y \times_X Y) \\ \downarrow & & \downarrow \\ (L(T) \times_{L(Z)} L(T)/G) \amalg L(Y) & \longrightarrow & L(Y) \times_{L(X)} L(Y)/G \end{array}$$

in which the vertical maps are obviously epimorphic.

Assume now that X is normal and that Z is nowhere dense in X, and let us prove that the canonical map $L(Y)/G \to L(X)$ is an isomorphism of qfh-sheaves. This is equivalent to prove that, for any qfh-sheaf of sets F, the map $f^* : F(X) \to F(Y)$ induces a bijection

$$F(X) \simeq F(Y)^G .$$

Let F be a qfh-sheaf. The map $f^* : F(X) \to F(Y)$ is injective because f is a qfh-cover, and it is clear that the image of f^* lies in $F(Y)^G$.

Let a be a section of F over Y which is invariant under the action of G. Denote by $pr_1, pr_2 : Y \times_X Y \to Y$ the two canonical projections. With the notations introduced in Lemma 3.3.18, we have

$$pr_1^*(a)|_{Y_g} = a = a.g = pr_2^*(a)|_{Y_g}$$

for every element g in G. As Z does not contain any generic point of X, the scheme $T \times_Z T$ does not contain any generic point of $Y \times_X Y$ neither: as any irreducible component of Y dominates an irreducible component of X, and, as X is normal, the finite map $Y \to X$ is universally open; in particular, the projection $pr_1 : Y \times_X Y \to Y$ is universally open, which implies that any generic point of $Y \times_X Y$ lies over a generic point of Y. By virtue of [**Voe96**, prop. 3.1.4], Lemma 3.3.18 thus gives a qfh-cover of $Y \times_X Y$ by closed subschemes of shape

$$Y \times_X Y = \bigcup_{g \in G} Y_g .$$

This implies that

$$pr_1^*(a) = pr_2^*(a) .$$

The morphism $Y \to X$ being a qfh-cover and F a qfh-sheaf, we deduce that the section a lies in the image of f^*. □

Corollary 3.3.20. *For any* qfh*-distinguished square of group G* (3.3.15.1)*, we get a bicartesian square of* qfh*-sheaves of abelian groups*

$$\begin{array}{ccc}\mathbf{Z}_{\mathrm{qfh}}(T)_G & \longrightarrow & \mathbf{Z}_{\mathrm{qfh}}(Y)_G \\ \downarrow & & \downarrow \\ \mathbf{Z}_{\mathrm{qfh}}(Z) & \longrightarrow & \mathbf{Z}_{\mathrm{qfh}}(X)\end{array}$$

(where the subscript G stands for the coinvariants under the action of G). In other words, there is a canonical short exact sequence of sheaves of abelian groups

$$0 \to \mathbf{Z}_{\mathrm{qfh}}(T)_G \to \mathbf{Z}_{\mathrm{qfh}}(Z) \oplus \mathbf{Z}_{\mathrm{qfh}}(Y)_G \to \mathbf{Z}_{\mathrm{qfh}}(X) \to 0\,.$$

Proof. As the abelianization functor preserves colimits and monomorphisms, the preceding proposition implies formally that we have a short exact sequence of shape

$$\mathbf{Z}_{\mathrm{qfh}}(T)_G \to \mathbf{Z}_{\mathrm{qfh}}(Z) \oplus \mathbf{Z}_{\mathrm{qfh}}(Y)_G \to \mathbf{Z}_{\mathrm{qfh}}(X) \to 0\,,$$

while the left exactness follows from the fact that $Z \to X$ being a monomorphism, the map obtained by pullback, $L(T)/G \to L(Y)/G$, is a monomorphism as well. □

3.3.21. Let $\mathscr{V}$ be a $\mathbf{Q}$-linear stable model category (see 3.2.14).

Consider a finite group G, and an object E of $\mathscr{V}$, endowed with an action of G. By viewing G as a category with one object we can see E as functor from G to $\mathscr{V}$ and take its homotopy limit in $\mathrm{Ho}(\mathscr{V})$, which we denote by E^{hG} (in the literature, E^{hG} is called the *object of homotopy fixed points* under the action of G on E). One the other hand, the category $\mathrm{Ho}(\mathscr{V})$ is, by assumption, a $\mathbf{Q}$-linear triangulated category with small sums, and, in particular, a $\mathbf{Q}$-linear pseudo-abelian category so that we can define E^G as the object of $\mathrm{Ho}(\mathscr{V})$ defined by

(3.3.21.1) $$E^G = \operatorname{Im} p\,,$$

where $p : E \to E$ is the projector defined in $\mathrm{Ho}(\mathscr{V})$ by the formula

(3.3.21.2) $$p(x) = \frac{1}{\#G} \sum_{g \in G} g.x\,.$$

The inclusion $E^G \to E$ induces a canonical isomorphism

(3.3.21.3) $$E^G \xrightarrow{\sim} E^{hG}$$

in $\mathrm{Ho}(\mathscr{V})$: to see this, by virtue of Theorem 3.2.15, we can assume that $\mathscr{V}$ is the model category of complexes of $\mathbf{Q}$-vector spaces, in which case it is obvious.

Corollary 3.3.22. *Let C be a presheaf of complexes of $\mathbf{Q}$-vector spaces on the category of S-schemes. Then, for any* qfh*-distinguished square of group G* (3.3.15.1)*, the commutative square*

$$\begin{array}{ccc}\mathbf{R}\Gamma_{\mathrm{qfh}}(X, C_{\mathrm{qfh}}) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{qfh}}(Y, C_{\mathrm{qfh}})^G \\ \downarrow & & \downarrow \\ \mathbf{R}\Gamma_{\mathrm{qfh}}(Z, C_{\mathrm{qfh}}) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{qfh}}(T, C_{\mathrm{qfh}})^G\end{array}$$

is a homotopy pullback square in the derived category of $\mathbf{Q}$-vector spaces. In particular, we get a long exact sequence of shape

$$H^n_{\mathrm{qfh}}(X, C_{\mathrm{qfh}}) \to H^n_{\mathrm{qfh}}(Z, C_{\mathrm{qfh}}) \oplus H^n_{\mathrm{qfh}}(Y, C_{\mathrm{qfh}})^G \to H^n_{\mathrm{qfh}}(T, C_{\mathrm{qfh}})^G \to H^{n+1}_{\mathrm{qfh}}(X, C_{\mathrm{qfh}})$$

If furthermore X is normal and Z is nowhere dense in X, then the maps

$$H^n_{\mathrm{qfh}}(X, C_{\mathrm{qfh}}) \to H^n_{\mathrm{qfh}}(Y, C_{\mathrm{qfh}})^G \quad \text{and} \quad H^n_{\mathrm{qfh}}(Z, C_{\mathrm{qfh}}) \to H^n_{\mathrm{qfh}}(T, C_{\mathrm{qfh}})^G$$

are isomorphisms for any integer n.

PROOF. Let $C_{\mathrm{qfh}} \to C'$ be a fibrant resolution in the qfh-local injective model category structure on the category of qfh-sheaves of complexes of $\mathbf{Q}$-vector spaces; see for instance [**Ayo07a**, Cor. 4.4.42]. Then for $U = Y, T$, we have a natural isomorphism of complexes

$$\mathrm{Hom}(\mathbf{Q}_{\mathrm{qfh}}(U)_G, C') = C'(U)^G$$

which gives an isomorphism

$$\mathbf{R}\mathrm{Hom}(\mathbf{Q}_{\mathrm{qfh}}(U)_G, C_{\mathrm{qfh}}) \simeq \mathbf{R}\Gamma_{\mathrm{qfh}}(U, C_{\mathrm{qfh}})^G$$

in the derived category of the abelian category of $\mathbf{Q}$-vector spaces. This corollary thus follows formally from Corollary 3.3.20 by evaluating at the derived functor $\mathbf{R}\mathrm{Hom}(-, C_{\mathrm{qfh}})$.

If furthermore X is normal, then one deduces the isomorphism $H^n_{\mathrm{qfh}}(X, C_{\mathrm{qfh}}) \simeq H^n_{\mathrm{qfh}}(Y, C_{\mathrm{qfh}})^G$ from the fact that $L(Y)/G \simeq L(Y/G) \simeq X$ (Proposition 3.3.19), which implies that $\mathbf{Z}_{\mathrm{qfh}}(Y)_G \simeq \mathbf{Z}_{\mathrm{qfh}}(X)$. The isomorphism $H^n_{\mathrm{qfh}}(Z, C_{\mathrm{qfh}}) \simeq H^n_{\mathrm{qfh}}(T, C_{\mathrm{qfh}})^G$ then comes as a byproduct of the long exact sequence above. □

THEOREM 3.3.23. *Let X be a scheme, and C be a presheaf of complexes of $\mathbf{Q}$-vector spaces on the small étale site of X. Then C satisfies étale descent if and only if it has the following properties.*

(a) The complex C satisfies Nisnevich descent.
(b) For any étale X-scheme U and any Galois cover $V \to U$ of group G, the map $C(U) \to C(V)^G$ is a quasi-isomorphism.

PROOF. These are certainly necessary conditions. To prove that they are sufficient, note that the Nisnevich cohomological dimension and the rational étale cohomological dimension of a noetherian scheme are bounded by the dimension; see [**MV99**, proposition 1.8, page 98] and [**Voe96**, Lemma 3.4.7]. By virtue of [**SV00a**, Theorem 0.3], for $\tau = \mathrm{Nis}, \text{ét}$, we have strongly convergent spectral sequences

$$E_2^{p,q} = H^p_\tau(U, H^q(C)_\tau) \Rightarrow H^{p+q}_\tau(U, C_\tau)\,.$$

Condition (a) gives isomorphisms $H^{p+q}(C(U)) \simeq H^{p+q}_{\mathrm{Nis}}(U, C_{\mathrm{Nis}})$, so that it is sufficient to prove that, for each of the cohomology presheaves $F = H^q(C)$, we have

$$H^p_{\mathrm{Nis}}(U, F_{\mathrm{Nis}}) \simeq H^p_{\text{ét}}(U, F_{\text{ét}})\,.$$

As the rational étale cohomology of any henselian scheme is trivial in non-zero degrees, it is sufficient to prove that, for any local henselian scheme U (obtained as the henselisation of an étale X-scheme at some point), $F_{\mathrm{Nis}}(U) \simeq F_{\text{ét}}(U)$. Let G be the absolute Galois group of the closed point of U. Then we have

$$F_{\mathrm{Nis}}(U) = F(U) \quad \text{and} \quad F_{\text{ét}}(U) = \varinjlim_\alpha F(U_\alpha)^{G_\alpha}\,,$$

where the U_α's run over all the Galois covers of U corresponding to the finite quotients $G \to G_\alpha$. But it follows from (b) that $F(U) \simeq F(U_\alpha)^{G_\alpha}$ for any α, so that $F_{\mathrm{Nis}}(U) \simeq F_{\text{ét}}(U)$. □

LEMMA 3.3.24. *Any* qfh*-cover admits a refinement of the form $Z \to Y \to X$, where $Z \to Y$ is a finite surjective morphism, and $Y \to X$ is an étale cover.*

PROOF. This property being clearly local on X with respect to the étale topology, we can assume that X is strictly henselian, in which case this follows from [**Voe96**, Lemma 3.4.2]. □

THEOREM 3.3.25. *A presheaf of complexes of $\mathbf{Q}$-vector spaces C on the category of S-schemes satisfies* qfh*-descent if and only if it has the following two properties:*

(a) the complex C satisfies Nisnevich descent;

(b) for any pseudo-Galois qfh*-distinguished square of group G (3.3.15.1), the commutative square*

$$\begin{array}{ccc} C(X) & \longrightarrow & C(Y)^G \\ \downarrow & & \downarrow \\ C(Z) & \longrightarrow & C(T)^G \end{array}$$

is a homotopy pullback square in the derived category of $\mathbf{Q}$-vector spaces.

Proof. Any complex of presheaves of $\mathbf{Q}$-vector spaces satisfying qfh-descent satisfies properties (a) and (b): property (a) follows from the fact that the qfh-topology is finer than the étale topology; property (b) is Corollary 3.3.22.

Assume now that C satisfies these two properties. Let $\varphi : C \to C'$ be a morphism of presheaves of complexes of $\mathbf{Q}$-vector spaces which is a quasi-isomorphism locally for the qfh-topology, and such that C' satisfies qfh-descent (such a morphism exists thanks to the qfh-local model category structure on the category of presheaves of complexes of $\mathbf{Q}$-vector spaces; see Proposition 3.2.10). Then the cone of φ also satisfies conditions (a) and (b). Hence it is sufficient to prove the theorem in the case where C is acyclic locally for the qfh-topology.

Assume from now on that C_{qfh} is an acyclic complex of qfh-sheaves, and denote by $H^n(C)$ the nth cohomology presheaf associated to C. We know that the associated qfh-sheaves vanish, and we want to deduce that $H^n(C) = 0$.

We shall prove by induction on d that, for any S-scheme X of dimension d and for any integer n, the group $H^n(C)(X) = H^n(C(X))$ vanishes. The case where $d < 0$ follows from the fact, that by (a), the presheaves $H^n(C)$ send finite sums to finite direct sums, so that, in particular, $H^n(C)(\varnothing) = 0$. Before going further, notice that condition (b) implies $H^n(C)(X_{red}) = H^n(C)(X)$ for any S-scheme X (consider the case where, in the diagram (3.3.15.1), $Z = Y = T = X_{red}$), so that it is always harmless to replace X by its reduction. Assume now that $d \geq 0$, and that the vanishing of $H^n(C)(X)$ is known whenever X is of dimension $< d$ and for any integer n. Under this inductive assumption, we have the following reduction principle.

Consider a pseudo-Galois qfh-distinguished square of group G (3.3.15.1). If Z and T are of dimension $< d$, then by condition (b), the map $H^n(C)(X) \to H^n(C)(Y)^G$ is an isomorphism: indeed, we have an exact sequence of shape

$$H^{n-1}(C)(T)^G \to H^n(C)(X) \to H^n(C)(Z) \oplus H^n(C)(Y)^G \to H^n(C)(T)^G \,,$$

which implies our assertion by induction on d.

We shall prove now the vanishing of $H^n(C)(T)$ for normal S-schemes T of dimension d. Let a be a section of $H^n(C)$ over such a T. As $H^n(C)_{\mathrm{qfh}}(T) = 0$, there exists a qfh-cover $g : Y \to T$ such that $g^*(a) = 0$. But, by virtue of Lemma 3.3.24, we can assume g is the composition of a finite surjective morphism $f : Y \to X$ and of an étale cover $e : X \to T$. We claim that $e^*(a) = 0$. To prove it, as, by (a), the presheaf $H^n(C)$ sends finite sums to finite direct sums, we can assume that X is normal and connected. Refining f further, we can assume that Y is the normalization of X in a finite extension of $k(X)$, and that $k(Y)$ is a Galois extension of group G over the inseparable closure of $k(X)$ in $k(Y)$. By virtue of Proposition 3.3.16, we get by the reduction principle the identification $H^n(C)(X) = H^n(C)(Y)^G$, whence $e^*(a) = 0$. As a consequence, the restriction of the presheaf of complexes C to the category of normal S-schemes of dimension $\leq d$ is acyclic locally for the étale topology (note that this is quite meaningful, as any étale scheme over a normal scheme is normal; see [**EGA4**, Prop. 18.10.7]). But C satisfies étale descent (by virtue of Theorem 3.3.23 this follows formally from property (a) and from property (b) for $Z = \varnothing$), so that $H^n(C)(T) = H^n_{\text{ét}}(T, C_{\text{ét}}) = 0$ for any normal S-scheme T of dimension $\leq d$ and any integer n.

Consider now a reduced S-scheme X of dimension $\leq d$. Let $p : T \to X$ be the normalization of X. As p is birational (see [**EGA2**, Cor. 6.3.8]) and finite surjective (because X is quasi-excellent), we can apply the reduction principle and see that the pullback map $p^* : H^n(C)(X) \to H^n(C)(T) = 0$ is an isomorphism for any integer n, which achieves the induction and the proof. □

LEMMA 3.3.26. *Étale coverings are finite étale coverings locally for the Nisnevich topology: any étale cover admits a refinement of the form $Z \to Y \to X$, where $Z \to Y$ is a finite étale cover and $Y \to X$ is a Nisnevich cover.*

PROOF. This property being local on X for the Nisnevich topology, it is sufficient to prove this in the case where X is local henselian. Then, by virtue of [**EGA4**, Cor. 18.5.12 and Prop. 18.5.15], we can even assume that X is the spectrum of field, in which case this is obvious. $\square$

LEMMA 3.3.27. *Any* qfh*-cover admits a refinement of the form $Z \to Y \to X$, where $Z \to Y$ is a finite surjective morphism, and $Y \to X$ is a Nisnevich cover.*

PROOF. As finite surjective morphisms are stable by pullback and composition, this follows immediately from lemmata 3.3.24 and 3.3.26. $\square$

LEMMA 3.3.28. *Any* h*-cover of an integral scheme X admits a refinement of the form*

$$U \to Z \to Y \to X\,,$$

where $U \to Z$ is a finite surjective morphism, $Z \to Y$ is a Nisnevich cover, $Y \to X$ is a proper surjective birational map, and Y is normal.

PROOF. By virtue of [**Voe96**, Theorem 3.1.9], any h-cover admits a refinement of shape

$$W \to V \to X\,,$$

where $W \to V$ is a qfh-cover, and $V \to X$ is a proper surjective birational map. By replacing V by its normalization Y, we get a refinement of shape

$$W \times_V Y \to Y \to X$$

where $W \times_V Y \to Y$ is a qfh-cover, and $Y \to X$ is proper surjective birational map. We conclude by Lemma 3.3.27. $\square$

LEMMA 3.3.29. *Let C be a presheaf of complexes of $\mathbf{Q}$-vector spaces on the category of S-schemes satisfying* qfh*-descent. Then, for any finite surjective morphism $f : Y \to X$ with X normal, the map $f^* : H^n(C)(X) \to H^n(C)(Y)$ is a monomorphism.*

PROOF. It is clearly sufficient to prove this when X is connected. Then, up to refinement, we can assume that f is a map as in Proposition 3.3.16. In this case, by virtue of Corollary 3.3.22, the $\mathbf{Q}$-vector space $H^n(C)(X) \simeq H^n(C)(Y)^G$ is a direct factor of $H^n(C)(Y)$. $\square$

THEOREM 3.3.30. *A presheaf of complexes of $\mathbf{Q}$-vector spaces on the category of S-schemes satisfies* h*-descent if and only if it satisfies* qfh*-descent and* cdh*-descent.*

PROOF. This is certainly a necessary condition, as the h-topology is finer than the qfh-topology and the cdh-topology. For the converse, as in the proof of Theorem 3.3.25, it is sufficient to prove that any presheaf of complexes of $\mathbf{Q}$-vector spaces C on the category of S-schemes satisfying qfh-descent and cdh-descent, and which is acyclic locally for the h-topology, is acyclic. We shall prove by noetherian induction that, given such a complex C, for any integer n, and any S-scheme X, for any section a of $H^n(C)$ over X, there exists a cdh-cover $X' \to X$ on which a vanishes. In other words, we shall get that C is acyclic locally for the cdh-topology, and, as C satisfies cdh-descent, this will imply that $H^n(C)(X) = H^n_{\mathrm{cdh}}(X, C_{\mathrm{cdh}}) = 0$ for any integer n and any S-scheme X. Note that the presheaves $H^n(C)$ send finite sums to finite direct sums (which follows, for instance, from the fact that C satisfies Nisnevich descent). In particular, $H^n(C)(\varnothing) = 0$ for any integer n.

Let X be an S-scheme, and $a \in H^n(C)(X)$. We have a cdh-cover of X of shape $X' \amalg X'' \to X$, where X' is the sum of the irreducible components of X_{red} and X'' is a nowhere dense closed subscheme of X, so that we can assume X is integral. Let a be a section of the presheaf $H^n(C)$ over X. As $H^n(C)_h = 0$, by virtue of Lemma 3.3.28, there exists a proper surjective birationnal map $p : Y \to X$ with Y normal, a Nisnevich cover $q : Z \to Y$, and a surjective finite morphism $r : U \to Z$ such that $r^*(q^*(p^*(a))) = 0$ in $H^n(C)(U)$. But then, Z is normal as well (see [**EGA4**, Prop. 18.10.7]), so that, by Lemma 3.3.29, we have $q^*(p^*(a)) = 0$ in $H^n(C)(Z)$. Let T be a nowhere dense closed subscheme of X such that p is an isomorphism over $X - T$. By noetherian induction, there exists a cdh-cover $T' \to T$ such that $a|_{T'}$ vanishes. Hence the section a vanishes on the cdh-cover $T' \amalg Z \to X$. □

3.3.d. *Proper descent with rational coefficients II: separation.* From now on, we assume that $\mathrm{Ho}(\mathscr{M})$ is $\mathbf{Q}$-linear.

Proposition 3.3.31. *Let $f : Y \to X$ be a morphism of schemes in $\mathscr{S}$, and G a finite group acting on Y over X. Denote by $\mathscr{Y}$ the scheme Y considered a functor from G to the category of S-schemes, and denote by $\varphi : (\mathscr{Y}, G) \to X$ the morphism induced by f. Then, for any object M of $\mathrm{Ho}(\mathscr{M})(X)$, there are canonical isomorphisms*

$$(\mathbf{R}f_*\,\mathbf{L}f^*(M))^G \simeq (\mathbf{R}f_*\,\mathbf{L}f^*(M))^{hG} \simeq \mathbf{R}\varphi_*\,\mathbf{L}\varphi^*(M)\,.$$

Proof. The second isomorphism comes from Proposition 3.1.15, and the first, from (3.3.21.3). □

Theorem 3.3.32. *If $\mathrm{Ho}(\mathscr{M})$ satisfies Nisnevich descent, the following conditions are equivalent:*

(i) *$\mathrm{Ho}(\mathscr{M})$ satisfies étale descent.*
(ii) *for any finite étale cover $f : Y \to X$, the functor*

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(Y)$$

is conservative;
(iii) *for any finite Galois cover $f : Y \to X$ of group G, and for any object M of $\mathrm{Ho}(\mathscr{M})(X)$, the canonical map*

$$M \to (\mathbf{R}f_*\,\mathbf{L}f^*(M))^G$$

is an isomorphism.

Proof. The equivalence between (i) and (iii) follows from Theorem 3.3.23 by corollaries 3.2.17 and 3.2.18, and Proposition 3.2.8 shows that (i) implies (ii). It is thus sufficient to prove that (ii) implies (iii). Let $f : Y \to X$ be a finite Galois cover of group G. As the functor $f^* = \mathbf{L}f^*$ is conservative by assumption, it is sufficient to check that the map $M \to (\mathbf{R}f_*\,\mathbf{L}f^*(M))^G$ becomes an isomorphism after applying f^*. By virtue of Proposition 3.1.17, this just means that it is sufficient to prove (iii) when f has a section, i.e. when Y is isomorphic to the trivial G-torsor over X. In this case, we have the (equivariant) identification $\bigoplus_{g \in G} M \simeq \mathbf{R}f_*\,\mathbf{L}f^*(M)$, where G acts on the left term by permuting the factors. Hence $M \simeq (\mathbf{R}f_*\,\mathbf{L}f^*(M))^G$. □

Proposition 3.3.33. *Assume that $\mathrm{Ho}(\mathscr{M})$ has the localization property. The following conditions are equivalent:*

(i) *$\mathrm{Ho}(\mathscr{M})$ is separated.*
(ii) *$\mathrm{Ho}(\mathscr{M})$ is semi-separated and satisfies étale descent.*

Proof. This follows from Proposition 2.3.9 and Theorem 3.3.32. □

Corollary 3.3.34. *Assume that all the residue fields of $\mathcal{S}$ are of characteristic zero, and that $\mathrm{Ho}(\mathscr{M})$ has the property of localization. Then the following conditions are equivalent:*

(i) *$\mathrm{Ho}(\mathscr{M})$ is separated.*
(ii) *$\mathrm{Ho}(\mathscr{M})$ satisfies étale descent.*

Proof. Consider a radicial finite surjective morphism $f : Y \to X$ in $\mathscr{S}$. To prove that the functor $\mathbf{L}f^*$ is conservative, as $\mathrm{Ho}(\mathscr{M})$ has the property of localization, by noetherian induction, we may replace X by any dense open subscheme U (and Y by $U \times_X Y$). The residue fields of X being of characteristic zero, this means that we may assume that f induces an isomorphism after reduction $Y_{red} \simeq X_{red}$. But it is clear that, by the localization property, such a morphism f induces an equivalence of categories $\mathbf{L}f^*$, so that $\mathrm{Ho}(\mathscr{M})$ is automatically semi-separated. We conclude by Proposition 3.3.33. □

PROPOSITION 3.3.35. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is separated, satisfies the localization property the proper transversality property. Then, for any pseudo-Galois cover* $f : Y \to X$ *of group* G*, and for any object* M *of* $\mathrm{Ho}(\mathscr{M})(X)$*, the canonical map*

$$M \to (\mathbf{R}f_*\,\mathbf{L}f^*(M))^G$$

is an isomorphism.

PROOF. By Proposition 3.3.33, this is an easy consequence of Proposition 2.1.9 and of condition (iii) of Theorem 3.3.32. □

3.3.36. From now on, we assume furthermore that any scheme in $\mathscr{S}$ is quasi-excellent.

THEOREM 3.3.37. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *satisfies the localization and proper transversality properties. Then the following conditions are equivalent:*

(i) $\mathrm{Ho}(\mathscr{M})$ *is separated;*
(ii) $\mathrm{Ho}(\mathscr{M})$ *satisfies* h-*descent;*
(iii) $\mathrm{Ho}(\mathscr{M})$ *satisfies* qfh-*descent;*
(iv) for any qfh-*distinguished square* (3.3.15.1) *of group* G*, if we write* $a = fh = ig : T \to X$ *for the composed map, then, for any object* M *of* $\mathrm{Ho}(\mathscr{M})(X)$*, the commutative square*

(3.3.37.1)
$$\begin{array}{ccc} M & \longrightarrow & (\mathbf{R}f_*\,\mathbf{L}f^*(M))^G \\ \downarrow & & \downarrow \\ \mathbf{R}i_*\,\mathbf{L}i^*(M) & \longrightarrow & (\mathbf{R}a_*\,\mathbf{L}a^*(M))^G \end{array}$$

is homotopy cartesian;
(v) the same as condition (iv), but only for pseudo-Galois qfh-*distinguished squares.*

PROOF. As $\mathscr{M}$ satisfies cdh-descent (Theorem 3.3.10), the equivalence between conditions (ii) and (iii) follows from Theorem 3.3.30 by Corollary 3.2.18. Similarly, Theorem 3.3.25 and corollaries 3.3.22, 3.2.17 and 3.2.18 show that conditions (iii), (iv) and (v) are equivalent. As étale surjective morphisms as well as finite radicial epimorphisms are qfh-coverings, it follows from Proposition 3.2.8, Theorem 3.3.32 and Proposition 3.3.33, that condition (iii) implies condition (i). It thus remains to prove that condition (i) implies condition (v). So let us consider a pseudo-Galois qfh-distinguished square (3.3.15.1) of group G, and prove that (3.3.37.1) is homotopy cartesian. Using proper transversality, we see that the image of (3.3.37.1) by the functor $\mathbf{L}i^*$ is (isomorphic to) the homotopy pullback square

$$\begin{array}{ccc} \mathbf{L}i^*(M) & \longrightarrow & (\mathbf{R}g_*\,\mathbf{L}g^*\,\mathbf{L}i^*(M))^G \\ \| & & \| \\ \mathbf{L}i^*(M) & \longrightarrow & (\mathbf{R}g_*\,\mathbf{L}g^*\,\mathbf{L}i^*(M))^G \end{array} .$$

Write $j : U \to X$ for the complement open immersion of i, and $b : f^{-1}(U) \to U$ for the map induced by f. As j is étale, we see, using Proposition 3.1.17, that the image of (3.3.9.1) by $j^* = \mathbf{L}j^*$ is (isomorphic to) the square

$$\begin{array}{ccc} j^*(M) & \longrightarrow & (\mathbf{R}b_*\,\mathbf{L}b^*\,j^*(M))^G \\ \downarrow & & \downarrow \\ 0 & = & 0 \end{array} .$$

in which the upper horizontal map is an isomorphism by Proposition 3.3.35. Hence it is a homotopy pullback square. Thus, because the pair of functors $(\mathbf{L}i^*, j^*)$ is conservative on $\mathrm{Ho}(\mathscr{M})(X)$, the square (3.3.37.1) is homotopy cartesian. □

Corollary 3.3.38. *Assume that all the residue fields of $\mathcal{S}$ are of characteristic zero, and that* $\mathrm{Ho}(\mathscr{M})$ *has the localization and proper transversality properties. Then* $\mathrm{Ho}(\mathscr{M})$ *satisfies* h*-descent if and only if it satisfies étale descent.*

Proof. This follows from Corollary 3.3.34 and Theorem 3.3.37. □

Corollary 3.3.39. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is separated and has the localization and proper transversality properties. Let $f : Y \to X$ be a finite surjective morphism, with X normal, and G a group acting on Y over X, such that the map $Y/G \to X$ is generically radicial (i.e. radicial over a dense open subscheme of X). Consider at last a pullback square of the following shape.*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

Then, for any object M of $\mathrm{Ho}(\mathscr{M})(X')$*, the natural map*

$$M \to (\mathbf{R}f'_*\, \mathbf{L}f'^*(M))^G$$

is an isomorphism.

Proof. For any presheaf C of complexes of $\mathbf{Q}$-vector spaces on $\mathscr{S}/X$, one has an isomorphism

$$\mathbf{R}\Gamma_{\mathrm{qfh}}(X', C_{\mathrm{qfh}}) \simeq \mathbf{R}\Gamma_{\mathrm{qfh}}(Y', C_{\mathrm{qfh}})^G\,.$$

This follows from the fact that we have an isomorphism of qfh-sheaves of sets $L(Y)/G \simeq L(X)$ (the map $Y \to Y/G$ being generically flat, this is Proposition 3.3.19), which implies that the map $L(Y')/G \to L(X')$ is an isomorphism of qfh-sheaves (by the universality of colimits in topoi), and implies this assertion (as in the proof of 3.3.22).

By virtue of Theorem 3.3.37, $\mathrm{Ho}(\mathscr{M})$ satisfies qfh-descent, so that the preceding computations imply the result by corollaries 3.2.17 and 3.2.18. □

Corollary 3.3.40. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is separated and has the localization and proper transversality properties. Then for any finite surjective morphism $f : Y \to X$ with X normal, the morphism*

$$M \to \mathbf{R}f_*\, \mathbf{L}f^*(M)$$

is a monomorphism and admits a functorial splitting in $\mathrm{Ho}(\mathscr{M})(X)$*. Furthermore, this remains true after base change by any map $X' \to X$.*

Proof. It is sufficient to treat the case where X is connected. We may replace Y by a normalization of X in a suitable finite extension of its field of function, and assume that a finite group G acts on Y over X, so that the properties described in the preceding corollary are fulfilled (see 3.3.16). □

Remark 3.3.41. The condition (iv) of Theorem 3.3.37 can be reformulated in a more global way as follows (this won't be used in these notes, but this might be useful for the reader who might want to formulate all this in terms of (pre-)algebraic derivators [**Ayo07a**, Def. 2.4.13]). Given a qfh-distinguished square (3.3.15.1) of group G, we can form a functor $\mathscr{F}$ from category $I = \ulcorner$ (3.3.11.1) to the category of diagrams of S-schemes corresponding to the diagram of diagrams of S-schemes

$$\begin{array}{ccc} (\mathscr{T}, G) & \xrightarrow{(h, 1_G)} & (\mathscr{Y}, G) \\ {\scriptstyle g}\big\downarrow & & \\ Z & & \end{array}$$

in which $\mathscr{T}$ and $\mathscr{Y}$ correspond to T anf Y respectively, seen as functor from G to $\mathscr{S}/X$. The construction of 3.1.22 gives a diagram of X-schemes $(\int \mathscr{F}, I_{\mathscr{F}})$ which can be described explicitely as follows. The category $I_{\mathscr{F}}$ is the cofibred category over $\ulcorner$ associated to the functor from $\ulcorner$ to the category of small categories defined by the diagram

$$\begin{array}{ccc} G & \xrightarrow{1_G} & G \\ \downarrow & & \\ e & & \end{array}$$

in which e stands for the terminal category, and G for the category with one object associated to G. It has thus three objects a, b, c (see (3.3.11.1)), and the morphisms are determined by

$$\mathrm{Hom}_{I_{\mathscr{F}}}(x,y) = \begin{cases} * & \text{if } y = c; \\ \varnothing & \text{if } x \neq y \text{ and } x = b, c; \\ \mathrm{G} & \text{otherwise.} \end{cases}$$

The functor $\mathscr{F}$ sends a, b, c to T, Y, Z respectively, and simply encodes the fact that the diagram

$$\begin{array}{ccc} T & \xrightarrow{h} & Y \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array}$$

is G-equivariant, the action on Z being trivial. Now, by propositions 3.1.23 and 3.3.31, if $\varphi : (\mathscr{F}, I_{\mathscr{F}}) \to (X, \ulcorner)$ denotes the canonical map, for any object M of $\mathrm{Ho}(\mathscr{M})(X)$, the object $\mathbf{R}\varphi_* \, \mathbf{L}\varphi^*(M)$ is the functor from $\lrcorner = \ulcorner^{op}$ to $\mathscr{M}(X)$ corresponding to the diagram below (of course, this is well defined only in the homotopy category of the category of functors from $\lrcorner$ to $\mathscr{M}(X)$).

$$\begin{array}{ccc} & & (\mathbf{R}f_* \, \mathbf{L}f^*(M))^G \\ & & \downarrow \\ \mathbf{R}i_* \, \mathbf{L}i^*(M) & \longrightarrow & (\mathbf{R}a_* \, \mathbf{L}a^*(M))^G \end{array}$$

As a consequence, if $\psi : (\int \mathscr{F}, I_{\mathscr{F}}) \to X$ denotes the structural map, the object $\mathbf{R}\psi_* \, \mathbf{L}\psi^*(M)$ is simply the homotopy homotopy limit of the diagram of $\mathscr{M}(X)$ above, so that condition (iv) of Theorem 3.3.37 can now be reformulated by saying that the map

$$M \to \mathbf{R}\psi_* \, \mathbf{L}\psi^*(M)$$

is an isomorphism, i.e. that the functor

$$\mathbf{L}\psi^* : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(\mathscr{M})(\textstyle\int \mathscr{F}, I_{\mathscr{F}})$$

is fully faithful.

4. Constructible motives

4.0. Consider as in 2.0 a base scheme $\mathcal{S}$ and a sub-category $\mathscr{S}$ of the category of $\mathcal{S}$-schemes. In section 4.4, and for the main theorem of section 4.2, we will assume:

(a) Any scheme in $\mathscr{S}$ is quasi-excellent.[54]

Apart in Definition 4.3.2 and the subsequent proposition, where we will consider an abstract situation, we will be concerned with the study of a fixed premotivic triangulated category $\mathscr{T}$ over $\mathscr{S}$ (recall Definition 2.4.45) such that:

(b) $\mathscr{T}$ is motivic (see Definition 2.4.45).
(c) $\mathscr{T}$ is endowed with a set of twists τ (see Paragraph 1.4.4) which is stable under Tate twists $\mathbb{1}(p)[q]$, for $p, q \in \mathbf{Z}$.

[54] See Paragraph 4.1.1. The reader can safely restrict his attention to the more classical notion of an excellent scheme ([**EGA4**, IV, 7.8.5]).

(d) $\mathscr{T}$ is the homotopy category associated with a stable combinatorial *Sm*-fibred model category $\mathscr{M}$ over $\mathscr{S}$.[55]

As usual, the geometric section of $\mathscr{T}$ will be denoted by M.

Unless explicitly referring to the underlying model category $\mathscr{M}$, we will not indicate in the notation of the six operations that the functors are derived functors.

4.1. Resolution of singularities. The aim of this subsection is to gather the results from the theory of resolution of singularities that will be used subsequently.

4.1.1. In [**EGA4**, IV, 7.8.2], Grothendieck defined the notion of an *excellent ring*. Matsumura introduced in [**Mat70**] the weaker notion of a *quasi-excellent* ring A. Recall A is quasi-excellent if the following conditions hold:

(1) A is noetherian.
(2) For any prime ideal $\mathfrak{p}$, $\hat{A}_{\mathfrak{p}}$ being the completion of A at $\mathfrak{p}$, the canonical morphism $A \to \hat{A}_{\mathfrak{p}}$ is regular (see 4.1.4 below).
(3) For any A-algebra B of finite type, the regular locus of $\operatorname{Spec}(B)$ is open.

Then a ring A is excellent if it is quasi-excellent and universally catenary. Following Gabber, we say a scheme X is *quasi-excellent* (*excellent*) if it admits an open cover by affine schemes whose rings are quasi-excellent (excellent, respectively).

THEOREM 4.1.2 (Gabber's weak local uniformisation). *Let X be a quasi-excellent scheme, and $Z \subset X$ a nowhere dense closed subscheme. Then there exists a finite* h*-cover $\{f_i : Y_i \to X\}_{i\in I}$ such that for all i in I, f_i is a morphism of finite type, the scheme Y_i is regular, and $f_i^{-1}(Z)$ is either empty or the support of a strict normal crossing divisor in Y_i.*

See [**Ill08**] for a sketch of proof. A complete argument can be found in [**ILO**]. Note that, if we are only interested in schemes of finite type over $\operatorname{Spec}(R)$, for R either a field, a complete discrete valuation ring, or a Dedekind domain whose field of functions is a global field, this is an immediate consequence of de Jong's resolution of singularities by alterations; see [**dJ96**]. One can also deduce the case of schemes of finite type over an excellent noetherian scheme of dimension lesser or equal to 2 from [**dJ97**]; see Theorem 4.1.10 and Corollary 4.1.11 below for a precise statement.

REMARK 4.1.3. This theorem will be used in the proof of Lemma 4.2.14 which is the key point for the proof of Theorem 4.2.16.

4.1.4. Recall that a morphism of rings $u : A \to B$ is *regular* if it is flat, and if, for any prime ideal $\mathfrak{p}$ in A, with residue field $\kappa(\mathfrak{p})$, the $\kappa(\mathfrak{p})$-algebra $\kappa(\mathfrak{p}) \otimes_A B$ is geometrically regular (equivalently, this means that, for any prime ideal $\mathfrak{q}$ of B, the A-algebra $B_{\mathfrak{q}}$ is formally smooth for the $\mathfrak{q}$-adic topology). We recall the following great generalization of Neron's desingularisation theorem:

THEOREM 4.1.5 (Popescu-Spivakovsky). *A morphism of noetherian rings $u : A \to B$ is regular if and only if B is a filtered colimit of smooth A-algebras of finite type.*

PROOF. See [**Spi99**, theorems 1.1 and 1.2]. □

4.1.6. Recall that an *alteration* is a proper surjective morphism $p : X' \to X$ which is generically finite, i.e. such that there exists a dense open subscheme $U \subset X$ over which p is finite.

DEFINITION 4.1.7 (de Jong). Let X be a noetherian scheme endowed with an action of a finite group G. A *Galois alteration* of the couple (X, G) is the data of a finite group G', of a surjective morphism of groups $G' \to G$, of an alteration $X' \to X$, and of an action of G' on X', such that:

(i) the map $X' \to X$ is G'-equivariant;
(ii) for any irreducible component T of X, there exists a unique irreducible component T' of X' over T, and the corresponding finite field extension

$$k(T)^G \subset k(T')^{G'}$$

is purely inseparable.

[55] We use this assumption to use freely the descent theory described in section 3.3.

In practice, we shall keep the morphism of groups $G' \to G$ implicit, and we shall say that $(X' \to X, G')$ is a Galois alteration of (X, G).

Given a noetherian scheme X, a *Galois alteration* of X is a Galois alteration $(X' \to X, G)$ of (X, e), where e denotes the trivial group. In this case, we shall say that $X' \to X$ is a *Galois alteration of X of group G*.

REMARK 4.1.8. If $p : X' \to X$ is a Galois alteration of group G over X, then, if X and X' are normal, irreducible and quasi-excellent, p can be factored as a radicial finite surjective morphism $X'' \to X$, followed by a Galois alteration $X' \to X''$ of group G, such that $k(X'') = k(X')^G$ (just define X'' as the normalization of X in $k(X')^G$). In other words, up to a radicial finite surjective morphism, X is generically the quotient of X' under the action of G.

DEFINITION 4.1.9. A noetherian scheme S *admits canonical dominant resolution of singularities up to quotient singularities* if, for any Galois alteration $S' \to S$ of group G, and for any G-equivariant nowhere dense closed subscheme $Z' \subset S'$, there exists a Galois alteration $(p : S'' \to S', G')$ of (S', G), such that S'' is regular and projective over S, and such that the inverse image of Z' in S'' is contained in a G'-equivariant strict normal crossing divisor (i.e. a strict normal crossing divisor whose irreducible components are stable under the action of G').

A noetherian scheme S admits *canonical resolution of singularities up to quotient singularities* if any integral closed subscheme of S admits canonical dominant resolution of singularities up to quotient singularities.

A noetherian scheme S admits *wide resolution of singularities up to quotient singularities* if, for any separated S-scheme of finite type X, and any nowhere dense closed subscheme $Z \subset X$, there exists a projective Galois alteration $p : X' \to X$ of group G, with X' regular, such that, in each connected component of X', $Z' = p^{-1}(Z)$ is either empty, either the support of a strict normal crossing divisor.

THEOREM 4.1.10 (de Jong). *If an excellent noetherian scheme of finite dimension S admits canonical resolution of singularities up to quotient singularities, then any separated S-scheme of finite type admits canonical resolution of singularities up to quotient singularities.*

PROOF. Let X be an integral separated S-scheme of finite type. There exists a finite morphism $S' \to S$, with S' integral, an integral dominant S'-scheme X' and a radicial extension $X' \to X$ over S, such that X' has a geometrically irreducible generic fiber over S'. It follows then from (the proof of) [**dJ97**, theorem 5.13] that X' admits canonical dominant resolution of singularities up to quotient singularities, which implies that X has the same property. □

COROLLARY 4.1.11 (de Jong). *Let S be an excellent noetherian scheme of dimension lesser or equal to* 2. *Then any separated scheme of finite type over S admits canonical resolution of singularities up to quotient singularities. In particular, S admits wide resolution of singularities up to quotient singularities.*

PROOF. See [**dJ97**, corollary 5.15]. □

4.2. Finiteness theorems. The aim of this section is to study the notion of τ-constructibility in the triangulated motivic case and to study its stability properties under Grothendieck six operations. Recall the following particular case of Definition 1.4.9:

DEFINITION 4.2.1. For a scheme X in $\mathscr{S}$, we denote by $\mathscr{T}_c(X)$ the thick triangulated subcategory of $\mathscr{T}(X)$ generated by premotives of the form $M_X(Y)\{i\}$ for a smooth X-scheme Y and a twist $i \in \tau$. We will say that a premotive in $\mathscr{T}_c(X)$ is *τ-constructible*, or, simply, *constructible.*

REMARK 4.2.2. Let us mention that our main examples:

- the stable homotopy category SH (cf. Example 1.4.3),
- the category of Voevodsky motives DM (cf. Definition 11.1.1),
- the category of Beilinson motives DM_{B} (cf. Definition 14.2.1)

are all generated by the Tate twists (*i.e.* $\tau = \mathbf{Z}$). Recall also Proposition 1.4.11: it applies to all these examples so that constructible premotives coincides with compact objects.[56]

[56]Notice however this fact is not true for étale motivic complexes.

Proposition 4.2.3. *If M and N are constructible in $\mathscr{T}(X)$, so is $M \otimes_X N$.*

Proof. For a fixed M, the full subcategory of $\mathscr{T}(X)$ spanned by objects such that $M \otimes_X N$ is constructible is a thick triangulated subcategory of $\mathscr{T}(X)$. In the case M is of shape $M_X(Y)\{n\}$ for Y smooth over X and $n \in \tau$, this proves that $M \otimes_X N$ is constructible whenever N is. By the same argument, using the symmetry of the tensor product, we get to the general case. □

Similarly, one has the following conservation property.

Proposition 4.2.4. *For any morphism $f : X \to Y$ of schemes, the functor*

$$f^* : \mathscr{T}(Y) \to \mathscr{T}(X)$$

preserves constructible objects. If moreover f is smooth, the functor

$$f_\sharp : \mathscr{T}(X) \to \mathscr{T}(Y)$$

also preserves constructible objects.

Corollary 4.2.5. *The categories $\mathscr{T}_c(X)$ form a thick triangulated monoidal Sm-fibred subcategory of $\mathscr{T}$.*

Proposition 4.2.6. *Let X a scheme, and $X = \bigcup_{i \in I} U_i$ a cover of X by open subschemes. An object M of $\mathscr{T}(X)$ is constructible if and only if its restriction to U_i is constructible in $\mathscr{T}(U_i)$ for all $i \in I$.*

Proof. This is a necessary condition by 4.2.4. For the converse, as X is noetherian, it is sufficient to treat the case where I is finite. Proceeding by induction on the cardinal of I it is sufficient to treat the case of a cover by two open subschemes $X = U \cup V$. For an open immersion $j : W \to X$, write $M_W = j_\sharp\, j^*(M)$. If the restrictions of M to U and V are constructible, then so is its restriction to $U \cap V$. According to Proposition 3.3.4, we get a distinguished triangle

$$M_{U\cap V} \to M_U \oplus M_V \to M \to M_{U\cap V}[1]$$

in which M_W is constructible for $W = U, V, U \cap V$ (using 4.2.4 again). Thus the premotive M is constructible. □

Corollary 4.2.7. *For any scheme X and any vector bundle E over X, the functors $\mathcal{T}h(E)$ and $\mathcal{T}h(-E)$ preserve constructible objects in $\mathscr{T}(X)$.*

Proof. To prove that $\mathcal{T}h(E)$ and $\mathcal{T}h(-E)$ preserves constructible objects, by virtue of the preceding proposition, we may assume that E is trivial of rank r. It is thus sufficient to prove that $M(r)$ is constructible whenever M is so for any integer r. For we may assume that $M = \mathbb{1}_X\{n\}$ for some $n \in \tau$ (using 4.2.4), this is true by assumption on τ; see 4.0(c). □

Corollary 4.2.8. *Let $f : X \to Y$ a morphism of finite type. The property that the functor*

$$f_* : \mathscr{T}(X) \to \mathscr{T}(Y)$$

preserves constructible objects is local on Y with respect to the Zariski topology.

Proof. Consider a finite Zariski cover $\{v_i : Y_i \to Y\}_{i\in I}$, and write $f_i : X_i \to Y_i$ for the pullback of f along v_i for each i in I. Assume that the functors $f_{i,*}$ preserves constructible objects; we shall prove that f_* has the same property. Let M be a constructible object in $\mathscr{T}(X)$. Then for $i \in I$, using the smooth base change isomorphism (for open immersions), we see that the restriction of $f_*(M)$ to Y_i is isomorphic to the image by $f_{i,*}$ of the restriction of M to X_i, hence is constructible. The preceding proposition thus implies that $f_*(M)$ is constructible. □

Proposition 4.2.9. *For any closed immersion $i : Z \to X$, the functor*

$$i_* : \mathscr{T}(Z) \to \mathscr{T}(X)$$

preserves constructible objects.

PROOF. It is sufficient to prove that for any smooth Z-scheme Y_0 and any twist $n \in \tau$, the premotive $i_*(M_Z(Y_0)\{n\})$ is constructible in $\mathscr{T}(X)$. According to the Mayer-Vietoris triangle (see Remark 3.3.6), this assertion is local in X. Thus we can assume there exists a smooth X-scheme Y such that $Y_0 = Y \times_X Z$ (apply [**EGA4**, 18.1.1]). Put $U = X - Z$ and let $j : U \to X$ be the obvious open immersion. From the localization property, we get a distinguished triangle

$$M_X(Y \times_X U)\{n\} \to M_X(Y)\{n\} \to i_*(M_Z(Y_0)\{n\}) \to M_X(Y \times_X U)\{n\}[1]$$

and this concludes. □

COROLLARY 4.2.10. *Let $i : Z \to X$ be a closed immersion with open complement $j : U \to X$. an object M of $\mathscr{T}(X)$ is constructible if and only if $j^*(M)$ and $i^*(M)$ are constructible in $\mathscr{T}(U)$ and $\mathscr{T}(Z)$ respectively.*

PROOF. We have a distinguished triangle

$$j_\sharp\, j^*(M) \to M \to i_*\, i^*(M) \to j_\sharp\, j^*(M)[1]\,.$$

Hence this assertion follows from propositions 4.2.4 and 4.2.9. □

PROPOSITION 4.2.11. *If $f : X \to Y$ is proper, then the functor*

$$f_* : \mathscr{T}(X) \to \mathscr{T}(Y)$$

preserves constructible objects.

PROOF. We shall first consider the case where f is projective. As this property is local on Y (Corollary 4.2.8), we may assume that f factors as a closed immersion $i : X \to \mathbf{P}^n_Y$ followed by the canonical projection $p : \mathbf{P}^n_Y \to Y$. By virtue of Proposition 4.2.9, we can assume that $f = p$. In this case, the functor p_* is isomorphic to $p_\sharp$ composed with the quasi-inverse of the Thom endofunctor associated to the cotangent bundle of p; see 2.4.50 (3). Therefore, the functor p_* preserves constructible objects by virtue of Proposition 4.2.4 and of Corollary 4.2.7. The case where f is proper follows easily from the projective case, using Chow's lemma and cdh-descent (the homotopy pullback squares (3.3.9.1)), by induction on the dimension of X. □

COROLLARY 4.2.12. *If $f : X \to Y$ is separated of finite type, then the functor*

$$f_! : \mathscr{T}(X) \to \mathscr{T}(Y)$$

preserves constructible objects.

PROOF. It is sufficient to treat the case where f is either an open immersion, either a proper morphism, which follows respectively from 4.2.4 and 4.2.11. □

PROPOSITION 4.2.13. *Let X be a scheme. The category of constructible objects in $\mathscr{T}(X)$ is the smallest thick triangulated subcategory which contains the objects of shape $f_*(\mathbb{1}_{X'}\{n\})$, where $f : X' \to X$ is a (strictly) projective morphism, and $n \in \tau$.*

PROOF. Let $\mathscr{T}_p(X)$ be the smallest thick triangulated subcategory which contains the objects of shape $f_*(\mathbb{1}_{X'}\{n\})$, where $f : X' \to X$ is a (strictly) projective morphism, and $n \in \tau$. Proposition 4.2.11 shows that $\mathscr{T}_p(X) \subset \mathscr{T}_c(X)$, to that it is sufficient to prove the reverse inclusion. Note that, for any separated smooth morphism f, locally for the Zariski topology, $f_\sharp$ coincides with $f_!$ up to a Tate twist. In other words, it is sufficient to prove that, for any separated morphism of finite type $f : Y \to X$, $f_!(\mathbb{1}_Y)$ belongs to $\mathscr{T}_p(X)$. If we factor f into an open immersion $j : Y \to X'$ followed by a proper morphism $p : X' \to X$, we see that is sufficient to prove that $j_\sharp(\mathbb{1}_Y)$ belongs to $\mathscr{T}_p(X')$. This follows straight away from the localization property. □

The following lemma is the key geometrical point for the finiteness Theorem 4.2.16

LEMMA 4.2.14. *Let $j : U \to X$ be a dense open immersion such that X is quasi-excellent. Then, there exists the following data:*

(i) *a finite h-cover* $\{f_i : Y_i \to X\}_{i\in I}$ *such that for all* i *in* I, f_i *is a morphism of finite type, the scheme* Y_i *is regular, and* $f_i^{-1}(U)$ *is either* Y_i *itself or the complement of a strict normal crossing divisor in* Y_i*; we shall write*

$$f : Y = \coprod_{i\in I} Y_i \to X$$

for the induced global h-cover;

(ii) *a commutative diagram*

(4.2.14.1)
$$\begin{array}{ccccc} X''' & & \xrightarrow{\quad g \quad} & & Y \\ {\scriptstyle q}\downarrow & & & & \downarrow{\scriptstyle f} \\ X'' & \xrightarrow{u} & X' & \xrightarrow{p} & X \end{array}$$

in which: p *is a proper birational morphism,* X' *is normal,* u *is a Nisnevich cover, and* q *is a finite surjective morphism.*

Let T *(resp.* T'*) be a closed subscheme of* X *(resp.* X'*) and assume that for any irreducible component* T_0 *of* T*, the following inequality is satisfied:*

(4.2.14.2)
$$\operatorname{codim}_{X'}(T') \geq \operatorname{codim}_X(T_0),$$

Then, possibly after shrinking X *in an open neighbourhood of the generic points of* T *in* X*, one can replace* X'' *by an open cover and* X''' *by its pullback along this cover, in such a way that we have in addition the following properties:*

(iii) $p(T') \subset T$ *and the induced map* $T' \to T$ *is finite and pseudo-dominant;*[57]

(iv) *if we write* $T'' = u^{-1}(T')$*, the induced map* $T'' \to T'$ *is an isomorphism.*

Proof. The existence of $f : Y \to X$ as in (i) follows from Gabber's weak uniformisation theorem (see 4.1.2), while the commutative diagram (4.2.14.1) satisfying property (ii) is ensured by Lemma 3.3.28.

Consider moreover closed subschemes $T \subset X$ and $T' \subset X'$ satisfying (4.2.14.2).

We first show that, by shrinking X in an open neighbourhood of the generic points of T and by replacing the diagram (4.2.14.1) by its pullback over this neighbourhood, we can assume that condition (iii) is satisfied. Note that shrinking X in this way does not change the condition (4.2.14.2) because $\operatorname{codim}_X(T_0)$ does not change and $\operatorname{codim}_{X'}(T')$ can only increase.[58]

Note first that, by shrinking X, we can assume that any irreducible component T'_0 of T' dominates an irreducible component T_0 of T. In fact, given an irreducible component T'_0 which does not satisfies this condition, $p(T'_0)$ is a closed subscheme of X disjoint from the set of generic points of T and replacing X by $X - f(T'_0)$, we can throw out T'_0.

Further, shrinking X again, we can assume that for any pair (T'_0, T_0) as in the preceding paragraph, $p(T'_0) \subset T_0$. In fact, in any case, as $p(T'_0)$ is closed we get that $T_0 \subset p(T'_0)$. Let Z be the closure of $p(T'_0) - T_0$ in X. Then Z does not contain any generic point of T (because $p(T'_0)$ is irreducible), and $p(T'_0) \cap (X - Z) \subset T_0$. Thus it is sufficient to replace X by $X - Z$ to ensure this assumption.

Consider again a pair (T'_0, T_0) as in the two preceding paragraphs and the induced commutative square:

(4.2.14.3)
$$\begin{array}{ccc} T'_0 & \longrightarrow & X' \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\ T_0 & \longrightarrow & X \end{array}$$

[57] Recall from 8.1.3 that this means that any irreducible component of T' dominates an irreducible component of T.

[58] Remember that for any scheme X, $\operatorname{codim}_X(\varnothing) = +\infty$.

We show that the map p_0 is generically finite. In fact, this will conclude the first step, because if it is true for any irreducible component T'_0 of T', we can shrink X again so that the dominant morphism $p_0 : T'_0 \to T_0$ becomes finite.

Denote by c' (resp. c) the codimension of T_0 in X (resp. T'_0 in X'). Note that (4.2.14.2) gives the inequality $c' \geq c$. Let t_0 be the generic point of T_0, Ω the localization of X at t_0 and consider the pullback of (4.2.14.3):

(4.2.14.4)
$$\begin{array}{ccc} W' & \longrightarrow & \Omega' \\ {\scriptstyle q_0}\downarrow & & \downarrow{\scriptstyle q} \\ \{t_0\} & \longrightarrow & \Omega. \end{array}$$

We have to prove that $\dim(W') = 0$. Consider an irreducible component Ω'_0 of Ω' containing W'. As q is still proper birational, Ω'_0 corresponds to a unique irreducible component Ω_0 of Ω such that q induces a proper birational map $\Omega'_0 \to \Omega_0$. According to [**EGA4**, 5.6.6], we get the inequality

$$\dim(\Omega'_0) \leq \dim(\Omega_0).$$

Thus, we obtain the following inequalities:

$$\dim(W') \leq \dim(\Omega'_0) - \mathrm{codim}_{\Omega'_0}(W') \leq \dim(\Omega_0) - \mathrm{codim}_{\Omega'_0}(W') \leq \dim(\Omega) - \mathrm{codim}_{\Omega'_0}(W').$$

As this is true for any irreducible component Ω'_0 of Ω', we finally obtain:

$$\dim(W') \leq \dim(\Omega) - \mathrm{codim}_{\Omega'}(W') \leq c - c'$$

and this concludes the first step.

Keeping T' and T as above, as the map from T'' to T' is a Nisnevich cover, it is a split epimorphism in a neighbourhood of the generic points of T' in X'. Hence, as the map $X' \to X$ is proper and birational, we can find a neighbourhood of the generic points of T in X over which the map $T'' \to T'$ admits a section $s : T' \to T''$. Let S be a closed subset of X'' such that $T'' = s(T') \amalg S$ (which exists because $X'' \to X'$ is étale). The map $(X'' - T'') \amalg (X'' - S) \to X'$ is then a Nisnevich cover. Replacing X'' by $(X'' - T'') \amalg (X'' - S)$ (and X''' by the pullback of $X''' \to X''$ along $(X'' - T'') \amalg (X'' - S) \to X'$), we may assume that the induced map $T'' \to T'$ is an isomorphism, without modifying further the data f, p, T and T'. This gives property (iv) and ends the proof the lemma. □

4.2.15. Let $\mathscr{T}_0$ be a full *Open*-fibred subcategory of $\mathscr{T}$ (where *Open* stands for the class of open immersions). We assume that $\mathscr{T}_0$ has the following properties.

(a) for any scheme X in $\mathscr{S}$, $\mathscr{T}_0(X)$ is a thick subcategory of $\mathscr{T}(X)$ which contains the objects of the form $\mathbb{1}_X\{n\}$, $n \in \tau$;
(b) for any separated morphism of finite type $f : X \to Y$ in $\mathscr{S}$, $\mathscr{T}_0$ is stable under $f_!$;
(c) for any dense open immersion $j : U \to X$, with X regular, which is the complement of a strict normal crossing divisor, $j_*(\mathbb{1}_U\{n\})$ is in $\mathscr{T}_0(U)$ for any $n \in \tau$.

Properties (a) and (b) have the following consequences: any constructible object belongs to $\mathscr{T}_0$; given a closed immersion $i : Z \to X$ with complement open immersion $j : U \to X$, an object M of $\mathscr{T}(X)$ belongs to $\mathscr{T}_0(X)$ if and only if $j^*(M)$ and $i^*(M)$ belongs to $\mathscr{T}_0(U)$ and $\mathscr{T}_0(Z)$ respectively; for any scheme X in $\mathscr{S}$, the condition that an object of $\mathscr{T}(X)$ belongs to $\mathscr{T}_0(X)$ is local on X for the Zariski topology.

Theorem 4.2.16. *Consider the above hypothesis and assume that $\mathscr{T}$ is $\mathbf{Q}$-linear and separated. Let Y be a quasi-excellent scheme and $f : X \to Y$ be a morphism of finite type.*
Then for any constructible object M of $\mathscr{T}(X)$, the object $f_*(M)$ belongs to $\mathscr{T}_0(Y)$.

Proof. It is sufficient to prove that, for any dense open immersion $j : U \to X$, and for any $n \in \tau$, the object $j_*(\mathbb{1}_U\{n\})$ is in $\mathscr{T}_0$. Indeed, assume this is known. We want to prove that $f_*(M)$ is in $\mathscr{T}_0(Y)$ whenever M is constructible. We deduce from property (b) of 4.2.15 and from Proposition 4.2.13 that it is sufficient to consider the case where $M = \mathbb{1}_X\{n\}$, with $n \in \tau$. Then, as this property is assumed to be known for dense open immersions, by an easy Mayer-Vietoris argument, we see that the condition that $f_*(\mathbb{1}_X\{n\})$ belongs to $\mathscr{T}_0$ is local on X with respect to the Zariski topology. Therefore, we may assume that f is separated. Consider a compactification of f, i.e. a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & \bar{Y} \\ {\scriptstyle f}\big\downarrow & \swarrow_{\bar{f}} & \\ X & & \end{array}$$

with j a dense open immersion, and $\bar{f}$ proper. By property (b) of 4.2.15, we may assume that $f = j$ is a dense open immersion.

Let $j : U \to X$ be a dense open immersion. We shall prove by induction on the dimension of X that , for any $n \in \tau$, the object $j_*(\mathbb{1}_U\{n\})$ is in $\mathscr{T}_0$. The case where X is of dimension ≤ 0 follows from the fact the map j is then an isomorphism, which implies that $j_\sharp \simeq j_*$, and allows to conclude (because $\mathscr{T}_0$ is *Open*-fibred).

Assume that $\dim X > 0$. Following an argument used by Gabber [**Ill07**] in the context of ℓ-adic sheaves, we shall prove by induction on $c \geq 0$ that there exists a closed subscheme $T \subset X$ of codimension $> c$ such that, for any $n \in \tau$, the restriction of $j_*(\mathbb{1}_U\{n\})$ to $X - T$ is in $\mathscr{T}_0(X - T)$. As X is of finite dimension, this will obviously prove Theorem 4.2.16.

The case where $c = 0$ is clear: we can choose T such that $X - T = U$. If $c > 0$, we choose a closed subscheme T of X, of codimension $> c - 1$, such that the restriction of $j_*(\mathbb{1}_U\{n\})$ to $X - T$ is in $\mathscr{T}_0$. It is then sufficient to find a dense open subscheme V of X, which contains all the generic points of T, and such that the restriction of $j_*(\mathbb{1}_U\{n\})$ to V is in $\mathscr{T}_0$: for such a V, we shall obtain that the restriction of $j_*(\mathbb{1}_U\{n\})$ to $V \cup (X - T)$ is in $\mathscr{T}_0$, the complement of $V \cup (X - T)$ being the support of a closed subscheme of codimension $> c$ in X. In particular, using the smooth base change isomorphism (for open immersions), we can always replace X by a generic neighbourhood of T. It is sufficient to prove that, possibly after shrinking X as above, the pullback of $j_*(\mathbb{1}_U\{n\})$ along $T \to X$ is in $\mathscr{T}_0$ (as we already know that its restriction to $X - T$ is in $\mathscr{T}_0$).

We may assume that T is purely of codimension c. We may assume that we have data as in points (i) and (ii) of Lemma 4.2.14. We let $j' : U' \to X'$ denote the pullback of j along $p : X' \to X$. Then, we can find, by induction on c, a closed subscheme T' in X', of codimension $> c - 1$, such that the restriction of $j'_*(\mathbb{1}_{U'}\{n\})$ to $X' - T'$ is in $\mathscr{T}_0$. By shrinking X, we may assume that conditions (iii) and (iv) of Lemma 4.2.14 are fulfilled as well.

For an X-scheme $w : W \to X$ and a closed subscheme $Z \subset W$, we shall write

$$\varphi(W, Z) = w_*\, i_*\, i^*\, j_{W,*}\, j_W^*(\mathbb{1}_W\{n\})\,,$$

where $i : Z \to W$ denotes the inclusion, and $j_W : W_U \to W$ stands for the pullback of j along w. This construction is functorial with respect to morphisms of pairs of X-schemes: if $W' \to W$ is a morphism of X-schemes, with Z' and Z two closed subschemes of W' and W respectively, such that Z' is sent to Z, then we get a natural map $\varphi(W, Z) \to \varphi(W', Z')$. Remember that we want to prove that $\varphi(X, T)$ is in $\mathscr{T}_0$. This will be done via the following lemmas (which hold assuming all the conditions stated in Lemma 4.2.14 as well as our inductive assumptions).

Lemma 4.2.17. *The cone of the map $\varphi(X, T) \to \varphi(X', T')$ is in $\mathscr{T}_0$.*

The map $\varphi(X, T) \to \varphi(X', T')$ factors as

$$\varphi(X, T) \to \varphi(X', p^{-1}(T)) \to \varphi(X', T')\,.$$

By the octahedral axiom, it is sufficient to prove that each of these two maps has a cone in $\mathscr{T}_0$.

We shall prove first that the cone of the map $\varphi(X', p^{-1}(T)) \to \varphi(X', T')$ is in $\mathscr{T}_0$. Given an immersion $a : S \to X'$, we shall write

$$M_S = a_!\, a^*(M)\,.$$

We then have distinguished triangles

$$M_{p^{-1}(T) - T'} \to M_{p^{-1}(T)} \to M_{T'} \to M_{p^{-1}(T) - T'}[1]\,.$$

For $M = j'_*(\mathbb{1}_{U'}\{n\})$ (recall j' is the pullback of j along p) the image of this triangle by p_* gives a distinguished triangle

$$p_*(M_{p^{-1}(T)-T'}) \to \varphi(X', p^{-1}(T)) \to \varphi(X', T') \to p_*(M_{p^{-1}(T)-T'})[1]\,.$$

As the restriction of $M = j'_*(\mathbb{1}_{U'}\{n\})$ to $X' - T'$ is in $\mathscr{T}_0$ by assumption on T', the object $M_{p^{-1}(T)-T'}$ is in $\mathscr{T}_0$ as well (by property (b) of 4.2.15 and because $\mathscr{T}_0$ is *Open*-fibred), from which we deduce that $p_*(M_{p^{-1}(T)-T'})$ is in $\mathscr{T}_0$ (using the condition (iii) of Lemma 4.2.14 and the property (b) of 4.2.15).

Let V be a dense open subscheme of X such that $p^{-1}(V) \to V$ is an isomorphism. We may assume that $V \subset U$, and write $i : Z \to U$ for the complement closed immersion. Let $p_U : U' = p^{-1}(U) \to U$ be the pullback of p along j, and let $\bar{Z}$ be the reduced closure of Z in X. We thus get the commutative squares of immersions below,

$$\begin{CD} Z @>{k}>> \bar{Z} \\ @V{i}VV @VV{l}V \\ U @>>{j}> X \end{CD} \qquad \text{and} \qquad \begin{CD} Z' @>{k'}>> \bar{Z}' \\ @V{i'}VV @VV{l'}V \\ U' @>>{j'}> X' \end{CD}$$

where the square on the right is obtained from the one on the left by pulling back along $p : X' \to X$. As p is an isomorphism over V, we get by cdh-descent (Proposition 3.3.10) the homotopy pullback square below.

$$\begin{CD} \mathbb{1}_U\{n\} @>>> p_{U,*}(\mathbb{1}_{U'}\{n\}) \\ @VVV @VVV \\ i_*\, i^*(\mathbb{1}_Z\{n\}) @>>> i_*\, i^*\, p_{U,*}(\mathbb{1}_{U'}\{n\}) \end{CD}$$

If $a : T \to X$ denotes the inclusion, applying the functor $a_*\, a^*\, j_*$ to the commutative square above, we see from the proper base change formula and from the identification $j_*\, i_* \simeq l_*\, k_*$ that we get a commutative square isomorphic to the following one

$$\begin{CD} \varphi(X,T) @>>> \varphi(X', p^{-1}(T)) \\ @VVV @VVV \\ \varphi(\bar{Z}, \bar{Z} \cap T) @>>> \varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T))\,, \end{CD}$$

which is thus homotopy cartesian as well. It is sufficient to prove that the two objects $\varphi(\bar{Z}, \bar{Z} \cap T)$ and $\varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T))$ are in $\mathscr{T}_0$. It follows from the proper base change formula that the object $\varphi(\bar{Z}, \bar{Z} \cap T)$ is canonically isomorphic to the restriction to T of $l_*\, k_*(\mathbb{1}_Z\{n\})$. As $\dim \bar{Z} < \dim X$, we know that the object $k_*(\mathbb{1}_Z\{n\})$ is in $\mathscr{T}_0$. By property (b) of 4.2.15, we obtain that $\varphi(\bar{Z}, \bar{Z} \cap T)$ is in $\mathscr{T}_0$. Similarly, the object $\varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T))$ is canonically isomorphic to the restriction of $p_*\, l'_*\, k'_*(\mathbb{1}_{Z'}\{n\})$ to T, and, as $\dim \bar{Z}' < \dim X'$ (because, p being an isomorphism over the dense open subscheme V of X, $\bar{Z}'$ does not contain any generic point of X'), $k'_*(\mathbb{1}_{Z'}\{n\})$ is in $\mathscr{T}_0$. We deduce again from property (b) of 4.2.15 that $\varphi(\bar{Z}', p^{-1}(\bar{Z} \cap T))$ is in $\mathscr{T}_0$ as well, which achieves the proof of the lemma.

Lemma 4.2.18. *The map $\varphi(X', T') \to \varphi(X'', T'')$ is an isomorphism in $\mathscr{T}(X)$.*

Condition (iv) of Lemma 4.2.14 can be reformulated by saying that we have the Nisnevich distinguished square below.

$$\begin{CD} X'' - T'' @>>> X'' \\ @VVV @VV{v}V \\ X' - T' @>>> X' \end{CD}$$

This lemma follows then by Nisnevich excision (Proposition 3.3.4) and smooth base change (for étale maps).

LEMMA 4.2.19. *Let T''' be the pullback of T'' along the finite surjective morphism $X''' \to X''$. The map $\varphi(X'', T'') \to \varphi(X''', T''')$ is a split monomorphism in $\mathscr{T}(X)$.*

We have the following pullback squares

$$\begin{array}{ccccc} T''' & \xrightarrow{t} & X''' & \xleftarrow{j'''} & U''' \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_U} \\ T'' & \xrightarrow{s} & X'' & \xleftarrow{j''} & U' \end{array}$$

in which j'' and j''' denote the pullback of j along pu and puq respectively, while s and t are the inclusions. By the proper base change formula applied to the left hand square, we see that the map $\varphi(X'', T'') \to \varphi(X''', T''')$ is isomorphic to the image of the map

$$j''_*(\mathbb{1}_{U''}\{n\}) \to q_* \, q^* \, j''_*(\mathbb{1}_{U''}\{n\}) \to q_* \, j'''_*(\mathbb{1}_{U'''}\{n\}) \, .$$

by $f_* \, s^*$, where $f : T'' \to T$ is the map induced by p (note that f is proper as $T'' \simeq T'$ by assumption). As $q_* \, j'''_* \simeq j''_* \, q_{U,*}$, we are thus reduced to prove that the unit map

$$\mathbb{1}_{U''}\{n\} \to q_{U,*}(\mathbb{1}_{U'''}\{n\})$$

is a split monomorphism. As X'' is normal (because X' is so by assumption, while $X'' \to X'$ is étale), this follows immediately from Corollary 3.3.40.

Now, we can finish the proof of Theorem 4.2.16. Consider the Verdier quotient

$$D = \mathscr{T}(X)/\mathscr{T}_0(X) \, .$$

We want to prove that, under the conditions stated in Lemma 4.2.14, we have $\varphi(X, T) \simeq 0$ in D. Let $\pi : T''' \to T$ be the map induced by $puq : X''' \to X$. If $a : T''' \to Y$ denotes the map induced by $g : X''' \to Y$, and $j_Y : Y_U \to Y$ the pullback of j by f, we have the commutative diagram below.

$$\begin{array}{ccc} \varphi(X,T) & \longrightarrow & \varphi(X''', T''') \\ & \searrow \quad \nearrow & \\ & \pi_* \, a^* \, j_{Y,*}(\mathbb{1}_{Y_U}\{n\}) & \end{array}$$

By virtue of lemmas 4.2.17, 4.2.19, and 4.2.18, the horizontal map is a split monomorphism in D. It is thus sufficient to prove that this map vanishes in D, for which it will be sufficient to prove that $\pi_* \, a^* \, j_{Y,*}(\mathbb{1}_{Y_U}\{n\})$ is in $\mathscr{T}_0$. The morphism π is finite (by construction, the map $T'' \to T'$ is an isomorphism, while the maps $T''' \to T''$ and $T' \to T$ are finite). Under this condition, $\mathscr{T}_0$ is stable under the operations π_* and a^*. To finish the proof of the theorem, it remains to check that $j_{Y,*}(\mathbb{1}_{Y_U}\{n\})$ is in $\mathscr{T}_0$, which follows from property (c) of 4.2.15 (and additivity). □

DEFINITION 4.2.20. We shall say that $\mathscr{T}$ is *τ-compatible* if it satisfies the following two conditions.

(a) For any closed immersion $i : Z \to X$ between regular schemes in $\mathscr{S}$, the image of $\mathbb{1}_X\{n\}$, $n \in \tau$, by the exceptional inverse image functor $i^! : \mathscr{T}(X) \to \mathscr{T}(Z)$ is constructible.

(b) For any scheme X, any $n \in \tau$, and any constructible object M in $\mathscr{T}(X)$, the object $\mathrm{Hom}_X(\mathbb{1}_X\{n\}, M)$ is constructible.

As usual, when τ is the monoid generated by the Tate twist, we say *compatible with Tate twists.*

REMARK 4.2.21. Condition (b) of the definition above will come essentially for free if the objects $\mathbb{1}_X\{n\}$ are $\otimes$-invertible with constructible $\otimes$-quasi-inverse (which will hold in practice, essentially by definition).

EXAMPLE 4.2.22. In pratice, condition (a) of the definition above will be a consequence of the *absolute purity theorem.* In particular, the category of Beilinson motives DM_{B} is compatible with Tate twist as a corollary of the fact the Tate twist is invertible and Theorem 14.4.1.

Lemma 4.2.23. *Assume that $\mathscr{T}$ is τ-compatible. Let $i : Z \to X$ be a closed immersion, with X regular, and Z the support of a strict normal crossing divisor. Then $i^!(\mathbb{1}_X\{n\})$ is constructible for any $n \in \tau$. As a consequence, if $j : U \to X$ denotes the complement open immersion, then $j_*(\mathbb{1}_U\{n\})$ is constructible for any $n \in \tau$.*

Proof. The first assertion follows easily by induction on the number of irreducible components of Z, using Proposition 4.2.6. The second assertion follows from the distinguished triangles

$$i_*\, i^!(M) \to M \to j_*\, j^*(M) \to i_*\, i^!(M)[1]$$

and from Lemma 4.2.9. □

Theorem 4.2.24. *Assume that $\mathscr{T}$ is $\mathbf{Q}$-linear, separated, and τ-compatible.*
Then, for any morphism of finite type $f : X \to Y$ such that Y is quasi-excellent, the functor

$$f_* : \mathscr{T}(X) \to \mathscr{T}(Y)$$

preserves constructible objects.

Proof. By virtue of propositions 4.2.4 and 4.2.11 as well as of Lemma 4.2.23, if $\mathscr{T}$ is τ-compatible, we can apply Theorem 4.2.16, where $\mathscr{T}$ stands for the subcategory of constructible objects. □

Corollary 4.2.25. *Under the assumptions of the above theorem, for any quasi-excellent scheme X, and for any couple of constructible objects M and N in $\mathscr{T}(X)$, the object $\mathit{Hom}_X(M,N)$ is constructible.*

Proof. It is sufficient to treat the case where $M = f_\sharp(\mathbb{1}_Y\{n\})$, for $n \in \tau$ and $f : Y \to X$ a smooth morphism. But then, we have, by transposition of the *Sm*-projection formula, a natural isomorphism:

$$\mathit{Hom}_X(M,N) \simeq f_*\, \mathit{Hom}(\mathbb{1}_Y\{n\}, f^*(N))\,.$$

This corollary follows then immediately from Proposition 4.2.4 and from Theorem 4.2.24. □

Corollary 4.2.26. *Under the assumptions of the above theorem, for any closed immersion $i : Z \to X$ such that X is quasi-excellent, the functor*

$$i^! : \mathscr{T}(X) \to \mathscr{T}(Z)$$

preserves constructible objects.

Proof. Let $j : U \to X$ be the complement open immersion. For an object M of $\mathscr{T}(X)$, we have the following distinguished triangle.

$$i_*\, i^!(M) \to M \to j_*\, j^*(M) \to i_*\, i^!(M)[1]\,.$$

By virtue of Proposition 4.2.6 and Theorem 4.2.24, if M is constructible, then $j_*\, j^*(M)$ have the same property, which allows to conclude. □

Lemma 4.2.27. *Let $f : X \to Y$ be a separated morphism of finite type. The condition that the functor $f^!$ preserves constructible objects in $\mathscr{T}$ is local over X and over Y for the Zariski topology.*

Proof. If $u : X' \to X$ is a Zariski cover, then we have, by definition, $u^* = u^!$, so that, by Proposition 4.2.6, the condition that $f^!$ preserves τ-constructibility is equivalent to the condition that $u^*\, f^! \simeq (fu)^!$ preserves τ-constructibility. Let $v : Y' \to Y$ be a Zariski cover, and consider the following pullback square.

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[v]{} & Y \end{array}$$

We then have a natural isomorphism $u^*\, f^! \simeq g^!\, v^*$, and, as u is still a Zariski cover, we deduce again from Proposition 4.2.6 that, if $g^!$ preserves τ-constructibility, so does $f^!$. □

Corollary 4.2.28. *Under the assuptions of the above theorem, for any separated morphism of finite type $f : X \to Y$, the functor*

$$f^! : \mathscr{T}(Y) \to \mathscr{T}(X)$$

preserves constructible objects.

Proof. By virtue of the preceding lemma, we may assume that f is affine. We can then factor f as an immersion $i : X \to \mathbf{A}^n_Y$ followed by the canonical projection $p : \mathbf{A}^n_Y \to Y$. The case of an immersion is reduced to the case of an open immersion (4.2.4) and to the case of a closed immersion (4.2.26). Thus we may assume that $f = p$, in which case $p^! \simeq p^*(-)(n)[2n]$ (according to point (3) of Theorem 2.4.50), so that we conclude by 4.2.4 and 4.2.9. □

In conclusion, we have proved the following finiteness theorem:

Theorem 4.2.29. *Assume the motivic triangulated category $\mathscr{T}$ is $\mathbf{Q}$-linear, separated and τ-compatible.*[59]

Then constructible objects of $\mathscr{T}$ are closed under the six operations of Grothendieck when restricted to the subcategory $\mathscr{S}'$ of $\mathscr{S}$ made of quasi-excellent schemes and morphisms of finite type. In particular, $\mathscr{T}_c$ is a τ-generated motivic category over $\mathscr{S}'$.

4.3. Continuity.

4.3.1. For the next definition, we consider an admissible class $\mathscr{P}$ of morphisms in $\mathscr{S}$ and an abstract symmetric monoidal $\mathscr{P}$-fibred model category $\mathscr{M}$ over $\mathscr{S}$.

Let $(S_\alpha)_{\alpha \in A}$ be a projective system of schemes in $\mathscr{S}$, with affine dominant transition maps, and such that $S = \varprojlim_{\alpha \in A} S_\alpha$ is representable in $\mathscr{S}$ (we assume that A is a partially ordered set to keep the notations simple). For each index α, we denote by $p_\alpha : S \to S_\alpha$ the canonical projection. Given an index $\alpha_0 \in A$ and an object E_{α_0} in $\mathrm{Ho}(\mathscr{M})(S_{\alpha_0})$, we write E_α for the pullback of E_{α_0} along the map $S_\alpha \to S_{\alpha_0}$, and put $E = \mathbf{L}p^*_\alpha(E_\alpha)$.

Definition 4.3.2. Consider the assumptions above and let τ be a set of twists of $\mathrm{Ho}(\mathscr{M})$.

We say that $\mathrm{Ho}(\mathscr{M})$ is *τ-continuous*, or *continuous* (if τ is clearly specified by the context), if it is τ-generated and if, given any projective system of schemes $\{S_\alpha\}$ as above, for any index α_0, any object E_{α_0} in $\mathrm{Ho}(\mathscr{M})(S_{\alpha_0})$, and any twist $n \in \tau$, the canonical map

$$\varinjlim_{\alpha \geq \alpha_0} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S_\alpha)}(\mathbb{1}_{S_\alpha}\{n\}, E_\alpha) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(\mathbb{1}_S\{n\}, E),$$

is bijective.

Example 4.3.3. The main examples of τ-continuous categories will be seen afterwards:

- the $\mathbf{A}^1$-derived category $\mathrm{D}_{\mathbf{A}^1,\Lambda}$ (Example 6.1.13);
- the category of motivic complexes DM_Λ, and its effective counterpart $\mathrm{DM}^{\mathit{eff}}_\Lambda$ (Theorem 11.1.24);
- the motivic category DM_{B} of Beilinson motives (Proposition 14.3.1).

The interest of this property is to allow a description of constructible objects over S in terms of constructible objects over the S_α's.

Proposition 4.3.4. *Assume, under the hypothesis of 4.3.1, that $\mathrm{Ho}(\mathscr{M})$ is τ-continuous. Consider a scheme S in $\mathscr{S}$, as well as a projective system of schemes $\{S_\alpha\}$ in $\mathscr{S}$ with affine transition maps and such that $S = \varprojlim_\alpha S_\alpha$.*

Then, for any index α_0, and for any objects C_{α_0} and E_{α_0} in $\mathrm{Ho}(\mathscr{M})(S_{\alpha_0})$, if C_{α_0} is constructible, then the canonical map

$$\varinjlim_{\alpha \geq \alpha_0} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S_\alpha)}(C_\alpha, E_\alpha) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(C, E) \tag{4.3.4.1}$$

[59] Remember also that $\mathscr{T}$ is associated with a combinatorial stable premotivic model category.

is bijective. Moreover, the canonical functor

$$2\text{-}\varinjlim_{\alpha} \mathrm{Ho}(\mathscr{M})_c(S_\alpha) \to \mathrm{Ho}(\mathscr{M})_c(S) \tag{4.3.4.2}$$

is an equivalence of monoidal triangulated categories.

Proof. To prove the first assertion, we may assume, without loss of generality, that $C_{\alpha_0} = M_{S_{\alpha_0}}(X_{\alpha_0})\{n\}$ for some some smooth S_{α_0}-scheme of finite type X_{α_0}, and $n \in \tau$. Consider an object E_{α_0} in $\mathrm{Ho}(\mathscr{M})(S_{\alpha_0})$. For $\alpha \geq \alpha_0$, write X_α (resp. E_α) for the pullback of X_{α_0} (resp. of E_{α_0}) along the map $S_\alpha \to S_{\alpha_0}$. Similarly, write X (resp. E) for the pullback of X_{α_0} (resp. of E_{α_0}) along the map $S \to S_{\alpha_0}$. We shall also write E'_α (resp. E') for the pullback of E_α (resp. E) along the smooth map $X_\alpha \to S_\alpha$ (resp. $X \to S$). Then, $\{X_\alpha\}$ is a projective system of schemes in $\mathscr{S}$, with affine transition maps, such that $X = \varprojlim_\alpha X_\alpha$. Therefore, by continuity, we have the following natural isomorphism, which proves the first assertion.

$$\begin{aligned}\varinjlim_{\alpha} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S_\alpha)}(M_{S_\alpha}(X_\alpha)\{n\}, E_\alpha) \simeq & \varinjlim_{\alpha} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X_\alpha)}(\mathbb{1}_{X_\alpha}\{n\}, E'_\alpha) \\ \simeq & \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbb{1}_X\{n\}, E') \\ \simeq & \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(M_S(X)\{n\}, E)\end{aligned}$$

Note that the first assertion implies that the functor (4.3.4.2) is fully faithful. Note that pseudo-abelian triangulated categories are stable by filtered 2-colimits. In particular, the source of the functor (4.3.4.2) can be seen as a thick subcategory of $\mathrm{Ho}(\mathscr{M})(S)$. The essential surjectivity of (4.3.4.2) follows from the fact that, for any smooth S-scheme of finite type X, there exists some index α, and some smooth S_α-scheme X_α, such that $X \simeq S \times_{S_\alpha} X_\alpha$; see [**EGA4**, 8.8.2 and 17.7.8]: this implies that the essential image of the fully faithful functor (4.3.4.2) contains all the objects of shape $M_S(X)\{n\}$ for $n \in \tau$ and X smooth over S, so that it contains $\mathrm{Ho}(\mathscr{M})_c(S)$, by definition. □

4.3.5. Before showing how the assumption of continuity can be used in the case of motivic categories, we state a proposition which later on will allow us to show continuity in concrete cases.

We consider again the assumptions and notations of 4.3.1 assuming the transition maps of the pro-scheme (S_α) are $\mathscr{P}$-morphisms. For each index $\alpha \in A$, we choose a small set I_α (resp. J_α) of generating cofibrations (resp. of generating trivial cofibration) in $\mathrm{Ho}(\mathscr{M})(S_\alpha)$. We also choose a small set I (resp. J) of generating cofibrations (resp. of generating trivial cofibration) in $\mathrm{Ho}(\mathscr{M})(S)$.

Consider the following assumptions:

(a) We have $I \subset \bigcup_{\alpha\in A} p^*_\alpha(I_\alpha)$ and $J \subset \bigcup_{\alpha \in A} p^*_\alpha(J_\alpha)$.
(b) For any index α_0, if C_{α_0} and E_{α_0} are two objects of $\mathscr{M}(S_{\alpha_0})$, with C_{α_0} either a source or a target of a map in $I_{\alpha_0} \cup J_{\alpha_0}$, the natural map

$$\varinjlim_{\alpha\in A} \mathrm{Hom}_{\mathscr{M}(S_\alpha)}(C_\alpha, E_\alpha) \to \mathrm{Hom}_{\mathscr{M}(S)}(C, E)$$

is bijective.

Proposition 4.3.6. *Under the assumptions of 4.3.5, for any index $\alpha_0 \in A$, the pullback functor $p^*_{\alpha_0} : \mathscr{M}(S_{\alpha_0}) \to \mathscr{M}(S)$ preserves fibrations and trivial fibrations. Moreover, given an index $\alpha_0 \in A$, as well as two objects C_{α_0} and E_{α_0} in $\mathscr{M}(S_{\alpha_0})$, if C_{α_0} belongs to smallest full subcategory of* $\mathrm{Ho}(\mathscr{M})(S_{\alpha_0})$ *which is closed under finite homotopy colimits and which contains the source and targets of I_{α_0}, then, the canonical map*

$$\varinjlim_{\alpha\in A} \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S_\alpha)}(C_\alpha, E_\alpha) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(S)}(C, E)$$

is bijective.

Proof. We shall prove first that, for any index $\alpha_0 \in A$, the pullback functor $p^*_{\alpha_0}$ preserves fibrations and trivial fibrations. By assumption, for any $\alpha \geq \alpha_0$, the pullback functor along the $\mathscr{P}$-morphism $S_\alpha \to S_{\alpha_0}$ is both a left Quillen functor and a right Quillen functor. Let $E_{\alpha_0} \to F_{\alpha_0}$

be a trivial fibration (resp. a fibration) of $\mathscr{M}(S_{\alpha_0})$. Let $i : C \to D$ a generating cofibration (resp. a generating trivial cofibration) in $\mathscr{M}(S)$. By condition (a) of 4.3.5, we may assume that there exists $\alpha_1 \in A$, a cofibration (resp. a trivial cofibration) $i_{\alpha_1} : C_{\alpha_1} \to D_{\alpha_1}$, such that $i = p^*_{\alpha_1}(i_{\alpha_1})$. We want to prove that the map

$$\mathrm{Hom}(D, E) \to \mathrm{Hom}(C, E) \times_{\mathrm{Hom}(C,F)} \mathrm{Hom}(D, F)$$

is surjective. But, by condition (b) of 4.3.5, this map is isomorphic to the filtered colimit of the surjective maps

$$\mathrm{Hom}(D_\alpha, E_\alpha) \to \mathrm{Hom}(C_\alpha, E_\alpha) \times_{\mathrm{Hom}(C_\alpha,F_\alpha)} \mathrm{Hom}(D_\alpha, F_\alpha)$$

with $\alpha \geq \sup(\alpha_0, \alpha_1)$, which proves the first assertion.

To prove the second assertion, we may assume that C_{α_0} is cofibrant and that E_{α_0} if fibrant. The set of maps from a cofibrant object to a fibrant object in the homotopy category of a model category can be described as homotopy classes of maps. Therefore, using the fact that $p^*_{\alpha_0}$ preserves cofibrations and fibrations, as well as the trivial ones, we see it is sufficient to prove that the map

$$\varinjlim_{\alpha \in A} \mathrm{Hom}_{\mathscr{M}(S_\alpha)}(C_\alpha, E_\alpha) \to \mathrm{Hom}_{\mathscr{M}(S)}(C, E)$$

is bijective for some nice cofibrant replacement of C_{α_0}. But the assumptions on C_{α_0} imply that it is weakly equivalent to an object C'_{α_0} such that the map $\varnothing \to C'_{\alpha_0}$ belongs to the smallest class of maps in $\mathscr{M}(S_{\alpha_0})$, which contains I_{α_0}, and which is closed under pushouts and (finite) compositions. We may thus assume that $C_{\alpha_0} = C'_{\alpha_0}$. In that case, C_{α_0} is in particular contained in the smallest full subcategory of $\mathscr{M}(S_{\alpha_0})$ which is stable by finite colimits and which contains the source and targets of I_{α_0}. As filtered colimits commute with finite limits in the category of sets, we conclude by using again condition (a) of 4.3.5. □

We now go back to the situation of a motivic triangulated category $\mathscr{T}$ satisfying our general assumptions 4.0

Lemma 4.3.7. *Let $a : X \to Y$ be a morphism in $\mathscr{S}$. Assume that $X = \varprojlim_\alpha X_\alpha$, where $\{X_\alpha\}$ is a projective system of smooth affine Y-schemes. If $\mathscr{T}$ is τ-continuous, then, for any objects E and F in $\mathscr{T}(Y)$, with E constructible, there is a canonical isomorphism*

$$a^*\,\mathit{Hom}_Y(E, F) \simeq \mathit{Hom}_X(a^*(E), a^*(F))\,.$$

Proof. We have

$$a_*\,\mathit{Hom}_X(a^*(E), a^*(F)) \simeq \mathit{Hom}_Y(E, a_*\,a^*(F))\,,$$

so that the map $F \to a_*\,a^*(F))$ induces a map

$$\mathit{Hom}_Y(E, F) \to a_*\,\mathit{Hom}_X(a^*(E), a^*(F))\,,$$

hence, by adjunction, a map

$$a^*\,\mathit{Hom}_Y(E, F) \to \mathit{Hom}_X(a^*(E), a^*(F))\,.$$

We already know that the later is an isomorphism whenever a is smooth.

Let us write $a_\alpha : X_\alpha \to Y$ for the structural maps. Let C be a constructible object in $\mathscr{T}(X)$. By Proposition 4.3.4, we may assume that there exists an index α_0, and a constructible object C_{α_0} in $\mathscr{T}(X_{\alpha_0})$, such that, if we write C_α for the pullback of C_{α_0} along the map $X_\alpha \to X_{\alpha_0}$ for $\alpha \geq \alpha_0$, we have isomorphisms:

$$\begin{aligned}
\mathrm{Hom}(C, a^*\,\mathit{Hom}_Y(E, F)) &\simeq \varinjlim_\alpha \mathrm{Hom}(C_\alpha, a^*_\alpha\,\mathit{Hom}_Y(E, F)) \\
&\simeq \varinjlim_\alpha \mathrm{Hom}(C_\alpha, \mathit{Hom}_X(a^*_\alpha(E), a^*_\alpha(F))) \\
&\simeq \varinjlim_\alpha \mathrm{Hom}(C_\alpha \otimes_{X_\alpha} a^*_\alpha(E), a^*_\alpha(F)) \\
&\simeq \mathrm{Hom}(C \otimes_X a^*(E), a^*(F)) \\
&\simeq \mathrm{Hom}(C, \mathit{Hom}_X(a^*(E), a^*(F)))\,.
\end{aligned}$$

As constructible objects generate $\mathscr{T}(X)$, this proves the lemma. □

4.3.8. Let X be a scheme in $\mathscr{S}$. Assume that, for any point x of X, the corresponding morphism $i_x : \operatorname{Spec}\left(\mathscr{O}^h_{X,x}\right) \to X$ is in $\mathscr{S}$ (where $\mathscr{O}^h_{X,x}$ denotes the henselisation of $\mathscr{O}_{X,x}$). Consider at last a scheme of finite type Y over X, and write

$$a_x : Y_x = \operatorname{Spec}\left(\mathscr{O}^h_{X,x}\right) \times_X Y \to Y$$

for the morphism obtained by pullback. Finally, for an object E of $\mathscr{T}(Y)$, let us write

$$E_x = a_x^*(E)\,.$$

Proposition 4.3.9. *Under the assumptions of 4.3.8, if moreover $\mathscr{T}$ is τ-continuous, then, the family of functors*

$$\mathscr{T}(Y) \to \mathscr{T}(Y_x)\ ,\quad E \longmapsto E_x\ ,\quad x \in X\,,$$

is conservative.

Proof. Let E be an object of $\mathscr{T}(Y)$ such that $E_x \simeq 0$ for any point x of X. For any constructible object C of $\mathscr{T}(Y)$, we have a presheaf of S^1-spectra on the small Nisnevich site of X:

$$F : U \longmapsto F(U) = \operatorname{Hom}(M_Y(U \times_X Y), \mathit{Hom}_Y(C, E))\,.$$

It is sufficient to prove that $F(X)$ is acyclic. As $\mathscr{T}$ satisfies Nisnevich descent (3.3.4), it is sufficient to prove that F is acyclic locally for the Nisnevich topology, i.e. that, for any point x of X, the spectrum $F(\operatorname{Spec}\left(\mathscr{O}^h_{X,x}\right))$ is acyclic. Writing $\operatorname{Spec}\left(\mathscr{O}^h_{X,x}\right)$ as the projective limit of the Nisnevich neighbourhoods of x in X, we see easily, using Proposition 4.3.4 and Lemma 4.3.7, that, for any integer i, $\pi_i(F(\operatorname{Spec}\left(\mathscr{O}^h_{X,x}\right)) \simeq \operatorname{Hom}(C_x, E_x[i]) \simeq 0$. □

Proposition 4.3.10. *Let S be a quasi-excellent noetherian and henselian scheme. Write $\hat{S}$ for its completion along its closed point, and assume that both S and $\hat{S}$ are in $\mathscr{S}$. Consider an S-scheme of finite type X, and write $i : \hat{S} \times_S X \to X$ for the induced map. If $\mathscr{T}$ is τ-continuous, then the pullback functor*

$$i^* : \mathscr{T}(X) \to \mathscr{T}(\hat{S} \times_S X)$$

is conservative.

Proof. As S is quasi-excellent, the map $\hat{S} \to S$ is regular. By Popescu's theorem, we can then write $\hat{S} = \varprojlim_\alpha S_\alpha$, where $\{S_\alpha\}$ is a projective system of schemes with affine transition maps, and such that each scheme S_α is smooth over S. Moreover, as $\hat{S}$ and S have the same residue field, and as S is henselian, each map S_α has a section. Write $X_\alpha = S_\alpha \times_S X$, so that we have $X = \varprojlim_\alpha X_\alpha$. Consider a constructible object C and an object E in $\mathscr{T}(X)$. Then, as the maps $X_\alpha \to X$ have sections, it follows from the first assertion of Proposition 4.3.4 that the map

$$\operatorname{Hom}_{\mathscr{T}(X)}(C, E) \to \operatorname{Hom}_{\mathscr{T}(\hat{S} \times_S X)}(i^*(C), i^*(E))$$

is a monomorphism (as a filtered colimit of such things). Hence, if $i^*(E) \simeq 0$, for any constructible object C in $\mathscr{T}(X)$, we have $\operatorname{Hom}_{\mathscr{T}(X)}(C, E) \simeq 0$. Therefore, as τ-constructible objects generate $\mathscr{T}(X)$, we get $E \simeq 0$. □

Proposition 4.3.11. *Let $a : X \to Y$ be a regular morphism in $\mathscr{S}$. If $\mathscr{T}$ is τ-continuous, then, for any objects E and F in $\mathscr{T}(Y)$, with E constructible, there is a canonical isomorphism*

$$a^*\, \mathit{Hom}_Y(E, F) \simeq \mathit{Hom}_X(a^*(E), a^*(F))\,.$$

Proof. We want to prove that the canonical map

$$a^*\, \mathit{Hom}_Y(E, F) \to \mathit{Hom}_X(a^*(E), a^*(F))$$

is an isomorphism, while we already know it is so whenever a is smooth. Therefore, to prove the general case, we see that the problem is local on X and on Y with respect to the Zariski topology. In particular, we may assume that both X and Y are affine. By Pospecu's Theorem 4.1.5, we thus have $X = \varprojlim_\alpha X_\alpha$, where $\{X_\alpha\}$ is a projective system of smooth affine Y-schemes. We conclude by Lemma 4.3.7. □

Proposition 4.3.12. *Consider the following pullback square in $\mathscr{S}$.*

$$\begin{array}{ccc} X' & \xrightarrow{a} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[b]{} & Y \end{array}$$

Assume that f is separated of finite type and that b is regular. Then, if $\mathscr{T}$ is τ-continuous, for any object E in $\mathscr{T}(Y)$, there is a canonical isomorphism in $\mathscr{T}(X')$:

$$a^*\, f^!(E) \simeq g^!\, b^*(E)\,.$$

Proof. We have a canonical map

$$f^!(E) \to a_*\, g^!\, b^*(E) \simeq f^!\, b_*\, b^*(E)\,,$$

which gives, by adjunction, a natural morphism

$$a^*\, f^!(E) \to g^!\, b^*(E)\,.$$

The latter is invertible whenever b is smooth: this is obvious in the case of an open immersion, so that, by Zariski descent, it is sufficient to treat the case where b is smooth with trivial cotangent bundle of rank d; in this case, by relative purity (2.4.50 (3)), this reduces to the canonical isomorphism $a^! f^! \simeq g^! b^!$ evaluated at $E(-d)[-2d]$. To prove the general case, as the condition is local on X and on Y for the Zariski topology, we may assume that f factors as an immersion $X \to \mathbf{P}^n_Y$, followed by the canonical projection $\mathbf{P}^n_Y \to Y$. We deduce from there that it is sufficient to treat the case where f is either a closed immersion, either a smooth morphism of finite type. The case where f (hence also g) is smooth follows by relative purity (2.4.50): we can then replace $f^!$ and $g^!$ by f^* and g^* respectively, and the formula follows from the fact that $a^* f^* \simeq g^* b^*$. We may thus assume that f is a closed immersion. As g is a closed immersion as well, the functor $g_!$ is conservative (it is fully faithful). Therefore, it is sufficient to prove that the map

$$b^*\, f_!\, f^!(E) \simeq g_!\, a^*\, f^!(E) \to g_!\, g^!\, b^*(E)$$

is invertible. Then, using Proposition 4.3.11 (which makes sense because $f_!$ preserves τ-constructibility by 4.2.11), and the projection formula, we have

$$\begin{aligned} b^*\, f_!\, f^!(E) &\simeq b^* \mathit{Hom}_Y(f_!(\mathbb{1}_X), E) \\ &\simeq \mathit{Hom}_{Y'}(b^*\, f_!(\mathbb{1}_X), b^*(E)) \\ &\simeq \mathit{Hom}_{Y'}(g_!(\mathbb{1}_{X'}), b^*(E)) \\ &\simeq g_!\, g^!\, b^*(E)\,, \end{aligned}$$

which achieves the proof. □

Lemma 4.3.13. *Let $f : X \to Y$ be a morphism in $\mathscr{S}$. Assume that $X = \varprojlim_\alpha X_\alpha$ and $Y = \varprojlim_\alpha Y_\alpha$, where $\{X_\alpha\}$ and $\{Y_\alpha\}$ are projective systems fo schemes with affine transition maps, while f is induced by a system of morphisms $f_\alpha : X_\alpha \to Y_\alpha$. Let α_0 be some index, C_{α_0} a constructible object of $\mathscr{T}(Y_{\alpha_0})$, and E_{α_0} an object of $\mathscr{T}(X_{\alpha_0})$. If $\mathscr{T}$ is τ-continuous, then we have a natural isomorphism of abelian groups*

$$\varinjlim_{\alpha \geq \alpha_0} \operatorname{Hom}_{\mathscr{T}(Y_\alpha)}(C_\alpha, f_{\alpha,*}(E_\alpha)) \simeq \operatorname{Hom}_{\mathscr{T}(Y)}(C, f_*(E))\,.$$

Proof. By virtue of Proposition 4.3.4, we have a natural isomorphism

$$\varinjlim_{\alpha \geq \alpha_0} \operatorname{Hom}_{\mathscr{T}(X_\alpha)}(f^*_\alpha(C_\alpha), E_\alpha) \simeq \operatorname{Hom}_{\mathscr{T}(Y)}(f^*(C), E)\,.$$

The expected formula follows by adjunction. □

Proposition 4.3.14. *Consider the following pullback square in $\mathscr{S}$.*

$$\begin{array}{ccc} X' & \xrightarrow{a} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[b]{} & Y \end{array}$$

with b regular. If $\mathscr{T}$ is τ-continuous, then, for any object E in $\mathscr{T}(X)$, there is a canonical isomorphism in $\mathscr{T}(Y')$:

$$b^* f_*(E) \simeq g_* a^*(E)\,.$$

Proof. This proposition is true in the case where b is smooth (by definition of *Sm*-fibred categories), from which we deduce, by Zariski separation, that this property is local on Y and on Y' for the Zariski topology. In particular, we may assume that both Y and Y' are affine. Then, by Popescu's Theorem 4.1.5, we may assume that $Y' = \varprojlim_\alpha Y'_\alpha$, where $\{Y'_\alpha\}$ is a projective system of smooth Y-algebras. Then, using the preceding lemma as well as Proposition 4.3.4, we reduce easily the proposition to the case where b is smooth. □

Proposition 4.3.15. *Assume that $\mathscr{T}$ is τ-continuous, $\mathbf{Q}$-linear and semi-separated, and consider a field k, with inseparable closure k', such that both* Spec (k) *and* Spec (k') *are in $\mathscr{S}$. Given a k-scheme X write $X' = k' \otimes_k X$, and $f : X' \to X$ for the canonical projection. Then the functor*

$$f^* : \mathscr{T}(X) \to \mathscr{T}(X')$$

is an equivalence of categories.

Proof. It follows immediately from Proposition 4.3.4 and from Proposition 2.1.9 that the functor

$$f^* : \mathscr{T}_c(X) \to \mathscr{T}_c(X')$$

is an equivalence of categories. Similarly, for any objects C and E in $\mathscr{T}(X)$, if C is constructible, the map

$$\mathrm{Hom}_{\mathscr{T}(X)}(C, E) \to \mathrm{Hom}_{\mathscr{T}(X)}(f^*(C), f^*(E))$$

is bijective. As constructible objects generate $\mathscr{T}(X)$, this implies that the functor

$$f^* : \mathscr{T}(X) \to \mathscr{T}(X')$$

is fully faithful. As the latter is essentially surjective on a set of generators, this implies that it is an equivalence of categories (see 1.3.21). □

4.4. Duality. The aim of this section is to prove a local duality theorem in $\mathrm{Ho}(\mathscr{M})$ (see 4.4.21 and 4.4.24).

If we work with rational coefficients, resolution of singularities up to quotient singularities is almost as good as classical resolution of singularities: we have the following replacement of the blow-up formula.

Theorem 4.4.1. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is $\mathbf{Q}$-linear and separated. Let X be a scheme in $\mathscr{S}$. Consider a proper surjective morphism $p : X' \to X$ and a finite group G acting on X' over X. Assume that there is a closed subscheme $Z \subset X$ such that $U = X - Z$ is normal, while the induced map $p_U : U' = p^{-1}(U) \to U$ is finite, and the map $U'/G \to U$ is generically radicial (i.e. is radicial over an open dense subscheme of U) — e.g. this situation occurs when p is a Galois alteration. Then the pullback square*

(4.4.1.1)
$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & X \end{array}$$

induces an homotopy pullback square

(4.4.1.2)
$$\begin{array}{ccc} M & \longrightarrow & (\mathbf{R}p_* \, \mathbf{L}p^*(M))^G \\ \downarrow & & \downarrow \\ \mathbf{R}i_* \, \mathbf{L}i^*(M) & \longrightarrow & (\mathbf{R}i_* \mathbf{R}q_* \, \mathbf{L}q^* \, \mathbf{L}i^*(M))^G \end{array}$$

for any object M of $\mathrm{Ho}(\mathscr{M})(X)$.

Proof. We already know that, for any object N of $\mathrm{Ho}(\mathscr{M})(U)$, the map

$$N \to (\mathbf{R}p_{U*} \, \mathbf{L}p_U^*(N))^G$$

is an isomorphism (Corollary 3.3.39). The proof is then similar to the proof of condition (iv) of Theorem 3.3.37. □

Remark 4.4.2. Under the assumptions of the preceding theorem, applying the total derived functor $\mathbf{R}\mathrm{Hom}_X(-,E)$ to the homotopy pullback square (4.4.1.2) for $M = \mathbb{1}_X$, we obtain the homotopy pushout square

(4.4.2.1)
$$\begin{array}{ccc} (i_! \, q_! \, q^! \, i^!(E))_G & \longrightarrow & (p_! \, p^!(E))_G \\ \downarrow & & \downarrow \\ i_! \, i^!(E) & \longrightarrow & E \end{array}$$

for any object E of $\mathrm{Ho}(\mathscr{M})(X)$.

Corollary 4.4.3. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* $\mathbf{Q}$*-linear and separated. Let B be a scheme in $\mathscr{S}$, admitting wide resolution of singularities up to quotient singularities. Consider a separated B-scheme of finite type S, endowed with a closed subscheme $T \subset S$. The category of constructible objects in* $\mathrm{Ho}(\mathscr{M})(S)$ *is the smallest thick triangulated subcategory which contains the objects of shape $\mathbf{R}f_*(\mathbb{1}_X\{n\})$ for $n \in \tau$, and for $f : X \to S$ a projective morphism, with X regular and connected, such that $f^{-1}(T)_{red}$ is either empty, either X itself, either the support of a strict normal crossing divisor.*

Proof. Let $\mathrm{Ho}(\mathscr{M})(S)'$ be the smallest thick triangulated subcategory of $\mathrm{Ho}(\mathscr{M})(S)$ which contains the objects of shape $\mathbf{R}f_*(\mathbb{1}_X\{n\})$ for $n \in \tau$ and $f : X \to S$ a projective morphism with X regular and connected, while $f^{-1}(T)_{red}$ is empty, or X itself, or the support of a strict normal crossing divisor. We clearly have $\mathrm{Ho}(\mathscr{M})(S)' \subset \mathrm{Ho}(\mathscr{M})_c(S)$ (Proposition 4.2.11). To prove the reverse inclusion, by virtue of Proposition 4.2.13, it is sufficient to prove that, for any $n \in \tau$, and any projective morphism $f : X \to S$, the object $\mathbf{R}f_*(\mathbb{1}_X\{n\})$ belongs to $\mathrm{Ho}(\mathscr{M})(S)'$. We shall proceed by induction on the dimension of X. If X is of dimension ≤ 0, we may replace it by its reduction, which is regular. If X is of dimension > 0, by assumption on B, there exists a Galois alteration $p : X' \to X$ of group G, with X' regular and projective over S (and in which T becomes either empty, either X' itself, either the support of a strict normal crossing divisor, in each connected component of X'). Choose a closed subscheme $Z \subset X$, such that $U = X - Z$ is a normal dense open subscheme, and such that the induced map $r : U' = p^{-1}(U) \to U$ is a finite morphism, and consider the pullback square (4.4.1.1). As Z and $Z' = p^{-1}(Z)$ are of dimension smaller than the dimension of X, we conclude from the homotopy pullback square obtained by appyling the functor $\mathbf{R}f_*$ to (4.4.1.2) for $M = \mathbb{1}_X\{n\}$, $n \in \tau$. □

Definition 4.4.4. Let S be a scheme in $\mathscr{S}$. An object R of $\mathrm{Ho}(\mathscr{M})(S)$ is *τ-dualizing* if it satisfies the following conditions.

(i) The object R is constructible.
(ii) For any constructible object M of $\mathrm{Ho}(\mathscr{M})(S)$, the natural map

$$M \to \mathbf{R}\mathit{Hom}_S(\mathbf{R}\mathit{Hom}_S(M,R),R)$$

is an isomorphism.

REMARK 4.4.5. If $\mathrm{Ho}(\mathscr{M})$ is τ-compatible, $\mathbf{Q}$-linear and separated, then, in particular, the six operations of Grothendieck preserve τ-constructibility in $\mathrm{Ho}(\mathscr{M})$ (4.2.29). Under this assumtion, for any scheme X in $\mathscr{S}$, and any $\otimes$-invertible object U in $\mathrm{Ho}(\mathscr{M})(X)$ which is constructible, its quasi-inverse is constructible: the quasi-inverse of U is simply its dual $U^\wedge = \mathbf{R}\mathit{Hom}(U, \mathbb{1}_X)$, which is constructible by virtue of 4.2.25.

PROPOSITION 4.4.6. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* τ*-compatible,* $\mathbf{Q}$*-linear and separated, and consider a scheme* X *in* $\mathscr{S}$*.*

(i) Let R *be a* τ*-dualizing object, and* U *be a constructible* $\otimes$*-invertible object in* $\mathrm{Ho}(\mathscr{M})(X)$*. Then* $U \otimes^{\mathbf{L}}_S R$ *is* τ*-dualizing.*

(ii) Let R *and* R' *be two* τ*-dualizing objects in* $\mathrm{Ho}(\mathscr{M})(X)$*. Then the evaluation map*

$$\mathbf{R}\mathit{Hom}_S(R, R') \otimes^{\mathbf{L}}_S R \to R'$$

is an isomorphism.

PROOF. This follows immediately from [**Ayo07a**, 2.1.139]. □

PROPOSITION 4.4.7. *Consider an open immersion* $j : U \to X$ *in* $\mathscr{S}$*. If* R *is a* τ*-dualizing object in* $\mathrm{Ho}(\mathscr{M})(X)$*, then* $j^!(R)$ *is* τ*-dualizing in* $\mathrm{Ho}(\mathscr{M})(U)$*.*

PROOF. If M is a constructible object in $\mathrm{Ho}(\mathscr{M})(U)$, then $j_!(M)$ is constructible, and the map

$$j_!(M) \to \mathbf{R}\mathit{Hom}_X(\mathbf{R}\mathit{Hom}_X(j_!(M), R), R) \tag{4.4.7.1}$$

is an isomorphism. Using the isomorphisms of type

$$M \simeq j^*\, j_!(M) = j^!\, j_!(M) \quad \text{and} \quad j^*\mathbf{R}\mathit{Hom}_X(A, B) \simeq \mathbf{R}\mathit{Hom}_U(j^*(A), j^*(B))\,,$$

we see that the image of the map (4.4.7.1) by the functor $j^* = j^!$ is isomorphic to the map

$$M \to \mathbf{R}\mathit{Hom}_U(\mathbf{R}\mathit{Hom}_U(M, j^!(R)), j^!(R))\,, \tag{4.4.7.2}$$

which proves the proposition. □

PROPOSITION 4.4.8. *Let* X *be a scheme in* $\mathscr{S}$*, and* R *an object in* $\mathrm{Ho}(\mathscr{M})(X)$*. Assume there exists an open cover* $X = \bigcup_{i\in I} U_i$ *such that the restriction of* R *on each of the open subschemes* U_i *is* τ*-dualizing in* $\mathrm{Ho}(\mathscr{M})(U_i)$*. Then* R *is* τ*-dualizing.*

PROOF. We already know that the property of τ-constructibility is local with respect to the Zariski topology (4.2.6). Denote by $j_i : U_i \to X$ the corresponding open immersions, and put $R_i = j_i^!(R)$. Let M be a constructible object in $\mathrm{Ho}(\mathscr{M})(X)$. Then, for all $i \in I$, the image by $j_i^* = j_i^!$ of the map

$$M \to \mathbf{R}\mathit{Hom}_X(\mathbf{R}\mathit{Hom}_X(M, R), R)$$

is isomorphic to the map

$$j_i^*(M) \to \mathbf{R}\mathit{Hom}_{U_i}(\mathbf{R}\mathit{Hom}_{U_i}(j_i^*(M), R_i), R_i)\,.$$

This proposition thus follows from the property of separation with respect to the Zariski topology. □

COROLLARY 4.4.9. *Let* $f : X \to Y$ *be a separated morphism of finite type in* $\mathscr{S}$*. Given an object* R *of* $\mathrm{Ho}(\mathscr{M})(Y)$*, the property for* $f^!(R)$ *of being a* τ*-dualizing object in* $\mathrm{Ho}(\mathscr{M})(X)$ *is local over* X *and over* Y *for the Zariski topology.*

PROPOSITION 4.4.10. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* τ*-compatible. Let* $i : Z \to X$ *be a closed immersion and* R *be a* τ*-dualizing object in* $\mathrm{Ho}(\mathscr{M})(X)$*. Then* $i^!(R)$ *is* τ*-dualizing in* $\mathrm{Ho}(\mathscr{M})(Z)$*.*

PROOF. As $\mathrm{Ho}(\mathscr{M})$ is τ-compatible, we already know that $i^!(R)$ is constructible. For any objects M and R of $\mathrm{Ho}(\mathscr{M})(Z)$ and $\mathrm{Ho}(\mathscr{M})(X)$ respectively, we have the identification:

$$i_!\, \mathbf{R}\mathit{Hom}_Z(M, i^!(R)) \simeq \mathbf{R}\mathit{Hom}_X(i_!(M), R)\,.$$

Let $j : U \to X$ be the complement immersion. Then we have

$$j^! \mathbf{R}Hom_X(i_!(M), R) \simeq \mathbf{R}Hom_U(j^* i_!(M), j^!(R)) \simeq 0\,,$$

so that

$$\mathbf{R}Hom_X(i_!(M), R) \simeq i_!\, \mathbf{L}i^* \mathbf{R}Hom_X(i_!(M), R)\,.$$

As $i_!$ is fully faithful, this provides a canonical isomorphism

$$\mathbf{L}i^* \mathbf{R}Hom_X(i_!(M), R) \simeq i^! \mathbf{R}Hom_X, (i_!(M), R)\,.$$

Under this identification, we see easily that the map

$$i_!(M) \to \mathbf{R}Hom_X(\mathbf{R}Hom_X(i_!(M), R), R)$$

is isomorphic to the image by $i_!$ of the map

$$M \to \mathbf{R}Hom_Z(\mathbf{R}Hom_Z(M, i^!(R)), i^!(R))\,.$$

As $i_!$ is fully faithful, it is conservative, and this ends the proof. □

Proposition 4.4.11. *Assume that* $\mathrm{Ho}(\mathscr{M})$ *is* τ*-compatible,* $\mathbf{Q}$*-linear and separated, and consider a scheme* B *in* $\mathscr{S}$ *which admits wide resolution of singularities up to quotient singularities. Consider a separated* B*-scheme of finite type* S*, and a constructible object* R *in* $\mathrm{Ho}(\mathscr{M})(S)$*. The following conditions are equivalent.*

(i) *For any separated morphism of finite type* $f : X \to S$*, the object* $f^!(R)$ *is* τ*-dualizing.*
(ii) *For any projective morphism* $f : X \to S$*, the object* $f^!(R)$ *is* τ*-dualizing.*
(iii) *For any projective morphism* $f : X \to S$*, with* X *regular, the object* $f^!(R)$ *is* τ*-dualizing.*
(iv) *For any projective morphism* $f : X \to S$*, with* X *regular, and for any* $n \in \tau$*, the map*

$$\mathbb{1}_X\{n\} \to \mathbf{R}Hom_X(\mathbf{R}Hom_X(\mathbb{1}_X\{n\}, f^!(R)), f^!(R)) \tag{4.4.11.1}$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(X)$*.*

If, furthermore, for any regular separated B*-scheme of finite type* X*, and for any* $n \in \tau$*, the object* $\mathbb{1}_X\{n\}$ *is* $\otimes$*-invertible, then these conditions are equivalent to the following one.*

(v) *For any projective morphism* $f : X \to S$*, with* X *regular, the map*

$$\mathbb{1}_X \to \mathbf{R}Hom_X(f^!(R), f^!(R)) \tag{4.4.11.2}$$

is an isomorphism in $\mathrm{Ho}(\mathscr{M})(X)$*.*

Proof. It is clear that (i) implies (ii), which implies (iii), which implies (iv). Let us check that condition (ii) also implies condition (i). Let $f : X \to S$ be a morphism of separated B-schemes of finite type, with S regular. We want to prove that $f^!(\mathbb{1}_S)$ is τ-dualizing, while we already know it is true whenever f is projective. In the general case, by virtue of Corollary 4.4.9, we may assume that f is quasi-projective, so that $f = pj$, where p is projective, and j is an open immersion. As $f^! \simeq j^!\, p^!$, we conclude with Proposition 4.4.7. Under the additional assumption, the equivalence between (iv) and (v) is obvious. It thus remains to prove that (iv) implies (ii). It is in fact sufficient to prove that, under condition (iv), the object R itself is τ-dualizing. To prove that the map

$$M \to \mathbf{R}Hom_X(\mathbf{R}Hom_X(M, R), R) \tag{4.4.11.3}$$

is an isomorphism for any constructible object M of $\mathrm{Ho}(\mathscr{M})(S)$, it is sufficient to consider the case where $M = \mathbf{R}f_*(\mathbb{1}_X\{n\}) = f_!(\mathbb{1}_X\{n\})$, where $n \in \tau$ and $f : X \to S$ is a projective morphism with X regular (Corollary 4.4.3). For any object A of $\mathrm{Ho}(\mathscr{M})(X)$, we have canonical isomorphisms

$$\begin{aligned}\mathbf{R}Hom_S(f_!(A), R) &\simeq \mathbf{R}f_*\, \mathbf{R}Hom_X(A, f^!(R))\\ &= f_!\, \mathbf{R}Hom_X(A, f^!(R))\,,\end{aligned}$$

from which we get a natural isomorphism:

$$\mathbf{R}Hom_S(\mathbf{R}Hom_S(f_!(A), R), R) \simeq f_!\, \mathbf{R}Hom_X(\mathbf{R}Hom_X(A, f^!(R)), f^!(R))\,.$$

Under these identifications, the map (4.4.11.3) for $M = f_!(\mathbb{1}_X\{n\})$ is the image of the map (4.4.11.1) by the functor $f_!$. As (4.4.11.1) is invertible by assumption, this proves that R is τ-dualizing. □

Lemma 4.4.12. *Let X be a scheme in $\mathscr{S}$, and R be an object of $\mathrm{Ho}(\mathscr{M})(X)$. The property for R of being $\otimes$-invertible is local over X with respect to the Zariski topology.*

Proof. Let $R^\wedge = \mathbf{R}Hom(R, \mathbb{1}_X)$ be the dual of R. The object R is $\otimes$-invertible if and only if the evaluation map

$$R^\wedge \otimes^{\mathbf{L}}_X R \to \mathbb{1}_X$$

is invertible. Let $j : U \to X$ be an open immersion. Then, for any objects M and N in $\mathrm{Ho}(\mathscr{M})(X)$, we have the identification

$$j^*\mathbf{R}Hom_X(M, N) \simeq \mathbf{R}Hom_U(j^*(M), j^*(N))\,.$$

In particular, we have $j^*(R^\wedge) \simeq j^*(R)^\wedge$. As j^* is monoidal, the lemma follows from the fact that $\mathrm{Ho}(\mathscr{M})$ has the property of separation with respect to the Zariski topology. □

Definition 4.4.13. We shall say that $\mathrm{Ho}(\mathscr{M})$ is *τ-dualizable* if it satisfies the following conditions:

(i) $\mathrm{Ho}(\mathscr{M})$ is τ-compatible (4.2.20);
(ii) for any closed immersion between regular schemes $i : Z \to S$ in $\mathscr{S}$, the object $i^!(\mathbb{1}_S)$ is $\otimes$-invertible (i.e. the functor $i^!(\mathbb{1}_S) \otimes^{\mathbf{L}}_S (-)$ is an equivalence of categories);
(ii) for any regular scheme X in $\mathscr{S}$, and for any $n \in \tau$, the map

$$\mathbb{1}_X\{n\} \to \mathbf{R}Hom_X(\mathbf{R}Hom_X(\mathbb{1}_X\{n\}, \mathbb{1}_X), \mathbb{1}_X)$$

is an isomorphism.

As in other similar situations, we simply say *dualizable with respect to Tate twist* when the set of twists τ is generated by the Tate twist.

Example 4.4.14. In practice, the property of being dualizable with respect to Tate twist is a consequence of the absolute purity theorem. Our main example is the motivic category $\mathrm{DM}_{\mathcal{B}}$ of Beilinson motives over excellent noetherian schemes, as a consequence of Theorem 14.4.1.

Remark 4.4.15. Note that, whenever the set of twists τ consists of rigid objects (which will be the case in practice), conditions (i) and (ii) of the preceding definition are equivalent to the condition that $i^!(\mathbb{1}_X)$ is constructible and $\otimes$-invertible for any closed immersion i between regular separated schemes in $\mathscr{S}$, while condition (iii) is then automatic. This principle gives easily the property of τ-purity when $\mathscr{S}$ is made of schemes of finite type over some field:

Proposition 4.4.16. *Assume that $\mathscr{S}$ consists exactly of schemes of finite type over a field k, and that one of the following conditions is satisfied:*

(a) the field k is perfect;
(b) $\mathrm{Ho}(\mathscr{M})$ is semi-separated (2.1.7).

If the objects $\mathbb{1}\{n\}$ are rigid in $\mathrm{Ho}(\mathscr{M})(\mathrm{Spec}\,(k))$ for all $n \in \tau$, then $\mathrm{Ho}(\mathscr{M})$ is τ-dualizable.

Proof. For any k-scheme of finite type $f : X \to \mathrm{Spec}\,(k)$, as the functor $\mathbf{L}f^*$ is symmetric monoidal, the objects $\mathbb{1}_X\{n\}$ are rigid in $\mathrm{Ho}(\mathscr{M})(X)$ for all $n \in \tau$. Therefore, as stated in remark 4.4.15, we have only to prove that, for any closed immersion $i : Z \to X$ between regular k-schemes of finite type, the object $i^!(\mathbb{1}_X)$ is $\otimes$-invertible and constructible. We may assume that X and Z are smooth (under condition (a), this is clear, and under condition (b), by virtue of Proposition 2.1.9, we may replace k by any of its finite extensions). Using 4.4.12 and 4.2.6, we may also assume that X is quasi-projective and that Z is purely of codimension c in X, while the normal bundle of i is trivial. This proposition is then a consequence of relative purity (2.4.50), which gives a canonical isomorphism $i^!(\mathbb{1}_X) \simeq \mathbb{1}_Z(-c)[-2c]$. □

Proposition 4.4.17. *Assume that $\mathscr{S}$ consists of schemes of finite type over a field k and that $\mathrm{Ho}(\mathscr{M})$ has the following properties:*

(a) it is τ-dualizable;
(b) for any $n \in \tau$, $\mathbb{1}\{n\}$ is rigid;
(c) either k is perfect, either $\mathrm{Ho}(\mathscr{M})$ is continuous.

Then, any constructible object of $\mathrm{Ho}(\mathscr{M})(k)$ is rigid.

Proof. By 4.3.15, it is sufficient to treat the case where k is perfect. It is well known that rigid objects form a thick subcategory of $\mathrm{Ho}(\mathscr{M})$. Thus we conclude easily from Corollary 4.4.3 and Proposition 2.4.31. □

Lemma 4.4.18. *Assume that $\mathrm{Ho}(\mathscr{M})$ is τ-dualizable. Then, for any projective morphism $f : X \to S$ between regular schemes in $\mathscr{S}$, the object $f^!(\mathbb{1}_S)$ is $\otimes$-invertible and constructible.*

Proof. As, for any open immersion $j : U \to X$, one has $j^* = j^!$, we deduce easily from Lemma 4.4.12 (resp. Proposition 4.2.6) that the property for $f^!(\mathbb{1}_S)$ of being $\otimes$-invertible (resp. constructible) is local on S for the Zariski topology. Therefore, we may assume that S is separated over B and that f factors as a closed immersion $i : X \to \mathbf{P}^n_S$ followed by the canonical projection $p : \mathbf{P}^n_S \to S$. Using relative purity for p, we have the following computations:

$$f^!(\mathbb{1}_S) \simeq i^! p^!(\mathbb{1}_S) \simeq i^!(\mathbb{1}_{\mathbf{P}^n_S}(n)[2n]) \simeq i^!(\mathbb{1}_{\mathbf{P}^n_S})(n)[2n]\,.$$

As i is a closed immersion between regular schemes, the object $i^!(\mathbb{1}_{\mathbf{P}^n_S})$ is $\otimes$-invertible and constructible by assumption on $\mathrm{Ho}(\mathscr{M})$, which implies that $f^!(\mathbb{1}_S)$ is $\otimes$-invertible and constructible as well. □

Definition 4.4.19. Let B a scheme in $\mathscr{S}$. We shall say that *local duality holds over B in* $\mathrm{Ho}(\mathscr{M})$ if, for any separated morphism of finite type $f : X \to S$, with S regular and of finite type over B, the object $f^!(\mathbb{1}_S)$ is τ-dualizing in $\mathrm{Ho}(\mathscr{M})(X)$.

Remark 4.4.20. By definition, if $\mathrm{Ho}(\mathscr{M})$ is τ-compatible, and if local duality holds over B in $\mathrm{Ho}(\mathscr{M})$, then the restriction of $\mathrm{Ho}(\mathscr{M})$ to the category of B-schemes of finite type is τ-dualizable. A convenient sufficient condition for local duality to hold in $\mathrm{Ho}(\mathscr{M})$ is the following (in particular, using the result below as well as Proposition 4.4.16, local duality holds almost systematically over fields).

Theorem 4.4.21. *Assume that $\mathrm{Ho}(\mathscr{M})$ is τ-dualizable, $\mathbf{Q}$-linear and separated, and consider a scheme B in $\mathscr{S}$ which admits wide resolution of singularities up to quotient singularities (e.g. B might be any scheme which is separated and of finite type over an excellent noetherian scheme of dimension lesser or equal to 2 in $\mathscr{S}$; see 4.1.11). Then local duality holds over B in $\mathrm{Ho}(\mathscr{M})$.*

Proof. Let S be a regular separated B-scheme of finite type. Then, for any separated morphism of finite type $f : X \to S$, the object $f^!(\mathbb{1}_S)$ is τ-dualizing: Lemma 4.4.18 implies immediately condition (iv) of Proposition 4.4.11. The general case (without the separation assumption on S) follows easily from Corollary 4.4.8. □

Proposition 4.4.22. *Consider a scheme B in $\mathscr{S}$. Assume that $\mathrm{Ho}(\mathscr{M})$ is τ-dualizable, and that local duality holds over B in $\mathrm{Ho}(\mathscr{M})$. Consider a regular B-scheme of finite type S.*

(i) An object of $\mathrm{Ho}(\mathscr{M})(S)$ is τ-dualizing if and only if it is constructible and $\otimes$-invertible.
(ii) For any separated morphism of S-schemes of finite type $f : X \to Y$, and for any τ-dualizing object R in $\mathrm{Ho}(\mathscr{M})(Y)$, the object $f^!(R)$ is τ-dualizing in $\mathrm{Ho}(\mathscr{M})(X)$.

Proof. As the unit of $\mathrm{Ho}(\mathscr{M})(S)$ is τ-dualizing by assumption, Proposition 4.4.6 implies that an object of $\mathrm{Ho}(\mathscr{M})(S)$ is τ-dualizing if and only if it is constructible and $\otimes$-invertible.

Consider a regular B-scheme of finite type S, as well as a separated morphism of S-schemes of finite type $f : X \to Y$, as well as a τ-dualizing object R in $\mathrm{Ho}(\mathscr{M})(Y)$. To prove that $f^!(R)$ is τ-dualizing, by virtue of Corollary 4.4.8, we may assume that Y is separated over S. Denote by u and v the structural maps from X and Y to S respectively. As we already know that $v^!(\mathbb{1}_S)$ is τ-dualizing, by virtue of Proposition 4.4.6, there exists a constructible and $\otimes$-invertible object U

in $\mathrm{Ho}(\mathscr{M})(Y)$ such that $U \otimes^{\mathbf{L}}_Y R \simeq v^!(\mathbb{1}_S)$. As the functor $\mathbf{L}f^*$ is symmetric monoidal, it preserves $\otimes$-invertible objects and their duals, from which we deduce the following isomorphisms:

$$\begin{aligned} u^!(\mathbb{1}_S) &\simeq f^!\, v^!(\mathbb{1}_S) \\ &\simeq f^!(U \otimes^{\mathbf{L}}_Y R) \\ &\simeq f^!\, \mathbf{R}\mathit{Hom}_Y(U^\wedge, R) \\ &\simeq \mathbf{R}\mathit{Hom}_X(\mathbf{L}f^*(U^\wedge), f^!(R)) \\ &\simeq \mathbf{R}\mathit{Hom}_X(\mathbf{L}f^*(U)^\wedge, f^!(R)) \\ &\simeq \mathbf{L}f^*(U) \otimes^{\mathbf{L}}_X f^!(R)\,. \end{aligned}$$

The object $a^!(\mathbb{1}_S)$ being τ-dualizing, while $\mathbf{L}f^*(U)$ is constructible and invertible, we deduce from Proposition 4.4.6 that $f^!(R)$ is τ-dualizing as well. □

4.4.23. Assume that $\mathrm{Ho}(\mathscr{M})$ is τ-dualizable, $\mathbf{Q}$-linear and separated.

Consider a scheme B in $\mathscr{S}$, such that local duality holds over B in $\mathrm{Ho}(\mathscr{M})$ – this is the case if B admits wide resolution of singularities up to quotient singularities according to the above Theorem. Consider a fixed regular B-scheme of finite type S, as well as a constructible and $\otimes$-invertible object R in $\mathrm{Ho}(\mathscr{M})(S)$ (in the case S is of pure dimension d, it might be wise to consider $R = \mathbb{1}_S(d)[2d]$, but an arbitrary R as above is eligible by 4.4.22). Then, for any separated S-scheme of finite type $f : X \to S$, we define the *local duality functor*

$$D_X : \mathrm{Ho}(\mathscr{M})(X)^{op} \to \mathrm{Ho}(\mathscr{M})(X)$$

by the formula

$$D_X(M) = \mathbf{R}\mathit{Hom}_X(M, f^!(R))\,.$$

This functor D_X is right adjoint to itself.

Corollary 4.4.24. *Under the above assumptions, we have the following properties of the motivic triangulated category* $\mathrm{Ho}(\mathscr{M})$*:*

- *(a) For any separated S-scheme of finite type X, the functor D_X preserves constructible objects.*
- *(b) For any separated S-scheme of finite type X, the natural map*

$$M \to D_X(D_X(M))$$

is an isomorphism for any constructible object M in $\mathrm{Ho}(\mathscr{M})(X)$*.*

- *(c) For any separated S-scheme of finite type X, and for any objects M and N in* $\mathrm{Ho}(\mathscr{M})(X)$*, if N is constructible, then we have a canonical isomorphism*

$$D_X(M \otimes^{\mathbf{L}}_X D_X(N)) \simeq \mathbf{R}\mathit{Hom}_X(M, N)\,.$$

- *(d) For any morphism between separated S-schemes of finite type $f : Y \to X$, we have natural isomorphisms*

$$\begin{aligned} D_Y(f^*(M)) &\simeq f^!(D_X(M)) \\ f^*(D_X(M)) &\simeq D_Y(f^!(M)) \\ D_X(f_!(N)) &\simeq f_*(D_Y(N)) \\ f_!(D_Y(N)) &\simeq D_X(f_*(N)) \end{aligned}$$

for any constructible objects M and N in $\mathrm{Ho}(\mathscr{M})(X)$ *and* $\mathrm{Ho}(\mathscr{M})(Y)$ *respectively.*

This corollary sums up what must be called the *Grothendieck duality* property for the motivic triangulated category $\mathrm{Ho}(\mathscr{M})$ with respect to the set of twists τ.

Proof. Assertions (a) and (b) are only stated for the record[60]; see 4.2.25. To prove (c), we see that we have an obvious isomorphism

$$D_X(M \otimes^{\mathbf{L}}_X P) \simeq \mathbf{R}Hom_X(M, D_X(P))$$

for any objects M and P. If N is constructible, we may replace P by $D_X(N)$ and get the expected formula using (b). The identification $D_Y\, f^* \simeq f^!\, D_X$ is a special case of the formula

$$\mathbf{R}Hom_Y(f^*(A), f^!(B)) \simeq f^!\, \mathbf{R}Hom_X(A, B)\,.$$

Therefore, we also get:

$$f^*\, D_X \simeq D_Y^2\, f^*\, D_X \simeq D_Y\, f^!\, D_X^2 \simeq D_Y\, f^!\,.$$

The two other formulas of (d) follow by adjunction. □

Theorem 4.4.25. *Assume that $\mathscr{S}$ consists of schemes of finite type over a field k, and consider a separated τ'-generated motivic triangulated category $\mathscr{T}'$ over $\mathscr{S}$, as well as a premotivic morphism*

$$\varphi^* : \mathscr{T} = \mathrm{Ho}(\mathscr{M}) \to \mathscr{T}'\,.$$

We suppose that the following properties hold:

(a) $\mathscr{T}$ is τ-dualizable and $\mathbf{Q}$-linear and separated;
(b) the object $\mathbb{1}\{i\}$ is rigid in $\mathscr{T}(k)$ for any $i \in \tau$.

Then, the premotivic morphism

$$\varphi^* : \mathscr{T}_c \to \mathscr{T}'$$

commutes with the six operations.

Remark 4.4.26. Remark that, as a corollary, we obtain immediately, under the assumptions of the theorem that $\mathscr{T}'$ is $\varphi^*(\tau)$-dualizable and that the functor φ^* commutes with the duality functors on $\mathscr{T}$ and $\mathscr{T}'$, respectively obtained by applying the above corollary in the case $B = \mathrm{Spec}\,(k)$.

Proof. Given a morphism of finite type $f : X \to \mathrm{Spec}\,(k)$, let us consider the following property.

$(*)_f$ *For any constructible object M in $\mathscr{T}(X)$, the natural exchange map*

$$\varphi^*\, f_*(M) \to f_*\, \varphi^*(M)$$

is invertible.

We will first prove the theorem assuming that property $(*)_f$ holds for any f.

Let $u : X \to Y$ be a k-morphism of finite type. We claim that the exchange map

$$\varphi^*\, u_*(M) \to u_*\, \varphi^*(M)$$

is invertible for any τ-constructible object M of $\mathscr{T}(X)$.

It is sufficient to prove that, for any smooth separated k-morphism of finite type $g : T \to X$, any constructible object M in $\mathscr{T}(X)$ and any twist i in τ', the natural map

$$\mathrm{Hom}_{\mathscr{T}'(X)}(g_\sharp(\mathbb{1}_T)\{i\}, \varphi^*\, u_*(M)) \to \mathrm{Hom}_{\mathscr{T}'(X)}(g_\sharp(\mathbb{1}_T)\{i\}, u_*(M))$$

is bijective. Consider the following commutative diagram of morphisms of schemes:

$$\begin{array}{ccc} V & \xrightarrow{v} & T \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{u} & Y \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle b} & \\ & \mathrm{Spec}\,(k) & \end{array}$$

[60]We have put to a lot of assumptions here: in fact, if $\mathrm{Ho}(\mathscr{M})$ is τ-dualizable and if local duality holds over B in $\mathrm{Ho}(\mathscr{M})$, the six Grothendieck operations preserve constructible objects on the restriction of $\mathrm{Ho}(\mathscr{M})$ to B-schemes of finite type; we leave this as a formal exercice for the reader.

in which the square is cartesian. Recall that the functor v_* preserves constructible objects by virtue of Theorem 4.2.16. Then we conclude by the computations below:

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{T}'(Y)}(g_\sharp(\mathbb{1}_T)\{i\}, \varphi^* u_*(M)) &= \mathrm{Hom}_{\mathscr{T}'(T)}(\mathbb{1}_Y\{i\}, g^* \varphi^* u_*(M)) \\
&= \mathrm{Hom}_{\mathscr{T}'(T)}(\mathbb{1}_Y\{i\}, \varphi^* g^* u_*(M)) \\
&= \mathrm{Hom}_{\mathscr{T}'(T)}(g^* b^*(\mathbb{1}_k)\{i\}, \varphi^* g^* u_*(M)) \\
&= \mathrm{Hom}_{\mathscr{T}'(T)}(g^* b^*(\mathbb{1}_k)\{i\}, \varphi^* v_* h^*(M)) \\
&= \mathrm{Hom}_{\mathscr{T}'(k)}(\mathbb{1}_k\{i\}, (bg)_* \varphi^* v_* h^*(M)) \\
&= \mathrm{Hom}_{\mathscr{T}'(k)}(\mathbb{1}_k\{i\}, \varphi^* (bg)_* v_* h^*(M)) \quad (\text{by } (*)_{bg}) \\
&= \mathrm{Hom}_{\mathscr{T}'(k)}(\mathbb{1}_k\{i\}, (bgv)_* \varphi^* h^*(M)) \quad (\text{by } (*)_{bgv}) \\
&= \mathrm{Hom}_{\mathscr{T}'(k)}(\mathbb{1}_k\{i\}, (bg)_* g^* u_* \varphi^*(M)) \\
&= \mathrm{Hom}_{\mathscr{T}'(Y)}(g_\sharp(\mathbb{1}_T)\{i\}, u_* \varphi^*(M))
\end{aligned}$$

From there, we see that, for any k-scheme of finite type X and any τ-constructible objects M and N of $\mathscr{T}(X)$, the natural map

$$\varphi^*(\mathit{Hom}_X(M,N)) \to \mathit{Hom}_X(\varphi^*(M), \varphi^*(N))$$

is invertible in $\mathscr{T}'(X)$. For this, we may assume that $M = f_\sharp(\mathbb{1}_Y\{i\})$ for a smooth morphism of finite type $f : Y \to X$ and a twist i, in which case we have

$$\varphi^*(\mathit{Hom}_X(M,N)) = \varphi^* f_* f^*(N) \simeq f_* f^* \varphi^*(N) = \mathit{Hom}_X(\varphi^*(M), \varphi^*(N)).$$

It remains to prove that for any separated k-morphism $f : X \to Y$ of finite type and any constructible object N in $\mathscr{T}(X)$, the exchange map:

$$\varphi^* f^!(N) \to f^! \varphi^*(N)$$

is an isomorphism. It is easy to see that this property is local for the Zariski topology, both on X and on Y, so that we may assume that f is affine. Therefore, it is sufficient to consider the situation where f i either a closed immersion or a separated smooth map. In the smooth case, as $f^!$ is of the form $f^*(d)[2d]$, this is obvious. If $f = i$ is a closed immersion with open complement j, as we already know that φ^* commutes with u_* for any morphism u, this property follows straight away from the localization distinguished triangles

$$i_* i^! \to 1 \to j_* j^* \to .$$

It remains to prove property $(*)_f$ for any morphism f of finite type.

We claim it is sufficient to prove that, for any k-scheme of finite type X with structural morphism f, the following property holds:

$(**)_X$ *For any twist $i \in \tau$, the natural exchange map*

$$\varphi^* f_*(\mathbb{1}_X\{i\}) \to f_* \varphi^*(\mathbb{1}_X\{i\})$$

is invertible.

Indeed, by virtue of Theorem 4.2.13, we may assume that $M = w_*(\mathbb{1}_W\{i\})$ for $w : W \to X$ a projective k-morphism, and $i \in \tau$. As the exchange map $\varphi^* w_* \to w_* \varphi^*$ is invertible (Proposition 2.4.53), we see that we may assume that $M = \mathbb{1}_X\{i\}$ for some twist i.

Let us prove property $(**)_X$ in the case X is in addition smooth over k. As φ^* is monoidal, for any rigid object M of $\mathscr{T}(k)$, we get the identification:

$$\varphi^*(M^\vee) = \varphi^*(M)^\vee.$$

On the other hand, according to assumption (b), the object $f_\sharp(\mathbb{1}_X)$ is rigid in $\mathscr{T}(k)$ as well as in $\mathscr{T}'(k)$ (because the functor φ^* is symmetric monoidal and commutes with the operations of the form $f_\sharp$ for f smooth). Thus we get:

$$f_*(\mathbb{1}_X\{i\}) = \mathit{Hom}_k(f_\sharp(\mathbb{1}_X), \mathbb{1}_k\{i\}) = f_\sharp(\mathbb{1}_X)^\vee\{i\}.$$

Then property $(**)_X$ readily follows.

We finally prove property $(**)_X$ for any algebraic k-scheme X. We will proceed by induction on the dimension of X.

In case $\dim(X) < 0$, the result is obvious. Let us assume $\dim(X) \geq 0$. According to the localization property, we can assume that X is reduced. Let $\bar{k}$ be an inseparable closure of k and $\bar{X} = X \otimes_k \bar{k}$. According to De Jong theorem applied to $\bar{X}$ (see Th. 4.1.10 for $S = \operatorname{Spec}\left(\bar{k}\right)$), there exists a Galois alteration $\bar{X}' \to \bar{X}$ of group G such that $\bar{X}'$ is smooth over $\bar{k}$.

We can assume that such a smooth alteration exists over a finite inseparable extension field E/k. Because $\mathscr{T}$ (resp. $\mathscr{T}'$) is $\mathbf{Q}$-linear and separated, the base change functor π^* associated with the finite morphism $\pi : \operatorname{Spec}(E) \to \operatorname{Spec}(k)$ and relative to the premotivic category $\mathscr{T}$ (resp. $\mathscr{T}'$) is an equivalence of categories (see Proposition 2.1.9). Thus we can replace k by E and assume that there exists a Galois alteration $p : X' \to X$ of group G such that X' is a smooth k-scheme. Using the localization property, we can assume X is reduced. Then there exists a nowhere dense closed subscheme $\nu : Z \to X$ such that $U = X - Z$ is regular (thus normal) and the induced map $p|_U : p^{-1}(U) \to U$ is finite. Thus we can apply Theorem 4.4.1 to the cartesian square:

$$\begin{array}{ccc} Z' & \xrightarrow{\nu'} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{\nu} & X \end{array}$$

and we get the distinguished triangle in $\mathscr{T}(X)$ (thus in $\mathscr{T}'(X)$ as well, as the functor φ^* is monoidal and commutes with the operations of the form u_* for any proper morphism u) of the form:

$$\mathbb{1}_X\{i\} \to p_*(\mathbb{1}_{X'}\{i\})^G \oplus \nu_*(\mathbb{1}_Z\{i\}) \to (\nu q)_*(\mathbb{1}_{Z'}\{i\})^G \xrightarrow{+1}$$

for any twist i. If we consider the triangles in $\mathscr{T}(k)$ and $\mathscr{T}'(k)$ obtained by applying the functor f_*, where f is the structural morphism of X/k, we deduce that property $(**)_X$ follows from properties $(**)_{X'}$, $(**)_Z$, $(**)_{Z'}$. Thus we can conclude applying either the case of a smooth k-scheme treated above or the induction hypothesis as $\dim(Z) = \dim(Z') < \dim(X)$. □

Part 2

Construction of fibred categories

5. Fibred derived categories

5.0. In this entire section, we fix a full subcategory $\mathscr{S}$ of the category of noetherian $\mathcal{S}$-schemes satisfying the following properties:

(a) $\mathscr{S}$ is closed under finite sums and pullback along morphisms of finite type.
(b) For any scheme S in $\mathscr{S}$, any quasi-projective S-scheme belongs to $\mathscr{S}$.

We fix an admissible class of morphisms $\mathscr{P}$ of $\mathscr{S}$. All our $\mathscr{P}$-premotivic categories (*cf.* definition 1.4.2) are defined over $\mathscr{S}$. Moreover, for any abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ in this section, we assume the following:

(c) $\mathscr{A}$ is a *Grothendieck* abelian $\mathscr{P}$-premotivic category (see definition 1.3.8 and the recall below).
(d) $\mathscr{A}$ is given with a generating set of twists τ. We sometimes refer to it as *the twists of* $\mathscr{A}$.
(e) We will denote by $M_S(X, \mathscr{A})$, or simply by $M_S(X)$, the geometric section over a $\mathscr{P}$-scheme X/S.

Without precision, any scheme will be assumed to be an object of $\mathscr{S}$.

In section 5.2, except possibly for 5.2.a, we assume further:

(f) $\mathscr{P}$ contains the class of smooth finite type morphisms.

In section 5.3, we assume (f) and instead of (d) above.

5.0.27. We will refer sometimes to the canonical dg-structure of the category of complexes $\mathrm{C}(\mathscr{A})$ over an abelian category $\mathscr{A}$. Recall that to any complexes K and L over $\mathscr{A}$, we associate a complex of abelian groups $\mathrm{Hom}^\bullet_{\mathscr{A}}(K, L)$ whose component in degree $n \in \mathbf{Z}$ is

$$\prod_{p \in \mathbf{Z}} \mathrm{Hom}_{\mathscr{A}}(K^p, L^{p+n})$$

and whose differential in degree $n \in \mathbf{Z}$ is defined by the formula:

$$(f_p)_{p \in \mathbf{Z}} \mapsto \big(d_L \circ f_p - (-1)^n . f_{p+1} \circ d_K)\big)_{p \in \mathbf{Z}}.$$

In other words, this is the image of the bicomplex $\mathrm{Hom}_{\mathscr{A}}(K, L)$ by the Tot-product functor which we denote by Tot^π. Of course, the associated homotopy category is the category $\mathrm{K}(\mathscr{A})$ of complexes up to chain homotopy equivalence.

5.1. From abelian premotives to triangulated premotives.

5.1.a. *Abelian premotives: recall and examples.* Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$. According to the convention of 5.0, for any scheme S, $\mathscr{A}_S$ is a Grothendieck abelian closed symmetric monoidal category. Moreover, if τ denotes the twists of $\mathscr{A}$, the essentially small family

$$\big(M_S(X)\{i\}\big)_{X \in \mathscr{P}/S, i \in \tau}$$

is a family of generators of $\mathscr{A}_S$ in the sense of [**Gro57**].

Example 5.1.1. Consider a fixed ring Λ. Let $\mathrm{PSh}(\mathscr{P}/S, \Lambda)$ be the category of Λ-presheaves (i.e. presheaves of Λ-modules) on $\mathscr{P}/S$. For any $\mathscr{P}$-scheme X/S, we let $\Lambda_S(X)$ be the free Λ-presheaf on $\mathscr{P}/S$ represented by X. Then $\mathrm{PSh}(\mathscr{P}/S, \Lambda)$ is a Grothendieck abelian category generated by the essentially small family $\big(\Lambda_S(X)\big)_{X \in \mathscr{P}/S}$.

There is a unique symmetric closed monoidal structure on $\mathrm{PSh}(\mathscr{P}/S, \Lambda)$ such that

$$\Lambda_S(X) \otimes_S \Lambda_S(Y) = \Lambda_S(X \times_S Y).$$

Finally the existence of functors f^*, f_* and, in the case when f is a $\mathscr{P}$-morphism, of $f_\sharp$, follows from general sheaf theory (*cf.* [**SGA4**]).

Thus, $\mathrm{PSh}(\mathscr{P}, \Lambda)$ defines an abelian $\mathscr{P}$-premotivic category.

5.1.2. Consider an abstract abelian $\mathscr{P}$-premotivic category $\mathscr{A}$. To any premotive M of $\mathscr{A}_S$, we can associate a presheaf of abelian groups

$$X \mapsto \mathrm{Hom}_{\mathscr{A}_S}(M_S(X), M)$$

which we denote by $\gamma_*(M)$.
This defines a functor $\gamma_* : \mathscr{A}_S \rightarrow \mathrm{PSh}(\mathscr{P}/S, \mathbf{Z})$. It admits the following left adjoint:

$$\gamma^* : \mathrm{PSh}(\mathscr{P}/S, \mathbf{Z}) \rightarrow \mathscr{A}_S \ , \quad F \mapsto \varinjlim_{X/F} M_S(X, \mathscr{A})$$

where the colimit runs over the category of representable presheaves over F.

It is now easy to check we have defined a morphism of (complete) abelian $\mathscr{P}$-premotivic categories:

$$\gamma^* : \mathrm{PSh}(\mathscr{P}, \mathbf{Z}) \rightleftarrows \mathscr{A} : \gamma_*. \tag{5.1.2.1}$$

Moreover $\mathrm{PSh}(\mathscr{P}, \mathbf{Z})$ appears as the initial abelian $\mathscr{P}$-premotivic category.

Remark that the functor $\gamma_* : \mathscr{A}_S \rightarrow \mathrm{PSh}(\mathscr{P}/S, \mathbf{Z})$ is conservative if the set of twists τ of $\mathscr{A}$ is trivial.

Definition 5.1.3. A $\mathscr{P}$-admissible topology t is a Grothendieck pretopology t on the category $\mathscr{S}$, such that any t-covering family consists of $\mathscr{P}$-morphisms.

Note that, for any scheme S in $\mathscr{S}$, such a topology t induces a pretopology on $\mathscr{P}/S$ (which we denote by the same letter). For any morphism (resp. $\mathscr{P}$-morphism) $f : T \rightarrow S$, the functor f^* (resp. $f_\sharp$) preserves t-covering families.

As $\mathscr{P}$ is fixed in all this section, we will simply say *admissible* for $\mathscr{P}$-admissible.

Example 5.1.4. Let t be an admissible topology. We denote by $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ the category of t-sheaves of Λ-modules on $\mathscr{P}/S$. Given a $\mathscr{P}$-scheme X/S, we let $\Lambda_S^t(X)$ be the free Λ-linear t-sheaf represented by X. Then, $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ is an abelian Grothendieck category with generators $(\Lambda_S^t(X))_{X \in \mathscr{P}/S}$.

As in the preceding example, the category $\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ admits a unique closed symmetric monoidal structure such that $\Lambda_S^t(X) \otimes_S \Lambda_S^t(Y) = \Lambda_S^t(X \times_S Y)$. Finally, for any morphism $f : T \rightarrow S$ of schemes, the existence of functors f^*, f_* (resp. $f_\sharp$ when f is a $\mathscr{P}$-morphism) follows from the general theory of sheaves (see again [**SGA4**]: according to our assumption on t and [**SGA4**, III, 1.6], the functors $f^* : \mathscr{P}/S \rightarrow \mathscr{P}/T$ and $f_\sharp : \mathscr{P}/T \rightarrow \mathscr{P}/S$ (for f in $\mathscr{P}$) are continuous).

Thus, $\mathrm{Sh}_t(\mathscr{P}, \Lambda)$ defines an abelian $\mathscr{P}$-premotivic category (with trivial set of twists).

The associated t-sheaf functor induces a morphism

$$a_t^* : \mathrm{PSh}(\mathscr{P}, \Lambda) \rightleftarrows \mathrm{Sh}_t(\mathscr{P}, \Lambda) : a_{t,*}. \tag{5.1.4.1}$$

Remark 5.1.5. Recall the abelian category $\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z})$ is a localization of the category $\mathrm{PSh}(S, \mathbf{Z})$ in the sense of Gabriel-Zisman. In particular, given an abstract abelian $\mathscr{P}$-premotivic category $\mathscr{A}$, the canonical morphism

$$\gamma^* : \mathrm{PSh}(\mathscr{P}/S, \mathbf{Z}) \rightleftarrows \mathscr{A}_S : \gamma_*$$

induces a unique morphism

$$\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z}) \rightleftarrows \mathscr{A}_S$$

if and only if for any presheaf of abelian groups F on $\mathscr{P}/S$ such that $a_t(F) = F_t = 0$, one has $\gamma^*(F) = 0$.

We leave to the reader the exercise which consists to formulate the universal property of the abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}, \mathbf{Z})$.[61]

5.1.b. *The t-descent model category structure.*

5.1.6. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ with set of twists τ.

We let $\mathrm{C}(\mathscr{A})$ be the $\mathscr{P}$-fibered abelian category over $\mathscr{S}$ whose fibers over a scheme S is the category $\mathrm{C}(\mathscr{A}_S)$ of (unbounded) complexes in $\mathscr{A}_S$. For any scheme S, we let $\iota_S : \mathscr{A}_S \rightarrow \mathrm{C}(\mathscr{A}_S)$ the embedding which sends an object of $\mathscr{A}_S$ to the corresponding complex concentrated in degree zero.

If $\mathscr{A}$ is τ-twisted, then the category $\mathrm{C}(\mathscr{A}_S)$ is obviously $(\mathbf{Z} \times \tau)$-twisted. The following lemma is straightforward :

[61] We will formulate a derived version in the paragraph on descent properties for derived premotives (*cf.* 5.2.9).

Lemma 5.1.7. *With the notations above, there is a unique structure of abelian $\mathscr{P}$-premotivic category on $\mathrm{C}(\mathscr{A})$ such that the functor $\iota : \mathscr{A} \to \mathrm{C}(\mathscr{A})$ is a morphism of abelian $\mathscr{P}$-premotivic categories.*

5.1.8. For a scheme S, let $(\mathscr{P}/S)^{\amalg}$ be the category introduced in 3.2.1. The functor $M_S(-)$ can be extended to $(\mathscr{P}/S)^{\amalg}$ by associating to a family $(X_i)_{i\in I}$ of $\mathscr{P}$-schemes over S the premotive

$$\bigoplus_{i\in I} M_S(X_i).$$

If $\mathcal{X}$ is a simplicial object of $(\mathscr{P}/S)^{\amalg}$, we denote by $M_S(\mathcal{X})$ the complex associated with the simplicial object of $\mathscr{A}_S$ obtained by applying degreewise the above extension of $M_S(-)$.

Definition 5.1.9. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category and t be an admissible topology.

Let S be a scheme and C be an object of $\mathrm{C}(\mathscr{A}_S)$:

(1) The complex C is said to be *local* (with respect to the geometric section) if, for any $\mathscr{P}$-scheme X/S and any pair $(n,i) \in \mathbf{Z} \times \tau$, the canonical morphism

$$\mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(M_S(X)\{i\}[n], C) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(M_S(X)\{i\}[n], C)$$

is an isomorphism.

(2) The complex C is said to be *t-flasque* if for any t-hypercover $\mathcal{X} \to X$ in $\mathscr{P}/S$, for any $(n,i) \in \mathbf{Z} \times \tau$, the canonical morphism

$$\mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(M_S(X)\{i\}[n], C) \to \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(M_S(\mathcal{X})\{i\}[n], C)$$

is an isomorphism.

We say the abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ satifies *cohomological t-descent* if for any t-hypercover $\mathcal{X} \to X$ of a $\mathscr{P}$-scheme X/S, and for any $i \in \tau$, the map

$$M_S(\mathcal{X})\{i\} \to M_S(X)\{i\}$$

is a quasi-isomorphism (or equivalently, if any local complex is t-flasque).

We say that $\mathscr{A}$ is *compatible with t* if $\mathscr{A}$ satisfies cohomological t-descent, and if, for any scheme S, any t-flasque complex of $\mathscr{A}_S$ is local.

Example 5.1.10. Consider the notations of 5.1.4.

Consider the canonical dg-structure on $\mathrm{C}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))$ (see 5.1.1). By definition, for any complexes D and C of sheaves, we get an equality:

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(D,C) = H^0(\mathrm{Hom}^{\bullet}_{\mathrm{Sh}_t(\mathscr{P}/S,\Lambda)}(D,C)) = H^0(\mathrm{Tot}^{\pi}\,\mathrm{Hom}_{\mathrm{Sh}_t(\mathscr{P}/S,\Lambda)}(D,C)).$$

In the case where $D = \Lambda^t_S(X)$ (resp. $D = \Lambda^t_S(\mathcal{X})$) for a $\mathscr{P}$-scheme X/S (resp. a simplicial $\mathscr{P}$-scheme over S) we obtain the following identification:

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\Lambda^t_S(X), C) = H^0(C(X)).$$

$$(\text{resp. } \mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\Lambda^t_S(\mathcal{X}), C) = H^0(\mathrm{Tot}^{\pi}\, C(\mathcal{X}))\,).$$

Thus, we get the following equivalences:

$$C \text{ is local} \Leftrightarrow \text{for any } \mathscr{P}\text{-scheme } X/S,\ H^n_t(X,C) \simeq H^n(C(X)).$$

$$C \text{ is } t\text{-flasque} \Leftrightarrow \text{for any } t\text{-hypercover } \mathcal{X} \to X,\ H^n(C(X)) \simeq H^n(\mathrm{Tot}^{\pi}\, C(\mathcal{X})).$$

According to the computation of cohomology with hypercovers (*cf.* [**Bro74**]), if the complex C is t-flasque, it is local. In other words, we have the expected property that the abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P},\Lambda)$ is compatible with t.

5.1.11. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ and an admissible topology t.

Fix a base scheme S. A morphism $p : C \to D$ of complexes on $\mathscr{A}_S$ is called a *t-fibration* if its kernel is a t-flasque complex and if for any $\mathscr{P}$-scheme X/S, any $i \in \tau$ and any integer $n \in \mathbf{Z}$, the map of abelian groups

$$\mathrm{Hom}_{\mathscr{A}_S}(M_S(X)\{i\}, C^n) \to \mathrm{Hom}_{\mathscr{A}_S}(M_S(X)\{i\}, D^n)$$

is surjective.

For any object A of $\mathscr{A}_S$, we let $S^n A$ (resp. $D^n A$) be the complex with only one non trivial term (resp. two non trivial terms) equal to A in degree n (resp. in degree n and $n+1$, with the identity as only non trivial differential). We define the class of *cofibrations* as the smallest class of morphisms of $\mathrm{C}(\mathscr{A}_S)$ which :

(1) contains the map $S^{n+1}M_S(X)\{i\} \to D^n M_S(X)\{i\}$ for any $\mathscr{P}$-scheme X/S, any $i \in \tau$, and any integer n;
(2) is stable by pushout, transfinite composition and retract.

A complex C is said to be *cofibrant* if the canonical map $0 \to C$ is a cofibration. For instance, for any $\mathscr{P}$-scheme X/S and any $i \in \tau$, the complex $M_S(X)\{i\}[n]$ is cofibrant.

Let $\mathcal{G}_S$ be the essentially small family made of premotives $M_S(X)\{i\}$ for a $\mathscr{P}$-scheme X/S and a twist $i \in \tau$, and $\mathcal{H}_S$ be the family of complexes of the form $\mathrm{Cone}(M_S(\mathcal{X})\{i\} \to M_S(X)\{i\})$ for any t-hypercover $\mathcal{X} \to X$ and any twist $i \in \tau$. By the very definition, as $\mathscr{A}$ is compatible with t (definition 5.1.9), $(\mathcal{G}_S, \mathcal{H}_S)$ is a descent structure on $\mathscr{A}_S$ in the sense of [**CD09**, def. 2.2]. Moreover, it is weakly flat in the sense of [**CD09**, par. 3.1]. Thus the following proposition is a particular case of [**CD09**, theorem 2.5, proposition 3.2, and corollary 5.5] :

PROPOSITION 5.1.12. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category, which we assume to be compatible with an admissible topology t. Then for any scheme S, the category $\mathrm{C}(\mathscr{A}_S)$ with the preceding definition of fibrations and cofibrations, with quasi-isomorphisms as weak equivalences is a proper symmetric monoidal model category.*

5.1.13. We will call this model structure on $\mathrm{C}(\mathscr{A}_S)$ the *t-descent model category structure* (over S). Note that, for any $\mathscr{P}$-scheme X/S and any twist $i \in \tau$, the complex $M_S(X)\{i\}$ concentrated in degree 0 is cofibrant by definition, as well as any of its suspensions and twists. They form a family of generators for the triangulated category $\mathrm{D}(A_S)$.
Observe also that the fibrant objects for the t-descent model category structure are exactly the t-flasque complexes in $\mathscr{A}_S$. Moreover, essentially by definition, a complex of $\mathscr{A}_S$ is local if and only if it is t-flasque (see [**CD09**, 2.5]).

5.1.14. Consider again the notations and hypothesis of 5.1.11.
Consider a morphism of schemes $f : T \to S$. Then the functor

$$f^* : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{A}_T)$$

sends $\mathcal{G}_S$ in $\mathcal{G}_T$, and $\mathcal{H}_S$ in $\mathcal{H}_T$ because the topology t is admissible. This means it satisfies descent according to the definition of [**CD09**, 2.4]. Applying theorem 2.14 of *op. cit.*, the functor f^* preserves cofibrations and trivial cofibrations, i.e. the pair of functors (f^*, f_*) is a Quillen adjunction with respect to the t-descent model category structures.

Assume that f is a $\mathscr{P}$-morphism. Then, similarly, the functor

$$f_\sharp : \mathrm{C}(\mathscr{A}_T) \to \mathrm{C}(\mathscr{A}_S)$$

sends $\mathcal{G}_S$ (resp. $\mathcal{H}_S$) in $\mathcal{G}_T$ (resp. $\mathcal{H}_T$) so that it $f_\sharp$ also satisfies descent in the sense of *op. cit.* Therefore, it preserves cofibrations and trivial cofibrations, and the pair of adjoint functors $(f_\sharp, f^*)$ is a Quillen adjunction for the t-descent model category structures.

In other words, we have obtained the following result.

COROLLARY 5.1.15. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. The $\mathscr{P}$-fibred category $\mathrm{C}(\mathscr{A})$ with the t-descent model category structure defined in 5.1.12 is a symmetric monoidal $\mathscr{P}$-fibred model category. Moreover, it is stable, proper and combinatorial.*

5.1.16. Recall the following consequences of this corollary (see also 1.3.24 for the general theory). Consider a morphism $f : T \to S$ of schemes. Then the pair of adjoint functors (f^*, f_*) admits total left/right derived functors

$$\mathbf{L}f^* : \mathrm{D}(\mathscr{A}_S) \rightleftarrows \mathrm{D}(\mathscr{A}_T) : \mathbf{R}f_*.$$

More precisely, f_* (resp. f^*) preserves t-local (resp. cofibrant) complexes. For any complex K on $\mathscr{A}_S$, $\mathbf{R}f_*(K) = f_*(K')$ (resp. $\mathbf{L}f^*(K) = f^*(K'')$) where $K' \to K$ (resp. $K \to K''$) is a t-local (resp. cofibrant) resolution of K.[62]

When f is a $\mathscr{P}$-morphism, the functor f^* is even exact and thus preserves quasi-isomorphisms. This implies that $\mathbf{L}f^* = f^*$. The functor $f_\sharp$ admits a total left derived functor

$$\mathbf{L}f_\sharp : \mathrm{D}(\mathscr{A}_T) \rightleftarrows \mathrm{D}(\mathscr{A}_S) : \mathbf{R}f^*$$

defined by the formula $\mathbf{L}f_\sharp(K) = f_\sharp(K'')$ for a complex K on $\mathscr{A}_T$ and a cofibrant resolution $K'' \to K$.

Note also that the tensor product (resp. internal Hom) of $\mathrm{C}(\mathscr{A}_S)$ admits a total left derived functor (resp. total right derived functor). For any complexes K and L on $\mathscr{A}_S$, this derived functors are defined by the formula:

$$K \otimes_S^{\mathbf{L}} L = K'' \otimes_S L''$$
$$\mathbf{R}\mathit{Hom}_S(K, L) = \mathit{Hom}_S(K'', L')$$

where $K \to K''$ and $L \to L''$ are cofibrant resolutions and $L' \to L$ is a t-local resolution.

It is now easy to check that these functors define a triangulated $\mathscr{P}$-premotivic category $\mathrm{D}(\mathscr{A})$, which is τ-generated according to 5.1.13.

Definition 5.1.17. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t.

The triangulated $\mathscr{P}$-premotivic category $\mathrm{D}(\mathscr{A})$ defined above is called the *derived $\mathscr{P}$-premotivic category* associated with $\mathscr{A}$.[63]

The geometric section of a $\mathscr{P}$-scheme X/S in the category $\mathrm{D}(\mathscr{A})$ is the complex concentrated in degree 0 equal to the object $M_S(X)$. The triangulated $\mathscr{P}$-fibred category is τ-generated and well generated in the sense of 1.3.16. Recall this means that $\mathrm{D}(\mathscr{A}_S)$ is equal to the localizing[64] subcategory generated by the family

(5.1.17.1) $$\{M_S(X)\{i\}; X/S\ \mathscr{P}\text{-scheme}, i \in \tau\}.$$

Example 5.1.18. Given any admissible topology t, the abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}, \Lambda)$ introduced in example 5.1.4 is compatible with t (*cf.* 5.1.10) and defines the derived $\mathscr{P}$-premotivic category $\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \Lambda))$.

Remark also that the abelian $\mathscr{P}$-premotivic category $\mathrm{PSh}(\mathscr{P}, \Lambda)$ introduced in example 5.1.1 is compatible with the coarse topology and gives the derived $\mathscr{P}$-premotivic category $\mathrm{D}(\mathrm{PSh}(\mathscr{P}, \Lambda))$.

Remark 5.1.19. Recall from 5.0.27 there exists a canonical dg-structure on $\mathrm{C}(\mathscr{A}_S)$. Then we can define a derived dg-structure by defining for any complexes K and L of $\mathscr{A}_S$, the complex of morphisms:

$$\mathbf{R}\mathrm{Hom}_{\mathscr{A}_S}(K, L) = \mathrm{Hom}^\bullet_{\mathscr{A}_S}(Q(K), R(L))$$

where R and Q are respectively some fibrant and cofibrant (functorial) resolutions for the t-descent model structure. The homotopy category associated with this new dg-structure on $\mathrm{C}(\mathscr{A}_S)$ is the derived category $\mathrm{D}(\mathscr{A}_S)$. Moreover, for any morphism (resp. $\mathscr{P}$-morphism) of schemes f, the pair $(\mathbf{L}f^*, \mathbf{R}f_*)$ (resp. $(\mathbf{L}f_\sharp, f^*)$) is a dg-adjunction. The same is true for the pair of bifunctors $(\otimes_S^{\mathbf{L}}, \mathbf{R}\mathit{Hom}_S)$.

5.1.20. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with a topology t. According to section 3.1.b, the 2-functor $\mathrm{D}(\mathscr{A})$ can be extended to the category of $\mathscr{S}$-diagrams: to any diagram of schemes $\mathscr{X} : I \to \mathscr{S}$ indexed by a small category I, we can associate a symmetric monoidal closed triangulated category $\mathrm{D}(\mathscr{A})(\mathscr{X}, I)$ which coincides with $\mathrm{D}(\mathscr{A})(X)$ when $I = e$, $\mathscr{X} = X$ for a scheme X.

[62] Recall also that fibrant/cofibrant resolutions can be made functorially, because our model categories are cofibrantely generated, so that the left or right derived functors are in fact defined at the level of complexes.

[63] Indeed remark that $\mathrm{D}(\mathscr{A})$ does not depend on the topology t.

[64] *i.e.* triangulated and stable by sums.

Let us be more specific. The fibred category $\mathscr{A}$ admits an extension to $\mathscr{S}$-diagrams: a section of $\mathscr{A}$ over a diagram of schemes $\mathscr{X} : I \to \mathscr{S}$, indexed by a small category I, is the following data:

(1) A family $(A_i)_{i \in I}$ such that A_i is an object of $\mathscr{A}_{X_i}$.
(2) A family $(a_u)_{u \in Fl(I)}$ such that for any arrow $u : i \to j$ in I, $a_u : u^*(A_j) \to A_i$ is a morphism in $\mathscr{A}_{X_i}$ and this family of morphisms satisfies a cocyle condition (see paragraph 3.1.1).

Then, $\mathrm{D}(\mathscr{A})(\mathscr{X}, I)$ is the derived category of the abelian category $\mathscr{A}(\mathscr{X}, I)$. In particular, objects of $\mathrm{D}(\mathscr{A})(\mathscr{X}, I)$ are complexes of sections of $\mathscr{A}$ over $(\mathscr{X}, I)$ (or, what amount to the same thing, families of complexes $(K_i)_{i \in I}$ with transition maps (a_u) as above, relative to the fibred category $\mathrm{C}(\mathscr{A})$).

Recall that a morphism of $\mathscr{S}$-diagrams $\varphi : (\mathscr{X}, I) \to (\mathscr{Y}, J)$ is given by a functor $f : I \to J$ and a natural transformation $\varphi : \mathscr{X} \to \mathscr{Y} \circ f$. We say that φ is a $\mathscr{P}$-morphism if for any $i \in I$, $\varphi_i : \mathscr{X}_i \to \mathscr{Y}_{f(i)}$ is a $\mathscr{P}$-morphism. For any morphism (resp. $\mathscr{P}$-morphism) φ, we have defined in 3.1.3 adjunctions of (abelian) categories:

$$\varphi^* : \mathscr{A}(\mathscr{Y}, J) \rightleftarrows \mathscr{A}(\mathscr{X}, I) : \varphi_*$$
$$\text{resp. } \varphi_\sharp : \mathscr{A}(\mathscr{X}, I) \rightleftarrows \mathscr{A}(\mathscr{Y}, J) : \varphi^*$$

which extends the adjunctions we had on trivial diagrams.

According to Proposition 3.1.11, these respective adjunctions admits left/right derived functors as follows:

$$\mathbf{L}\varphi^* : \mathrm{D}(\mathscr{A})(\mathscr{Y}, J) \rightleftarrows \mathrm{D}(\mathscr{A})(\mathscr{X}, I) : \mathbf{R}\varphi_* \tag{5.1.20.1}$$
$$\text{resp. } \mathbf{L}\varphi_\sharp : \mathrm{D}(\mathscr{A})(\mathscr{X}, I) \rightleftarrows \mathrm{D}(\mathscr{A})(\mathscr{Y}, J) : \mathbf{L}\varphi^* = \varphi^* \tag{5.1.20.2}$$

Again, these adjunctions coincide on trivial diagrams with the map we already had.

Note also that the symmetric closed monoidal structure on $\mathrm{C}(\mathscr{A}(\mathscr{X}, I))$ can be derived and induces a symmetric monoidal structure on $\mathrm{D}(\mathscr{A})(\mathscr{X}, I)$ (see Proposition 3.1.24).[65]

Recall from 3.2.5 and 3.2.7 that, given a topology t' (not necessarily admissible) over $\mathscr{S}$, we say that $\mathrm{D}(\mathscr{A})$ satisfies t'-descent if for any t'-hypercover $p : \mathscr{X} \to X$ (here $\mathscr{X}$ is considered as a $\mathscr{S}$-diagram), the functor

$$\mathbf{L}p^* : \mathrm{D}(\mathscr{A})(X) \to \mathrm{D}(\mathscr{A})(\mathscr{X}) \tag{5.1.20.3}$$

is fully faithful (see Corollary 3.2.7).

Proposition 5.1.21. *Consider the notations and hypothesis introduced above. Let t' be an admissible topology on $\mathscr{S}$. Then the following conditions are equivalent:*

(i) $\mathrm{D}(\mathscr{A})$ *satisfies t'-descent.*
(ii) $\mathscr{A}$ *satisfies cohomological t'-descent.*

Proof. We prove (i) implies (ii). Consider a t'-hypercover $p : \mathscr{X} \to X$ in $\mathscr{P}/S$. This is a $\mathscr{P}$-morphism. Thus, by the fully faithfulness of (5.1.20.3), the counit map $\mathbf{L}p_\sharp p^* \to 1$ is an isomorphism. By applying the latter to the unit object $\mathbb{1}_X$ of $\mathrm{D}(\mathscr{A}_X)$, we thus obtain that

$$M_X(\mathscr{X}) \to \mathbb{1}_X$$

is an isomorphism in $\mathrm{D}(\mathscr{A}_X)$. If $\pi : X \to S$ is the structural $\mathscr{P}$-morphism, by applying the functor $\mathbf{L}\pi_\sharp$ to this isomorphism, we obtain that

$$M_S(\mathscr{X}) \to M_S(X)$$

is an isomorphism in $\mathrm{D}(\mathscr{A}_S)$ and this concludes.

Reciprocally, to prove (i), we can restrict to t'-hypercovers $p : \mathscr{X} \to X$ which are $\mathscr{P}$-morphisms because t' is admissible. Because $\mathbf{R}p^* = p^*$ admits a left adjoint $\mathbf{L}p_\sharp$, we have to prove that the counit

$$\mathbf{L}p_\sharp p^* \to 1$$

[65] In fact, $\mathrm{D}(\mathscr{A})$ is then a monoidal $\mathscr{P}_{cart}$-fibred category over the category of $\mathscr{S}$-diagrams (remark 3.1.21).

is an isomorphism. This is a natural transformation between triangulated functors which commutes with small sums. Thus, according to (5.1.17.1), we have only to check this is an isomorphism when evaluated at a complex of the form $M_X(Y)\{i\}$ for a $\mathscr{P}$-scheme Y/X and a twist $i \in \tau$. But the resulting morphism is then $M_X(\mathscr{X} \times_X Y)\{i\} \to M_X(Y)\{i\}$ and we can conclude because $\mathscr{X} \times_X Y \to Y$ is a t'-hypercover in $\mathscr{P}/S$ (again because t' is admissible). □

5.1.22. . Consider the situation of 5.1.20 Let S be a scheme. An interesting particular case is given for constant $\mathscr{S}$-diagrams over S; for a small category I, we let I_S be the constant $\mathscr{S}$-diagram $I \to \mathscr{S}, i \mapsto S, u \mapsto 1_S$. Then the adjunctions (5.1.20.1) for this kind of diagrams define a *Grothendieck derivator*

$$I \mapsto \mathrm{D}(\mathscr{A})(I_S).$$

Recall that, if $f : I \to e$ is the canonical functor to the terminal category and $\varphi = f_X : I_X \to X$ the corresponding morphism of $\mathscr{S}$-diagrams, for any I-diagram $K_\bullet = (K_i)_{i \in I}$ of complexes over $\mathscr{A}_S$, we get right derived limits and left derived colimits:

$$\mathbf{R}\varphi_*(K_\bullet) = \mathbf{R} \varprojlim_{i \in I} K_i.$$
$$\mathbf{L}\varphi_\sharp(K_\bullet) = \mathbf{L} \varinjlim_{i \in I} K_i.$$

5.1.23. The associated derived $\mathscr{P}$-premotivic category is functorial in the following sense.

Consider an adjunction

$$\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$$

of abelian $\mathscr{P}$-premotivic categories. Let τ (resp. τ') be the set of twists of $\mathscr{A}$ (resp. $\mathscr{B}$), and recall that φ induces a morphisms of monoid $\tau \to \tau'$ still denoted by φ. Consider two topologies t and t' such that t' is finer than t. Suppose $\mathscr{A}$ (resp. $\mathscr{B}$) is compatible with t (resp. t') and let $(\mathcal{G}^{\mathscr{A}}_S, \mathcal{H}^{\mathscr{A}}_S)$ (resp. $(\mathcal{G}^{\mathscr{B}}_S, \mathcal{H}^{\mathscr{B}}_S)$) be the descent structure on $\mathscr{A}_S$ (resp. $\mathscr{B}_S$) defined in 5.1.11.

For any scheme S, consider the evident extensions

$$\varphi_S : \mathrm{C}(\mathscr{A}_S) \rightleftarrows \mathrm{C}(\mathscr{B}_S) : \psi_S$$

of the above adjoint functors to complexes. Recall that for any $\mathscr{P}$-scheme X/S and any twist $i \in \tau$, $\varphi_S(M_S(X, \mathscr{A})\{i\}) = M_S(X, \mathscr{B})\{\varphi(i)\}$ by definition. Thus, φ_S sends $\mathcal{G}^{\mathscr{A}}_S$ to $\mathcal{G}^{\mathscr{A}}_S$. Because t' is finer than t, it sends also $\mathcal{H}^{\mathscr{A}}_S$ to $\mathcal{H}^{\mathscr{B}}_S$. In other words, it satifies descent in the sense of [**CD09**, par. 2.4] so that the pair (φ_S, ψ_S) is a Quillen adjunction with respect to the respective t-descent and t'-descent model structure on $\mathrm{C}(\mathscr{A}_S)$ and $\mathrm{C}(\mathscr{B}_S)$.

Considering the derived functors, it is now easy to check we have obtained a $\mathscr{P}$-premotivic adjunction[66]

$$\mathbf{L}\varphi : \mathrm{D}(\mathscr{A}) \rightleftarrows \mathrm{D}(\mathscr{B}) : \mathbf{R}\psi.$$

EXAMPLE 5.1.24. Let t be an admissible topology. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with t. Then the morphism of abelian $\mathscr{P}$-premotivic categories (5.1.2.1) induces a morphism of triangulated $\mathscr{P}$-premotivic categories:

$$\mathbf{L}\gamma^* : \mathrm{D}(\mathrm{PSh}(\mathscr{P}, \mathbf{Z})) \rightleftarrows \mathrm{D}(\mathscr{A}) : \mathbf{R}\gamma_* \tag{5.1.24.1}$$

Similarly, the morphism (5.1.4.1) induces a morphism of triangulated $\mathscr{P}$-premotivic categories

$$a_t^* : \mathrm{D}(\mathrm{PSh}(\mathscr{P}, \Lambda)) \rightleftarrows \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \Lambda)) : \mathbf{R}a_{t,*}. \tag{5.1.24.2}$$

Note that $a_t^* = \mathbf{L}a_t^*$ on objects, because the functor a_t^* is exact.

[66]Remark also that this adjunction extends on $\mathscr{S}$-diagrams considering the situation described in 5.1.20: for any diagram $\mathcal{X} : I \to \mathscr{S}$, we get an adjunction

$$\mathbf{L}\varphi_{\mathcal{X}} : \mathrm{D}(\mathscr{A})(\mathcal{X}) \rightleftarrows \mathrm{D}(\mathscr{B})(\mathcal{X}) : \mathbf{R}\psi_{\mathcal{X}}$$

and this defines a morphism of triangulated monoidal $\mathscr{P}_{cart}$-fibred categories over the $\mathscr{S}$-diagrams (*cf.* Proposition 3.1.32).

EXAMPLE 5.1.25. Consider an admissible topology t. Let $\varphi : \Lambda \to \Lambda'$ be a morphism of rings. For any scheme S, it induces a pair of adjoint functors:

$$\varphi^* : \mathrm{Sh}_t(\mathscr{P}_S, \Lambda) \leftrightarrows \mathrm{Sh}_t(\mathscr{P}_S, \Lambda') : \varphi_* \tag{5.1.25.1}$$

such that φ^* (resp. φ_*) is induced by the obvious extension (resp. restriction) of scalars functor. By definition, for any $\mathscr{P}$-scheme X/S, the functor φ^* sends the representable sheaf of Λ-modules $\Lambda_S^t(X)$ to the representable sheaf of Λ'-modules $\Lambda_S'^t(X)$. Thus (φ^*, φ_*) defines an adjunction of abelian $\mathscr{P}$-premotivic categories. Applying the results of Paragraph 5.1.23, one deduces a $\mathscr{P}$-premotivic adjunction:

$$\mathbf{L}\varphi^* : \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \Lambda)) \leftrightarrows \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \Lambda')) : \mathbf{R}\varphi_*.$$

The functor φ_* is exact so that $\mathbf{R}\varphi_* = \varphi_*$. Similarly when Λ'/Λ is flat, $\mathbf{L}\varphi^* = \varphi^*$.

The following result can be used to check the compatibility to a given admissible topology:

PROPOSITION 5.1.26. *Let t be an admissible topology. Consider a morphism of abelian $\mathscr{P}$-premotivic categories*

$$\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$$

such that:

(a) For any scheme S, ψ_S is exact.
(b) The morphism φ induces an isomorphism of the underlying set of twists of $\mathscr{A}$ and $\mathscr{B}$.

According to the last property, we identify the set of twists of $\mathscr{A}$ and $\mathscr{B}$ to a monoid τ in such a way that φ acts on τ by the identity.

Assume that $\mathscr{A}$ is compatible with t. Then the following conditions are equivalent:

(i) $\mathscr{B}$ is compatible with t.
(ii) $\mathscr{B}$ satisfies cohomological t-descent,

PROOF. The fact (i) implies (ii) is clear from the definition and we prove the converse using the following lemma :

LEMMA 5.1.27. *Consider a morphism of $\mathscr{P}$-premotivic abelian categories*

$$\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$$

satisfying conditions (a) and (b) of the above proposition and a base scheme S.

Given a simplicial $\mathscr{P}$-scheme $\mathcal{X}$ over S, a twist $i \in \tau$ and a complex C over $\mathscr{B}_S$, we denote by

$$\epsilon_{\mathcal{X},i,C} : \mathrm{Hom}_{\mathrm{C}(\mathscr{B}_S)}\big(M_S(\mathcal{X}, \mathscr{B})\{i\}, C\big) \to \mathrm{Hom}_{\mathrm{C}(\mathscr{A}_S)}\big(M_S(\mathcal{X}, \mathscr{A})\{i\}, \psi_S(C)\big)$$

the adjunction isomorphism obtained for the adjoint pair (φ_S, ψ_S).
Then there exists a unique isomorphism $\epsilon'_{\mathcal{X},i,C}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{C}(\mathscr{B}_S)}\big(M_S(\mathcal{X}, \mathscr{B})\{i\}, C\big) & \xrightarrow{\epsilon_{\mathcal{X},i,C}} & \mathrm{Hom}_{\mathrm{C}(\mathscr{A}_S)}\big(M_S(\mathcal{X}, \mathscr{A})\{i\}, \psi_S(C)\big) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{K}(\mathscr{B}_S)}\big(M_S(\mathcal{X}, \mathscr{B})\{i\}, C\big) & \xrightarrow{\epsilon'_{\mathcal{X},i,C}} & \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}\big(M_S(\mathcal{X}, \mathscr{A})\{i\}, \psi_S(C)\big). \end{array}$$

Assume moreover that $\mathscr{B}$ satisfies cohomological t-descent.
Then there exists an isomorphism $\epsilon''_{\mathcal{X},i,C}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{K}(\mathscr{B}_S)}\big(M_S(\mathcal{X}, \mathscr{B})\{i\}, C\big) & \xrightarrow{\epsilon'_{\mathcal{X},i,C}} & \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}\big(M_S(\mathcal{X}, \mathscr{A})\{i\}, \psi_S(C)\big) \\ {\scriptstyle \pi^{\mathscr{B}}_{\mathcal{X},i,C}}\downarrow & & \downarrow{\scriptstyle \pi^{\mathscr{A}}_{\mathcal{X},i,C}} \\ \mathrm{Hom}_{\mathrm{D}(\mathscr{B}_S)}\big(M_S(\mathcal{X}, \mathscr{B})\{i\}, C\big) & \xrightarrow{\epsilon''_{\mathcal{X},i,C}} & \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}\big(M_S(\mathcal{X}, \mathscr{A})\{i\}, \psi_S(C)\big), \end{array} \tag{5.1.27.1}$$

where $\pi^{\mathscr{A}}_{\mathcal{X},i,C}$ and $\pi^{\mathscr{B}}_{\mathcal{X},i,C}$ are induced by the obvious localization functors.

The existence and unicity of isomorphism $\epsilon'_{\mathcal{X},i,C}$ follows from the fact that the functors φ_S and ψ_S are additive. Indeed, this implies that the isomorphism $\epsilon_{\mathcal{X},i,C}$ is compatible with chain homotopies.

Consider the injective model structure on $\mathrm{C}(\mathscr{A}_S)$ and $\mathrm{C}(\mathscr{B}_S)$ (see for example [**CD09**, 1.2] for the definition). We first treat the case when C is fibrant for this model structure on $\mathrm{C}(\mathscr{B}_S)$. Because the premotive $M_S(\mathcal{X},\mathscr{B})\{i\}$ is cofibrant for the injective model structure, we obtain that the canonical map $\pi^{\mathscr{B}}_{\mathcal{X},i,C}$ is an isomorphism. This implies there exists a unique map $\epsilon''_{\mathcal{X},i,C}$ making diagram (5.1.27.1) commutative. On the other hand, the isomorphism $\epsilon'_{\mathcal{X},i,C}$ obtained previously is obviously functorial in $\mathcal{X}$. Thus, because $\mathscr{B}$ satisfies t-descent, we obtain that $\psi_S(C)$ is t-flasque. Because $\mathscr{A}$ is compatible with t, this implies $\psi_S(C)$ is t-local, and because $M_S(\mathcal{X},\mathscr{B})\{i\}$ is cofibrant for the t-descent model structure on $\mathrm{C}(\mathscr{A}_S)$, this implies $\pi^{\mathscr{B}}_{\mathcal{X},i,C}$ is an isomorphism. Thus finally, $\epsilon''_{\mathcal{X},i,C}$ is an isomorphism as required.

To treat the general case, we consider a fibrant resolution $C \to D$ for the injective model structure on $\mathrm{C}(\mathscr{B}_S)$. Because ψ_S is exact, it preserves isomorphisms. Using the previous case, We define $\epsilon''_{\mathcal{X},i,C}$ by the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{D}(\mathscr{B}_S)}\big(M_S(\mathcal{X},\mathscr{B})\{i\},C\big) & \xrightarrow{\epsilon''_{\mathcal{X},i,C}} & \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}\big(M_S(\mathcal{X},\mathscr{A})\{i\},\psi_S(C)\big) \\
\sim\downarrow & & \downarrow\sim \\
\mathrm{Hom}_{\mathrm{D}(\mathscr{B}_S)}\big(M_S(\mathcal{X},\mathscr{B})\{i\},D\big) & \xrightarrow{\epsilon''_{\mathcal{X},i,D}} & \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}\big(M_S(\mathcal{X},\mathscr{A})\{i\},\psi_S(D)\big).
\end{array}$$

The required property for $\epsilon''_{\mathcal{X},i,C}$ then follows easily and the lemma is proved.

To finish the proof that (ii) implies (i), we note the lemma immediately implies, under (ii), that the following two conditions are equivalent :

- C is t-flasque (resp. local) in $\mathrm{C}(\mathscr{B}_S)$;
- $\psi_S(C)$ is t-flasque (resp. local) in $\mathrm{C}(\mathscr{A}_S)$.

This concludes. □

5.1.c. *Constructible premotivic complexes.*

Definition 5.1.28. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. We will say that t is *bounded in* $\mathscr{A}$ if for any scheme S, there exists an essentially small family $\mathcal{N}^t_S$ of bounded complexes which are direct factors of finite sums of objects of type $M_S(X)\{i\}$ in each degree, such that, for any complex C of $\mathscr{A}_S$, the following conditions are equivalent.

(i) C is t-flasque.
(ii) For any H in $\mathcal{N}^t_S$, the abelian group $\mathrm{Hom}_{K(\mathscr{A}_S)}(H,C)$ vanishes.

In this case, we say the family $\mathcal{N}^t_S$ is a *bounded generating family for t-hypercoverings in $\mathscr{A}_S$.*

Example 5.1.29. (1) Assume $\mathscr{P}$ contains the open immersions so that the Zariski topology is admissible. Let MV_S to be the family of complexes of the form

$$\Lambda_S(U\cap V) \xrightarrow{l_*-k_*} \Lambda_S(U)\oplus\Lambda_S(V) \xrightarrow{i_*+j_*} \Lambda_S(X)$$

for any open cover $X = U \cup V$, where i,j,k,l denotes the obvious open immersions. It follows then from [**BG73**] that MV_S is a bounded generating family of Zariski hypercovers in $\mathrm{Sh}_{\mathrm{Zar}}(\mathscr{P}/S,\Lambda)$.

(2) Assume $\mathscr{P}$ contains the étale morphisms so that the Nisnevich topology is admissible. We let BG_S be the family of complexes of the form

$$\Lambda_S(W) \xrightarrow{g_*-l_*} \Lambda_S(U)\oplus\Lambda_S(V) \xrightarrow{j_*+f_*} \Lambda_S(X)$$

for a Nisnevich distinguished square in $\mathscr{S}$ (*cf.* 2.1.11)

$$\begin{array}{ccc} W & \xrightarrow{l} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X. \end{array}$$

Then, by applying 3.3.2, we see that BG_S is a bounded generating family for Nisnevich hypercovers in $\mathrm{Sh}_{\mathrm{Nis}}(\mathscr{P}/S, \Lambda)$.

(3) Assume that $\mathscr{P} = \mathscr{S}^{ft}$ is the class of morphisms of finite type in $\mathscr{S}$. We let $PCDH_S$ be the family of complexes of the form

$$\Lambda_S(T) \xrightarrow{g_* - k_*} \Lambda_S(Z) \oplus \Lambda_S(Y) \xrightarrow{i_* + f_*} \Lambda_S(X)$$

for a cdh-distinguished square in $\mathscr{S}$ (*cf.* 2.1.11)

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X. \end{array}$$

Then, by virtue of 3.3.8, $CDH_S = BG_S \cup PCDH_S$ is a bounded generating family for cdh-hypercovers in $\mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft}/S, \Lambda\big)$.

(4) The étale topology is not bounded in $\mathrm{Sh}_{\text{ét}}(Sm, \Lambda)$ for an arbitray ring Λ. However, if $\Lambda = \mathbf{Q}$, it is bounded: by virtue of Theorem 3.3.23, a bounded generating family for étale hypercovers in $\mathrm{Sh}_{\text{ét}}(Sm, \mathbf{Q})_S$ is the union of the class BG_S and that of complexes of the form $\mathbf{Q}_S(Y)_G \to \mathbf{Q}_S(X)$ for any Galois cover $Y \to X$ of group G.

(5) As in the case of étale topology, the qfh-topology is not bounded in general, but it is so with rational coefficients. Let $PQFH_S$ be the family of complexes of the form

$$\mathbf{Q}_S(T)_G \xrightarrow{g_* - k_*} \mathbf{Q}_S(Z) \oplus \mathbf{Q}_S(Y)_G \xrightarrow{i_* + f_*} \mathbf{Q}_S(X)$$

for a qfh-distinguished square of group G in $\mathscr{S}$ (*cf.* 3.3.15)

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X. \end{array}$$

Then, by virtue of Theorem 3.3.25, $QFH_S = PQFH_S \cup BG_S$ is a bounded generating family for qfh-hypercovers in $\mathrm{Sh}_{\mathrm{qfh}}\big(\mathscr{S}^{ft}/S, \mathbf{Q}\big)$.

(6) Similarly, by Theorem 3.3.30, $H_S = CDH_S \cup QFH_S$ is a bounded generating family for h-hypercovers in $\mathrm{Sh}_{\mathrm{h}}\big(\mathscr{S}^{ft}/S, \mathbf{Q}\big)$.

PROPOSITION 5.1.30. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. We make the following assumptions:*

(a) t is bounded in $\mathscr{A}$;

(b) for any $\mathscr{P}$-morphism $X \to S$ and any $n \in \tau$, the functor $\mathrm{Hom}_{\mathscr{A}_S}(M_S(X)\{n\}, -)$ preserves filtered colimits.

Then t-local complexes are stable by filtering colimits.

PROOF. Let $\mathcal{N}_S^t$ is a bounded generating family for t-hypercovers in $\mathscr{A}_S$. Then a complex C of $\mathscr{A}_S$ is t-flasque if and only if for any $H \in \mathcal{N}_S^t$, the abelian group $\mathrm{Hom}_{K(\mathscr{A}_S)}(H, C)$ is trivial. Hence it is sufficient to prove that the functor

$$C \mapsto \mathrm{Hom}_{K(\mathscr{A}_S)}(H, C)$$

preserves filtering colimits of complexes. This will follow from the fact that the functor

$$C \mapsto \mathrm{Hom}_{\mathrm{C}(\mathscr{A}_S)}(H, C)$$

preserves filtering colimits. As H a is bounded complex that is degreewise compact, this latter property is obvious. $\square$

5.1.31. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with an admissible topology t, with generating set of twists τ. Assume that t is bounded in $\mathscr{A}$ and consider a bounded generating family $\mathscr{N}_S^t$ for t-hypercovers in $\mathscr{A}_S$.

Let $M(\mathscr{P}/S, \mathscr{A})$ be the full subcategory of $\mathscr{A}_S$ spanned by direct factors of finite sums of premotives of shape $M_S(X)\{i\}$ for a $\mathscr{P}$-scheme X/S and a twist $i \in \tau$. This category is additive and we can associate with it its category of complexes up to chain homotopy. We get an obvious triangulated functor

$$K^b\big(M(\mathscr{P}/S, \mathscr{A})\big) \to \mathrm{D}(\mathscr{A}_S). \tag{5.1.31.1}$$

Then the previous functor induces a triangulated functor

$$K^b\big(M(\mathscr{P}/S, \mathscr{A})\big)/\mathcal{N}_S^t \to \mathrm{D}(\mathscr{A}_S)$$

where the left hand side stands for the Verdier quotient of $K^b\big(M(\mathscr{P}/S, \mathscr{A})\big)$ by the thick subcategory generated by $\mathcal{N}_S^t$.

The category $K^b\big(M(\mathscr{P}/S, \mathscr{A})\big)/\mathcal{N}_S^t$ may not be pseudo-abelian while the aim of the previous functor is. Thus we can consider its pseudo-abelian envelope and the induced functor

$$\Big(K^b\big(M(\mathscr{P}/S, \mathscr{A})\big)/\mathcal{N}_S^t\Big)^\natural \to \mathrm{D}(\mathscr{A}_S). \tag{5.1.31.2}$$

According to Definition 1.4.9, the image of this functor is the subcategory of τ-constructible premotives of the triangulated $\mathscr{P}$-premotivic category $\mathrm{D}(\mathscr{A}_S)$. Then the following proposition is a corollary of [**CD09**, theorem 6.2] :

Proposition 5.1.32. *Consider the hypothesis and notations above.*

If $\mathscr{A}$ is finitely τ-presented then $\mathrm{D}(\mathscr{A})$ is compactly τ-generated. Moreover, the functor (5.1.31.2) *is fully faithful.*

Let us denote by $\mathrm{D}_c(\mathscr{A})$ the subcategory of $\mathrm{D}(\mathscr{A})$ made of τ-constructible premotives in the sense of Definition 1.4.9. Taking into account Porposition 1.4.11, the previous proposition admits the following corollary:

Corollary 5.1.33. *Consider the situation of 5.1.31, and assume that $\mathscr{A}$ is finitely τ-presented. For any premotive $\mathcal{M}$ in $\mathrm{D}(\mathscr{A}_S)$, the following conditions are equivalent:*

(i) $\mathcal{M}$ is compact.
(ii) $\mathcal{M}$ is τ-constructible.

Moreover, the functor (5.1.31.2) *induces an equivalence of categories:*

$$\Big(K^b\big(M(\mathscr{P}/S, \mathscr{A})\big)/\mathcal{N}_S^t\Big)^\natural \to \mathrm{D}_c(\mathscr{A}_S).$$

Example 5.1.34. According to example 5.1.29, we get the following examples:

(1) Let $\Lambda(Sm/S) = M(Sm/S, \mathscr{A})$ for $\mathscr{A} = \mathrm{Sh}_{\mathrm{Nis}}(Sm/S, \Lambda)$. We obtain a fully faithful functor
$$\Big(\mathrm{K}^b\left(\Lambda(Sm/S)\right)/BG_S\Big)^\natural \to \mathrm{D}\left(\mathrm{Sh}_{\mathrm{Nis}}(Sm/S, \Lambda)\right).$$
which is essentially surjective on compact objects.

(2) Let $\Lambda(\mathscr{S}^{ft}/S) = M(Sm/S, \mathscr{A})$ for $\mathscr{A} = \mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft}/S, \Lambda\big)$. We obtain a fully faithful functor
$$\Big(\mathrm{K}^b\left(\Lambda(\mathscr{S}^{ft}/S)\right)/BG_S \cup CDH_S\Big)^\natural \to \mathrm{D}\Big(\mathrm{Sh}_{\mathrm{cdh}}(\mathscr{S}^{ft}/S, \Lambda)\Big).$$
which is essentially surjective on compact objects.

(3) Let $\mathbf{Q}_{\mathrm{\acute{e}t}}(Sm/S) = M(Sm/S, \mathscr{A})$ for $\mathscr{A} = \mathrm{Sh}_{\mathrm{\acute{e}t}}(Sm/S, \mathbf{Q})$. We obtain a fully faithful functor
$$\Big(\mathrm{K}^b\left(\mathbf{Q}_{\mathrm{\acute{e}t}}(Sm/S)\right)/BG_S\Big)^\natural \to \mathrm{D}\left(\mathrm{Sh}_{\mathrm{\acute{e}t}}(Sm/S, \mathbf{Q})\right).$$
which is essentially surjective on compact objects.

5.1.35. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$. We introduce the following property of $\mathscr{A}$:

(C) Consider a projective system $(S_\alpha)_{\alpha \in A}$ of schemes in $\mathscr{S}$ with affine transition maps such that $S = \varprojlim_{\alpha \in A} S_\alpha$ belongs to $\mathscr{S}$. For any index $\alpha_0 \in A$, any object A_{α_0} in $\mathscr{A}_{S_{\alpha_0}}$, and any twist $n \in \tau$, the canonical map

$$\varinjlim_{\alpha \in A/\alpha_0} \mathrm{Hom}_{\mathscr{A}_{S_\alpha}}(\mathbb{1}_{S_\alpha}\{n\}, A_\alpha) \to \mathrm{Hom}_{\mathscr{A}_S}(\mathbb{1}_S\{n\}, A)$$

is an isomorphism where A_α (resp. A) is the pullback of A_{α_0} along the canonical map $S_\alpha \to S_{\alpha_0}$ (resp. $S \to S_{\alpha_0}$).

Proposition 5.1.36. *Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with an admissible topology t and satisfying the assumption (C) above.*

Then the derived premotivic category $\mathrm{D}(\mathscr{A})$ is τ-continuous.

Proof. We use Proposition 4.3.6 applied to the t-descent model structure on $\mathrm{C}(\mathscr{A}_T)$ for $T = S$ or $T = S_\alpha$. (see Paragraph 5.1.13). Recall from Paragraph 5.1.11 that this model structure is associated with a descent structure. Thus according to [**CD09**, 2.3], there exist an explicit generating set I (resp. J) for cofibrations (resp. trivial cofibrations). Moreover, the source or target of any map in $I \cup J$ is a complex C satisfying the following assumption:

(rep) for any integer $i \in \mathbf{Z}$, C^i is a sum of premotives of the form $M_T(X)\{n\}$ where X/T is a $\mathscr{P}$-scheme and $n \in \tau$.

Thus, to check the assumption of 4.3.6 for $\mathrm{C}(\mathscr{A})$, we fix a projective system $(S_\alpha)_{\alpha \in A}$ satisfying the assumptions of property (C) above; we have to prove that for any index $\alpha_0 \in A$ and any complexes C_{α_0} and E_{α_0} such that C_{α_0} satisfies (rep), the natural map:

$$\varinjlim_{\alpha \in A/\alpha_0} \mathrm{Hom}_{\mathrm{C}(\mathscr{A}_{S_\alpha})}(C_\alpha, E_\alpha) \to \mathrm{Hom}_{\mathrm{C}(\mathscr{A}_S)}(C, E)$$

is bijective.

Given the definition of morphisms in a category of complexes, it is sufficient to check this when the Hom groups are computed as morphisms of $\mathbf{Z}$-graded objects. Thus it is sufficient to treat the case where C_{α_0} and E_{α_0} are concentrated in degree 0. Thus, as C_{α_0} satisfies property (rep), we are exactly reduced to assumption (C) on $\mathscr{A}$. □

Example 5.1.37. (1) Assume $\mathscr{P}$ is contained in the class of morphisms of finite type.

Then the abelian $\mathscr{P}$-premotivic category $\mathrm{PSh}(\mathscr{P}, \Lambda)$ of example 5.1.1 satisfies assumption (C). Indeed, property (C) when A is a representable presheaf follows from the assumption on $\mathscr{P}$: $\mathscr{P}$-schemes over some base S always are of finite presentation over S – S is noetherian according to our general assumption 5.0. Then the case of a general presheaf A follows because A is an inductive limit of representable presheaf and the global sections functor commutes with inductive limit of presheaves.

(2) Let $\mathscr{S}^{ft}$ be the class of morphisms of finite type and let t be one of the following topologies: $\mathrm{Nis}, \text{ét}, \mathrm{cdh}, \mathrm{qfh}, \mathrm{h}$.

Then the generalized abelian premotivic category $\mathrm{Sh}_t\left(\mathscr{S}^{ft}, \Lambda\right)$ of example 5.1.4 satisfies assumption (C).

Indeed, according to the preceding example, we have only to prove that for any morphism $f : X \to S$, the functor

$$f^* : \mathrm{PSh}(\mathscr{S}^{ft}_S, \Lambda) \to \mathrm{PSh}(\mathscr{S}^{ft}_T, \Lambda)$$

preserves the property of being a t-sheaf.

If f is a morphism of finite type, the functor f^* admits as a left adjoint the functor $f_\sharp$, which preserves t-covers. Thus the assertion is clear in that case.

In the general case, we use the fact that X/S is a projective limit of a projective system $(X_\alpha)_{\alpha \in A}$ where X_α is an S-scheme affine and of finite type over S. To check that for a t-sheaf F over S, the presheaf $f^*(F)$ is a t-sheaf, we fix a t-cover $(W_i)_{i \in I}$ of X in $\mathscr{S}^{ft}_X$. As X is noetherian, we can assume I is finite. Moreover, there exists an index $\alpha_0 \in A$ such that for the t-cover $(W_i)_{i \in I}$ can be lifted to X_{α_0}. Then, using property (C) of $\mathrm{PSh}(\mathscr{S}^{ft},\Lambda)$ applied to F and (X_α), we reduce to check that $f_\alpha^*(F)$ is a t-sheaf for $\alpha \geq \alpha_0$. This follows from the first case treated.

(3) Let Sm be the class of smooth morphisms and t be one of the topologies: Nis, ét. As we will see in Example 6.1.1, there exists a canonical enlargement of abelian premotivic categories (see (6.1.1.1)):
$$\rho_\sharp : \mathrm{Sh}_t(Sm,\Lambda) \rightleftarrows \mathrm{Sh}_t\big(\mathscr{S}^{ft},\Lambda\big) : \rho^*.$$
As the functor $\rho_\sharp$ is fully faithful and commutes with f^* for any morphism of schemes f, we deduce from the preceding point that the abelian premotivic category $\mathrm{Sh}_t(Sm,\Lambda)$ satisfies the above condition (C).

As an application of the previous proposition, we thus obtain that the derived premotivic category $\mathrm{D}(\mathrm{Sh}_t(Sm,\Lambda))$ is τ-continuous.

5.2. The $\mathbf{A}^1$-derived premotivic category.

5.2.a. *Localization of triangulated premotivic categories.*

5.2.1. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t and $\mathrm{D}(\mathscr{A})$ be the associated derived $\mathscr{P}$-premotivic category.

Suppose given an essentially small family of morphisms $\mathscr{W}$ in $\mathrm{C}(\mathscr{A})$ which is stable by the operations f^*, $f_\sharp$ (in other words, $\mathscr{W}$ is a sub-$\mathscr{P}$-fibred category of $\mathrm{C}(\mathscr{A})$). Remark that the localizing subcategory $\mathcal{T}$ of $\mathrm{D}(\mathscr{A})$ generated by the cones of arrows in $\mathscr{W}$ is again stable by these operations. Moreover, as for any $\mathscr{P}$-morphism $f : X \to S$ we have $f_\sharp f^* = M_S(X) \otimes_S (-)$, the category $\mathcal{T}$ is stable by tensor product with a geometric section.

We will say that a complex K over $\mathscr{A}_S$ is *$\mathscr{W}$-local* if for any object T of $\mathcal{T}$ and any integer $n \in \mathbf{Z}$, $\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(T, K[n]) = 0$. A morphism of complexes $p : C \to D$ over $\mathscr{A}_S$ is a *$\mathscr{W}$-equivalence* if for any $\mathscr{W}$-local complex K over $\mathscr{A}_S$, the induced map
$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(D, K) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(C, K)$$
is bijective.

A morphism of complexes over $\mathscr{A}_S$ is called a *$\mathscr{W}$-fibration* if it is a t-fibration with a $\mathscr{W}$-local kernel. A complex over $\mathscr{A}_S$ will be called *$\mathscr{W}$-fibrant* if it is t-local and $\mathscr{W}$-local.

As consequence of [**CD09**, 4.3, 4.11 and 5.6], we obtain :

Proposition 5.2.2. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t and $\mathscr{W}$ be an essentially small family of morphisms in $\mathrm{C}(\mathscr{A})$ stable by f^* and $f_\sharp$.*

Then the category $\mathrm{C}(\mathscr{A}_S)$ is a proper closed symmetric monoidal category with the $\mathscr{W}$-fibrations as fibrations, the cofibrations as defined in 5.1.11, and the $\mathscr{W}$-equivalences as weak equivalences.

The homotopy category associated with this model category will be denoted by $\mathrm{D}(\mathscr{A}_S)[\mathscr{W}_S^{-1}]$. It can be described as the Verdier quotient $\mathrm{D}(\mathscr{A}_S)/\mathcal{T}_S$.

In fact, the $\mathscr{W}$-local model category on $\mathrm{C}(\mathscr{A}_S)$ is nothing else than the left Bousfield localization of the t-local model category structure. As a consequence, we obtain an adjunction of triangulated categories:

$$\pi_S : \mathrm{D}(\mathscr{A}_S) \rightleftarrows \mathrm{D}(\mathscr{A}_S)[\mathscr{W}_S^{-1}] : \mathcal{O}_S \tag{5.2.2.1}$$

such that $\mathcal{O}_S$ is fully faithful with essential image the $\mathscr{W}$-local complexes. In fact, the model structure gives a functorial $\mathscr{W}$-fibrant resolution $1 \to R_{\mathscr{W}}$
$$R_{\mathscr{W}} : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{A}_S)\,,$$
which induces $\mathcal{O}_S$.
Note that the triangulated category $\mathrm{D}(\mathscr{A}_S)[\mathscr{W}_S^{-1}]$ is generated by the complexes concentrated in degree 0 of the form $M_S(X)\{i\}$ – or, equivalently, the $\mathscr{W}$-local complexes $R_{\mathscr{W}}(M_S(X)\{i\})$ – for a $\mathscr{P}$-scheme X and a twist $i \in \tau$.

Remark 5.2.3. Another very useful property is that $\mathscr{W}$-equivalences are stable by filtering colimits; see [**CD09**, prop. 3.8].

5.2.4. Recall from 5.1.14 that for any morphism (resp. $\mathscr{P}$-morphism) $f : T \to S$, the functor f^* (resp. $f_\sharp$) satisfies descent; as it also preserves $\mathscr{W}$, it follows from [**CD09**, 4.9] that the adjunction

$$f^* : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{A}_T) : f_*$$
$$(\text{resp. } f_\sharp : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{A}_T) : f^*)$$

is a Quillen adjunction with respect to the $\mathscr{W}$-local model structures. This gives the following corollary.

Corollary 5.2.5. *The $\mathscr{P}$-fibred category $\mathrm{C}(\mathscr{A})$ with the $\mathscr{W}$-local model structure on its fibers defined above is a monoidal $\mathscr{P}$-fibred model category, which is moreover stable, proper and combinatorial.*

We will denote by $\mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$ the triangulated $\mathscr{P}$-premotivic category whose fiber over a scheme S is the homotopy category of the $\mathscr{W}_S$-local model category $\mathrm{C}(\mathscr{A}_S)$. The adjunction (5.2.2.1) readily defines an adjuntion of triangulated $\mathscr{P}$-premotivic categories

$$\pi : \mathrm{D}(\mathscr{A}) \rightleftarrows \mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}] : \mathcal{O}. \tag{5.2.5.1}$$

The $\mathscr{P}$-fibred categories $\mathrm{D}(\mathscr{A})$ and $\mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$ are both τ-generated (and this adjunction is compatible with τ-twists in a strong sense).

Remark 5.2.6. For any scheme S, the category $\mathrm{D}(\mathscr{A}_S)[\mathscr{W}_S^{-1}]$ is well generated and has a canonical dg-structure (see also 5.1.19).

5.2.7. With the notations above, let us put $\mathscr{T} = \mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$ to clarify the following notations. As in 5.1.20, the fibred category $\mathscr{T}$ has a canonical extension to $\mathscr{S}$-diagrams $\mathscr{X} : I \to \mathscr{S}$.

If we define $\mathscr{W}_{\mathscr{X}}$ as the class of morphisms $(f_i)_{i\in I}$ in $\mathrm{C}(\mathscr{A}(\mathscr{X}, I))$ such that for any object i, f_i is a $\mathscr{W}$-equivalence, then $\mathscr{T}(X)$ is the triangulated category $\mathrm{D}(\mathscr{A}(\mathscr{X}, I))[\mathscr{W}_{\mathscr{X}}^{-1}]$.

Again, this triangulated category is symmetric monoidal closed and for any morphism (resp. $\mathscr{P}$-morphism) $\varphi : (\mathscr{X}, I) \to (\mathscr{Y}, J)$, we get (derived) adjunctions as in 5.1.20:

$$\mathbf{L}\varphi^* : \mathscr{T}(\mathscr{Y}, J) \rightleftarrows \mathscr{T}(\mathscr{X}, I) : \mathbf{R}\varphi_* \tag{5.2.7.1}$$
$$(\text{resp. } \mathbf{L}\varphi_\sharp : \mathscr{T}(\mathscr{X}, I) \rightleftarrows \mathscr{T}(\mathscr{Y}, J) : \mathbf{L}\varphi^* = \varphi^*) \tag{5.2.7.2}$$

In fact, $\mathscr{T}$ is then a complete monoidal $\mathscr{P}_{cart}$-fibred category over the category of diagrams of schemes and the adjunction (5.2.5.1) extends to an adjunction of complete monoidal $\mathscr{P}_{cart}$-fibred categories.

Example 5.2.8. Suppose we are under the hypothesis of example 5.1.24.2.

Let $\mathscr{W}_{t,S}$ denote the family of maps which are of the form $\Lambda_S(\mathcal{X}) \to \Lambda_S(X)$ for a t-hypercover $\mathcal{X} \to X$ in $\mathscr{P}/S$. Then $\mathscr{W}_t$ is obviously stable by f^* and $f_\sharp$.

Recall now that a complex of t-sheaves on $\mathscr{P}/S$ is local if and only if its t-hypercohomology and its hypercohomology computed in the coarse topology agree (*cf.* 5.1.10).

This readily implies the adjunction considered in example 5.1.24.2

$$a_t^* : \mathrm{D}(\mathrm{PSh}(\mathscr{P}, \Lambda)) \rightleftarrows \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \Lambda)) : \mathbf{R}a_{t,*}.$$

induces an equivalence of triangulated $\mathscr{P}$-premotivic categories

$$\mathrm{D}(\mathrm{PSh}(\mathscr{P}, \Lambda))[\mathscr{W}_t^{-1}] \rightleftarrows \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \Lambda)).$$

Recall $\mathbf{R}a_{t,*}$ is fully faithful and identifies $\mathrm{D}(\mathrm{Sh}_t(S, \Lambda))$ with the full subcategory of $\mathrm{D}(\mathrm{PSh}(S, \Lambda))$ made by t-local complexes.

5.2.9. A triangulated $\mathscr{P}$-premotivic category $(\mathscr{T}, M)$ such that there exists:

(1) an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with an admissible topology t_0 on Sm.
(2) an essentially small family $\mathscr{W}$ of morphisms in $\mathrm{C}(\mathscr{A})$ stable by f^* and $f_\sharp$
(3) an adjunction of triangulated $\mathscr{P}$-premotivic categories $\mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}] \simeq \mathscr{T}$

will be called for short a *derived $\mathscr{P}$-premotivic category.* According to convention 5.0(d) and from the above construction, $\mathscr{T}$ is τ-generated for some set of twists τ. [67]

Let us denote simply by $M_S(X)$ the geometric sections of $\mathscr{T}$. In this case, using the morphisms (5.1.24.1) and (5.2.5.1), we get a canonical morphism of triangulated $\mathscr{P}$-premotivic categories:

$$\varphi^* : \mathrm{D}(\mathrm{PSh}(\mathscr{P}, \mathbf{Z})) \rightleftarrows \mathscr{T} : \varphi_*. \tag{5.2.9.1}$$

By definition, for any premotive $\mathcal{M}$, any scheme X and any integer $n \in \mathbf{Z}$, we get a canonical identification:

$$\mathrm{Hom}_{\mathscr{T}(S)}(M_S(X), \mathcal{M}[n]) = H^n\Gamma(X, \varphi_*(\mathcal{M})). \tag{5.2.9.2}$$

Given any simplicial scheme $\mathcal{X}$, we put $M_S(\mathcal{X}) = \varphi^*\big(\mathbf{Z}_S(\mathcal{X})\big)$, so that we also obtain:

$$\mathrm{Hom}_{\mathscr{T}(S)}(M_S(\mathcal{X}), \mathcal{M}[n]) = H^n\big(\,\mathrm{Tot}^\pi\,\Gamma(\mathcal{X}, \mathbf{R}\gamma_*(\mathcal{M}))\big). \tag{5.2.9.3}$$

PROPOSITION 5.2.10. *Consider the above notations and t an admissible topology. The following conditions are equivalent.*

(i) For any t-hypercover $\mathcal{X} \to X$ in $\mathscr{P}/S$, the induced map $M_S(\mathcal{X}) \to M_S(X)$ is an isomorphism in $\mathscr{T}(S)$.
(i′) For any t-hypercover $p : \mathcal{X} \to X$ in $\mathscr{P}/S$, the induced functor $\mathbf{L}p^* : \mathscr{T}(X) \to \mathscr{T}(\mathcal{X})$ is fully faithful.
(i″) $\mathscr{T}$ satisfies t-descent.
(ii) There exists an essentially unique map $\varphi_t^* : \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z})) \to \mathscr{T}(S)$ making the following diagram essentially commutative:

$$\begin{array}{ccc} \mathrm{D}(\mathrm{PSh}(\mathscr{P}/S, \mathbf{Z})) & \xrightarrow{\varphi^*} & \mathscr{T}(S) \\ {\scriptstyle a_t}\downarrow & \nearrow_{\varphi_t^*} & \\ \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z})) & & \end{array}$$

(ii′) For any complex $C \in \mathrm{C}(\mathrm{PSh}(\mathscr{P}/S, \mathbf{Z}))$ such that $a_t(C) = 0$, $\varphi^*(C) = 0$.
(ii″) For any map $f : C \to D$ in $\mathrm{C}(\mathrm{PSh}(\mathscr{P}/S, \mathbf{Z}))$ such that $a_t(f)$ is an isomorphism, $\varphi^*(f)$ is an isomorphism.
(iii) There exists an essentially unique map $\varphi_{t*} : \mathscr{T}(S) \to \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z}))$ making the following diagram essentially commutative:

$$\begin{array}{ccc} \mathrm{D}(\mathrm{PSh}(\mathscr{P}/S, \mathbf{Z})) & \xleftarrow{\varphi_*} & \mathscr{T}(S) \\ {\scriptstyle \mathbf{R}\mathcal{O}_t}\uparrow & \swarrow_{\varphi_{t*}} & \\ \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z})) & & \end{array}$$

(iii′) For any premotive $\mathcal{M}$ in $\mathscr{T}(S)$, the complex $\varphi_*(\mathcal{M})$ is local.
(iii″) For any premotive $\mathcal{M}$ in $\mathscr{T}(S)$, any $\mathscr{P}$-scheme X/S and any integer $n \in \mathbf{Z}$,

$$\mathrm{Hom}_{\mathscr{T}(S)}(M_S(X), \mathcal{M}[n]) = H^n_t(X, \varphi_*(\mathcal{M})).$$

When these conditions are fulfilled for any scheme S, the functors appearing in (ii) and (iii) induce a morphism of triangulated $\mathscr{P}$-premotivic categories:

$$\varphi_t^* : \mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \mathbf{Z})) \rightleftarrows \mathscr{T} : \varphi_{t*}.$$

PROOF. The equivalence between conditions (i), (i') and (i'') is clear (we proceed as in the proof of 5.1.21). The equivalences $(ii) \Leftrightarrow (ii') \Leftrightarrow (ii'')$ and $(iii) \Leftrightarrow (iii')$ follows from example 5.2.8 and the definition of a localization. The equivalence $(i) \Leftrightarrow (ii'')$ follows again from *loc. cit.* The equivalences $(i) \Leftrightarrow (iii') \Leftrightarrow (iii'')$ follows finally from (5.2.9.2), (5.2.9.3), and the characterisation of a local complex of sheaves (*cf.* 5.1.10). $\square$

[67] We will formulate in some remarks below universal properties of some derived $\mathscr{P}$-premotivic categories. When doing so, we will restrict to morphisms of derived $\mathscr{P}$-premotivic categories which can be written as

$$\mathbf{L}\varphi : \mathrm{D}(\mathscr{A}_1)[\mathscr{W}_1^{-1}] \to \mathrm{D}(\mathscr{A}_2)[\mathscr{W}_2^{-1}]$$

for a morphism $\varphi : \mathscr{A}_1 \to \mathscr{A}_2$ of abelian $\mathscr{P}$-premotivic categories compatible with suitable topologies. More natural universal properties could be obtained if one considers the framework of dg-categories or triangulated derivator.

Remark 5.2.11. The preceding proposition express the fact that the category $\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}, \mathbf{Z}))$ is the universal derived $\mathscr{P}$-premotivic category satisfying t-descent.

5.2.12. We end this section by making explicit two particular cases of the descent property for derived $\mathscr{P}$-premotivic categories.

Consider a derived $\mathscr{P}$-premotivic category $\mathscr{T}$ with geometric sections M. Considering any diagram $\mathcal{X} : I \to \mathscr{P}/S$ of $\mathscr{P}$-schemes over S, with projection $p : \mathcal{X} \to S$, we can associate a premotive in $\mathscr{T}$:

$$M_S(\mathcal{X}) = \mathbf{L}p_\sharp(\mathbb{1}_S) = \mathbf{L}\varinjlim_{i \in I} M_S(\mathcal{X}_i).$$

In particular, when I is the category $\bullet \to \bullet$, we associate to every S-morphism $f : Y \to X$ of $\mathscr{P}$-schemes over S a canonical[68] *bivariant premotive*

$$M_S(X \xrightarrow{f} Y).$$

When f is an immersion, we will also write $M_S(Y/X)$ for this premotive. Note that in any case, there is a canonical distinguished triangle in $\mathscr{T}(S)$:

$$M_S(X) \xrightarrow{f_*} M_S(Y) \xrightarrow{\pi_f} M_S(X \xrightarrow{f} Y) \xrightarrow{\partial_f} M_S(X)[1].$$

This triangle is functorial in the arrow f – with respect to *commutative* squares.

Given a commutative square of $\mathscr{P}$-schemes over S

(5.2.12.1)
$$\begin{array}{ccc} B & \xrightarrow{e'} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{e} & X \end{array}$$

we will say that the image square in $\mathscr{T}(S)$

$$\begin{array}{ccc} M_S(B) & \xrightarrow{e'_*} & M_S(Y) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ M_S(A) & \xrightarrow{e_*} & M_S(X) \end{array}$$

is *homotopy cartesian*[69] if the premotive associated with diagram 5.2.12.1 is zero.

Proposition 5.2.13. *Consider a derived $\mathscr{P}$-premotivic category $\mathscr{T}$. We assume that $\mathscr{P}$ contains the étale morphisms (resp. $\mathscr{P} = \mathscr{S}^{ft}$). Then, with the above definitions, the following conditions are equivalent:*

(i) $\mathscr{T}$ satisfies Nisnevich (resp. proper cdh*) descent.*

(ii) For any scheme S and any Nisnevich (resp. proper cdh*) distinguished square Q of S-schemes, the square $M_S(Q)$ is homotopy cartesian in $\mathscr{T}(S)$.*

(iii) For any Nisnevich (resp. proper cdh*) distinguished square of shape* (5.2.12.1)*, the canonical map $M_S(Y/B) \xrightarrow{(f/g)_*} M_S(X/A)$ is an isomorphism.*

Moreover, under these conditions, to any Nisnevich (resp. proper cdh*) distinguished square Q of shape* (5.2.12.1)*, we associate a map*

$$\partial_Q : M_S(X) \xrightarrow{\pi_e} M_S(X/A) \xrightarrow{(f/g)_*^{-1}} M_S(Y/B) \xrightarrow{\partial_{e'}} M_S(Y)[1]$$

which defines a distinguished triangle in $\mathscr{T}(S)$:

$$M_S(B) \xrightarrow{\binom{e'_*}{-g_*}} M_Z(Y) \oplus M_S(A) \xrightarrow{(f_*, e_*)} M_S(X) \xrightarrow{\partial_Q} M_S(Y)[1].$$

[68]In fact, if $\mathscr{T} = \mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$ for an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$, then we can define $M_S(X \to Y)$ as the cone of the morphism of complexes (concentrated in degree 0) $M_S(X) \xrightarrow{f_*} M_S(Y)$.

[69]If $\mathscr{T} = \mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$, this amount to say that the diagram obtained of complexes by applying the functor $M_S(-)$ is homotopy cartesian in the $\mathscr{W}$-local model category $\mathrm{C}(\mathscr{A})$.

Proof. The equivalence of (i) and (ii) follows from the theorem of Morel-Voevodsky 3.3.2 (resp. the theorem of Voevodsky 3.3.8). To prove the equivalence of (ii) and (iii), we assume $\mathscr{T} = \mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$. Then, the homotopy colimit of a square of shape 5.2.12.1 is given by the complex

$$\mathrm{Cone}\big(\mathrm{Cone}(M_S(B) \to M_S(Y)) \to \mathrm{Cone}(M_S(A) \to M_S(X))\big).$$

This readily proves the needed equivalence, together with the remaining assertion. □

Remark 5.2.14. In the first of the respective cases of the proposition, condition (ii) is what we usually called the *Brown-Gersten* property (BG) for $\mathscr{T}$, whereas condition (iii) can be called the *excision property*. In the second respective case, condition (ii) will be called the *proper* cdh property for the *generalized premotivic category* $\mathscr{T}$. We say also that $\mathscr{T}$ satisfies the (cdh) property if it satisfies condition (ii) with respect to any cdh distinguished square Q.

5.2.b. *The homotopy relation.*

5.2.15. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t.

We consider $\mathscr{W}_{\mathbf{A}^1}$ to be the family of morphisms $M_S(\mathbf{A}^1_X)\{i\} \to M_S(X)\{i\}$ for a $\mathscr{P}$-scheme X/S and an twist i in τ. The family $\mathscr{W}_{\mathbf{A}^1}$ is obviously stable by f^* and $f_\sharp$.

Definition 5.2.16. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. With the notation above, we define $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}) = \mathrm{D}(\mathscr{A})[\mathscr{W}_{\mathbf{A}^1}^{-1}]$ and refer to it as the (effective) *$\mathscr{P}$-premotivic $\mathbf{A}^1$-derived category* with coefficients in $\mathscr{A}$.

By definition, the category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ satisfies the homotopy property (Htp) (see 2.1.3). According to the general facts about localization of derived premotivic categories, the triangulated premotivic category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ is τ-generated.

Example 5.2.17. We can divide our examples into two types:
1) Assume $\mathscr{P} = \mathit{Sm}$:

Consider the admissible topology $t = \mathrm{Nis}$. Following F. Morel, we define the *(effective)* $\mathbf{A}^1$-*derived category over* S to be $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\mathrm{Nis}}(\mathit{Sm}/S, \Lambda))$. Indeed we get a triangulated premotivic category (see also the construction of [**Ayo07b**]):

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda} := \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{Nis}}(\mathit{Sm}, \Lambda)\right). \tag{5.2.17.1}$$

We shall also write its fibres

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(S, \Lambda) := \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda}(S) = \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{Nis}}(\mathit{Sm}/S, \Lambda)\right) \tag{5.2.17.2}$$

for a scheme S. For $\Lambda = \mathbf{Z}$, we shall often write simply

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1} := \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{Nis}}(\mathit{Sm}, \mathbf{Z})\right). \tag{5.2.17.3}$$

Another interesting case is when $t = \text{ét}$; we get a triangulated premotivic category of *effective étale premotives*:

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\text{ét}}(\mathit{Sm}, \Lambda)\right).$$

In each of these cases, we denote by $\Lambda^t_S(X)$ the premotive associated with a smooth S-scheme X.
2) Assume $\mathscr{P} = \mathscr{S}^{ft}$:

Consider the admissible topology $t = \mathrm{h}$ (resp. $t = \mathrm{qfh}$). In [**Voe96**], Voevodsky has introduced the category of h-motives (resp. qfh-motives). In our formalism, one defines the category of *effective* h-*motives* (resp. *effective* h-*motives*) over S with coefficients in Λ as:

$$\underline{DM}^{\mathit{eff}}_{\mathrm{h}}(S, \Lambda) = \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\big(\mathrm{Sh}_{\mathrm{h}}\big(\mathscr{S}^{ft}/S, \Lambda\big)\big),$$
$$\text{resp. } \underline{DM}^{\mathit{eff}}_{\mathrm{qfh}}(S, \Lambda) = \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\big(\mathrm{Sh}_{\mathrm{qfh}}\big(\mathscr{S}^{ft}/S, \Lambda\big)\big).$$

In other words, this is the $\mathbf{A}^1$-derived category of h-sheaves (resp. qfh-sheaves) of Λ-modules. Moreover, these categories for various schemes S are the fibers of a generalized premotivic triangulated category. What we have added to the construction of Voevodsky is the functors of the generalized premotivic structure.

We will denote simply by $\underline{\Lambda}_S^t(X)$ the corresponding premotive associated with X in $\underline{DM}_t^{eff}(S,\Lambda)$.

Another interesting case is obtained when $t = \mathrm{cdh}$. We get an $\mathbf{A}^1$-derived generalized premotivic category $\mathrm{D}_{\mathbf{A}^1}^{eff}\big(\mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft},\Lambda\big)\big)$ whose premotives are simply denoted by $\underline{\Lambda}_S^{\mathrm{cdh}}(X)$ for any finite type S-scheme X.

5.2.18. Let C be a complex with coefficients in $\mathscr{A}_S$. According to the general case, we say that C is $\mathbf{A}^1$-*local* if for any $\mathscr{P}$-scheme X/S and any $(i,n) \in \tau \times \mathbf{Z}$, the map induced by the canonical projection

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(M_S(X)\{i\}[n],C) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(M_S(\mathbf{A}_X^1)\{i\}[n],C)$$

is an isomorphism. The adjunction (5.2.2.1) defines a morphism of triangulated $\mathscr{P}$-premotivic categories

$$\mathrm{D}(\mathscr{A}) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}^{eff}(\mathscr{A})$$

such that for any scheme S, $\mathrm{D}_{\mathbf{A}^1}^{eff}(\mathscr{A}_S)$ is identified with the full subcategory of $\mathrm{D}(\mathscr{A}_S)$ made of $\mathbf{A}^1$-local complexes.

Fibrant objects for the model category structure on $\mathrm{C}(\mathscr{A}_S)$ appearing in Proposition 5.2.2 relatively to $\mathscr{W}_{\mathbf{A}^1}$, simply called $\mathbf{A}^1$-*fibrant* objects, are the t-flasque and $\mathbf{A}^1$-local complexes.

We say a morphism $f : C \to D$ of complexes of $\mathscr{A}_S$ is an $\mathbf{A}^1$-*equivalence* if it becomes an isomorphism in $\mathrm{D}_{\mathbf{A}^1}^{eff}(\mathscr{A}_S)$. Considering moreover two morphisms $f,g : C \to D$ of complexes of $\mathscr{A}_S$, we say they are $\mathbf{A}^1$-*homotopic* if there exists a morphism of complexes

$$H : M_S(\mathbf{A}_S^1) \otimes_S C \to D$$

such that $H \circ (s_0 \otimes 1_C) = f$ and $H \circ (s_1 \otimes 1_C) = g$, where s_0 and s_1 are respectively induced by the zero and the unit section of $\mathbf{A}_S^1/S$. When f and g are $\mathbf{A}^1$-homotopic, they are equal as morphisms of $\mathrm{D}_{\mathbf{A}^1}^{eff}(\mathscr{A}_S)$. We say the morphism $p : C \to D$ is a *strong* $\mathbf{A}^1$-*equivalence* if there exists a morphism $q : D \to C$ such that the morphisms $p \circ q$ and $q \circ p$ are $\mathbf{A}^1$-homotopic to the identity. A complex C is $\mathbf{A}^1$-*contractible* if the map $C \to 0$ is a strong $\mathbf{A}^1$-equivalence.

As an example, for any integer $n \in \mathbf{N}$, and any $\mathscr{P}$-scheme X/S, the map

$$p_* : M_S(\mathbf{A}_X^n) \to M_S(X)$$

induced by the canonical projection is a strong $\mathbf{A}^1$-equivalence with inverse the zero section $s_{0,*} : M_S(X) \to M_S(\mathbf{A}_X^n)$.

5.2.19. The category $\mathrm{D}_{\mathbf{A}^1}^{eff}(\mathscr{A})$ is functorial in $\mathscr{A}$.

Let $\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$ be an adjunction of abelian $\mathscr{P}$-premotivic categories. Consider two topologies t and t' such that t' is finer than t. Suppose $\mathscr{A}$ (resp. $\mathscr{B}$) is compatible with t (resp. t').

For any scheme S, consider the evident extensions $\varphi_S : \mathrm{C}(\mathscr{A}_S) \rightleftarrows \mathrm{C}(\mathscr{B}_S) : \psi_S$ of the above adjoint functors to complexes. We easily check that the functor ψ_S preserves $\mathbf{A}^1$-local complexes. Thus, applying 5.1.23, the pair (φ_S, ψ_S) is a Quillen adjunction for the respective $\mathbf{A}^1$-localized model structure on $\mathrm{C}(\mathscr{A}_S)$ and $\mathrm{C}(\mathscr{B}_S)$; see [**CD09**, 3.11]. Considering the derived functors, it is now easy to check we have obtained an adjunction

$$\mathbf{L}\varphi : \mathrm{D}_{\mathbf{A}^1}^{eff}(\mathscr{A}) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}^{eff}(\mathscr{B}) : \mathbf{R}\psi$$

of triangulated $\mathscr{P}$-premotivic categories.

Example 5.2.20. Consider the notations of 5.2.17. In the case where $\mathscr{P} = Sm$, we get from the adjunction of (5.1.24.2) the following adjunction of triangulated premotivic categories

$$a_{\text{ét}}^* : \mathrm{D}_{\mathbf{A}^1,\Lambda}^{eff} \rightleftarrows \mathrm{D}_{\mathbf{A}^1}^{eff}\left(\mathrm{Sh}_{\text{ét}}(Sm,\Lambda)\right) : \mathbf{R}a_{\text{ét},*}.$$

Example 5.2.21. Let $\mathscr{T}$ be a derived $\mathscr{P}$-premotivic category as in 5.2.9. If $\mathscr{T}$ satisfies the property (Htp), then the canonical morphism (5.2.9.1) induces a morphism

$$\mathrm{D}_{\mathbf{A}^1}^{eff}(\mathrm{PSh}(\mathscr{P},\mathbf{Z})) \rightleftarrows \mathscr{T}.$$

If moreover $\mathscr{T}$ satisfies t-descent for an admissible topology t, we further obtain as in 5.2.10 a morphism

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P},\mathbf{Z})) \rightleftarrows \mathscr{T}.$$

Particularly interesting cases are given by $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}$ (resp. $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\big(\mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft},\mathbf{Z}\big)\big)$) which is the universal derived premotivic category (resp. generalized premotivic category), *i.e.* initial premotivic category satisfying Nisnevich descent (resp. cdh descent) and the homotopy property.

5.2.22. As in Example 5.1.25, let t be an admissible topology and $\varphi : \Lambda \to \Lambda'$ be an extension of rings. Then, from the $\mathscr{P}$-premotivic adjunction (5.1.25.1) and according to Paragraph 5.2.19, we get an adjunction of triangulated $\mathscr{P}$-premotivic categories:

$$\mathbf{L}\varphi_* : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\big(\,\mathrm{Sh}_t(\mathscr{P},\Lambda)\,\big) \leftrightarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\big(\,\mathrm{Sh}_t(\mathscr{P},\Lambda')\,\big) : \mathbf{R}\varphi_*.$$

Consider also complexes C and D of t-sheaves of Λ-modules over $\mathscr{P}_S$. Then there exists a canonical morphism of Λ'-modules:

$$\mathrm{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}_S,\Lambda))}\big(C,D\big)\otimes_\Lambda \Lambda' \longrightarrow \mathrm{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}_S,\Lambda'))}\big(\mathbf{L}\varphi^*(C),\mathbf{L}\varphi^*(D)\big) \tag{5.2.22.1}$$

There are two notable cases where this map is an isomorphism:

PROPOSITION 5.2.23. *Consider the above assumptions. Then the map* (5.2.22.1) *is an isomorphism in the two following cases:*

(1) If Λ' is a free Λ-module and C is compact;
(2) If Λ' is a free Λ-module of finite rank.

PROOF. Note that in any case, the functor φ_* admits a right adjoint $\varphi^!$.[70]
We can assume that $\Lambda' = I.\Lambda$ for a set I. In this case, we get for any sheaf F of Λ-modules:

$$\varphi_*\varphi^*(F) = F \otimes_\Lambda \Lambda' = I.F.$$

Moreover, for any $\mathscr{P}$-scheme X/S, we get:

$$\varphi_*(\Lambda'^t_S(X)) = \Lambda'^t_S(X) = I.\Lambda^t_S(X).$$

In particular, the functor $\varphi_* : \mathrm{C}(\mathrm{Sh}_t(\mathscr{P}_S,\Lambda')) \to \mathrm{C}(\mathrm{Sh}_t(\mathscr{P}_S,\Lambda))$ satisfies descent in the sense of [**CD09**, 2.4] and preserves the family $\mathscr{W}_{\mathbf{A}^1}$. Thus it is a left Quillen functor with respect to the $\mathbf{A}^1$-local model structures. In particular, because it is also a right Quillen functor, we get: $\mathbf{R}\varphi_* = \varphi_* = \mathbf{L}\varphi_*$. In particular, we get in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}_S,\Lambda))$:

$$\mathbf{R}\varphi_*\mathbf{L}\varphi^*(D) = \mathbf{L}\varphi_*\mathbf{L}\varphi^*(D) = \mathbf{L}(\varphi_*\varphi^*)(D) = I.D.$$

Thus the Proposition follows as the functor $\mathrm{Hom}(C,-)$ commutes with direct sums if C is compact and with finite direct sums in any case. □

We remark the following useful property.

PROPOSITION 5.2.24. *Consider a morphism*

$$\varphi^* : \mathscr{A} \rightleftarrows \mathscr{B} : \varphi_*$$

of abelian $\mathscr{P}$-premotivic categories such that $\mathscr{A}$ (resp. $\mathscr{B}$) is compatible with an admissible topology t (resp. t'). Assume t' is finer than t.

Let S be a base scheme. Assume that $\varphi_* : \mathscr{A}_S \to \mathscr{B}_S$ commutes with colimits[71]*. Then $\varphi_* : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{B}_S)$ respects $\mathbf{A}^1$-equivalences.*

In other words, the right derived functor $\mathbf{R}\varphi_* : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{B}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$ satisfies the relation $\mathbf{R}\varphi_* = \varphi_*$.

[70]It is defined by the formula:

$$\varphi^!(F) = \mathit{Hom}_\Lambda(\Lambda', F)$$

equipped with its canonical structure of sheaf of Λ'-modules.

[71]This amounts to ask that φ_* is exact and commutes with direct sums.

PROOF. In this proof, we write φ_* for $\varphi_{*,S}$. We first prove that φ_* preserves strong $\mathbf{A}^1$-equivalences (see 5.2.18).

Consider two maps $u, v : K \to L$ in $\mathrm{C}(\mathscr{B}_S)$. To give an $\mathbf{A}^1$-homotopy $H : M_S(\mathbf{A}^1_S, \mathscr{B}) \otimes_S K \to L$ between u and v is equivalent by adjunction to give a map $H' : K \to \mathit{Hom}_{\mathscr{B}_S}(M_S(\mathbf{A}^1_S, \mathscr{B}), L)$ which fits into the following commutative diagram:

$$\begin{array}{ccccc} & & K & & \\ & {}^{u}\swarrow & \downarrow{\scriptstyle H'} & \searrow^{v} & \\ L & \xleftarrow[s_0^*]{} & \mathit{Hom}_{\mathscr{B}_S}(M_S(\mathbf{A}^1_S, \mathscr{B}), L) & \xrightarrow[s_1^*]{} & L \end{array}$$

where s_0 and s_1 are the respective zero and unit section of $\mathbf{A}^1_S/S$.

Because $M_S(\mathbf{A}^1_S, \mathscr{B}) = \varphi_S^*(M_S(\mathbf{A}^1_S, \mathscr{A}))$, we get a canonical isomorphism (see paragraph 1.2.9)

$$\varphi_*(\mathit{Hom}_{\mathscr{B}_S}(M_S(\mathbf{A}^1_S, \mathscr{B}), L)) \simeq \mathit{Hom}_{\mathscr{B}_S}(M_S(\mathbf{A}^1_S, \mathscr{A}), \varphi_*(L)).$$

Thus, applying φ_* to the previous commutative diagram and using this identification, we obtain that $\varphi_*(u)$ is $\mathbf{A}^1$-homotopic to $\varphi_*(v)$.

As a consequence, for any $\mathscr{P}$-scheme X over S, and any $\mathscr{B}$-twist i, the map

$$\varphi_*(M_S(\mathbf{A}^1_X, \mathscr{B})\{i\}) \to \varphi_*(M_S(X, \mathscr{B})\{i\})$$

induced by the canonical projection is a strong $\mathbf{A}^1$-equivalence, thus an $\mathbf{A}^1$-equivalence.

The functor $\varphi_* : \mathscr{B}_S \to \mathscr{A}_S$ commutes with colimits. Thus it admits a right adjoint that we will denote by $\varphi^!$. Consider the injective model structure on $\mathrm{C}(A_S)$ and $\mathrm{C}(\mathscr{B}_S)$ (see [**CD09**, 2.1]). Because φ_* is exact, it is a left Quillen functor for these model structures. Thus, the right derived functor $\mathbf{R}\varphi^!$ is well defined. From the result we just get, we see that $\mathbf{R}\varphi^!$ preserves $\mathbf{A}^1$-local objects, and this readily implies $\mathbf{L}\varphi_* = \varphi_*$ preserves $\mathbf{A}^1$-equivalences. $\square$

5.2.25. To relate the category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(S)$ with the homotopy category of schemes of Morel and Voevodsky [**MV99**], we have to consider the category of simplicial Nisnevich sheaves of sets denoted by $\Delta^{op}\,\mathrm{Sh}(\mathit{Sm}/S)$. Considering the free abelian sheaf functor, we obtain an adjunction of categories

$$\Delta^{op}\,\mathrm{Sh}(\mathit{Sm}/S) \rightleftarrows \mathrm{C}(\mathrm{Sh}(\mathit{Sm}/S, \mathbf{Z})).$$

If we consider Blander's projective $\mathbf{A}^1$-model structure [**Bla03**] on the category $\Delta^{op}\,\mathrm{Sh}(\mathit{Sm}/S)$, we can easily see that this is a Quillen pair, so that we obtain a $\mathscr{P}$-premotivic adjunction of simple $\mathscr{P}$-premotivic categories

$$N : \mathscr{H} \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1} : K.$$

Note that the functor N sends cofiber sequences in $\mathscr{H}(S)$ to distinguished triangles in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(S)$.

5.2.c. *Explicit* $\mathbf{A}^1$*-resolution.*

5.2.26. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with an admissible topology t.

Consider the canonically split exact sequence

$$0 \to \mathbb{1}_S \xrightarrow{s_0} M_S(\mathbf{A}^1_S) \to U \to 0$$

where the map $s_0 : \mathbb{1}_S \to M_S(\mathbf{A}^1_S)$ is induced by the zero section of $\mathbf{A}^1$. The section corresponding to 1 in $\mathbf{A}^1$ defines another map

$$s_1 : \mathbb{1}_S \to M_S(\mathbf{A}^1_S)$$

which does not factor through s_0, so that we get canonically a non trivial map $u : \mathbb{1}_S \to U$. This defines for any complex C of $\mathscr{A}_S$ a map, called the *evaluation at* 1,

$$\mathit{Hom}(U, C) = \mathbb{1}_S \otimes_S \mathit{Hom}(U, C) \xrightarrow{u \otimes 1} U \otimes_S \mathit{Hom}(U, C) \xrightarrow{ev} C.$$

We define the complex $R^{(1)}_{\mathbf{A}^1}(C)$ to be

$$R^{(1)}_{\mathbf{A}^1}(C) = \mathrm{Cone}\big(\mathit{Hom}(U, C) \to C\big).$$

We have by construction a map

$$r_C : C \to R^{(1)}_{\mathbf{A}^1}(C).$$

This defines a morphism of functors from the identity functor to $R^{(1)}_{\mathbf{A}^1}$. For an integer $n \geq 1$, we define by induction a complex

$$R^{(n+1)}_{\mathbf{A}^1}(C) = R^{(1)}_{\mathbf{A}^1}(R^{(n)}_{\mathbf{A}^1}(C)),$$

and a map

$$r_{R^{(n)}_{\mathbf{A}^1}(C)} : R^{(n)}_{\mathbf{A}^1}(C) \to R^{(n+1)}_{\mathbf{A}^1}.$$

We finaly define a complex $R_{\mathbf{A}^1}(C)$ by the formula

$$R_{\mathbf{A}^1}(C) = \varinjlim_n R^{(n)}_{\mathbf{A}^1}(C).$$

We have a functorial map

$$C \to R_{\mathbf{A}^1}(C).$$

Lemma 5.2.27. *With the above hypothesis and notations, the map $C \to R_{\mathbf{A}^1}(C)$ is an $\mathbf{A}^1$-equivalence.*

Proof. For any closed symmetric monoidal category $\mathscr{C}$ and any objects A, B, C and I in $\mathscr{C}$, we have

$$\begin{aligned}\mathrm{Hom}(I \otimes \mathit{Hom}(B,C), \mathit{Hom}(A,C)) &= \mathrm{Hom}(\mathit{Hom}(B,C), \mathit{Hom}(I, \mathit{Hom}(A,C))) \\ &= \mathrm{Hom}(\mathit{Hom}(B,C), \mathit{Hom}(I \otimes A, C)).\end{aligned}$$

Hence any map $I \otimes A \to B$ induces a map $I \otimes \mathit{Hom}(B,C) \to \mathit{Hom}(A,C)$ for any object C. If we apply this to $\mathscr{C} = \mathrm{C}(\mathscr{A}_S)$ and $I = M_S(\mathbf{A}^1)$, we see immediately that the functor $\mathit{Hom}(-,C)$ preserves strong $\mathbf{A}^1$-homotopy equivalences. In particular, for any complex C, the map $C \to \mathit{Hom}(M_S(\mathbf{A}^1_X), C)$ is a strong $\mathbf{A}^1$-homotopy equivalence. This implies that $\mathit{Hom}(U,C) \to 0$ is an $\mathbf{A}^1$-equivalence, so that the map r_C is an $\mathbf{A}^1$-equivalence as well. As $\mathbf{A}^1$-equivalences are stable by filtering colimits, this implies our result. □

Proposition 5.2.28. *Consider the above notations and hypothesis, and assume that t is bounded in $\mathscr{A}$.*

For any t-flasque complex C of $\mathscr{A}_S$, the complex $R_{\mathbf{A}^1}(C)$ is t-flasque and $\mathbf{A}^1$-local. Moreover, the morphism $C \to R_{\mathbf{A}^1}(C)$ is an $\mathbf{A}^1$-equivalence. If furthermore C is t-flasque, so is $R_{\mathbf{A}^1}(C)$.

Proof. The last assertion is a particular case of Lemma 5.2.27. The functor $R^{(1)}_{\mathbf{A}^1}$ preserves t-flasque complexes. By virtue of 5.1.30, the functor $R_{\mathbf{A}^1}$ has the same gentle property. It thus remains to prove that the functor $R_{\mathbf{A}^1}$ sends t-flasque complexes on $\mathbf{A}^1$-local ones. We shall use that the derived category $\mathrm{D}(\mathscr{A}_S)$ is compactly generated; see 5.1.30.

Let C be a t-flasque complex of $\mathscr{A}_S$. To prove $R_{\mathbf{A}^1}(C)$ is $\mathbf{A}^1$-local, we are reduced to prove that the map

$$R_{\mathbf{A}^1}(C) \to \mathit{Hom}(M_S(\mathbf{A}^1_X), R_{\mathbf{A}^1}(C))$$

is a quasi-isomorphism, or, equivalently, that the complex $\mathit{Hom}(U, R_{\mathbf{A}^1}(C))$ is acyclic. As U is a direct factor of $M_S(\mathbf{A}^1_X, \mathscr{A})$, for any $\mathscr{P}$-scheme X over S and any i in I, the object $\mathbf{Z}_S(X;\mathscr{A})\{i\} \otimes_S U$ is compact. This implies that the canonical map

$$\varinjlim_n \mathit{Hom}(U, R^{(n)}_{\mathbf{A}^1}(C)) \to \mathit{Hom}(U, R_{\mathbf{A}^1}(C))$$

is an isomorphism of complexes. As filtering colimits preserve quasi-isomorphisms, the complex $\mathit{Hom}(U, R_{\mathbf{A}^1}(C))$ (resp. $R_{\mathbf{A}^1}(C)$) can be considered as the homotopy colimit of the complexes $\mathit{Hom}(U, R^{(n)}_{\mathbf{A}^1}(C))$ (resp. $R^{(n)}_{\mathbf{A}^1}(C)$). In particular, for any compact object K of $\mathrm{D}(\mathscr{A}_S)$, the canonical morphisms

$$\varinjlim_n \mathrm{Hom}(K, \mathit{Hom}(U, R^{(n)}_{\mathbf{A}^1}(C))) \to \mathrm{Hom}(K, \mathit{Hom}(U, R_{\mathbf{A}^1}(C)))$$

$$\varinjlim_n \mathrm{Hom}(K, R^{(n)}_{\mathbf{A}^1}(C)) \to \mathrm{Hom}(K, R_{\mathbf{A}^1}(C))$$

are bijective.

By construction, we have distinguished triangles

$$Hom(U, R^{(n)}_{\mathbf{A}^1}(C)) \to R^{(n)}_{\mathbf{A}^1}(C) \to R^{(n+1)}_{\mathbf{A}^1}(C) \to Hom(U, R^{(n)}_{\mathbf{A}^1}(C))[1].$$

This implies that the evaluation at 1 morphism

$$ev_1 : Hom(U, R_{\mathbf{A}^1}(C)) \to R_{\mathbf{A}^1}(C)$$

induces the zero map

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(K, Hom(U, R_{\mathbf{A}^1}(C))) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(K, R_{\mathbf{A}^1}(C))$$

for any compact object K of $\mathrm{D}(\mathscr{A}_S)$. Hence the induced map

$$a = Hom(U, ev_1) : Hom(U, Hom(U, R_{\mathbf{A}^1}(C))) \to Hom(U, R_{\mathbf{A}^1}(C))$$

has the same property: for any compact object K, the map

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(K, Hom(U, Hom(U, R_{\mathbf{A}^1}(C)))) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(K, Hom(U, R_{\mathbf{A}^1}(C)))$$

is zero.

The multiplication map $\mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ induces a map

$$\mu : U \otimes_S U \to U$$

such that the composition of

$$\mu^* : Hom(U, R_{\mathbf{A}^1}(C)) \to Hom(U \otimes_S U, R_{\mathbf{A}^1}(C)) = Hom(U, Hom(U, R_{\mathbf{A}^1}(C)))$$

with a is the identity of $Hom(U, R_{\mathbf{A}^1}(C))$. As $\mathrm{D}(\mathscr{A}_S)$ is compactly generated, this implies that $Hom(U, R_{\mathbf{A}^1}(C)) = 0$ in the derived category $\mathrm{D}(\mathscr{A}_S)$. □

Remark 5.2.29. Consider a t-flasque resolution functor (*i.e.* a fibrant resolution for the t-local model structure) $R_t : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathscr{A}_S)$, $1 \to R_t$. As a corollary of the proposition, the composite functor $R_{\mathbf{A}^1} \circ R_t$ is a resolution functor by t-local and $\mathbf{A}^1$-local complexes.

Example 5.2.30. Consider an admissible topology t and the $\mathscr{P}$-premotivic $\mathbf{A}^1$-derived category $D = \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}, \Lambda))$. Suppose that t is bounded for abelian t-sheaves (for example, this is the case for the Zariski and the Nisnevich topologies, see 5.1.29).

Let C be a complex of abelian t-sheaves on $\mathscr{P}/S$. If C is $\mathbf{A}^1$-local, then

$$\mathrm{Hom}_{D(S)}(\Lambda^t_S(X), C) = \mathrm{H}^n_t(X; C)$$

(this is true without any condition on t).

Consider a t-local resolution C_t of C in $\mathrm{C}\big(\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)\big)$. Then we get the following formula:

$$\mathrm{Hom}_{D(S)}\big(\Lambda^t_S(X), C[n]\big) = \mathrm{H}^n\big(\Gamma\big(X, R_{\mathbf{A}^1}(C_t)\big)\big).$$

Corollary 5.2.31. *Consider a morphism of abelian $\mathscr{P}$-premotivic categories*

$$\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$$

Suppose there are admissible topologies t and t', with t' finer than t, such that the following conditions are verified.

(i) $\mathscr{A}$ is compatible with t and $\mathscr{B}$ is compatible with t'.
(ii) $\mathscr{B}$ and $\mathrm{D}(\mathscr{B})$ are compactly τ-generated.
(iii) For any scheme S, the functor $\psi_S : \mathscr{B}_S \to \mathscr{A}_S$ preserves filtering colimits.

Then, $\psi_S : \mathrm{C}(\mathscr{B}_S) \to \mathrm{C}(\mathscr{A}_S)$ preserves $\mathbf{A}^1$-equivalences between t'-flasque objects. If moreover ψ_S is exact, the functor ψ_S preserves $\mathbf{A}^1$-equivalences.

Proof. We already know that ψ_S is a right Quillen functor, so that it preserves local objects and $\mathbf{A}^1$-fibrant objects. This implies also that ψ_S preserves $\mathbf{A}^1$-equivalences between $\mathbf{A}^1$-fibrant objects (this is Ken Brown's lemma [**Hov99**, 1.1.12]). Let D be a t'-flasque complex of $\mathscr{B}_S$. Then $\psi_S(D)$ is a t-flasque complex of $\mathscr{A}_S$. It follows from Proposition 5.2.28 that $R_{\mathbf{A}^1}(D)$ is $\mathbf{A}^1$-local and that $D \to R_{\mathbf{A}^1}(D)$ is an $\mathbf{A}^1$-equivalence. Lemma 5.2.27 implies the map

$$\psi_S(D) \to R_{\mathbf{A}^1}(\psi_S(D)) = \psi_S(R_{\mathbf{A}^1}(D))$$

is a an $\mathbf{A}^1$-equivalence. This implies the first assertion.

The last assertion is a direct consequence of the first one. □

5.2.32. Consider the usual cosimplicial scheme $\Delta^\bullet$ defined by

$$\Delta^n = \operatorname{Spec}\left(\mathbf{Z}[t_0, \ldots, t_n]/(t_1 + \cdots + t_n - 1)\right) \simeq \mathbf{A}^n$$

(see [**MV99**]). For any scheme S, we get a cosimplicial object of $\mathscr{A}_S$, namely $M_S(\Delta^\bullet_S)$. Given any complex C of $\mathscr{A}_S$, we define its associated *Suslin singular complex* as

$$\underline{C}^*(C) = \operatorname{Tot}^{\oplus} \mathit{Hom}(M_S(\Delta^\bullet_S), C), \tag{5.2.32.1}$$

where $\mathit{Hom}(M_S(\Delta^\bullet_S), C)$ is considered as a bicomplex by the Dold-Kan correspondence. The canonical map $M_S(\Delta^\bullet_S) \to \mathbb{1}_S$ induces a map

$$C \to \underline{C}^*(C).$$

Lemma 5.2.33. *For any complex C of $\mathscr{A}_S$, the map*

$$\underline{C}^*(C) \to \mathit{Hom}(M_S(\mathbf{A}^1_S), \underline{C}^*(C)) = \underline{C}^*(\mathit{Hom}(M_S(\mathbf{A}^1_S), C))$$

is a chain homotopy equivalence.

Proof. The composite morphism

$$(s_0 p \times Id)_* : M_S(\mathbf{A}^1 \times \Delta^\bullet_S) \to M_S(\mathbf{A}^1 \times \Delta^\bullet_S),$$

where s_0 is the map induced by the zero section, and p is the map induced by the obvious projection of $\mathbf{A}^1$ on its base, is chain homotopic to the identity. Indeed, the homotopy relation is given by the formula

$$s_n = \sum_{i=0}^{n} (-1)^i.(1 \otimes_S \psi_i) : M_S(\mathbf{A}^1 \times \Delta^{n+1}_S) \to M_S(\mathbf{A}^1 \times \Delta^n_S)$$

where 1 is the identity of $M_S(\mathbf{A}^1_S)$, and ψ_i is induced by the map $\Delta^{n+1}_S \to \mathbf{A}^1 \times \Delta^n_S$ which sends the j-th vertex $v_{j,n+1}$ to either $0 \times v_{j,n}$, if $j \leq i$, or to $1 \times v_{j-1,n}$ otherwise. This implies the lemma. □

Lemma 5.2.34. *For any t-flasque complex C of $\mathscr{A}_S$, we have a canonical isomorphism*

$$\underline{C}^*(C) \simeq \mathbf{L}\varinjlim_n \mathbf{R}\mathit{Hom}(M_S(\Delta^n_S), C)$$

in $\mathrm{D}(\mathscr{A}_S)$.

This is a variation on the Dold-Kan correspondence. As a direct consequence, we get:

Lemma 5.2.35. *For any complex C of $\mathscr{A}_S$, the map $C \to \underline{C}^*(C)$ is an $\mathbf{A}^1$-equivalence.*

Proposition 5.2.36. *If t is bounded in $\mathscr{A}$, then, for any t-flasque complex C of $\mathscr{A}_S$, $\underline{C}^*(C)$ is $\mathbf{A}^1$-local.*

Proof. Using the first premotivic adjunction of example 5.2.21 and the fact that $\mathrm{D}(\mathscr{A})$ is compactly generated (5.1.30), we can reduce the proposition to the case where $\mathscr{A}_S$ is the category of presheaves of abelian groups over $\mathscr{P}/S$, in which case this is well known. □

5.2.d. *Constructible $\mathbf{A}^1$-local premotives.*

5.2.37. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ compatible with an admissible topology t. Assume that t is bounded in $\mathscr{A}$ (see Definition 5.1.28) and consider a bounded generating family $\mathscr{N}^t_S$ for t-hypercovers in $\mathscr{A}_S$.

Let $\mathcal{T}_{\mathbf{A}^1_S}$ be the family of complexes of $\mathrm{C}(\mathscr{A}_S)$ of shape

$$M_S(\mathbf{A}^1_X)\{i\} \to M_S(X)\{i\}$$

for a $\mathscr{P}$-scheme X over S and a twist $i \in I$. Then the functor (5.1.31.1) obviously induces the following functor

$$\left(K^b\big(M(\mathscr{P}/S, \mathscr{A})\big)/\mathscr{N}^t_S \cup \mathcal{T}_{\mathbf{A}^1_S}\right)^\natural \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S), \tag{5.2.37.1}$$

where the category on the left is the pseudo-abelian category associated to the Verdier quotient of $K^b\big(M(\mathscr{P}/S,\mathscr{A})\big)$ by the thick subcategory generated by $\mathcal{N}_S^t \cup \mathcal{T}_{\mathbf{A}^1_S}$. Applying Thomason's localization theorem [**Nee01**], we get from Proposition 5.1.32 the following result:

Proposition 5.2.38. *Consider the previous hypothesis and notations and assume that $\mathscr{A}$ is finitely τ-presented.*

Then $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ is compactly τ-generated. Moreover, the functor (5.2.37.1) *is fully faithful.*

Let us denote by $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}(\mathscr{A})$ the subcategory of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ made of τ-constructible premotives in the sense of Definition 1.4.9. Taking into account Proposition 1.4.11, we deduce from the above proposition the following corollary:

Corollary 5.2.39. *Under the assumptions of 5.2.38, for any premotive $\mathcal{M}$ in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$, the following conditions are equivalent:*

(i) $\mathcal{M}$ is compact;
(ii) $\mathcal{M}$ is τ-constructible.

Moreover, the functor (5.2.37.1) *induces an equivalence of categories:*

$$\Big(K^b\big(M(\mathscr{P}/S,\mathscr{A})\big)/\mathcal{N}_S^t \cup \mathcal{T}_{\mathbf{A}^1_S}\Big)^\natural \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}(\mathscr{A}_S).$$

Example 5.2.40. With the notations of 5.1.34, we get the following equivalences of categories:

$$\Big(\mathrm{K}^b\left(\Lambda(Sm/S)\right)/(BG_S \cup \mathcal{T}_{\mathbf{A}^1_S})\Big)^\natural \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}(S,\Lambda).$$
$$\Big(\mathrm{K}^b\left(\Lambda(\mathscr{S}^{ft}/S)\right)/CDH_S \cup \mathcal{T}_{\mathbf{A}^1_S}\Big)^\natural \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}\Big(\mathrm{Sh}_{\mathrm{cdh}}(\mathscr{S}^{ft}/S,\Lambda)\Big).$$

This statement is the analog of the embedding theorem [**VSF00**, chap. 5, 3.2.6].

Proposition 5.2.41. *Assume $\mathscr{P} = \mathscr{S}^{ft}$ is the class of finite type (resp. separated and of finite type) morphisms.*

Let $\mathscr{A}$ be an abelian generalized premotivic category compatible with an admissible topology t and satisfying the property (C) of Paragraph 5.1.35.

Then the triangulated generalized premotivic category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ is τ-continuous.

Proof. The proof relies on the following lemma:

Lemma 5.2.42. *Under the assumptions of the preceding proposition, for any morphism of schemes $f : T \to S$, the functor*

$$\mathbf{L}f^* : \mathrm{D}(\mathscr{A}_S) \to \mathrm{D}(\mathscr{A}_T)$$

preserves $\mathbf{A}^1$-local complexes.

When f is a morphism of finite type (resp. separated of finite type), the functor $\mathbf{L}f^*$ admits $\mathbf{L}f_\sharp$ as a left adjoint and the lemma is clear. In the general case, one can write f as a projective limit of a projective system of morphisms of scheme $(f_\alpha : T_\alpha \to S)_{\alpha\in A}$ such that f_α is affine of finite type. Recall from Proposition 5.1.36, $\mathrm{D}(\mathscr{A})$ is τ-continuous. Thus, to check that for an $\mathbf{A}^1$-local complexe C in $\mathrm{D}(\mathscr{A}_S)$, the complex $\mathbf{L}f^*(C)$ is $\mathbf{A}^1$-local, we thus are reduced to prove that $\mathbf{L}f^*_\alpha(C)$ is $\mathbf{A}^1$-local which follows from the first treated case. The lemma is proven.

Given the full embedding $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}) \to \mathrm{D}(\mathscr{A})$ whose image is made of $\mathbf{A}^1$-local complexes, the proposition now directly follows from the previous lemma and the fact $\mathrm{D}(\mathscr{A})$ is τ-continuous. □

Example 5.2.43. Taking into account the second point of Example 5.1.37, the previous proposition can be applied to the category $\mathrm{Sh}_t\big(\mathscr{S}^{ft},\mathbf{Z}\big)$ where $t = \mathrm{Nis}, \mathrm{\acute{e}t}, \mathrm{cdh}, \mathrm{qfh}, \mathrm{h}$.

Remark 5.2.44. The previous proposition will be extended to the (non generalized) premotivic case in Corollary 6.1.12.

5.3. The stable $\mathbf{A}^1$-derived premotivic category.

5.3.a. *Modules.* Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category with generating set of twists τ.

A *cartesian commutative monoid* R of $\mathscr{A}$ is a cartesian section of the fibred category $\mathscr{A}$ over $\mathscr{S}$ such that for any scheme S, R_S has a commutative monoid structure in $\mathscr{A}_S$ and for any morphism of schemes $f : T \to S$, the structural transition maps $\phi_f : f^*(R_S) \to R_T$ are isomorphisms of monoids.

Let us fix a cartesian commutative monoid R of $\mathscr{A}$.

Consider a base scheme S. We denote by $R_S\text{-}\mathrm{mod}$ the category of modules in the monoidal category $\mathscr{A}_S$ over the monoid R_S. For any $\mathscr{P}$-scheme X/S and any twist $i \in \tau$, we put

$$R_S(X)\{i\} = R_S \otimes_S M_S(X)\{i\}$$

endowed with its canonical R_S-module structure. The category $R_S\text{-}\mathrm{mod}$ is a Grothendieck abelian category such that the forgetful functor $U_S : R_S\text{-}\mathrm{mod} \to \mathscr{A}_S$ is exact and conservative. A family of generators for $R_S\text{-}\mathrm{mod}$ is given by the modules $R_S(X)\{i\}$ for a $\mathscr{P}$-scheme X/S and a twist $i \in \tau$. As A_S is commutative, $R_S\text{-}\mathrm{mod}$ has a unique symmetric monoidal structure such that the free R_S-module functor is symmetric monoidal. We denote by $\otimes_R$ this tensor product. Note that $R_S(X) \otimes_R R_S(Y) = R_S(X \times_S Y)$. Finally the categories of modules $R_S\text{-}\mathrm{mod}$ form a symmetric monoidal $\mathscr{P}$-fibred category, such that the following proposition holds (see 7.2.10).

Proposition 5.3.1. *Let $\mathscr{A}$ be a τ-generated abelian $\mathscr{P}$-premotivic category and R be a cartesian commutative monoid of $\mathscr{A}$.*

Then the category $R\text{-}\mathrm{mod}$ equipped with the structures introduced above is a τ-generated abelian $\mathscr{P}$-premotivic category.

Moreover, we have an adjunction of abelian $\mathscr{P}$-premotivic categories:

$$R \otimes (-) : \mathscr{A} \rightleftarrows R\text{-}\mathrm{mod} : U\,. \tag{5.3.1.1}$$

Remark 5.3.2. With the hypothesis of the preceding proposition, for any morphism of schemes $f : T \to S$, the exchange transformation $f^*U_S \to U_T f^*$ is an isomorphism by construction of $R\text{-}\mathrm{mod}$ (7.2.10).

Proposition 5.3.3. *Let $\mathscr{A}$ be a τ-generated abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. Consider a cartesian commutative monoid R of $\mathscr{A}$ such that for any scheme S, tensoring quasi-isomorphisms between cofibrant complexes by R_S gives quasi-isomorphisms (e.g. R_S might be cofibrant (as a complex concentrated in degree zero), or flat). Then the abelian $\mathscr{P}$-premotivic category $R\text{-}\mathrm{mod}$ is compatible with t.*

Proof. In view of Proposition 5.1.26, we have only to show that $R\text{-}\mathrm{mod}$ satisfies cohomological t-descent. Consider a t-hypercover $p : \mathcal{X} \to X$ in $\mathscr{P}/S$. We prove that the map $p_* : R_S(\mathcal{X}) \to R_S(X)$ is a quasi-isomorphism in $\mathrm{C}(R_S\text{-}\mathrm{mod})$. The functor U_S is conservative, and $U_S(p_*)$ is equal to the map:

$$R_S \otimes_S M_S(\mathcal{X}) \to R_S \otimes_S M_S(X).$$

But this is a quasi-isomorphism in $\mathrm{C}(\mathscr{A}_S)$ by assumption on R_S. □

Remark 5.3.4. According to Lemma 5.1.27, for any simplicial $\mathscr{P}$-scheme $\mathcal{X}$ over S, any twist $i \in \tau$ and any R_S-module C, we get canonical isomorphisms:

$$\mathrm{Hom}_{\mathrm{K}(R_S\text{-}\mathrm{mod})}\big(R_S(\mathcal{X})\{i\}, C\big) \simeq \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(M_S(\mathcal{X})\{i\}, C) \tag{5.3.4.1}$$

$$\mathrm{Hom}_{\mathrm{D}(R_S\text{-}\mathrm{mod})}(R_S(\mathcal{X})\{i\}, C) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(M_S(\mathcal{X})\{i\}, C). \tag{5.3.4.2}$$

5.3.b. *Symmetric sequences.* Let $\mathscr{A}$ be an abelian category.

Let G be a group. An action of G on an object $A \in \mathscr{A}_S$ is a morphism of groups $G \to \mathrm{Aut}_{\mathscr{A}}(A), g \mapsto \gamma_g^A$. We say that A is a G-object of $\mathscr{A}$. A G-equivariant morphism $A \xrightarrow{f} B$ of G-objects of $\mathscr{A}$ is a morphism f in $\mathscr{A}$ such that $\gamma_g^B \circ f = f \circ \gamma_g^A$.

If E is any object of $\mathscr{A}$, we put $G \times E = \bigoplus_{g \in G} E$ considered as a G-object via the permutation isomorphisms of the summands.

If H is a subgroup of G, and E is an H-object, $G \times E$ has two actions of H : the first one, say γ, is obtained via the inclusion $H \subset G$, and the second one denoted by γ', is obtained using the structural action of H on E. We define $G \times_H E$ as the coequalizer of the family of morphisms $(\gamma_\sigma - \gamma'_\sigma)_{\sigma \in H}$, and consider it equipped with its induced action of G.

Definition 5.3.5. Let $\mathscr{A}$ be an abelian category.

A symmetric sequence of $\mathscr{A}$ is a sequence $(A_n)_{n \in \mathbf{N}}$ such that for each $n \in \mathbf{N}$, A_n is a $\mathfrak{S}_n$-object of $\mathscr{A}$. A morphism of symmetric sequences of $\mathscr{A}$ is a collection of $\mathfrak{S}_n$-equivariant morphism $(f_n : A_n \to B_n)_{n \in \mathbf{N}}$.

We let $\mathscr{A}^{\mathfrak{S}}$ be the category of symmetric sequences of $\mathscr{A}$.

It is straightforward to check $\mathscr{A}^{\mathfrak{S}}$ is abelian. For any integer $n \in \mathbf{N}$, we define the *n-th evaluation functor* as follows:

$$ev_n : \mathscr{A}^{\mathfrak{S}} \to \mathscr{A}, A_* \mapsto A_n.$$

Any object A of $\mathscr{A}$ can be considered as the trivial symmetric sequence $(A, 0, \ldots)$. The functor $i_0 : A \mapsto (A, 0, \ldots)$ is obviously left adjoint to ev_0 and we obtain an adjunction

$$i_0 : \mathscr{A} \rightleftarrows \mathscr{A}^{\mathfrak{S}} : ev_0. \tag{5.3.5.1}$$

Remark i_0 is also right adjoint to ev_0. Thus i_0 preserves every limits and colimits.

For any integer $n \in \mathbf{N}$ and any symmetric sequence A_* of $\mathscr{A}$, we put

$$(A_*\{-n\})_m = \begin{cases} \mathfrak{S}_m \times_{\mathfrak{S}_{m-n}} A_{m-n} & \text{if } m \geq n \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.5.2}$$

This define an endofunctor on $\mathscr{A}^{\mathfrak{S}}$, and we have $A_*\{-n\}\{-m\} = A_*\{-n-m\}$ (through a canonical isomorphism). Remark finally that for any integer $n \in \mathbf{N}$, the functor

$$i_n : \mathscr{A} \to \mathscr{A}^{\mathfrak{S}}, A \mapsto (i_0(A))\{-n\}$$

is left adjoint to ev_n.

Remark 5.3.6. Let $\mathfrak{S}$ be the category of finite sets with bijective maps as morphisms. Then the category of symmetric sequences is canonically equivalent to the category of functors $\mathfrak{S} \to \mathscr{A}$. This presentation is useful to define a tensor product on $\mathscr{A}^{\mathfrak{S}}$.

Definition 5.3.7. Let $\mathscr{A}$ be a symmetric closed monoidal abelian category.

Given two functors $A_*, B_* : \mathfrak{S} \to \mathscr{A}$, we put:

$$\begin{array}{rcl} E \otimes^{\mathfrak{S}} F : \mathfrak{S} & \mapsto & \mathscr{A} \\ N & \mapsto & \bigoplus_{N = P \sqcup Q} E(P) \otimes F(Q). \end{array}$$

If $\mathbb{1}_{\mathscr{A}}$ is the unit object of the monoidal category $\mathscr{A}$, the category $\mathscr{A}^{\mathfrak{S}}$ is then a symmetric closed monoidal category with unit object $i_0(\mathbb{1}_{\mathscr{A}})$.

5.3.8. Let A be an object of $\mathscr{A}$. Then the n-th tensor power $A^{\otimes n}$ of A is endowed with a canonical action of the group $\mathfrak{S}_n$ through the structural permutation isomorphism of the symmetric structure on $\mathscr{A}$. Thus the sequence $\mathrm{Sym}(A) = (A^{\otimes n})_{n \in \mathbf{N}}$ is a symmetric sequence.

Moreover, the isomorphism $A^{\otimes n} \otimes A^{\otimes m} \to A^{\otimes n+m}$ is $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant. Thus it induces a morphism $\mu : \mathrm{Sym}(A) \otimes^{\mathfrak{S}} \mathrm{Sym}(A) \to \mathrm{Sym}(A)$ of symmetric sequences. We also consider the obvious morphism $\eta : i_0(\mathbb{1}_{\mathscr{A}}) = i_0(A^{\otimes 0}) \to \mathrm{Sym}(A)$. One can check easily that $\mathrm{Sym}(A)$ equipped with the multiplication μ and the unit η is a commutative monoid in the monoidal category $\mathscr{A}^{\mathfrak{S}}$.

Definition 5.3.9. Let $\mathscr{A}$ be an abelian symmetric monoidal category. The commutative monoid $\mathrm{Sym}(A)$ of $\mathscr{A}^{\mathfrak{S}}$ defined above will be called the symmetric monoid generated by A.

Remark 5.3.10. One can describe $\mathrm{Sym}(A)$ by a universal property: given a commutative monoid R in $\mathscr{A}^{\mathfrak{S}}$, to give a morphism of commutative monoids $\mathrm{Sym}(A) \to R$ is equivalent to give a morphism $A \to R_1$ in $\mathscr{A}$.

5.3.11. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$.

Consider a base scheme S. According to the previous paragraph, the category $\mathscr{A}_S^{\mathfrak{S}}$ is an abelian category, endowed with a symmetric tensor product $\otimes_S^{\mathfrak{S}}$. For any $\mathscr{P}$-scheme X/S and any integer $n \in \mathbf{N}$, using (5.3.5.2), we put

$$M_S(X, \mathscr{A}^{\mathfrak{S}})\{-n\} = i_0(M_S(X, \mathscr{A}))\{-n\}.$$

It is immediate that the class of symmetric sequences of the form $M_S(X, \mathscr{A}^{\mathfrak{S}})\{-n\}$ for a smooth S-scheme X and an integer $n \geq 0$ is a generating family for the abelian category $\mathscr{A}_S^{\mathfrak{S}}$ which is therefore a Grothendieck abelian category. It is clear that for any $\mathscr{P}$-scheme X and Y over S,

$$M_S(X, \mathscr{A}^{\mathfrak{S}})\{-n\} \otimes_S^{\mathfrak{S}} M_S(Y, \mathscr{A}^{\mathfrak{S}})\{-n\} = M_S(X \times_S Y, \mathscr{A}^{\mathfrak{S}})\{-n\}.$$

Given a morphism (resp. $\mathscr{P}$-morphism) of schemes $f : T \to S$ and a symmetric sequence A_* of $\mathscr{A}_S$, we put $f_{\mathfrak{S}}^*(A_*) = (f^* A_n)_{n \in \mathbf{N}}$ (resp. $f_\sharp^{\mathfrak{S}}(A_*) = (f_\sharp A_n)_{n \in \mathbf{N}}$). This defines a functor $f_{\mathfrak{S}}^* : \mathscr{A}_S^{\mathfrak{S}} \to \mathscr{A}_T^{\mathfrak{S}}$ (resp. $f_\sharp^{\mathfrak{S}} : A_T^{\mathfrak{S}} \to \mathscr{A}_S^{\mathfrak{S}}$) which is obviously right exact. Thus the functor $f_{\mathfrak{S}}^*$ admits a right adjoint which we denote by $f_*^{\mathfrak{S}}$. When f is in $\mathscr{P}$, we check easily the functor $f_\sharp^{\mathfrak{S}}$ is left adjoint to $f_{\mathfrak{S}}^*$.

From criterion 1.1.42 and Lemma 1.2.13, we check easily the following proposition:

Proposition 5.3.12. *Consider the previous hypothesis and notations.*

The association $S \mapsto \mathscr{A}_S^{\mathfrak{S}}$ together with the structures introduced above defines an $\mathbf{N} \times \tau$-generated abelian $\mathscr{P}$-premotivic category.

Moreover, the different adjunctions of the form (5.3.5.1) *over each fibers over a scheme S define an adjunction of $\mathscr{P}$-premotivic categories:*

$$i_0 : \mathscr{A} \rightleftarrows \mathscr{A}^{\mathfrak{S}} : ev_0 \tag{5.3.12.1}$$

Indeed, i_0 is trivially compatible with twists.

Proposition 5.3.13. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category, and t be an admissible topology. If $\mathscr{A}$ is compatible with t then $\mathscr{A}^{\mathfrak{S}}$ is compatible with t.*

Proof. This is based on the following lemma (see [**CD09**, 7.5, 7.6]):

Lemma 5.3.14. *For any complex C of $\mathscr{A}_S$, any complex E of $\mathscr{A}_S^{\mathfrak{S}}$ and any integer $n \geq 0$, there are canonical isomorphisms:*

$$\mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S^{\mathfrak{S}})}(i_0(C)\{-n\}, E) \simeq \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(C, E_n) \tag{5.3.14.1}$$

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S^{\mathfrak{S}})}(i_0(C)\{-n\}, E) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(C, E_n) \tag{5.3.14.2}$$

If $\mathscr{A}$ is compatible with t, this implies that E is local (resp. t-flasque) if and only if for any $n \geq 0$, E_n is local (resp. t-flasque). This concludes. □

5.3.c. *Symmetric Tate spectra.*

5.3.15. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$.

For any scheme S, the unit point of $\mathbf{G}_{m,S}$ defines a split monomorphism of $\mathscr{A}$-premotives $\mathbb{1}_S \to M_S(\mathbf{G}_{m,S})$. We denote by $\mathbb{1}_S\{1\}$ the cokernel of this monomorphism and call it the *suspended Tate S-premotive* with coefficients in $\mathscr{A}$. The collection of these objects for any scheme S is a cartesian section of $\mathscr{A}$ denoted by $\mathbb{1}\{1\}$. For any integer $n \geq 0$, we denote by $\mathbb{1}\{n\}$ its n-the tensor power.

With the notations of 5.3.9, we define the *symmetric Tate spectrum* over S as the symmetric sequence $\mathbb{1}_S\{*\} = Sym(\mathbb{1}_S\{1\})$ in $\mathscr{A}_S^{\mathfrak{S}}$. The corresponding collection defines a cartesian commutative monoid of the fibred category $\mathscr{A}^{\mathfrak{S}}$, called the *absolute Tate spectrum.*

Definition 5.3.16. Consider an abelian $\mathscr{P}$-premotivic category $\mathscr{A}$.

We denote by $\mathrm{Sp}(\mathscr{A})$ the abelian $\mathscr{P}$-premotivic category of modules over $\mathbb{1}\{*\}$ in the category $\mathscr{A}^{\mathfrak{S}}$. The objects of $\mathrm{Sp}(\mathscr{A})$ are called the abelian (symmetric) Tate spectra.[72]

[72]As we will almost never consider non symmetric spectra, we will cancel the word "symmetric" in our terminology.

The category $\mathrm{Sp}(\mathscr{A})$ is $(\mathbf{N}\times\tau)$-generated. Composing the adjunctions (5.3.1.1) and (5.3.12.1), we get an adjuntion

$$\Sigma^\infty : \mathscr{A} \rightleftarrows \mathrm{Sp}(\mathscr{A}) : \Omega^\infty \tag{5.3.16.1}$$

of abelian $\mathscr{P}$-premotivic categories.

Let us explicit the definition. An abelian Tate spectrum (E, σ) is the data of :

(1) for any $n \in \mathbf{N}$, an object E_n of $\mathscr{A}_S$ endowed with an action of $\mathfrak{S}_n$
(2) for any $n \in \mathbf{N}$, a morphism $\sigma_n : E_n\{1\} \to E_{n+1}$ in $\mathscr{A}_S$

such that the composite map

$$E_m\{n\} \xrightarrow{\sigma_m\{n-1\}} E_{m+1}\{n-1\} \to \ldots \xrightarrow{\sigma_{m+n-1}} E_{m+n}$$

is $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant with respect to the canonical action of $\mathfrak{S}_n$ on $\mathbb{1}_S\{n\}$ and the structural action of $\mathfrak{S}_m$ on E_m. By definition, $ev_0(E) = E_0$. Recall that ev_0 is exact.

Given an object A of $\mathscr{A}_S$, the abelian Tate spectrum $\Sigma^\infty A$ is defined such that $(\Sigma^\infty A)_n = A\{n\}$ with the action of $\mathfrak{S}_n$ given by its action on $\mathbb{1}_S\{n\}$ by permutations of the factors.

Be careful we consider the category $\mathrm{Sp}(\mathscr{A}_S)$ as $\mathbf{N}$-twisted by negative twists. For any abelian Tate spectrum E_*, $(E_*\{-n\})_m = \mathfrak{S}_n \times_{\mathfrak{S}_{m-n}} E_{m-n}$ for $n \geq m$.

5.3.17. Consider a morphism

$$\varphi : \mathscr{A} \to \mathscr{B}$$

of abelian $\mathscr{P}$-premotivic categories. Then as $\varphi(\mathbb{1}^{\mathscr{A}}\{1\}) = \mathbb{1}^{\mathscr{B}}\{1\}$, φ can be extended to abelian Tate spectra in such a way that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\varphi} & \mathscr{B} \\ {\scriptstyle \Sigma^\infty_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle \Sigma^\infty_{\mathscr{B}}} \\ \mathrm{Sp}(\mathscr{A}) & \xrightarrow{\mathrm{Sp}(\varphi)} & \mathrm{Sp}(\mathscr{B}). \end{array}$$

(Of course the obvious diagram for the corresponding right adjoints also commutes.)

Definition 5.3.18. For any scheme S, a complex of abelian Tate spectra over S will be called simply a *Tate spectrum* over S.

A Tate spectrum E is a bigraded object. In the notation E_n^m, the index m corresponds to the (cochain) complex structure and the index n to the symmetric sequence structure.

From propositions 5.3.3 and 5.3.13, we get the following:

Proposition 5.3.19. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. Then* $\mathrm{Sp}(\mathscr{A})$ *is compatible with t.*

Note also that remark 5.3.4 and Lemma 5.3.14 implies that for any simplicial $\mathscr{P}$-scheme $\mathcal{X}$ over S, any integer $n \in \mathbf{N}$, and any Tate spectrum E, we have canonical isomorphisms:

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{Sp}(\mathscr{A}_S))}(\Sigma^\infty M_S(\mathcal{X}, \mathscr{A})\{-n\}, E) \simeq \mathrm{Hom}_{\mathrm{K}(\mathscr{A}_S)}(\Sigma^\infty M_S(\mathcal{X}, \mathscr{A}), E_n) \tag{5.3.19.1}$$

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Sp}(\mathscr{A}_S))}(\Sigma^\infty M_S(\mathcal{X}, \mathscr{A})\{-n\}, E) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_S)}(\Sigma^\infty M_S(\mathcal{X}, \mathscr{A}), E_n) \tag{5.3.19.2}$$

According to the proposition, the category $\mathrm{C}(\mathrm{Sp}(\mathscr{A}_S))$ of Tate spectra over S has a t-descent model structure. The previous isomorphisms allow to describe this structure as follows:

(1) For any simplicial $\mathscr{P}$-scheme $\mathcal{X}$ over S, and any integer $n \geq 0$, the Tate spectrum $\Sigma^\infty M_S(\mathcal{X}, \mathscr{A})\{-n\}$ is cofibrant.
(2) A Tate spectrum E over S is fibrant if and only if for any integer $n \geq 0$, the complex E_n over $\mathscr{A}_S$ is local (*i.e.* t-flasque).
(3) Let $f : E \to F$ be a morphism of Tate spectra over S. Then f is a fibration (resp. quasi-isomorphism) if and only if for any integer $n \geq 0$, the morphism $f_n : E_n \to F_n$ of complexes over $\mathscr{A}_S$ is a fibration (resp. quasi-isomorphism).

Note that properties (2) and (3) follows from (5.3.4.1) and (5.3.14.1).

5.3.20. We can also introduce the $\mathbf{A}^1$-localization of this model structure. The corresponding homotopy category is the $\mathbf{A}^1$-derived $\mathscr{P}$-premotivic category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))$ introduced in 5.2.16. The isomorphism (5.3.19.2) gives the following assertion: From the above, a Tate spectrum E is $\mathbf{A}^1$-local if and only if for any integer $n \geq 0$, E_n is $\mathbf{A}^1$-local.

(1) A Tate spectrum E over S is $\mathbf{A}^1$-local if and only if for any integer $n \geq 0$, the complex E_n over $\mathscr{A}_S$ is $\mathbf{A}^1$-local.
(2) Let $f : E \to F$ be a morphism of Tate spectra over S. Then f is a $\mathbf{A}^1$-local fibration (resp. weak $\mathbf{A}^1$-equivalence) if and only if for any integer $n \geq 0$, the morphism $f_n : E_n \to F_n$ of complexes over $\mathscr{A}_S$ is a $\mathbf{A}^1$-local fibration (resp. weak $\mathbf{A}^1$-equivalence).

As a consequence, the isomorphism (5.3.19.2) induces an isomorphism

(5.3.20.1) $$\mathrm{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}_S))}(\Sigma^\infty M_S(\mathcal{X},\mathscr{A})\{-n\}, E) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)}(\Sigma^\infty M_S(\mathcal{X},\mathscr{A}), E_n).$$

Similarly, the adjunction (5.3.16.1) induces an adjunction of triangulated $\mathscr{P}$-premotivic categories

(5.3.20.2) $$\mathbf{L}\Sigma^\infty : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A})) : \mathbf{R}\Omega^\infty.$$

5.3.d. *Symmetric Tate Ω-spectra.*

5.3.21. The final step is to localize further the category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))$. The aim is to relate the positive twists on $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ obtained by tensoring with $\mathbb{1}_S\{1\}$ and the negative twists on $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))$ induced by the consideration of symmetric sequences.

Let X be a $\mathscr{P}$-scheme over S. From the definition of Σ^∞, there is a canonical morphism of abelian Tate spectra:

$$\big[\Sigma^\infty\big(\mathbb{1}_S\{1\}\big)\big]\{-1\} \to \Sigma^\infty \mathbb{1}_S.$$

Tensoring this map by $\Sigma^\infty M_S(X,\mathscr{A})\{-n\}$ for any $\mathscr{P}$-scheme X over S and any integer $n \in \mathbf{N}$, we obtain a family of morphisms of Tate spectra concentrated in cohomological degree 0:

$$\big[\Sigma^\infty\big(M_S(X,\mathscr{A})\{1\}\big)\big]\{-n-1\} \to \Sigma^\infty M_S(X,\mathscr{A})\{-n\}.$$

We denote by $\mathscr{W}_\Omega$ this family and put $\mathscr{W}_{\Omega,\mathbf{A}^1} = \mathscr{W}_\Omega \cup \mathscr{W}_{\mathbf{A}^1}$. Obviously, $\mathscr{W}_{\Omega,\mathbf{A}^1}$ is stable by the operations f^* and $f_\sharp$.

Definition 5.3.22. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. With the notations introduced above, we define the *stable $\mathbf{A}^1$-derived $\mathscr{P}$-premotivic category with coefficients in $\mathscr{A}$* as the derived $\mathscr{P}$-premotivic category

$$\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}) := \mathrm{D}(\mathrm{Sp}(\mathscr{A}))[\mathscr{W}_{\Omega,\mathbf{A}^1}^{-1}]$$

defined in Corollary 5.2.5.

5.3.23. According to this definition, we get the following identification:

$$\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}) = \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))[\mathscr{W}_\Omega^{-1}].$$

Using the left Bousfield localization of the $\mathbf{A}^1$-local model structure on $\mathrm{C}(\mathrm{Sp}(\mathscr{A}))$, we thus obtain a canonical adjunction of triangulated $\mathscr{P}$-fibred premotivic categories

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A})) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))[\mathscr{W}_\Omega^{-1}]$$

which allows to describe $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$ as the full subcategory of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}_S))$ made of Tate spectra which are $\mathscr{W}_\Omega$-local in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}_S))$. Recall a Tate spectrum E is a sequence of complexes $(E_n)_{n\in\mathbf{N}}$ over $\mathscr{A}_S$ together with suspension maps in $\mathrm{C}(\mathscr{A}_S)$

$$\sigma_n : \mathbb{1}_S\{1\} \otimes E_n \to E_{n+1}.$$

From this, we deduce a canonical morphism $\mathbb{1}_S\{1\} \otimes^{\mathbf{L}} E_n \to E_{n+1}$ in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ whose adjoint morphism we denote by

(5.3.23.1) $$u_n : E_n \to \mathbf{R}\mathit{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)}(\mathbb{1}_S\{1\}, E_{n+1})$$

According to (5.3.20.1), the condition that E is $\mathscr{W}_\Omega$-local in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))$ is equivalent to ask that for any integer $n \geq 0$, the map (5.3.23.1) is an isomorphism in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}))$.

Considering the adjunction (5.3.20.2), we obtain finally an adjunction of triangulated $\mathscr{P}$-fibred categories:

$$\Sigma^\infty : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A})) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}) : \Omega^\infty. \tag{5.3.23.2}$$

Note that tautologically, the Tate spectrum $\Sigma^\infty(\mathbb{1}_S\{1\})$ has a tensor inverse given by the spectrum $(\Sigma^\infty \mathbb{1}_S)\{-1\}$ in $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$. Thus, we have obtained from the abelian premotivic category $\mathscr{A}$ a triangulated premotivic category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$ which satisfies the properties:

- the homotopy property (Htp);
- the stability property (Stab);
- the t-descent property.

As we will see in the followings, the construction satisfies a universality property that the reader can already guess.

Definition 5.3.24. Consider the assumptions of definition 5.3.22.

For any scheme S, we say that a Tate spectrum E over S is a *Tate Ω-spectrum* if the following conditions are fulfilled:

(a) For any integer $n \geq 0$, E_n is t-flasque and $\mathbf{A}^1$-local.
(b) For any integer $n \geq 0$, the adjoint of the structural suspension map

$$E_n \to \mathit{Hom}_{\mathrm{C}(\mathscr{A}_S)}(\mathbb{1}_S\{1\}, E_{n+1})$$

is a quasi-isomorphism.

In particular, a Tate Ω-spectrum is $\mathscr{W}_\Omega$-local in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}_S))$. In fact, it is also $\mathscr{W}_{\Omega,\mathbf{A}^1}$-local in the category $\mathrm{D}(\mathrm{Sp}(\mathscr{A}_S))$ so that the category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ is also equivalent to the full subcategory of $\mathrm{D}(\mathrm{Sp}(\mathscr{A}_S))$ spanned by Tate Ω-spectra.

Fibrant objects of the $\mathscr{W}_{\Omega,\mathbf{A}^1}$-local model category on $\mathrm{C}(\mathrm{Sp}(\mathscr{A}))$ obtained in definition 5.3.22 are exactly the Tate Ω-spectra.

Proposition 5.3.25. *Consider the above notations. Let S be a base scheme.*

(1) If the endofunctor

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S), C \mapsto \mathbf{R}\mathit{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)}(\mathbb{1}_S\{1\}, C)$$

is conservative, then the functor Ω^∞_S is conservative.

(2) If the Tate twist $E \mapsto E(1)$ is fully faithful in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$, then Σ^∞_S is fully faithful.

(3) If the Tate twist $E \mapsto E(1)$ induces an auto-equivalence of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$, then $(\Sigma^\infty_S, \Omega^\infty_S)$ are adjoint equivalences of categories.

Remark 5.3.26. Similar statements can be obtained for the derived categories rather than the $\mathbf{A}^1$-derived categories. We left their formulation to the reader.

Proof. Consider point (1). We have to prove that for any $\mathscr{W}_\Omega$-local Tate spectrum E in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sp}(\mathscr{A}_S))$, if $\mathbf{R}\Omega^\infty(E) = 0$, then $E = 0$. But $\mathbf{R}\Omega^\infty(E) = \Omega^\infty(E) = E_0$ (see 5.3.20). Because for any integer $n \geq 0$, the map (5.3.23.1) is an $\mathbf{A}^1$-equivalence, we deduce that for any integer $n \in \mathbf{Z}$, the complex E_n is (weakly) $\mathbf{A}^1$-acyclic. According to (5.3.20.1), this implies $E = 0$ – because $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$ is $\mathbf{N}$-generated.

Consider point (2). We want to prove that for any complex C over $\mathscr{A}_S$, the counit map $C \to \mathbf{R}\Omega^\infty \mathbf{L}\Sigma^\infty(C)$ is an isomorphism. It is enough to treat the case where C is cofibrant.

Considering the left adjoint $\mathbf{L}\Sigma^\infty$ of (5.3.20.2), we first prove that $\mathbf{L}\Sigma^\infty(C)$ is $\mathscr{W}_\Omega$-local. Because C is cofibrant, this Tate spectrum is equal in degree n to the complex $C\{n\}$ (with its natural action of $\mathfrak{S}_n$). Moreover, the suspension map is given by the isomorphism (in the monoidal category $\mathrm{C}(\mathscr{A}_S)$)

$$\sigma_n : \mathbb{1}_S\{1\} \otimes_S C\{n\} \to C\{n+1\}.$$

In particular, the corresponding map in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$

$$\sigma'_n : \mathbb{1}_S\{1\} \otimes^L_S C\{n\} \to C\{n+1\}.$$

is canonically isomorphic to

$$\mathbb{1}_S\{1\} \otimes^L_S C\{n\} \xrightarrow{1\otimes 1} \mathbb{1}_S\{1\} \otimes^L_S C\{n\}.$$

Thus, because the Tate twist is fully faithful in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$, the adjoint map to σ'_n is an $\mathbf{A}^1$-equivalence. In other words, $\mathbf{L}\Sigma^\infty(C)$ is $\mathscr{W}_\Omega$-local. But then, as C is cofibrant, $C = \Omega^\infty\Sigma^\infty(C) = \mathbf{R}\Omega^\infty\mathbf{L}\Sigma^\infty(C)$, and this concludes.

Point (3) is then a consequence of (1) and (2). □

Remark 5.3.27. (1) The construction of the triangulated category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ can also be obtained using the more general construction of [**CD09**, §7] – see also [**Hov01**, 7.11] and [**Ayo07b**, chap. 4] for even more general accounts. Here, we exploit the simplification arising from the fact that we invert a complex concentrated in degree 0: this allowed us to describe $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ simply as a Verdier quotient of the derived category of an abelian category. However, we can also consider the category of symmetric spectra in $\mathrm{C}(\mathscr{A}_S)$ with respect to one of the complexes $\mathbb{1}_S(1)[2]$ or $\mathbb{1}_S(1)$ and this leads to the equivalent categories; see [**Hov01**, 8.3].

(2) Point (3) of Proposition 5.3.25 is a particular case of [**Hov01**, 8.1].

5.3.28. Consider a morphism of abelian $\mathscr{P}$-premotivic categories

$$\varphi : \mathscr{A} \rightleftarrows \mathscr{B} : \psi$$

such that $\mathscr{A}$ (resp. $\mathscr{B}$) is compatible with a system of topology t (resp. t'). Suppose t' is finer than t. According to 5.3.17, we obtain an adjunction of abelian $\mathscr{P}$-premotivic categories

$$\varphi : \mathrm{C}(\mathrm{Sp}(\mathscr{A})) \rightleftarrows \mathrm{C}(\mathrm{Sp}(\mathscr{B})) : \psi.$$

The pair (φ_S, ψ_S) is a Quillen adjunction for the stable model structures (apply again [**CD09**, prop. 3.11]). Thus we obtain a morphism of triangulated $\mathscr{P}$-premotivic categories:

$$\mathbf{L}\varphi : \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}(\mathscr{B}) : \mathbf{R}\psi.$$

Remark 5.3.29. Under the light of Proposition 5.3.25, the category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ might be considered as the universal derived $\mathscr{P}$-premotivic category $\mathscr{T}$ with a morphism $\mathrm{D}(\mathscr{A}) \to \mathscr{T}$, and such that $\mathscr{T}$ satisfies the homotopy and the stability property. This can be made precise in the setting of algebraic derivators or of dg-categories (or any other kind of stable ∞-categories).

Proposition 5.3.30. *Let t and t' be two admissible topologies, with t' finer than t. Then $\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{t'}(\mathscr{P},\Lambda))$ is canonically equivalent to the the full subcategory of $\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P},\Lambda))$ spanned by the objects which satisfy t'-descent.*

Proof. It is sufficient to prove this proposition in the case where t is the coarse topology. We deduce from [**Ayo07b**, 4.4.42] that, for any scheme S in $\mathscr{S}$, we have

$$\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{t'}(\mathscr{P}/S,\Lambda)) = \mathrm{D}\left(\mathrm{PSh}(\mathscr{P}/S,\Lambda)\right)[\mathscr{W}^{-1}],$$

with $\mathscr{W} = \mathscr{W}_{t'} \cup \mathscr{W}_{\mathbf{A}^1} \cup \mathscr{W}_\Omega$, where $\mathscr{W}_{t'}$ is the set of maps of shape

$$\Sigma^\infty M_S(\mathcal{X})\{n\}[i] \to \Sigma^\infty M_S(X)\{n\}[i],$$

for any t'-hypercover $\mathcal{X} \to X$ and any integers $n \leq 0$ and i. The assertion is then a particular case of the description of the homotopy category of a left Bousfield localization. □

Example 5.3.31. We have the stable versions of the $\mathscr{P}$-premotivic categories introduced in example 5.2.17:
1) Consider the admissible topology $t = \mathrm{Nis}$. Following F. Morel, we define the *stable $\mathbf{A}^1$-derived premotivic category* as (see also the construction of [**Ayo07b**]):

$$\mathrm{D}_{\mathbf{A}^1,\Lambda} := \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\mathrm{Nis}}(Sm,\Lambda)) \quad \text{and} \quad \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda} := \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{Nis}}\left(\mathscr{S}^{ft},\Lambda\right)\right),$$

as well as the *generalized stable $\mathbf{A}^1$-derived premotivic category*[73]

$$\underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda} := \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{Nis}}\left(\mathscr{S}^{ft},\Lambda\right)\right). \tag{5.3.31.1}$$

[73] We will see in Example 6.1.10 that the generalized version contains the usual one as a full subcategory.

Given a scheme S, we shall also write:

(5.3.31.2) $$\mathrm{D}_{\mathbf{A}^1}(S,\Lambda) := \mathrm{D}_{\mathbf{A}^1,\Lambda}(S) \quad \text{and} \quad \underline{\mathrm{D}}_{\mathbf{A}^1}(S,\Lambda) := \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}(S).$$

In the case when $t = \text{ét}$, we get the triangulated premotivic categories of *étale premotives*:

$$\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Sm,\Lambda)) \quad \text{and} \quad \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\text{ét}}\left(\mathscr{S}^{ft},\Lambda\right)\right).$$

In each of these cases, we denote by $\Sigma^\infty \Lambda_S^t(X)$ the premotive associated with a smooth S-scheme X.

From the adjunction (5.1.24.2), we get an adjunction of triangulated premotivic categories:

$$a_{\text{ét}} : \mathrm{D}_{\mathbf{A}^1,\Lambda} \rightleftarrows \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Sm,\Lambda)) : \mathbf{R}O_{\text{ét}}.$$

2) Assume $\mathscr{P} = \mathscr{S}^{ft}$:

Consider the $\mathscr{S}^{ft}$-admissible topology $t = \mathrm{h}$ (resp. $t = \mathrm{qfh}$). In [**Voe96**], Voevodsky has introduced the category of effective h-motives (resp. qfh-motives). According to the theory presented above, one can extends this definition to the stable setting: one defines the category of stable h-*motives* (resp. qfh-*motives*) over S with coefficients in Λ as:

$$\underline{\mathrm{DM}}_{\mathrm{h}}(S,\Lambda) := \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{h}}\left(\mathscr{S}^{ft}/S,\Lambda\right)\right).$$
$$\text{resp. } \underline{\mathrm{DM}}_{\mathrm{qfh}}(S,\Lambda) := \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{qfh}}\left(\mathscr{S}^{ft}/S,\Lambda\right)\right).$$

In other words, this is the stable $\mathbf{A}^1$-derived category of h-sheaves (resp. qfh-sheaves) of Λ-modules. Moreover, we get the *generalized triangulated premotivic category of* h-*motives (resp.* qfh-*motives)* with coefficients in Λ over $\mathscr{S}$:

$$\underline{\mathrm{DM}}_{\mathrm{h},\Lambda} := \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{h}}\left(\mathscr{S}^{ft},\Lambda\right)\right).$$
$$\text{resp. } \underline{\mathrm{DM}}_{\mathrm{qfh},\Lambda} := \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}_{\mathrm{qfh}}\left(\mathscr{S}^{ft},\Lambda\right)\right).$$

For an S-scheme of finite type X, we will denote by $\Sigma^\infty \underline{\Lambda}_S^{\mathrm{h}}(X)$ (resp $\Sigma^\infty \underline{\Lambda}_S^{\mathrm{qfh}}(X)$) the corresponding premotive associated with X in $\underline{\mathrm{DM}}_t(S,\Lambda)$. Note that the h-sheafification functor induces a premotivic adjunction (see Paragraph 5.3.28):

(5.3.31.3) $$\underline{\mathrm{DM}}_{\mathrm{qfh},\Lambda} \rightleftarrows \underline{\mathrm{DM}}_{\mathrm{h},\Lambda}.$$

These generalized premotivic categories are too big to be reasonable (in particular for the localization property – see Remark 2.3.4). Therefore, we introduce the triangulated category $\mathrm{DM}_t(S,\Lambda)$ as the localizing subcategory of $\underline{\mathrm{DM}}_t(S,\Lambda)$ generated by objects of shape $\Sigma^\infty \Lambda_S^t(X)(p)[q]$ for any smooth S-scheme of finite type X and any integers p and q. The fibred category $\mathrm{DM}_{\mathrm{h},\Lambda}$ (resp. $\mathrm{DM}_{\mathrm{qfh},\Lambda}$) defined above is premotivic. We call it the *premotivic category of* h-*motives (resp.* qfh-*motives).* The family of inclusions

(5.3.31.4) $$\mathrm{DM}_t(S,\Lambda) \to \underline{\mathrm{DM}}_t(S,\Lambda)$$

indexed by a scheme S defines a premotivic morphism (the existence of right adjoints is ensured by the Brown representability theorem).

Remark 5.3.32. When $\Lambda = \mathbf{Q}$, we will show that the categories $\mathrm{DM}_{\mathrm{h},\mathbf{Q}}$ and $\mathrm{DM}_{\mathrm{qfh},\mathbf{Q}}$ are equivalent and satisfy the axioms of a motivic category. In fact, they are equivalent to the category of Beilinson motives. See Theorem 16.1.2 for all these results.

Proposition 5.3.33. *Consider the notations of the second point in the above example. Then the premotivic category* $\mathrm{DM}_{t,\Lambda}$ *satisfies* t*-descent.*

Proof. This is true for $\underline{\mathrm{DM}}_{t,\Lambda}$ by construction, which implies formally the assertion for $\mathrm{DM}_{t,\Lambda}$. □

Remark 5.3.34. According to Proposition 5.2.10 and Remark 5.3.29, for any admissible topology t, $\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P},\mathbf{Z}))$ is the universal derived $\mathscr{P}$-premotivic category satisfying t-descent as well as the homotopy and stability properties.

A crucial example for us: the stable $\mathbf{A}^1$-derived premotivic category $\mathrm{D}_{\mathbf{A}^1}$ is the universal derived premotivic category satisfying the properties of homotopy, of stability and of Nisnevich descent.

5.3.35. We assume $\mathscr{P} = Sm$.
Let $\mathrm{Sh}_\bullet(Sm)$ be the category of pointed Nisnevich sheaves of sets. Consider the pointed version of the adjunction of $\mathscr{P}$-premotivic categories

$$N : \Delta^{op}\,\mathrm{Sh}_\bullet(Sm) \rightleftarrows \mathrm{C}(\mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})) : K$$

constructed in 5.2.25.

If we consider on the left hand side the $\mathbf{A}^1$-model category defined by Blander [**Bla03**], (N_S, K_S) is a Quillen adjunction for any scheme S.

We consider $(\mathbf{G}_m, 1)$ as a constant pointed simplicial sheaf. The construction of symmetric $\mathbf{G}_m$-spectra respectively to the model category $\Delta^{op}\,\mathrm{Sh}_\bullet(Sm)$ can now be carried out following [**Jar00**] or [**Ayo07b**] and yields a symmetric monoidal model category whose homotopy category is the stable homotopy category of Morel and Voevodsky $\mathrm{SH}(S)$.

Using the functoriality statements [**Hov01**, th. 8.3 and 8.4], we finally obtain a $\mathscr{P}$-premotivic adjunction

(5.3.35.1) $$N : \mathrm{SH} \rightleftarrows \mathrm{D}_{\mathbf{A}^1} : K.$$

The functor K is the analog of the Eilenberg-Mac Lane functor in algebraic topology; in fact, this adjunction is actually induced by the Eilenberg-MacLane functor (see [**Ayo07b**, chap. 4]). In particular, as the rational model category of topological (symmetric) S^1-spectra is Quillen equivalent to the model category of complexes of $\mathbf{Q}$-vector spaces, we have a natural equivalence of premotivic categories

(5.3.35.2) $$\mathrm{SH}_{\mathbf{Q}} \rightleftarrows \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}\,,$$

(where $\mathrm{SH}_{\mathbf{Q}}(S)$ denotes the Verdier quotient of $\mathrm{SH}(S)$ by the localizing subcategory generated by compact torsion objects).

5.3.36. We can extend the considerations of Example 5.1.25 and Paragraph 5.2.22 on changing coefficients in categories of sheaves.

Let t be an admissible topology and $\varphi : \Lambda \to \Lambda'$ be an extension of rings. Using the $\mathscr{P}$-premotivic adjunction (5.1.25.1) and according to Paragraph 5.3.28, we get an adjunction of triangulated $\mathscr{P}$-premotivic categories:

$$\mathbf{L}\varphi_* : \mathrm{D}_{\mathbf{A}^1}\big(\mathrm{Sh}_t(\mathscr{P}, \Lambda)\big) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}\big(\mathrm{Sh}_t(\mathscr{P}, \Lambda')\big) : \mathbf{R}\varphi_*.$$

Givew two Tate spectra C and D of t-sheaves of Λ-modules over $\mathscr{P}_S$, we get a canonical morphism of Λ'-modules:

(5.3.36.1) $$\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}_S,\Lambda))}\big(C, D\big) \otimes_\Lambda \Lambda' \longrightarrow \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}_S,\Lambda'))}\big(\mathbf{L}\varphi^*(C), \mathbf{L}\varphi^*(D)\big)$$

Then the stable version of Proposition 5.2.23 holds (the proof is the same):

PROPOSITION 5.3.37. *Consider the above assumptions. Then the map* (5.3.36.1) *is an isomorphism in the two following cases:*

(1) If Λ' is a free Λ-module and C is compact;
(2) If Λ' is a free Λ-module of finite rank.

5.3.e. *Constructible premotivic spectra.*

LEMMA 5.3.38. *Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with a topology t and such that the category $\mathbf{A}^1$-derived category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ satisfies Nisnevich descent.*

Then, for any scheme S, the non trival cyclic permutation (123) *of order* 3 *acts as the identity on the premotive $\mathbb{1}_S\{1\}^{\otimes 3}$ in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$.*

PROOF. Using example 5.2.21, it is sufficient to prove this in $\mathrm{D}_{\mathbf{A}^1,\Lambda}(S)$, which is well known; see for example [**Ayo07b**, 4.5.65]. □

PROPOSITION 5.3.39. *Consider the hypothesis of the previous lemma and assume that the triangulated premotivic category* $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ *is compactly* τ*-generated.*

Then, for any scheme S*, any couple of integers* (i, a)*, any compact object* C *of* $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$ *and any Tate spectrum* E *in* $\mathscr{A}_S$*, we have a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)}(\mathbf{L}\Sigma^\infty(C)\{a\}, E[i]) \simeq \varinjlim_{r>>0} \mathrm{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)}(C\{a+r\}, E_r[i]).$$

PROOF. Given the previous lemma, this is a direct consequence of [**Ayo07b**, theorems 4.3.61 and 4.3.79]. □

COROLLARY 5.3.40. *Under the assumptions of the preceding proposition, the triangulated category* $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$ *is compactly* $(\mathbf{Z} \times \tau)$*-generated where the factor* $\mathbf{Z}$ *corresponds to the Tate twist.*

More precisely, if $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}(\mathscr{A}_S)$ *denotes the category of compact objects in* $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$*, then the category of compact objects in* $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$ *is canonically equivalent to the pseudo-abelian completion of the category obtained as the* 2*-colimit of the following diagram:*

$$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}(\mathscr{A}_S) \xrightarrow{\otimes \mathbb{1}_S\{1\}} \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}(\mathscr{A}_S) \longrightarrow \cdots \longrightarrow \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}(\mathscr{A}_S) \xrightarrow{\otimes \mathbb{1}_S\{1\}} \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,c}(\mathscr{A}_S) \longrightarrow \cdots$$

5.3.41. Let $\mathscr{A}$ be an abelian $\mathscr{P}$-premotivic category compatible with an admissible topology t. Assume that:

- The topology t is bounded in $\mathscr{A}$ (Definition 5.1.28).
- The abelian $\mathscr{P}$-premotivic category $\mathscr{A}$ is finitely τ-presented.

We will denote by $\mathscr{N}^t_S$ a bounded generating family for t-hypercovers in $\mathscr{A}_S$.

Recall from Proposition 5.2.38 that the category of compact objects of the triangulated category $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$ is canonically equivalent to the triangulated monoidal category:

$$\Big(K^b\big(\mathbf{Z}_S(Sm/S; \mathscr{A})\big)/(\mathscr{N}^t_S \cup \mathscr{T}_{\mathbf{A}^1_S})\Big)^\natural$$

Let us denote by $\mathrm{D}_{\mathbf{A}^1,gm}(\mathscr{A}_S)$ the category obtained from the monoidal category on the left hand side of the above functor by formally inverting the Tate twist $\mathbf{Z}^{\mathscr{A}}_S(1)$. Because $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ satisfies the stability property by construction, we readily obtains a canonical monoidal functor

$$\mathrm{D}_{\mathbf{A}^1,gm}(\mathscr{A}_S) \to \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S). \tag{5.3.41.1}$$

Then applying Proposition 5.2.38, the above corollary and Proposition 1.4.11, we deduce:

COROLLARY 5.3.42. *Consider the above hypothesis and notations.*

Then the triangulated premotivic category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ *is compactly* $(\mathbf{Z} \times \tau)$*-generated. For any premotive* $\mathcal{M}$ *in* $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$ *the following conditions are equivalent:*

(i) $\mathcal{M}$ *is compact;*
(ii) $\mathcal{M}$ *is* $(\mathbf{Z} \times \tau)$*-constructible.*

Moreover, the functor (5.3.41.1) *is fully faithful and has for essential image the compact (*i.e. τ*-constructible) objects of* $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$*.*

EXAMPLE 5.3.43. From the considerations of Example 5.2.40, we obtain that for any scheme S, the compact objects of the category $\mathrm{D}_{\mathbf{A}^1}(S, \Lambda)$ (resp. $\mathrm{D}_{\mathbf{A}^1}\Big(\mathrm{Sh}_{\mathrm{cdh}}(\mathscr{S}^{ft}/S, \Lambda)\Big)$) is obtained from the monoidal triangulated category

$$\mathrm{K}^b\left(\Lambda(Sm/S)\right) \text{ (resp. } \mathrm{K}^b\left(\Lambda(\mathscr{S}^{ft}/S)\right)\text{)}$$

by the following steps:

- one mods out by the triangulated subcategories $\mathcal{T}_{\mathbf{A}^1_S}$ and BG_S (resp. CDH_S) corresponding to the $\mathbf{A}^1$-homotopy property and the Brown-Gersten triangles (resp. cdh-triangles),
- one takes the pseudo-abelian envelope,
- one formally inverts the Tate twist.

Proposition 5.3.44. *Assume $\mathscr{P} = \mathscr{S}^{ft}$ is the class of finite type (resp. separated and of finite type) morphisms.*

Let $\mathscr{A}$ be an abelian generalized premotivic category compatible with an admissible topology t such that:

- *$\mathscr{A}$ satisfies property (C) of Paragraph 5.1.35.*
- *The $\mathbf{A}^1$-derived category $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A})$ is compactly τ-generated and satisfies Nisnevich descent.*

Then the stable $\mathbf{A}^1$-derived premotivic category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ is $(\mathbf{Z} \times \tau)$-continuous.

Proof. This is an immediate corollary of Proposition 5.2.41 combined with Proposition 5.3.39. □

Example 5.3.45. According to the previous proposition and the second point of Example 5.1.37, the generalized triangulated premotivic category $\underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}$ is continuous. We also refer the reader to Corollary 6.1.12 for an extension of this result to the non generalized case.

6. Localization and the universal derived example

6.0. In this section, $\mathscr{S}$ is an adequate category of $\mathcal{S}$-schemes as in 2.0. In sections 6.2 and 6.3, we assume in addition that the schemes in $\mathscr{S}$ are finite dimensional.

We will apply the definitions of the preceding section to the admissible class made of morphisms of finite type (resp. smooth morphisms of finite type) in $\mathscr{S}$, denoted by $\mathscr{S}^{ft}$ (resp. Sm).

Recall the general convention of section 1.4:

- *premotivic* means Sm-premotivic.
- *generalized premotivic* means $\mathscr{S}^{ft}$-premotivic.

6.1. Generalized derived premotivic categories.

Example 6.1.1. Let t be a $\mathscr{S}^{ft}$-admissible topology. For a scheme S, we denote by $\mathrm{Sh}_t\big(\mathscr{S}^{ft}/S, \Lambda\big)$ the category of sheaves of abelian groups on $\mathscr{S}^{ft}/S$ for the topology t_S. For an S-scheme of finite type X, we let $\underline{\Lambda}^t_S(X)$ be the free t-sheaf of Λ-modules represented by X. Recall $\mathrm{Sh}_t\big(\mathscr{S}^{ft}, \Lambda\big)$ is a generalized abelian premotivic category (see 5.1.4).
Let $\rho : Sm/S \to \mathscr{S}^{ft}/S$ be the obvious inclusion functor and let us denote by t_S the initial topology on Sm/S such that ρ is continuous. Then it induces (*cf.* [**SGA4**, IV, 4.10]) a sequence of adjoint functors

$$\mathrm{Sh}_t(Sm/S, \Lambda) \underset{\rho_*}{\overset{\rho_\sharp}{\rightrightarrows}} \xleftarrow{\rho^*} \mathrm{Sh}_t\big(\mathscr{S}^{ft}/S, \Lambda\big)$$

and we checked easily that this induces an enlargement of abelian premotivic categories:

$$\rho_\sharp : \mathrm{Sh}_t(Sm, \Lambda) \rightleftarrows \mathrm{Sh}_t\big(\mathscr{S}^{ft}, \Lambda\big) : \rho^*. \tag{6.1.1.1}$$

Remark 6.1.2. Note that for any scheme S, the abelian category $\mathrm{Sh}_t(Sm/S, \Lambda)$ can be described as the Gabriel quotient of the abelian category $\mathrm{Sh}_t\big(\mathscr{S}^{ft}/S, \Lambda\big)$ with respect to the sheaves $\underline{F}$ over $\mathscr{S}^{ft}/S$ such that $\rho^*(\underline{F}) = 0$.

An example of such a sheaf in the case where $t = \mathrm{Nis}$ and $\dim(S) > 0$ is the Nisnevich sheaf $\underline{\Lambda}_S(Z)$ on $\mathscr{S}^{ft}/S$ represented by a nowhere dense closed subscheme Z of S is zero when restricted to Sm/S.

6.1.3. Consider an abelian premotivic category $\mathscr{A}$ compatible with an admissible topology t on Sm and a generalized abelian premotivic category $\underline{\mathscr{A}}$ compatible with an admissible topology t' on $\mathscr{S}$. We denote by M (resp. $\underline{M}$) the geometric sections of $\mathscr{A}$ (resp. $\underline{\mathscr{A}}$). We assume that t' restricted to Sm is finer that t, and consider an adjunction of abelian premotivic categories:

$$\rho_\sharp : \mathscr{A} \rightleftarrows \underline{\mathscr{A}} : \rho^*.$$

Let S be a scheme in $\mathscr{S}$. The functors $\rho_\sharp$ and ρ^* induce a derived adjunction (see 5.2.19):

$$\mathbf{L}\rho_\sharp : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\underline{\mathscr{A}}_S) : \mathbf{R}\rho^*$$

(where $\underline{\mathscr{A}}$ is considered as an Sm-fibred category).

PROPOSITION 6.1.4. *Consider the previous hypothesis, and fix a scheme S. Assume furthermore that we have the following properties.*

(i) The functor $\rho_\sharp : \mathscr{A}_S \to \underline{\mathscr{A}}_S$ is fully faithful.
(ii) The functor $\rho^* : \underline{\mathscr{A}}_S \to \mathscr{A}_S$ commutes with small colimits.

Then, the following conditions hold :

(a) The induced functor
$$\rho^* : \mathrm{C}(\underline{\mathscr{A}}_S) \to \mathrm{C}(\mathscr{A}_S)$$
preserves $\mathbf{A}^1$-equivalences.
(b) The $\mathbf{A}^1$-derived functor $\mathbf{L}\rho_\sharp : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\underline{\mathscr{A}}_S)$ is fully faithful.

PROOF. Point (a) follows from Proposition 5.2.24. To prove (b), we have to prove that the unit map

$$M \to \rho^*\mathbf{L}\rho_\sharp(M)$$

is an isomorphism for any object M of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S)$. For this purpose, we may assume that M is cofibrant, so that we have

$$M \simeq \rho^*\rho_\sharp(M) \simeq \rho^*\mathbf{L}\rho_\sharp(M)$$

(where the first isomorphism holds already in $\mathrm{C}(\mathscr{A}_S)$). □

COROLLARY 6.1.5. *Consider the hypothesis of the previous proposition. Then the family of adjunctions $\mathbf{L}\rho_\sharp : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\underline{\mathscr{A}}_S) : \mathbf{R}\rho^*$ indexed by a scheme S induces an enlargement of triangulated premotivic categories*

$$\mathbf{L}\rho_\sharp : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}) \rightleftarrows \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\underline{\mathscr{A}}) : \mathbf{R}\rho^*.$$

EXAMPLE 6.1.6. Considering the situation of 6.1.1, we will be particularly interested in the case of the Nisnevich topology. We denote by $\underline{\mathrm{D}}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda}$ the generalized $\mathbf{A}^1$-derived premotivic category associated with $\mathrm{Sh}\big(\mathscr{S}^{ft}, \Lambda\big)$ (see also Example 5.3.31). The preceding corollary gives a canonical enlargement:

(6.1.6.1) $$\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda} \rightleftarrows \underline{\mathrm{D}}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda}$$

6.1.7. Consider again the hypothesis of 6.1.3. We denote simply by M (resp. $\underline{M}$) the geometric sections of the premotivic triangulated category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\underline{\mathscr{A}})$).

Recall from 5.3.15 that we have defined $\mathbb{1}_S\{1\}$ (resp. $\underline{\mathbb{1}}_S\{1\}$) as the cokernel of the canonical map $\mathbb{1}_S \to M_S(\mathbf{G}_{m,S})$ (resp. $\underline{\mathbb{1}}_S \to \underline{M}_S(\mathbf{G}_{m,S})$). Thus, it is obvious that we get a canonical identification $\rho_\sharp(\mathbb{1}_S\{1\}) = \underline{\mathbb{1}}_S\{1\}$. Therefore, the enlargement $\rho_\sharp$ can be extended canonically to an enlargement

$$\rho_\sharp : \mathrm{Sp}(\mathscr{A}) \rightleftarrows \mathrm{Sp}(\underline{\mathscr{A}}) : \rho^*$$

of abelian premotivic categories in such a way that for any scheme S, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A}_S & \xrightarrow{\rho_\sharp} & \underline{\mathscr{A}}_S \\ {\scriptstyle \Sigma^\infty_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle \Sigma^\infty_{\underline{\mathscr{A}}}} \\ \mathrm{Sp}(\mathscr{A}_S) & \xrightarrow{\rho_\sharp} & \mathrm{Sp}(\underline{\mathscr{A}}_S). \end{array}$$

According to Proposition 5.3.13, $\mathrm{Sp}(\mathscr{A})$ (resp. $\mathrm{Sp}(\underline{\mathscr{A}})$) is compatible with t (resp. t'), and we obtain an adjoint pair of functors (5.3.28):

$$\mathbf{L}\rho_\sharp : \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}(\underline{\mathscr{A}}_S) : \mathbf{R}\rho^*.$$

From the preceding commutative square, we get the identification:

(6.1.7.1) $$\mathbf{L}\rho_\sharp \circ \Sigma^\infty_{\mathscr{A}} = \Sigma^\infty_{\underline{\mathscr{A}}} \circ \mathbf{L}\rho_\sharp$$

As in the non effective case, we get the following result:

Proposition 6.1.8. *Keep the assumptions of Proposition 6.1.4, and suppose furthermore that both* $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ *and* $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\underline{\mathscr{A}})$ *are compactly* τ*-generated. Then the derived functor* $\mathbf{L}\rho_\sharp : \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}_{\mathbf{A}^1}(\underline{\mathscr{A}}_S)$ *is fully faithful.*

Proof. We have to prove that for any Tate spectrum E of $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$, the adjunction morphism

$$E \to \mathbf{L}\rho^* \mathbf{R}\rho_\sharp(E)$$

is an isomorphism. According to Proposition 1.3.20, the functor $\mathbf{L}\rho^*$ admits a right adjoint. Thus, applying Lemma 1.1.43, it is sufficient to consider the case where $E = M_S(X)\{i\}[n]$ for a smooth S-scheme X, and a couple $(n, i) \in \mathbf{Z} \times \tau$.

Moreover, it is sufficient to prove that for another smooth S-scheme Y and an integer $j \in \mathbf{Z}$, the induced morphism

$$\mathrm{Hom}(\Sigma^\infty M_S(Y)\{j\}, \Sigma^\infty M_S(X)\{i\}[n]) \to \mathrm{Hom}(\Sigma^\infty \underline{M}_S(Y)\{j\}, \Sigma^\infty \underline{M}_S(X)\{i\}[n])$$

is an isomorphism. Using the identification (6.1.7.1), propositions 5.3.39 and 6.1.4 allows to conclude. □

Corollary 6.1.9. *If the assumptions of Proposition 6.1.8 hold for any scheme S in $\mathscr{S}$, then we obtain an enlargement of triangulated premotivic categories*

$$\mathbf{L}\rho_\sharp : \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}) \rightleftarrows \mathrm{D}_{\mathbf{A}^1}(\underline{\mathscr{A}}) : \mathbf{R}\rho^*.$$

Example 6.1.10. Considering again the situation of 6.1.1, in the case of the Nisnevich topology. We denote by $\underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}$ the generalized stable $\mathbf{A}^1$-derived premotivic category associated with $\mathrm{Sh}\big(\mathscr{S}^{ft}, \Lambda\big)$. The preceding corollary gives a canonical enlargement:

$$\mathbf{L}\rho_\sharp : \mathrm{D}_{\mathbf{A}^1,\Lambda} \rightleftarrows \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda} : \mathbf{R}\rho^* \tag{6.1.10.1}$$

which is compatible with the enlargement (6.1.6.1) in the sense that the following diagram is essentially commutative:

$$\begin{array}{ccc} \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda} & \longrightarrow & \underline{\mathrm{D}}^{\mathit{eff}}_{\mathbf{A}^1,\Lambda} \\ {\scriptstyle \Sigma^\infty}\downarrow & & \downarrow{\scriptstyle \underline{\Sigma}^\infty} \\ \mathrm{D}_{\mathbf{A}^1,\Lambda} & \longrightarrow & \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda} \end{array}$$

Corollary 6.1.11. *Consider a Grothendieck topology t on our category of schemes $\mathscr{S}$. Let S be a scheme in $\mathscr{S}$, and M an object of* $\mathrm{D}_{\mathbf{A}^1,\Lambda}(S)$*. Then M satisfies t-descent in* $\mathrm{D}_{\mathbf{A}^1,\Lambda}(S)$ *if and only if* $\mathbf{L}\rho_\sharp(M)$ *satisfies t-descent in* $\underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}(S)$*.*

Proof. Let $f : \mathscr{X} \to S$ be a diagram of S-schemes of finite type. Define

$$H^q(\mathscr{X}, M(p)) = \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\Lambda}(S)}(\Lambda_{\mathscr{X}}, \mathbf{L}f^*(M)(p)[q])$$

$$\underline{H}^q(\mathscr{X}, M(p)) = \mathrm{Hom}_{\underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda}(S)}(\underline{\Lambda}_{\mathscr{X}}, \mathbf{L}f^*\, \mathbf{L}\rho_\sharp(M)(p)[q])$$

for any integers p and q. The full faithfulness of $\mathbf{L}\rho_\sharp$ ensures that the comparison map

$$H^q(\mathscr{X}, M(p)) \to \underline{H}^q(\mathscr{X}, M(p))$$

is always bijective. This proposition follows then from the fact that M (resp. $\mathbf{L}\rho_\sharp(M)$) satisfies t-descent if and only if, for any integers p and q, for any S-scheme of finite type X, and any t-hypercover $\mathscr{X} \to X$, the induced map

$$H^q(X, M(p)) \to H^q(\mathscr{X}, M(p)) \text{ (resp. } \underline{H}^q(X, M(p)) \to \underline{H}^q(\mathscr{X}, M(p))\text{)}$$

is bijective. □

We end-up this section with another interesting application of the preceding results.

Corollary 6.1.12. *Consider the hypothesis and assumptions of Proposition 6.1.4. We suppose furthermore that the generalized abelian premotivic category $\underline{\mathscr{A}}$ satisfies condition (C) of Paragraph 5.1.35.*

(1) *Then the triangulated premotivic category* $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A})$ *is* τ*-continuous.*
(2) *Assume furhtermore that* $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A})$ *and* $\mathrm{D}^{eff}_{\mathbf{A}^1}(\underline{\mathscr{A}})$ *are compactly* τ*-generated. Then the triangulated premotivic category* $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ *is* τ*-continuous.*

PROOF. According to Proposition 5.2.41, the category $\mathrm{D}^{eff}_{\mathbf{A}^1}(\underline{\mathscr{A}})$ is τ-continuous. According to Corollary 6.1.5, the functor $\mathbf{L}\rho_\sharp : \mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A}) \to \mathrm{D}^{eff}_{\mathbf{A}^1}(\underline{\mathscr{A}}) : \mathbf{R}\rho^*$ is fully faithful and commutes with $\mathbf{L}f^*$. Thus Point (1) follows.

In the assumption of Point (2), we deduce from Proposition 5.3.44 that $\mathrm{D}_{\mathbf{A}^1}(\underline{\mathscr{A}})$ is $(\mathbf{Z}\times\tau)$-continuous. Thus it is sufficient to apply Corollary 6.1.9 as in the effective case to get the assertion of Point (2). □

EXAMPLE 6.1.13. According to the second point of Example 5.1.37, we can apply this corollary to the enlargement

$$\mathrm{Sh}_{\mathrm{Nis}}(Sm,\Lambda) \to \mathrm{Sh}_{\mathrm{Nis}}\big(\mathscr{S}^{ft},\Lambda\big)\,.$$

Thus, we deduce that the triangulated premotivic categories $\mathrm{D}^{eff}_{\mathbf{A}^1,\Lambda}$ and $\mathrm{D}_{\mathbf{A}^1,\Lambda}$ both are continuous.

6.2. The fundamental example. Recall the following theorem of Ayoub [**Ayo07b**]:

THEOREM 6.2.1. *The triangulated premotivic categories* $\mathrm{D}^{eff}_{\mathbf{A}^1,\Lambda}$ *and* $\mathrm{D}_{\mathbf{A}^1,\Lambda}$ *satisfy the localization property.*

COROLLARY 6.2.2. (1) *The premotivic category* $\mathrm{D}_{\mathbf{A}^1,\Lambda}$ *is a motivic category.*
(2) *It is compactly generated by the Tate twist.*
(3) *Suppose that* $\mathscr{T}$ *is a derived premotivic category (see 5.2.9) which is a motivic category. Then there exists a canonical morphism of derived premotivic categories:*

$$\mathrm{D}_{\mathbf{A}^1,\mathbf{Z}} \to \mathscr{T}.$$

PROOF. The first assertion follows from the previous theorem and Remark 2.4.47. The second one follows from Corollary 5.3.42. The last one follows from Proposition 3.3.5 and Example 5.3.34. □

REMARK 6.2.3. Thus, Theorem 2.4.50 can be applied to $\mathrm{D}_{\mathbf{A}^1,\Lambda}$. In particular, for any separated morphism of finite type $f: T\to S$, there exists a pair of adjoint functors

$$f_! : \mathrm{D}_{\mathbf{A}^1,\Lambda}(T) \rightleftarrows \mathrm{D}_{\mathbf{A}^1,\Lambda}(S) : f^!$$

as in the theorem *loc. cit.* so that we have removed the quasi-projective assumption in [**Ayo07a**].

6.2.4. Because the cdh topology is finer than the Nisnevich topology, we get an adjunction of generalized premotivic categories:

$$a^*_{\mathrm{cdh}} : \underline{\mathrm{D}}_{\mathbf{A}^1,\Lambda} \rightleftarrows \mathrm{D}_{\mathbf{A}^1}\big(\mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft},\Lambda\big)\big) : \mathbf{R}a_{\mathrm{cdh},*}.$$

COROLLARY 6.2.5. *For any scheme* S*, the composite functor*

$$\mathrm{D}_{\mathbf{A}^1}(S,\Lambda) \to \underline{\mathrm{D}}_{\mathbf{A}^1}(S,\Lambda) \xrightarrow{a_{\mathrm{cdh}}} \mathrm{D}_{\mathbf{A}^1}\big(\mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft}/S,\Lambda\big)\big)$$

is fully faithful.

Moreover, it induces an enlargement of premotivic categories:

$$\mathrm{D}_{\mathbf{A}^1,\Lambda} \rightleftarrows \mathrm{D}_{\mathbf{A}^1}\big(\mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft},\Lambda\big)\big) \tag{6.2.5.1}$$

REMARK 6.2.6. This corollary is a generalisation in our derived setting of the main theorem of [**Voe10c**]. Note that if $\dim(S) > 0$, there is no hope that the above composite functor is essentially surjective because as soon as Z is a nowhere dense closed subscheme of S, the premotive $\underline{M}^{\mathrm{cdh}}_S(Z,\Lambda)$ does not belong to its image (*cf.* remark 6.1.2).

PROOF. According to Corollary 6.2.2 and Proposition 3.3.10, any Tate spectrum E of $\mathrm{D}_{\mathbf{A}^1}(S,\Lambda)$ satisfies cdh-descent in the derived premotivic category $\mathrm{D}_{\mathbf{A}^1,\Lambda}$, and this implies the first assertion by 5.3.30 and 6.1.11. The second one then follows from the fact the forgetful functor

$$\mathrm{D}_{\mathbf{A}^1}\big(\mathrm{Sh}_{\mathrm{cdh}}\big(\mathscr{S}^{ft}/S,\Lambda\big)\big) \to \underline{\mathrm{D}}_{\mathbf{A}^1}(S,\Lambda).$$

commutes with direct sums (its left adjoint preserves compact objects). □

6.3. Nearly Nisnevich sheaves.

6.3.1. In all this section, we fix an abelian premotivic category $\mathscr{A}$ and we consider the canonical premotivic adjunction (5.1.2.1) associated with $\mathscr{A}$.

We assume $\mathscr{A}$ satisfies the following properties.

(i) $\mathscr{A}$ is compatible with Nisnevich topology, so that we have from (5.1.2.1) a premotivic adjunction:

(6.3.1.1) $$\gamma^* : \mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z}) \rightleftarrows \mathscr{A} : \gamma_*.$$

(ii) $\mathscr{A}$ is finitely presented (*i.e.* the functors $\mathrm{Hom}_{\mathscr{A}_S}(M_S(X), -)$ preserve filtered colimits and form a conservative family, Def. 1.3.11).

(iii) For any scheme S, and for any open immersion $U \to X$ of smooth S-schemes, the map $M_S(U) \to M_S(X)$ is a monomorphism.

(iv) For any scheme S, the functor $\gamma_* : \mathscr{A}_S \to \mathrm{Sh}_{\mathrm{Nis}}(Sm/S, \mathbf{Z})$ is exact.

Note that the functor $\gamma_* : \mathscr{A}_S \to \mathrm{Sh}_{\mathrm{Nis}}(Sm/S, \mathbf{Z})$ is exact and conservative. As it also preserves filtered colimits, this functor preserves in fact small colimits.

Observe also that, according to these assumption, the abelian premotivic category of Tate spectra $\mathrm{Sp}(\mathscr{A})$ is compatible with Nisnevich topology, $\mathbf{N}$-generated. Moreover, we get a canonical premotivic adjunction

(6.3.1.2) $$\gamma^* : \mathrm{Sp}(\mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})) \rightleftarrows \mathrm{Sp}(\mathscr{A}) : \gamma_*$$

such that γ_* is conservative and preserves small colimits.

In the following, we show how one can deduce properties of the premotivic triangulated categories $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A})$ and $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ from the good properties of $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Z}}$ and $\mathrm{D}_{\mathbf{A}^1,\mathbf{Z}}$.

6.3.a. *Support property (effective case).*

Proposition 6.3.2. *For any scheme S, the functor $\gamma_* : \mathrm{C}(\mathscr{A}_S) \to \mathrm{C}(\mathrm{Sh}_{\mathrm{Nis}}(Sm/S, \mathbf{Z}))$ preserves and detects $\mathbf{A}^1$-equivalences.*

Proof. It follows immediately from Corollary 5.2.31 that γ_* preserves $\mathbf{A}^1$-equivalences. The fact it detects them can be rephrased by saying that the induced functor

$$\gamma_* : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Z}}(S)$$

is conservative. This is obviously true once we noticed that its left adjoint is essentially surjective on generators. □

Corollary 6.3.3. *The right derived functor*

$$\mathbf{R}\gamma_* = \gamma_* : \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Z}}(S)$$

is conservative.

Proposition 6.3.4. *Let $f : S' \to S$ be a finite morphism of schemes. Then the induced functor*

$$f_* : \mathrm{C}(\mathscr{A}_{S'}) \to \mathrm{C}(\mathscr{A}_S)$$

preserves colimits and $\mathbf{A}^1$-equivalences.

Proof. We first prove f_* preserves colimits. We know the functors γ_* preserve colimits and are conservative. As we have the identification $\gamma_* f_* = f_* \gamma_*$, it is sufficient to prove the property for $\mathscr{A} = \mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})$. Let X be a smooth S-scheme. It is sufficient to prove that, for any point x of X, if X^h_x denotes the henselianization of X at x, the functor

$$\mathrm{Sh}_{\mathrm{Nis}}(Sm/S', \mathbf{Z}) \to \mathscr{A}b \quad , \qquad F \mapsto f_*(F)(X^h_x) = F(S' \times_S X^h_x)$$

commutes to colimits. Moreover the scheme $S' \times_S X^h_x$ is finite over X^h_x, so that we have $S' \times_S X^h_x = \amalg_i Y_i$, where the Y_i's are a finite family of henselian local schemes over $S' \times_S X^h_x$. Hence we have to check that the functor $F \mapsto \bigoplus_i F(Y_i)$ preserves colimits. As colimits commute to sums, it is thus sufficient to prove that the functors $F \mapsto F(Y_i)$ commute to colimits. This follows from the fact that the local henselian schemes Y_i are points of the topos of sheaves over the small Nisnevich site of X.

We are left to prove that the functor $f_* : \mathrm{C}(\mathscr{A}_{S'}) \to \mathrm{C}(\mathscr{A}_S)$ respects $\mathbf{A}^1$-equivalences. For this, we shall study the behaviour of f_* with respect to the $\mathbf{A}^1$-resolution functor constructed in 5.2.26. Note that f_* commutes to limits because it has a left adjoint. In particular, we know that f_* is exact. Moreover, one checks easily that $f_* R^{(n)}_{\mathbf{A}^1} = f_* R^{(n)}_{\mathbf{A}^1}$. As f_* commutes to colimits, this gives the formula $f_* R_{\mathbf{A}^1} = R_{\mathbf{A}^1} f_*$. Let C be a complex of Nisnevich sheaves of abelian groups on Sm/S'. Choose a quasi-isomorphism $C \to C'$ with C' a Nis-flasque complex. Applying Proposition 5.2.28, we know that $R_{\mathbf{A}^1}(C')$ is $\mathbf{A}^1$-fibrant and that we get a canonical $\mathbf{A}^1$-equivalence

$$f_*(C) \to f_*(C') \to f_*(R_{\mathbf{A}^1}(C')) = R_{\mathbf{A}^1}(f_*(C')).$$

Hence we are reduced to prove that f_* preserves $\mathbf{A}^1$-equivalences between $\mathbf{A}^1$-fibrant objects. But such $\mathbf{A}^1$-equivalences are quasi-isomorphisms, so that we can conclude using the exactness of f_*. □

PROPOSITION 6.3.5. *For any open immersion of schemes $j : U \to S$, the exchange transformation $j_\sharp \gamma_* \to \gamma_* j_\sharp$ is an isomorphism of functors.*

PROOF. Let X be a scheme, and F a Nisnevich sheaf of abelian groups on Sm/X. Define the category $\mathscr{C}_F$ as follows. The objects are the couples (Y, s), where Y is a smooth scheme over X, and s is a section of F over Y. The arrows $(Y, s) \to (Y', s')$ are the morphisms $f \in \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{Nis}}(Sm/X,\mathbf{Z})}(\mathbf{Z}_X(Y), \mathbf{Z}_X(Y'))$ such that $f^*(s') = s$. We have a canonical functor

$$\varphi_F : \mathscr{C}_F \to \mathrm{Sh}_{\mathrm{Nis}}(Sm/X, \mathbf{Z})$$

defined by $\varphi_F(Y, s) = \mathbf{Z}_X(Y)$, and one checks easily that the canonical map

$$\varinjlim_{\mathscr{C}_F} \varphi_F = \varinjlim_{(Y,s)\in\mathscr{C}_F} \mathbf{Z}_X(Y) \to F$$

is an isomorphism in $\mathrm{Sh}_{\mathrm{Nis}}(Sm/X, \mathbf{Z})$ (this is essentially a reformulation of the Yoneda lemma).

Consider now an object F in the category $\mathscr{A}_U$. We get two categories $\mathscr{C}_{\gamma_*(F)}$ and $\mathscr{C}_{\gamma_*(j_\sharp(F))}$. There is a functor

$$i : \mathscr{C}_{\gamma_*(F)} \to \mathscr{C}_{\gamma_*(j_\sharp(F))}$$

which is defined by the formula $i(Y, s) = (Y, j_\sharp(s))$. To explain our notations, let us say that we see s as a morphism from $M_S(U, \mathscr{A})$ to F, so that $j_\sharp(s)$ is a morphism from $M_S(Y, \mathscr{A}) = j_\sharp M_S(U, \mathscr{A})$ to $j_\sharp(F)$. This functor i has right adjoint

$$i' : \mathscr{C}_{\gamma_*(j_\sharp(F))} \to \mathscr{C}_{\gamma_*(F)}$$

defined by $i'(Y, s) = (Y_U, s_U)$, where $Y_U = Y \times_S U$, and s_U is the section of $\gamma_*(F)$ over Y_U that corresponds to the section $j^*(s)$ of $j^* j_\sharp \gamma_*(F)$ over Y_U under the canonical isomorphism $\gamma_*(F) \simeq j^* j_\sharp \gamma_*(F)$ (here, we use strongly the fact the functor $j_\sharp$ is fully faithful). The existence of a right adjoint implies i is cofinal. This latter property is sufficient for the canonical morphism

$$\varinjlim_{C_{\gamma_*(F)}} \varphi_{\gamma_*(j_\sharp(F))} \circ i \to \varinjlim_{\mathscr{C}_{\gamma_*(j_\sharp(F))}} \varphi_{\gamma_*(j_\sharp(F))} = \gamma_*(j_\sharp(F))$$

to be an isomorphism. But the functor $\varphi_{\gamma_*(j_\sharp(F))} \circ i$ is exactly the composition of the functor $j_\sharp$ with $\varphi_{\gamma_*(F)}$. As the functor $j_\sharp$ commutes to colimits, we have

$$\varinjlim_{C_{\gamma_*(F)}} \varphi_{\gamma_*(j_\sharp(F))} \circ i = \varinjlim_{C_{\gamma_*(F)}} j_\sharp\, \varphi_{\gamma_*(F)} \simeq j_\sharp \varinjlim_{C_{\gamma_*(F)}} \varphi_{\gamma_*(F)} \simeq j_\sharp(\gamma_*(F)).$$

Hence we obtain a canonical isomorphism $j_\sharp(\gamma_*(F)) \simeq \gamma_*(j_\sharp(F))$. It is easily seen that the corresponding map $\gamma_*(F) \to j^*(\gamma_*(j_\sharp(F))) = \gamma_*(j^* j_\sharp(F))$ is the image by γ_* of the unit map $F \to j^* j_\sharp(F)$. This shows the isomorphism we have constructed is the exchange morphism. □

COROLLARY 6.3.6. *For any open immersion of schemes $j : U \to S$, the functor $j_\sharp : \mathscr{A}_U \to \mathscr{A}_S$ is exact. Moreover, the induced functor*

$$j_\sharp : \mathrm{C}(\mathscr{A}_U) \to \mathrm{C}(\mathscr{A}_S)$$

preserves $\mathbf{A}^1$*-equivalences.*

Proof. Using the fact γ_* is exact and conservative, and propositions 6.3.2 and 6.3.5, it is sufficient to prove this corollary when $\mathscr{A} = \mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})$. It is straightforward to prove exactness using Nisnevich points. The fact $j_\sharp$ preserves $\mathbf{A}^1$-equivalences follows from the exactness property and from the obvious fact it preserves strong $\mathbf{A}^1$-equivalences. □

Corollary 6.3.7. *Let* $j : U \to S$ *be an open immersion of schemes. For any object* M *of* $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathscr{A}_U)$ *the exchange morphism*

$$\mathbf{L}j_\sharp(\mathbf{R}\gamma_*(M)) \to \mathbf{R}\gamma_*(\mathbf{L}j_\sharp(M)) \tag{6.3.7.1}$$

is an isomorphism in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{Z})$*.*

6.3.b. *Support property (stable case).*

6.3.8. Recall from 5.3.17 that the premotivic adjunction (γ^*, γ_*) induces a canonical adjunction of abelian premotivic categories that we denote by:

$$\tilde{\gamma}^* : \mathrm{Sp}(\mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})) \rightleftarrows \mathrm{Sp}(\mathscr{A}_S) : \tilde{\gamma}_*$$

Proposition 6.3.9. *For any scheme* S*, the functor induced functor*

$$\tilde{\gamma}_* : \mathrm{C}\left(\mathrm{Sp}(\mathscr{A}_S)\right) \rightleftarrows \mathrm{C}\left(\mathrm{Sp}(\mathrm{Sh}_{\mathrm{Nis}}(Sm/S, \mathbf{Z}))\right)$$

preserves and detects stable $\mathbf{A}^1$*-equivalences.*

Proof. Using the equivalence between symmetric Tate spectra and non symmetric Tate spectra, we are reduced to prove this for complexes of non symmetric Tate spectra. Consider a non symmetric Tate spectrum $(E_n)_{n\in\mathbf{N}}$ with suspension maps $\sigma_n : E_n\{1\} \to E_{n+1}$. The non symmetric Tate spectrum $\tilde{\gamma}_*(E)$ is equal to $\gamma_*(E_n)$ in degree $n \in \mathbf{Z}$, and the suspension map is given by the composite:

$$\mathbb{1}_S\{1\} \otimes_S \gamma_*(E_n) \to \gamma_*(\gamma^*(\mathbb{1}_S\{1\}) \otimes_S E_n) = \gamma_*(E_n\{1\}) \xrightarrow{\gamma_*(\sigma_n)} E_{n+1}.$$

Thus, propositions 6.3.2 and 5.3.40 allows to conclude. □

Corollary 6.3.10. *The right derived functor*

$$\mathbf{R}\gamma_* = \gamma_* : \mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S) \to \mathrm{D}_{\mathbf{A}^1,\mathbf{Z}}(S)$$

is conservative.

Proposition 6.3.11. *Let* $j : U \to X$ *be an open immersion of schemes. For any object* M *of* $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_U)$*, the exchange morphism*

$$\mathbf{L}j_\sharp(\mathbf{R}\gamma_*(M)) \to \mathbf{R}\gamma_*(\mathbf{L}j_\sharp(M))$$

is an isomorphism in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Z}}(X)$*.*

Proof. From Corollary 6.3.6 and the $\mathscr{P}$-base change formula for the open immersion j, one deduces easily that $j_\sharp$ preserves stable $\mathbf{A}^1$-equivalences of (non symmetric) Tate spectra. Moreover, Proposition 6.3.5 shows that $j_\sharp\gamma_* = \gamma_* j_\sharp$ at the level of Tate spectra. This concludes. □

Corollary 6.3.12. *The triangulated premotivic category* $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ *satisfies the support property.*

Proof. According to corollary 6.3.10, the functor $\mathbf{R}\gamma_*$ is conservative. Thus, by virtue of the preceding proposition, to prove the support property in the case of $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ it is sufficient to prove it in the case where $\mathscr{A} = \mathrm{Sh}_{\mathrm{Nis}}(Sm, \mathbf{Z})$. This follows from theorems 6.2.1 and 2.4.50. □

6.3.c. *Localization for smooth schemes.*

LEMMA 6.3.13. *Let $i : Z \to S$ be a closed immersion which admits a smooth retraction $p : S \to Z$. Then the exchange transformation*

$$\mathbf{L}\gamma^* \mathbf{R}i_* \to \mathbf{R}i_* \mathbf{L}\gamma^*$$

is an isomorphism in $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A}_S)$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A}_S)$).

PROOF. We first remark that for any object C of $\mathrm{C}(\mathscr{A}_Z)$ (resp. $\mathrm{C}(\mathrm{Sp}(\mathscr{A}_Z))$) the canonical sequence

$$j_\sharp (pj)^*(C) \to p^*(C) \to i_*(C)$$

is a cofiber sequence in $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A}_S)$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})_S)$). Indeed, we can check this after applying the exact conservative functor γ_*. The sequence we obtain is canonically isomorphic through exchange transformations to

$$j_\sharp j^* p^*(\gamma_* C) \to p^*(\gamma_* C) \to i_* i^* p^*(\gamma_* C)$$

using Corollary 6.3.7, the commutation of γ_* with j^*, p^* and i_* (recall it is the right adjoint of a premotivic adjunction) and the relation $pi = 1$. But this last sequence is a cofiber sequence in $\mathrm{D}^{eff}_{\mathbf{A}^1,\mathbf{Z}}(S)$ (resp. $\mathrm{D}_{\mathbf{A}^1,\mathbf{Z}}(S)$) because it satisfies the localization property (see 6.2.1).

Using exchange transformations, we obtain a morphism of distinguished triangles in $\mathrm{DM}^{eff}_{\mathbf{Z}}(S)$

$$\begin{array}{ccccccc}
\gamma^* j_\sharp j^* p^*(C) & \longrightarrow & \gamma^* p^*(C) & \longrightarrow & \gamma^* i_*(C) & \longrightarrow & \gamma^* j_\sharp j^* p^*(C)[1] \\
\| & & \| & & \downarrow{\scriptstyle Ex(\gamma^*, i_*)} & & \| \\
j_\sharp j^* p^*(\gamma^* C) & \longrightarrow & p^*(\gamma^* C) & \longrightarrow & i_*(\gamma^* C) & \longrightarrow & j_\sharp j^* p^*(\gamma^* C)[1]
\end{array}$$

The first two vertical arrows are isomorphisms as γ^* is the left adjoint of a premotivic adjunction; thus the morphism $Ex(\gamma^*, i_*)$ is also an isomorphism. □

PROPOSITION 6.3.14. *Let $i : Z \to S$ be a closed immersion. If i admits a smooth retraction, then $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A})$ satisfies (Loc_i).*

PROOF. This follows from Proposition 2.3.19 and the preceding lemma. □

COROLLARY 6.3.15. *Let S be a scheme. Then the premotivic category $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$) satisfies localization with respect to any closed immersion between smooth S-schemes.*

PROOF. Let $i : Z \to X$ be closed immersion between smooth S-schemes. We want to prove that $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$) satisfies localization with respect to i. According to 2.3.18, it is sufficient to prove that for any smooth S-scheme S, the canonical map

$$M_S(X/X - X_Z) \to i_* M_Z(X_Z)$$

is an isomorphism where we use the notation of *loc. cit.* and $M(., \mathscr{A})$ denotes the geometric sections of $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$). But the premotivic triangulated category category $\mathrm{D}_{\mathbf{A}^1}(\mathscr{A})$ (resp. $\mathrm{D}^{eff}_{\mathbf{A}^1}(\mathscr{A})$) satisfies the Nisnevich separation property and the Sm-base change property. Thus, we can argue locally in S for the Nisnevich topology. Thus, the statement is reduced to the preceding proposition as i admits locally for the Nisnevich topology a smooth retraction (see for example [**Dég07**, 4.5.11]). □

7. Basic homotopy commutative algebra

7.1. Rings.

DEFINITION 7.1.1. A symmetric monoidal model category $\mathscr{V}$ satisfies the *monoid axiom* if, for any trivial cofibration $A \to B$ and any object X, the smallest class of maps of $\mathscr{V}$ which contains the map $X \otimes A \to X \otimes B$ and is stable by pushouts and transfinite compositions is contained in the class of weak equivalences.

7.1.2. Let $\mathscr{V}$ be a symmetric monoidal category. We denote by $Mon(\mathscr{V})$ the category of monoids in $\mathscr{V}$. If $\mathscr{V}$ has small colimits, the forgetful functor

$$U : Mon(\mathscr{V}) \to \mathscr{V}$$

has a left adjoint

$$F : \mathscr{V} \to Mon(\mathscr{V})\,.$$

THEOREM 7.1.3. *Let $\mathscr{V}$ a symmetric monoidal combinatorial model category which satisfies the monoid axiom. The category of monoids $Mon(\mathscr{V})$ is endowed with the structure of a combinatorial model category whose weak equivalences (resp. fibrations) are the morphisms of commutative monoids which are weak equivalences (resp. fibrations) in $\mathscr{V}$. In particular, the forgetful functor $U : Mon(\mathscr{V}) \to \mathscr{V}$ is a right Quillen functor. Moreover, if the unit object of $\mathscr{V}$ is cofibrant, then any cofibrant object of $Mon(\mathscr{V})$ is cofibrant as an object of $\mathscr{V}$.*

PROOF. This is very a particular case of the third assertion of [**SS00**, Theorem 4.1] (the fact that $Mon(\mathscr{V})$ is combinatorial whenever $\mathscr{V}$ is so comes for instance from [**Bek00**, Proposition 2.3]). □

DEFINITION 7.1.4. A symmetric monoidal model category $\mathscr{V}$ is *strongly* $\mathbf{Q}$*-linear* if the underlying category of $\mathscr{V}$ is additive and $\mathbf{Q}$-linear (i.e. all the objects of $\mathscr{V}$ are uniquely divisible).

REMARK 7.1.5. If $\mathscr{V}$ is a strongly $\mathbf{Q}$-linear stable model category, then it is $\mathbf{Q}$-linear in the sense of 3.2.14.

LEMMA 7.1.6. *Let $\mathscr{V}$ be a strongly $\mathbf{Q}$-linear model category, G a finite group, and $u : E \to F$ an equivariant morphism of representations of G in $\mathscr{V}$. Then, if u is a cofibration in $\mathscr{V}$, so is the induced map $E_G \to F_G$ (where the subscript G denotes the coinvariants under the action of the group G).*

PROOF. The map $E_G \to F_G$ is easily seen to be a direct factor (retract) of the cofibration $E \to F$. □

7.1.7. If $\mathscr{V}$ is a symmetric monoidal category, we denote by $Comm(\mathscr{V})$ the category of commutative monoids in $\mathscr{V}$. If $\mathscr{V}$ has small colimits, the forgetful functor

$$U : Comm(\mathscr{V}) \to \mathscr{V}$$

has a left adjoint

$$F : \mathscr{V} \to Comm(\mathscr{V})\,.$$

THEOREM 7.1.8. *Let $\mathscr{V}$ a symmetric monoidal combinatorial model category. Assume that $\mathscr{V}$ is left proper and tractable, satisfies the monoid axiom, and is strongly $\mathbf{Q}$-linear. Then the category of commutative monoids $Comm(\mathscr{V})$ is endowed with the structure of a combinatorial model category whose weak equivalences (resp. fibrations) are the morphisms of commutative monoids which are weak equivalences (resp. fibrations) in $\mathscr{V}$. In particular, the forgetful functor $U : Comm(\mathscr{V}) \to \mathscr{V}$ is a right Quillen functor.*

If moreover the unit object of $\mathscr{V}$ is cofibrant, then any cofibrant object of $Comm(\mathscr{V})$ is cofibrant as an object of $\mathscr{V}$.

PROOF. The preceding lemma implies immediately that $\mathscr{V}$ is freely powered in the sense of [**Lur12**, Definition 4.3.17], so that the existence of this model category structure follows from a general result of Lurie [**Lur12**, Proposition 4.3.21]. The second assertion is then true by definition. The last assertion is proved by a careful analysis of pushouts by free maps in $Comm(\mathscr{V})$ as follows. For two cofibrations $u : A \to B$ and $v : C \to D$ in $\mathscr{V}$, write $u \wedge v$ for the map

$$u \wedge v : A \otimes D \amalg_{A \otimes C} B \otimes C \to B \otimes D$$

(which is a cofibration by definition of monoidal model categories). By iterating this construction, we get, for a cofibration $u : A \to B$ in $\mathscr{V}$, a cofibration

$$\wedge^n(u) = \underbrace{u \wedge \cdots \wedge u}_{n \text{ times}} : \square^n(u) \to B^{\otimes n}\,.$$

Note that the symmetric group $\mathfrak{S}_n$ acts naturally on $B^{\otimes n}$ and $\square^n(u)$. We define

$$Sym^n(B) = (B^{\otimes n})_{\mathfrak{S}_n} \quad \text{and} \quad Sym^n(B, A) = \square^n(u)_{\mathfrak{S}_n}\,.$$

By virtue of Lemma 7.1.6, we get a cofibration of $\mathscr{V}$:

$$\sigma^n(u) : Sym^n(B, A) \to Sym^n(B)\,.$$

Consider now the free map $F(u) : F(A) \to F(B)$ can be filtered by $F(A)$-modules as follows. Define $D_0 = F(A)$. As $A = Sym^1(B, A)$, we have a natural morphism $F(A)\otimes Sym^1(B, A) \to F(A)$. The objects D_n are then defined by induction with the pushouts below.

$$\begin{array}{ccc} F(A) \otimes Sym^n(B, A) & \xrightarrow{1_{F(A)\otimes\sigma^n(u)}} & F(A) \otimes Sym^n(B) \\ \downarrow & & \downarrow \\ D_{n-1} & \longrightarrow & D_n \end{array}$$

We get natural maps $D_n \to F(B)$ which induce an isomorphism

$$\varinjlim_{n\geq 0} D_n \simeq F(B)$$

in such a way that the morphism $F(u)$ correspond to the canonical map

$$F(A) = D_0 \to \varinjlim_{n\geq 0} D_n\,.$$

Hence, if $F(A)$ is cofibrant, all the maps $D_{n-1} \to D_n$ are cofibrations, so that the map $F(A) \to F(B)$ is a cofibration in $\mathscr{V}$. In the particular case where A is the initial object of $\mathscr{V}$, we see that for any cofibrant object B of $\mathscr{V}$, the free commutative monoid $F(B)$ is cofibrant as an object of $\mathscr{V}$ (because the initial object of $Comm(\mathscr{V})$ is the unit object of $\mathscr{V}$). This also implies that, if u is a cofibration between cofibrant objects, the map $F(u)$ is a cofibration in $\mathscr{V}$.

This description of $F(u)$ also allows to compute the pushouts of $F(u)$ in $Comm(\mathscr{V})$ in $\mathscr{V}$ as follows. Consider a pushout

$$\begin{array}{ccc} F(A) & \xrightarrow{F(u)} & F(B) \\ \downarrow & & \downarrow \\ R & \xrightarrow[v]{} & S \end{array}$$

in $Comm(\mathscr{V})$. For $n \geq 0$, define R_n by the pushouts of $\mathscr{V}$:

$$\begin{array}{ccc} F(A) & \longrightarrow & D_n \\ \downarrow & & \downarrow \\ R & \longrightarrow & R_n \end{array}$$

We then have an isomorphism

$$\varinjlim_{n\geq 0} R_n \simeq S\,.$$

In particular, if u is a cofibration between cofibrant objects, the morphism of commutative monoids $v : R \to S$ is then a cofibration in $\mathscr{V}$. As the forgetful functor U preserves filtered colimits, conclude easily from there (with the small object argument [**Hov99**, Theorem 2.1.14]) that any cofibration of $Comm(\mathscr{V})$ is a cofibration of $\mathscr{V}$. Using again that the unit object of $\mathscr{V}$ is cofibrant in $\mathscr{V}$ (i.e. that the initial object of $Comm(\mathscr{V})$ is cofibrant in $\mathscr{V}$) this proves the last assertion of the theorem. □

COROLLARY 7.1.9. *Let $\mathscr{V}$ a symmetric monoidal combinatorial model category. Assume that $\mathscr{V}$ is left proper and tractable, satisfies the monoid axiom, and is strongly* $\mathbf{Q}$*-linear. Consider a small set H of maps of $\mathscr{V}$, and denote by $L_H\mathscr{V}$ the left Bousfield localization of $\mathscr{V}$ by H; see* [**Bar10**, Theorem 4.7]. *Define the class of H-equivalences in* $\mathrm{Ho}(\mathscr{V})$ *to be the class of maps which become invertible in* $\mathrm{Ho}(L_H\mathscr{V})$. *If H-equivalences are stable by (derived) tensor product in* $\mathrm{Ho}(\mathscr{V})$, *then $L_H\mathscr{V}$ is a symmetric monoidal combinatorial model category (which is again left proper and tractable, satisfies the monoid axiom, and is strongly* $\mathbf{Q}$*-linear).*

In particular, under these assumtions, there exists a morphism of commutative monoids $\mathbb{1} \to R$ in $\mathscr{V}$ which is a weak equivalence of $L_H\mathscr{V}$, with R a cofibrant and fibrant object of $L_H\mathscr{V}$.

Proof. The first assertion is a triviality. The last assertion follows immediately: the map $\mathbb{1} \to R$ is simply obtained as a fibrant replacement of $\mathbb{1}$ in the model category $Comm(L_H\mathscr{V})$ obtained from Theorem 7.1.8 applied to $L_H\mathscr{V}$. □

7.1.10. Consider now a category $\mathscr{S}$, as well as a closed symmetric monoidal bifibred category $\mathscr{M}$ over $\mathscr{S}$. We shall also assume that the fibers of $\mathscr{M}$ admit limits and colimits.

Then the categories $Mon(\mathscr{M}(X))$ (resp. $Comm(\mathscr{M}(X))$) define a bifibred category over $\mathscr{S}$ as follows. Given a morphism $f : X \to Y$, the functor

$$f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$$

is symmetric monoidal, so that it preserves monoids (resp. commutative monoids) as well as morphisms between them. It thus induces a functor

$$\begin{gathered} f^* : Mon(\mathscr{M}(Y)) \to Mon(\mathscr{M}(X)) \\ (\text{resp. } f^* : Comm(\mathscr{M}(Y)) \to Comm(\mathscr{M}(X))\,). \end{gathered} \tag{7.1.10.1}$$

As $f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$ is symmetric monoidal, its right adjoint f_* is lax monoidal: there is a natural morphism

$$\mathbb{1}_Y \to f_*(\mathbb{1}_X) = f_*\, f^*(\mathbb{1}_Y)\,, \tag{7.1.10.2}$$

and, for any objects A and B of $\mathscr{M}(X)$, there is a natural morphism

$$f_*(A) \otimes_Y f_*(B) \to f_*(A \otimes_X B) \tag{7.1.10.3}$$

which corresponds by adjunction to the map

$$f^*(f_*(A) \otimes_Y f_*(B)) \simeq f^*\, f_*(A) \otimes f^*\, f_*(B) \to A \otimes B\,.$$

Hence the functor f_* preserves also monoids (resp. commutative monoids) as well as morphisms between them, so that we get a functor

$$\begin{gathered} f_* : Mon(\mathscr{M}(X)) \to Mon(\mathscr{M}(Y)) \\ (\text{resp. } f_* : Comm(\mathscr{M}(X)) \to Comm(\mathscr{M}(Y))\,). \end{gathered} \tag{7.1.10.4}$$

By construction, the functor f^* of (7.1.10.1) is a left adjoint ot the functor f_* of (7.1.10.4). These constructions extend to morphisms of $\mathscr{S}$-diagrams in a similar way.

Proposition 7.1.11. *Let $\mathscr{M}$ be a symmetric monoidal combinatorial fibred model category over $\mathscr{S}$. Assume that, for any object X of $\mathscr{S}$, the model category $\mathscr{M}(X)$ satisfies the monoid axiom (resp. is left proper and tractable, satisfies the monoid axiom, and is strongly* $\mathbf{Q}$*-linear).*

(a) For any object X of $\mathscr{S}$, the category $Mon(\mathscr{M})(X)$ (resp. $Comm(\mathscr{M})(X)$) of monoids (resp. of commutative monoids) in $\mathscr{M}(X)$ is a combinatorial model category structure whose weak equivalences (resp. fibrations) are the morphisms of commutative monoids which are weak equivalences (resp. fibrations) in $\mathscr{M}(X)$. This turns $Mon(\mathscr{M})$ (resp. $Comm(\mathscr{M})$) into a combinatorial fibred model category over $\mathscr{S}$.

(b) For any morphism of $\mathscr{S}$-diagrams $\varphi : (\mathscr{X}, I) \to (Y, J)$, the adjunction

$$\varphi^* : Mon(\mathscr{M})(\mathscr{Y}, J) \rightleftarrows Mon(\mathscr{M})(\mathscr{X}, I) : \varphi_*$$

$$(\text{resp. } \varphi^* : Comm(\mathscr{M})(\mathscr{Y}, J) \rightleftarrows Comm(\mathscr{M})(\mathscr{X}, I) : \varphi_*)$$

is a Quillen adjunction (where the categories of monoids $Mon(\mathscr{M})(\mathscr{X}, I)$ (resp. of commutative monoids $Comm(\mathscr{M})(\mathscr{X}, I)$) are endowed with the injective model category structure obtained from Proposition 3.1.7 applied to $Mon(\mathscr{M})$ (resp. to $Comm(\mathscr{M})$).

(d) *If moreover, for any object X of $\mathscr{S}$, the unit $\mathbb{1}_X$ is cofibrant in $\mathscr{M}(X)$, then, for morphism of $\mathscr{S}$-diagrams $\varphi : (\mathscr{X}, I) \to (Y, J)$, the square*

$$\begin{array}{ccc} \mathrm{Ho}(Mon(\mathscr{M}))(\mathscr{Y}, J) & \xrightarrow{\mathbf{L}\varphi^*} & \mathrm{Ho}(Mon(\mathscr{M}))(\mathscr{X}, I) \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ \mathrm{Ho}(\mathscr{M})(\mathscr{Y}, J) & \xrightarrow{\mathbf{L}\varphi^*} & \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I) \end{array} \tag{7.1.11.1}$$

is essentially commutative. Similarly, in the respective case, the square

$$\begin{array}{ccc} \mathrm{Ho}(Comm(\mathscr{M}))(\mathscr{Y}, J) & \xrightarrow{\mathbf{L}\varphi^*} & \mathrm{Ho}(Comm(\mathscr{M}))(\mathscr{X}, I) \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ \mathrm{Ho}(\mathscr{M})(\mathscr{Y}, J) & \xrightarrow{\mathbf{L}\varphi^*} & \mathrm{Ho}(\mathscr{M})(\mathscr{X}, I) \end{array} \tag{7.1.11.2}$$

is essentially commutative.

Proof. Assertion (a) is an immediate consequence of Theorem 7.1.3 (resp. of Theorem 7.1.8), and assertion (b) is a particular case of Proposition 3.1.11 (beware that the injective model category structure on $Comm(\mathscr{M})(\mathscr{X}, I)$ does not necessarily coincide with the model category structure given by Theorem 7.1.3 (resp. of Theorem 7.1.8) applied to the injective model structure on $\mathscr{M}(\mathscr{X}, I)$). For assertion (d), we see by the second assertion of Proposition 3.1.6 that it is sufficient to prove it when $\varphi : X \to Y$ is simply a morphism of $\mathscr{S}$. In this case, by construction of the total left derived functor of a left Quillen functor, this follows from the fact that φ^* commutes with the forgetful functor and from the fact that, by virtue of the last assertion of Theorem 7.1.3 (resp. of Theorem 7.1.8), the forgetful functor U preserves weak equivalences and cofibrant objects. □

Remark 7.1.12. The main application of the preceding corollary will come from assertion (d): it says that, given a monoid (resp. a commutative monoid) R in $\mathscr{M}(Y)$ and a morphism $f : X \to Y$, the image of R by the functor

$$\mathbf{L}f^* : \mathrm{Ho}(\mathscr{M})(Y) \to \mathrm{Ho}(\mathscr{M})(X)$$

is canonically endowed with a structure of monoid (resp. of commutative monoid) in the strongest sense possible. Under the assumptions of assertion (c) of Proposition 7.1.11, we shall often make the abuse of saying that $\mathbf{L}f^*(R)$ is a monoid (resp. a commutative monoid) in $\mathscr{M}(X)$ without refereeing explicitely to the model category structure on $Mon(\mathscr{M})(X)$ (resp. on $Comm(\mathscr{M})(X)$). Similarly, for any monoid (resp. commutative monoid) R in $\mathscr{M}(X)$, $\mathbf{R}f_*(R)$ will be canonically endowed with a structure of a monoid (resp. a commutative monoid) in $\mathscr{M}(Y)$. In particular, for any monoid (resp. commutative monoid) R in $\mathscr{M}(Y)$, the adjunction map

$$R \to \mathbf{R}f_* \, \mathbf{L}f^*(R)$$

is a morphism of monoids (i.e. is a map in the homotopy category $\mathrm{Ho}(Mon(\mathscr{M}))(X)$ (resp. $\mathrm{Ho}(Comm(\mathscr{M}))(X)$)), and, for any monoid (resp. commutative monoid) R in $\mathscr{M}(X)$, the adjunction map

$$\mathbf{L}f^* \, \mathbf{R}f_*(R) \to R$$

is a morphism of monoids (i.e. is a map in the homotopy category $\mathrm{Ho}(Mon(\mathscr{M}))(Y)$ (resp. $\mathrm{Ho}(Comm(\mathscr{M}))(Y)$)).

Remark 7.1.13. In order to get a good homotopy theory of commutative monoids wihout the strongly $\mathbf{Q}$-linear assumption, we should replace commutative monoids by E_∞-algebras (i.e. objects endowed with a structure of commutative monoid up to a bunch of coherent homotopies). More generally, we should prove the analog of Theorem 7.1.3 and of Theorem 7.1.8 by replacing $Mon(\mathscr{V})$ by the category of algebras of some 'well behaved' operad, and then get as a consequence the analog of Proposition 7.1.11. All this is a consequence of the general constructions and results of [**Spi01, BM03, BM09**].

However, in the case we are interested in the homotopy theory of commutative monoids in some category of spectra $\mathscr{V}$, it seems that some version of Shipley's *positive stable model structure* (*cf.* [**Shi04**, Proposition 3.1]) would provide a good model category for commutative monoids, which, by Lurie's strictification theorem [**Lur12**, Theorem 4.4.4.7], would be equivalent to the homotopy theory of E_∞-algebras in $\mathscr{V}$. This kind of technics is available in the context of stable homotopy theory of schemes, which provides a good setting to speak of motivic commutative ring spectra; see [**Hor10, GG11**]. Therefore, Theorem 7.1.8 and Proposition 7.1.11 are in fact true in SH for genuine commutative monoids without any $\mathbf{Q}$-linearity assumption.

7.2. Modules.

7.2.1. Given a monoid R in a symmetric monoidal category $\mathscr{V}$, we shall write $R\text{-}\mathrm{mod}(\mathscr{V})$ for the category of (left) R-modules. The forgetful functor

$$U : R\text{-}\mathrm{mod}(\mathscr{V}) \to \mathscr{V}$$

is a left adjoint to the free R-module functor

$$R \otimes (-) : \mathscr{V} \to R\text{-}\mathrm{mod}(\mathscr{V})\,.$$

If $\mathscr{V}$ has enough small colimits, and if R is a commutative monoid, the category $R\text{-}\mathrm{mod}(\mathscr{V})$ is endowed with a unique symmetric monoidal structure such that the functor $R \otimes (-)$ is naturally symmetric monoidal. We shall denote by $\otimes_R$ the tensor product of $R\text{-}\mathrm{mod}(\mathscr{V})$.

Theorem 7.2.2. *Let $\mathscr{V}$ be a combinatorial symmetric model category which satisfies the monoid axiom.*

- *(i) For any monoid R in $\mathscr{V}$, the category of right (resp. left) R-modules is a combinatorial model category with weak equivalences (resp. fibrations) the morphisms of R-modules which are weak equivalences (resp. fibrations) in $\mathscr{V}$.*
- *(ii) For any commutative monoid R in $\mathscr{V}$, the model category of R-modules given by (i) is a combinatorial symmetric monoidal model category which satisfies the monoid axiom.*

Proof. Assertions (i) and (ii) are particular cases of the first two assertions of [**SS00**, Theorem 4.1]. □

Definition 7.2.3. A symmetric monoidal model category $\mathscr{V}$ is *perfect* if it has the following properties.

- (a) $\mathscr{V}$ is combinatorial and tractable (3.1.27);
- (b) $\mathscr{V}$ satisfies the monoid axiom;
- (c) For any weak equivalence of monoids $R \to S$, the functor $M \mapsto S \otimes_R M$ is a left Quillen equivalence from the category of left R-modules to the category of left S-modules.
- (d) weak equivalences are stable by small sums in $\mathscr{V}$.

Remark 7.2.4. If $\mathscr{V}$ is a perfect symmetric monoidal model category, then, for any commutative monoid R, the symmetric monoidal model category of R-modules in $\mathscr{V}$ given by Theorem 7.2.2 (ii) is also perfect: condition (c) is quite obvious, and condition (d) comes from the fact that the forgetful functor $U : R\text{-}\mathrm{mod} \to \mathscr{V}$ commutes with small sums, while it preserves and detects weak equivalences. Note that condition (d) implies that the functor $U : \mathrm{Ho}(R\text{-}\mathrm{mod}) \to \mathrm{Ho}(\mathscr{V})$ preserves small sums.

Remark 7.2.5. If $\mathscr{V}$ is a stable symmetric monoidal model category which satisfies the monoid axiom, then for any monoid R of $\mathscr{V}$, the model category of (left) R-modules given by Theorem 7.2.2 is stable as well: the suspension functor of $\mathrm{Ho}(R\text{-}\mathrm{mod})$ is given by the derived tensor product by the R-bimodule $R[1]$, which is clearly invertible with inverse $R[-1]$.

In this work, a basic example of perfect model categories are those coming from stable $\mathbf{A}^1$-derived premotivic categories (cf Def. 5.3.22):

Proposition 7.2.6. *Let t be an admissible topology. Then, for any scheme S in $\mathscr{S}$, the symmetric monoidal model structure on $\mathrm{C}(\mathrm{Sp}(\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z})))$ underlying the triangulated category $\mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z}))$ is perfect.*

PROOF. The generating family of $\mathrm{Sh}_t(\mathscr{P}/S, \mathbf{Z})$ is flat in the sense of [**CD09**, 3.1], so that, by virtue of [**CD09**, prop. 7.22 and cor. 7.24], the assumptions of Proposition 7.2.9 are fulfilled. □

PROPOSITION 7.2.7. *Let $\mathscr{V}$ be a stable perfect symmetric monoidal model category. Assume furthermore that $\mathrm{Ho}(\mathscr{V})$ admits a small family $\mathcal{G}$ of compact generators (as a triangulated category). For any monoid R in $\mathscr{V}$, the triangulated category $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$ admits the set $\{R \otimes^{\mathbf{L}} E \mid E \in \mathcal{G}\}$ as a family of compact generators.*

PROOF. We have a derived adjunction

$$R \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{V}) \rightleftarrows \mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V})) : U\,.$$

As the functor U preserves small sums the functor $R \otimes^{\mathbf{L}} (-)$ preserves compact objects. But U is also conservative, so that $\{R \otimes^{\mathbf{L}} E \mid E \in \mathcal{G}\}$ is a family of compact generators of $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$. □

REMARK 7.2.8. If $\mathscr{V}$ is a combinatorial symmetric model category which satisfies the monoid axiom, then there are two ways to derive the tensor product. The first one consists to derive the left Quillen bifunctor $(-) \otimes (-)$, which gives the usual derived tensor product

$$(-) \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{V}) \times \mathrm{Ho}(\mathscr{V}) \to \mathrm{Ho}(\mathscr{V})\,.$$

Remember that, by construction, $A \otimes^{\mathbf{L}} B = A' \otimes B'$, where A' and B' are cofibrant replacements of A and B respectively. On the other hand, the monoid axiom gives that, for any object A of $\mathscr{V}$, the functor $A \otimes (-)$ preserves weak equivalences between cofibrant objects, which implies that it has also a total left derived functor

$$A \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{V}) \to \mathrm{Ho}(\mathscr{V})\,.$$

Despite the fact we have adopted very similar (not to say identical) notations for these two derived functor, there is no reason they would coincide in general: by construction, the second one is defined by $A \otimes^{\mathbf{L}} B = A \otimes B'$, where B' is some cofibrant replacement of B. However, they coincide quite often in practice (e.g. for simplicial sets, for the good reason that all of them are cofibrant, or for symmetric S^1-spectra, or for complexes of quasi-coherent $\mathcal{O}_X$-modules over a quasi-compact and quasi-separated scheme X).

PROPOSITION 7.2.9. *Let $\mathscr{V}$ be a stable combinatorial symmetric monoidal model category which satisfies the monoid axiom. Assume furthermore that, for any cofibrant object A of $\mathscr{V}$, the functor $A \otimes (-)$ preserve weak equivalences (in other words, that the two ways to derive the tensor product explained in Remark 7.2.8 coincide), and that weak equivalences are stable by small sums in $\mathscr{V}$. Then the symmetric monoidal model category $\mathscr{V}$ is perfect.*

PROOF. We just have to check condition (c) of Definition 7.2.3. Consider a weak equivalence of monoids $R \to S$. We then get a derived adjunction

$$S \otimes^{\mathbf{L}}_R (-) : \mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V})) \rightleftarrows \mathrm{Ho}(S\text{-}\mathrm{mod}(\mathscr{V})) : U\,,$$

where $S \otimes^{\mathbf{L}}_R (-)$ is the left derived functor of the functor $M \mapsto S \otimes_R M$. We have to prove that, for any left R-module M, the map

$$M \to S \otimes^{\mathbf{L}}_R M$$

is an isomorphism in $\mathrm{Ho}(\mathscr{V})$. As this is a morphism of triangulated functors which commutes with sums, and as $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$ is well generated in the sense of Neeman [**Nee01**] (as the localization of a stable combinatorial model category), it is sufficient to check this when M runs over a small family of generators of $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$. Let us chose is a small family of generators $\mathcal{G}$ of $\mathrm{Ho}(\mathscr{V})$. As the forgetful functor from $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$ to $\mathrm{Ho}(\mathscr{V})$ is conservative, we see that $\{R \otimes^{\mathbf{L}} E \mid E \in \mathcal{G}\}$ is a small generating family of $\mathrm{Ho}(R\text{-}\mathrm{mod}(\mathscr{V}))$. We are thus reduced to prove that the map

$$R \otimes^{\mathbf{L}} E \to S \otimes^{\mathbf{L}}_R (R \otimes^{\mathbf{L}} E) \simeq S \otimes^{\mathbf{L}} E$$

is an isomorphism for any object E in $\mathcal{G}$. For this, we can assume that E is cofibrant, and this follows then from the fact that the functor $(-) \otimes E$ preserves weak equivalences by assumption. □

7.2.10. Let $\mathscr{S}$ be a category endowed with an admissible class of morphisms $\mathscr{P}$, and $\mathscr{M}$ a symmetric monoidal $\mathscr{P}$-fibred category. Consider a monoid R in the symmetric monoidal category $\mathscr{M}(1_{\mathscr{S}}, \mathscr{S})$ (i.e. a section of the fibred category $Mon(\mathscr{M})$ over $\mathscr{S}$). In other words, R consists of the data of a monoid R_X for each object X of $\mathscr{S}$, and of a morphism of monoids $a_f : f^*(R_Y) \to R_X$ for each map $f : X \to Y$ in $\mathscr{S}$, subject to coherence relations; see 3.1.2.

For an object X of $\mathscr{S}$, we shall write $R\text{-}\operatorname{mod}(X)$ for the category of (left) R_X-modules in $\mathscr{M}(X)$, i.e.

$$R\text{-}\operatorname{mod}(X) = R_X\text{-}\operatorname{mod}(\mathscr{M}(X)) .$$

This defines a fibred category $R\text{-}\operatorname{mod}$ over $\mathscr{S}$ as follows.

For a morphism $f : X \to Y$, the inverse image functor

$$f^* : R\text{-}\operatorname{mod}(Y) \to R\text{-}\operatorname{mod}(X) \tag{7.2.10.1}$$

is defined by

$$M \mapsto R_X \otimes_{f^*(R_Y)} f^*(M) \tag{7.2.10.2}$$

(where, on the right hand side, f^* stands for the inverse image functor in $\mathscr{M}$). The functor (7.2.10.1) has a right adjoint

$$f_* : R\text{-}\operatorname{mod}(X) \to R\text{-}\operatorname{mod}(Y) \tag{7.2.10.3}$$

which is simply the functor induced by $f_* : \mathscr{M}(X) \to \mathscr{M}(Y)$ (as the latter sends R_X-modules to $f_*(R_X)$-modules, which are themselves R_Y-modules via the map a_f).

If the map f is a $\mathscr{P}$-morphism, then, for any R_X-module M, the object $f_\sharp(M)$ has a natural structure of R_Y-module: using the map a_f, M has a natural structure of $f^*(R_Y)$-module

$$f^*(R_Y) \otimes_X M \to M ,$$

and applying $f_\sharp$, we get by the $\mathscr{P}$-projection formula (1.1.26) a morphism

$$R_Y \otimes f_\sharp(M) \simeq f_\sharp(f^*(R_Y) \otimes M) \to f_\sharp(M)$$

which defines a natural R_Y-module structure on $f_\sharp(M)$. For a $\mathscr{P}$-morphism $f : X \to Y$, we define a functor

$$f_\sharp : R\text{-}\operatorname{mod}(X) \to R\text{-}\operatorname{mod}(Y) \tag{7.2.10.4}$$

as the functor induced by $f_\sharp : \mathscr{M}(X) \to \mathscr{M}(Y)$. Note that the functor (7.2.10.4) is a left adjoint to the functor (7.2.10.1) whenever the map $a_f : f^*(R_Y) \to R_X$ is an isomorphism in $\mathscr{M}(X)$.

We shall say that R is a *cartesian monoid in $\mathscr{M}$ over $\mathscr{S}$* if R is a monoid of $\mathscr{M}(1_{\mathscr{C}}, \mathscr{C})$ such that all the structural maps $f^*(R_Y) \to R_X$ are isomorphisms (i.e. if R is a cartesian section of the fibred category $Mon(\mathscr{M})$ over $\mathscr{S}$)

If R is a cartesian monoid in $\mathscr{M}$ over $\mathscr{S}$, then $R\text{-}\operatorname{mod}$ is a $\mathscr{P}$-fibred category over $\mathscr{S}$: to see this, it remains to prove that, for any pullback square of $\mathscr{S}$

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[h]{} & Y \end{array}$$

in which f is a $\mathscr{P}$-morphism, and for any R_X-module M, the base change map

$$f'_\sharp\, g^*(M) \to h^*\, f_\sharp(M)$$

is an isomorphism, which follows immediately from the analogous formula for $\mathscr{M}$.

Similarly, we see that whenever R is a commutative monoid of $\mathscr{M}(1_{\mathscr{S}}, \mathscr{S})$ (i.e. R_X is a commutative monoid in $\mathscr{M}(X)$ for all X in $\mathscr{S}$), then $R\text{-}\operatorname{mod}$ is a symmetric monoidal $\mathscr{P}$-fibred category.

Proposition 7.2.11. *Let $\mathscr{M}$ be a combinatorial symmetric monoidal $\mathscr{P}$-fibred model category over $\mathscr{S}$ which satisfies the monoid axiom, and R a monoid in $\mathscr{M}(1_{\mathscr{S}}, \mathscr{S})$ (resp. a cartesian monoid in $\mathscr{M}$ over $\mathscr{S}$). Then 7.2.2 (i) applied termwise turns R-mod into a combinatorial fibred model category (resp. a combinatorial $\mathscr{P}$-fibred model category).*

If moreover R is commutative, then R-mod is a combinatorial symmetric monoidal fibred model category (resp. a combinatorial symmetric monoidal $\mathscr{P}$-fibred model category).

Proof. Choose, for each object X of $\mathscr{S}$, two small sets of maps I_X and J_X which generate the class of cofibrations and the class of trivial cofibrations in $\mathscr{M}(X)$ respectively. Then $R_X \otimes_X I_X$ and $R_X \otimes_X J_X$ generate the class of cofibrations and the class of trivial cofibrations in R-mod(X) respectively. For a map $f : X \to Y$ in $\mathscr{S}$, we see from formula (7.2.10.2) that the functor (7.2.10.1) sends these generating cofibrations and trivial cofibrations to cofibrations and trivial cofibrations respectively, from which we deduce that the functor (7.2.10.1) is a left Quillen functor. In the respective case, if f is a $\mathscr{P}$-morphism, then we deduce similarly from the projection formula (1.1.26) in $\mathscr{M}$ that the functor (7.2.10.4) sends generating cofibrations and trivial cofibrations to cofibrations and trivial cofibrations respectively. The last assertion follows easily by applying 7.2.2 (ii) termwise. □

Definition 7.2.12. Let $\mathscr{M}$ be a symmetric monoidal $\mathscr{P}$-fibred model category over $\mathscr{S}$. A *homotopy cartesian monoid* R in $\mathscr{M}$ will be a homotopy cartesian section of $Mon(\mathscr{M})$.

Proposition 7.2.13. *Let $\mathscr{M}$ be a perfect symmetric monoidal $\mathscr{P}$-fibred model category over $\mathscr{S}$, and consider a homotopy cartesian monoid R in $\mathscr{M}$ over $\mathscr{S}$.*

Then Ho(R-mod) *is a $\mathscr{P}$-fibred category over $\mathscr{S}$, and*

$$R \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(R\text{-}\mathrm{mod})$$

is a morphism of $\mathscr{P}$-fibred categories. In the case where R is commutative, Ho(R-mod) *is even a symmetric monoidal $\mathscr{P}$-fibred category.*

Moreover, for any weak equivalence between homotopy cartesian monoids $R \to S$ over $\mathscr{S}$, the Quillen morphism

$$S \otimes_R (-) : R\text{-}\mathrm{mod} \to S\text{-}\mathrm{mod}$$

induces an equivalence of $\mathscr{P}$-fibred categories over $\mathscr{S}$

$$S \otimes^{\mathbf{L}}_R (-) : \mathrm{Ho}(R\text{-}\mathrm{mod}) \to \mathrm{Ho}(S\text{-}\mathrm{mod})\,.$$

Proof. It is sufficient to prove these assertions by restricting everything over $\mathscr{S}/S$, where S runs over all the objects of $\mathscr{S}$. In particular, we may (and shall) assume that $\mathscr{S}$ has a terminal object S. As $\mathscr{M}$ is perfect, it follows from condition (c) of Definition 7.2.3 that we can replace R by any of its cofibrant resolution. In particular, we may assume that R_S is a cofibrant object of $Mon(\mathscr{M})(S)$. We can thus define a termwise cofibrant cartesian monoid R' as the family of monoids $f^*(R_S)$, where $f : X \to S$ runs over all the objects of $\mathscr{S} \simeq \mathscr{S}/S$. There is a canonical morphism of homotopy cartesian monoids $R' \to R$ which is a termwise weak equivalence. We thus get, by condition (c) of Definition 7.2.3, an equivalence of fibred categories

$$R \otimes^{\mathbf{L}}_{R'} (-) : \mathrm{Ho}(R'\text{-}\mathrm{mod}) \to \mathrm{Ho}(R\text{-}\mathrm{mod})\,.$$

We can thus replace R by R', which just means that we can assume that R is cartesian and termwise cofibrant. The first assertion follows then easily from Proposition 7.2.11. In the case where R is commutative, we prove that Ho(R-mod) is a $\mathscr{P}$-fibred symmetric monoidal category as follows. Let $f : X \to Y$ a morphism of $\mathscr{S}$. We would like to prove that, for any object M in Ho(R-mod)(X) and any object N in Ho(R-mod)(Y), the canonical map

$$\mathbf{L}f_\sharp(M \otimes^{\mathbf{L}}_R f^*(N)) \to \mathbf{L}f_\sharp(M) \otimes^{\mathbf{L}}_R N \tag{7.2.13.1}$$

is an isomorphism. By adjunction, this is equivalent to prove that, for any objects N and E in Ho(R-mod)(Y), the map

$$f^*\mathbf{R}Hom_R(N, E) \to \mathbf{R}Hom_R(f^*(N), f^*(E)) \tag{7.2.13.2}$$

is an isomorphism in $\mathrm{Ho}(R\text{-}\mathrm{mod})(X)$ (where $\mathbf{R}Hom_R$ stands for the internal Hom of $\mathrm{Ho}(R\text{-}\mathrm{mod})$). But the forgetful functors

$$U : \mathrm{Ho}(R\text{-}\mathrm{mod})(X) \to \mathrm{Ho}(\mathscr{M})(X)$$

are conservative, commute with f^* for any $\mathscr{P}$-morphism f, and commute with internal Hom: by adjunction, this follows immediately from the fact that the functors

$$R \otimes^{\mathbf{L}} (-) : \mathrm{Ho}(\mathscr{M})(X) \to \mathrm{Ho}(R\text{-}\mathrm{mod})(X) \simeq \mathrm{Ho}(R'\text{-}\mathrm{mod})(X)$$

are symmetric monoidal and define a morphism of $\mathscr{P}$-fibred categories (and thus, in particular, commute with $f_\sharp$ for any $\mathscr{P}$-morphism f). Hence, to prove that (7.2.13.2) is an isomorphism, it is sufficient to prove that its analog in $\mathrm{Ho}(\mathscr{M})$ is so, which follows immediately from the fact that the analog of (7.2.13.1) is an isomorphism in $\mathrm{Ho}(\mathscr{M})$ by assumption.

For the last assertion, we are also reduced to the case where R and S are cartesian and termwise cofibrant, in which case this follows easily again from condition (c) of Definition 7.2.3. □

Proposition 7.2.14. *Let $\mathscr{M}$ be a combinatorial symmetric monoidal model category over $\mathscr{S}$ which satisfies the monoid axiom. Then, for any cartesian monoid R in $\mathscr{M}$ over $\mathscr{S}$ we have a Quillen morphism*

$$R \otimes (-) : \mathscr{M} \to R\text{-}\mathrm{mod}\ .$$

If, for any object X of $\mathscr{S}$, the unit object $\mathbb{1}_X$ is cofibrant in $\mathscr{M}(X)$ and the monoid R_X is cofibrant in $Mon(\mathscr{M})(X)$, then the forgetful functors also define a Quillen morphism

$$U : R\text{-}\mathrm{mod} \to \mathscr{M}\ .$$

Proof. The first assertion is obvious. For the second one, note that, for any object X of $\mathscr{S}$, the monoid R_X is also cofibrant as an object of $\mathscr{M}(X)$; see Theorem 7.1.3. This implies that the forgetful functor

$$U : R_X\text{-}\mathrm{mod} \to \mathscr{M}(X)$$

is a left Quillen functor: by the small object argument and by definition of the model category structure of Theorem 7.2.2 (i), this follows from the trivial fact that the endofunctor

$$R_X \otimes (-) : \mathscr{M}(X) \to \mathscr{M}(X)$$

is a left Quillen functor itself whenever R_X is cofibrant in $\mathscr{M}(X)$. □

Remark 7.2.15. The results of the preceding proposition (as well as their proofs) are also true in terms of $\mathscr{P}_{cart}$-fibred categories (3.1.21) over the category of $\mathscr{S}/S$-diagrams for any object S of $\mathscr{S}$ (whence over all $\mathscr{S}$-diagrams whenever $\mathscr{S}$ has a terminal object).

7.2.16. Consider now a noetherian scheme S of finite dimension. We choose a full subcategory of the category of separated noetherian S-schemes of finite dimension which is stable by finite limits, contains separated S-schemes of finite type, and such that, for any étale S-morphism $Y \to X$, if X is in $\mathscr{S}/S$, so is Y. We denote by $\mathscr{S}/S$ this chosen category of S-schemes.

We also fix an admissible class $\mathscr{P}$ of morphisms of $\mathscr{S}/S$ which contains the class of étale morphisms.

Definition 7.2.17. A property P of $\mathrm{Ho}(\mathscr{M})$, for $\mathscr{M}$ a stable combinatorial $\mathscr{P}$-fibred model category over $\mathscr{S}/S$, is *homotopy linear* if the following implications are true.

(a) If $\gamma : \mathscr{M} \to \mathscr{M}'$ is a Quillen equivalence (i.e. a Quillen morphism which is termwise a Quillen equivalence) between stable combinatorial $\mathscr{P}$-fibred model category over $\mathscr{S}/S$, then $\mathscr{M}$ has property P is and only if $\mathscr{M}'$ has property P.
(b) If $\mathscr{M}$ is a stable combinatorial symmetric monoidal $\mathscr{P}$-model category which satisfies the monoid axiom, and such that the unit $\mathbb{1}_X$ of $\mathscr{M}(X)$ is cofibrant, then, for any cartesian and termwise cofibrant monoid R in $\mathscr{M}$ over $\mathscr{S}/S$, R-mod has property P.

Proposition 7.2.18. *The following properties are homotopy linear: $\mathbf{A}^1$-homotopy invariance, $\mathbf{P}^1$-stability, the localization property, the property of proper transversality, separability, semi-separability, t-descent (for a given Grothendieck topology t on $\mathscr{S}/S$).*

PROOF. Property (a) of the definition above is obvious. Property (b) comes from the fact that the forgetful functors

$$U : \mathrm{Ho}(R\text{-}\,\mathrm{mod}) \to \mathrm{Ho}(\mathscr{M})$$

are conservative and commute with all the operations: $\mathbf{L}f^*$ and $\mathbf{R}f_*$ for any morphism f, as well as $\mathbf{L}f_\sharp$ for any $\mathscr{P}$-morphism (by Proposition 7.2.14). Hence any property formulated in terms of equations involving only these operations is homotopy linear. □

Part 3

Motivic complexes and relative cycles

In this entire part, we adopt the special convention that smooth means smooth separated of finite type. This concerns also the framework of premotivic categories: we assume the admissible class *Sm* is made of smooth separated morphisms of finite type.

This assumption is required by the use of the theory of finite correspondences (see more precisely Example 9.1.4).

8. Relative cycles

8.0. In this entire section, $\mathscr{S}$ is the category of noetherian schemes; any scheme is assumed to be noetherian. We fix a subring $\Lambda \subset \mathbf{Q}$ which will be the ring of coefficients of the algebraic cycles considered in the following section.

8.1. Definitions.

8.1.a. *Category of cycles.*

8.1.1. Let X be a scheme. As usual, an element of the underlying set of X will be called a *point* and a morphism $\operatorname{Spec}(k) \to X$ where k is a field will be called a *geometric point*. We often identify a point $x \in X$ with the corresponding geometric point $\operatorname{Spec}(\kappa_x) \to X$. However, the explicit expression "the point $\operatorname{Spec}(k) \to X$" always refers to a geometric point.
As our schemes are assumed to be noetherian, any immersion $f : X \to Y$ is quasi-compact. Thus, according to [**EGA1**, 9.5.10], the *schematic closure* $\bar{X}$ of X in Y exists which gives a unique factorization of f

$$X \xrightarrow{j} \bar{X} \xrightarrow{i} Y$$

such that i is a closed immersion and j is an open immersion with dense image[74]. Note that when Y is reduced, $\bar{X}$ coincide with the topological closure of X in Y with its induced reduced subscheme structure. In this case, we simply call $\bar{Y}$ the closure of Y in X.

Definition 8.1.2. A Λ-*cycle* is a couple (X, α) such that X is a scheme and α is a Λ-linear combination of points of X. A generic point of (X, α) is a point which appears in the Λ-linear combination α with a non zero coefficient. The support $\operatorname{Supp}(\alpha)$ of α is the closure of the generic points of α.

A morphism of Λ-cycles $(Y, \beta) \to (X, \alpha)$ is a morphism of scheme $f : Y \to X$ such that $f(\operatorname{Supp}(\beta)) \subset \operatorname{Supp}(\alpha)$. We say this morphism is *pseudo-dominant* if for any generic point y of (Y, β), $f(y)$ is a generic point of (X, α).

When considering such a pair (X, α), we will denote it simply by α and refer to X as the *domain* of α. We also use the notation $\alpha \subset X$ to mean the domain of the cycle α is the scheme X.

The category of Λ-cycle is functorial in Λ with respect to morphisms of integral rings. In what follows, cycles are assumed to have coefficients in Λ unless explicitly stated.

8.1.3. Given a property $(\mathcal{P})$ of morphisms of schemes, we will say that a morphism $f : \beta \to \alpha$ of cycles satisfies property $(\mathcal{P})$ if the induced morphism $f|_{\operatorname{Supp}(\beta)}^{\operatorname{Supp}(\alpha)}$ satisfies property $(\mathcal{P})$.

Definition 8.1.4. Let X be a scheme. We denote by $X^{(0)}$ the set of generic points of X. We define as usual the *cycle associated with* X as the cycle with domain X :

$$\langle X \rangle = \sum_{x \in X^{(0)}} \lg(\mathcal{O}_{X,x}).x.$$

The integer $\lg(\mathcal{O}_{X,x})$, length of an artinian local ring, is called the *geometric multiplicity* of x in X.

When no confusion is possible, we usually omit the delimiters in the notation $\langle X \rangle$. As an example, we say that α *is a cycle over* X to mean the existence of a structural morphism of cycles $\alpha \to \langle X \rangle$.

[74]Recall the scheme $\bar{X}$ is characterized by the property of being the smallest sub-scheme of Y with the existence of such a factorization.

8.1.5. When Z is a closed subscheme of a scheme X, we denote by $\langle Z\rangle_X$ the cycle $\langle Z\rangle$ considered as a cycle with domain X.
Consider a cycle α with domain X. Let $(Z_i)_{i\in I}$ be the family of the reduced closure of generic points of α. Then we can write α uniquely as $\alpha=\sum_{i\in I} n_i.\langle Z_i\rangle_X$. We call this writing the *standard form* of α for short.

Definition 8.1.6. Let $\alpha=\sum_{i\in I} n_i.x_i$ be a cycle with domain X and $f : X\to Y$ be any morphism.

For any $i\in I$, put $y_i=f(x_i)$. Then f induces an extension field $\kappa(x_i)/\kappa(y_i)$ between the residue fields. We let d_i be the degree of this extension field in case it is finite and 0 otherwise.

We define the *pushforward* of α by f as the cycle with domain Y

$$f_*(\alpha)=\sum_{i\in I} n_i d_i.f(x_i).$$

Thus, when f is an immersion, $f_*(\alpha)$ is the same cycle as α but seen as a cycle with domain X. Remark also that we obtain the following equality

$$f_*\big(\langle X\rangle\big)=\big\langle \bar X\big\rangle_Y \tag{8.1.6.1}$$

where $\bar X$ is the schematic closure of X in Y (indeed X is a dense open subscheme in $\bar X$). When f is clear, we sometimes abusively put: $\langle X\rangle_Y := f_*(\langle X\rangle)$.

By transitivity of degrees, we obviously have $f_*g_*=(fg)_*$ for a composable pair of morphisms (f,g).

Definition 8.1.7. Let $\alpha=\sum_{i\in I} n_i.x_i$ be a cycle over a scheme S with domain $f : X\to S$ and $U\subset S$ be an open subscheme. Let $I'=\{i\in I \mid f(x_i)\in U\}$. We define the *restriction* of α over U as the cycle $\alpha|_U=\sum_{i\in I'} n_i.x_i$ with domain $X\times_S U$ considered as a cycle over U.

If $\alpha=\sum_{i\in I} n_i.\langle Z_i\rangle_X$, then obviously $\alpha|_U=\sum_{i\in I} n_i.\langle Z_i\times_S U\rangle_{X_U}$. We state the following obvious lemma for convenience :

Lemma 8.1.8. *Let S be a scheme, $U\subset S$ an open subscheme and X be an S-scheme. Let $j : X_U\to X$ be the obvious open immersion.*

- *(i) For any cycle (X_U,α'), $\big(j_*(\alpha')\big)|_U=\alpha'$.*
- *(ii) Assume $\bar U=S$. For any cycle (X,α) pseudo-dominant over S, $j_*(\alpha\,|_U)=\alpha$.*

8.1.b. *Hilbert cycles.*

8.1.9. Recall that a finite dimensional scheme X is equidimensional – we will say *absolutely equidimensional* – if its irreducible components have all the same dimension.

We will say that a flat morphism $f : X\to S$ is equidimensional if it is of finite type and for any connected component X' of X, there exists an integer $e\in\mathbf{N}$ such that for any generic point η in X', the fiber $f^{-1}[f(\eta)]$ is absolutely equidimensional of dimension e.

Definition 8.1.10. Let S be a scheme.

Let α be a cycle over S with domain X. We say that α is a *Hilbert cycle* over S if there exists a finite family $(Z_i)_{i\in I}$ of closed subschemes of X which are flat equidimensional over S and a finite family $(n_i)_{i\in I}\in\Lambda^I$ such that

$$\alpha=\sum_{i\in I} n_i.\langle Z_i\rangle_X.$$

Example 8.1.11. Any cycle over a field k is a Hilbert cycle over $\mathrm{Spec}\,(k)$. Let S be the spectrum of a discrete valuation ring. A cycle $\alpha=\sum_{i\in I} n_i.x_i$ over S is a Hilbert cycle if and only if each point x_i lies over the generic points of S. Indeed, an integral S-scheme is flat if and only if it is dominant.

The following lemma follows almost directly from a result of [**SV00b**]:

Lemma 8.1.12. *Let $f : S' \to S$ be a morphism of schemes and X be an S-scheme of finite type. Put $X' = X \times_S S'$.*

Let $(Z_i)_{i \in I}$ be a finite family of closed subschemes of X such that each Z_i is flat equidimensional over S. We assume the following relation:

$$\sum_{i \in I} n_i.\langle Z_i \rangle_X = 0 \tag{8.1.12.1}$$

Then we the following equality holds:

$$\sum_{i \in I} n_i.\langle Z_i \times_S S' \rangle_{X'} = 0.$$

Proof. When we assume that for any index $i \in I$, Z_i/S is equidimensional of dimension e, this lemma is exactly [**SV00b**, Prop. 3.2.2]. We show how to reduce to that case.

Up to adding more members to the family (Z_i), we can always assume that Z_i is connected. Then, because Z_i/S is equidimensional by assumption, there exists an integer e_i such that for any point $x \in Z_i^{(0)}$, the fiber $f^{-1}[f(x)]$ is absolutely equidimensional of dimension e_i. In particular the transcendence degree d_x of the residual extension $\kappa_x/\kappa_{f(x)}$ satisfies the relation: $d_x = e_i$.

For any integer $e \in \mathbf{N}$, we define the following subset of I:

$$I_e = \{i \in I \mid \forall x \in Z_i^{(0)}, d_x = e\}.$$

Thus $(I_e)_{e \in \mathbf{N}}$ is a partition of I.

One can rewrite the assumption (8.1.12.1) as follows: for any point $x \in X$,

$$\sum_{i \in I | x \in Z_i^{(0)}} n_i.\mathrm{lg}(\mathcal{O}_{Z_i,x}) = 0.$$

In particular, given any integer $e \in \mathbf{N}$, we deduce that the family $(Z_i)_{i \in I_e}$ still satisfies the relation (8.1.12.1). As any member of this family is equidimensional of dimension e, we can apply [**SV00b**, Prop. 3.2.2] to $(Z_i)_{i \in I_e}$. This concludes. □

8.1.13. Consider a Hilbert S-cycle $\alpha \subset X$ and a morphism of schemes $f : S' \to S$. Put $X' = X \times_S S'$. We choose a finite family $(Z_i)_{i \in I}$ of flat equidimensional S-schemes and a finite family $(n_i)_{i \in I} \in \Lambda^I$ such that $\alpha = \sum_{i \in I} n_i.\langle Z_i \rangle_X$. The previous lemma says exactly that the cycle

$$\sum_{i \in I} n_i.\langle Z_i \times_S S' \rangle_{X'}$$

depends only on α and not on the chosen families.

Definition 8.1.14. Adopting the preceding notations and hypothesis, we define the *pullback cycle* of α along the morphism $f : S' \to S$ as the cycle with domain X'

$$\alpha \otimes^\flat_S S' = \sum_{i \in I} n_i.\langle Z_i \times_S S' \rangle_{X'}.$$

In this setting the following lemma is obvious :

Lemma 8.1.15. *Let α be a Hilbert cycle over S, and $S'' \to S' \to S$ be morphisms of schemes. Then $(\alpha \otimes^\flat_S S') \otimes^\flat_{S'} S'' = \alpha \otimes^\flat_S S''$.*

We will use another important computation from [**SV00b**] (it is a particular case of *loc. cit.*, 3.6.1).

Proposition 8.1.16. *Let R be a discrete valuation ring with residue field k.*

Let $\alpha \subset X$ be a Hilbert cycle over $\mathrm{Spec}\,(R)$ *and $f : X \to Y$ a morphism over* $\mathrm{Spec}\,(R)$. *We denote by $f' : X' \to Y'$ the pullback of f over* $\mathrm{Spec}\,(k)$.

Suppose that the support of α is proper with respect to f.

Then $f_*(\alpha)$ is a Hilbert cycle over R and the following equality of cycles holds in X':

$$f'_*(\alpha \otimes^\flat_S k) = f_*(\alpha) \otimes^\flat_S k.$$

DEFINITION 8.1.17. Let $p : \tilde{S} \to S$ be a birational morphism. Let C be the minimal closed subset of S such that p induces an isomorphism $(\tilde{S} - \tilde{S} \times_S C) \to (S - C)$.

Consider $\alpha = \sum_{i \in I} n_i.\langle Z_i \rangle_X$ a cycle over S written in standard form.

We define the strict transform $\tilde{Z}_i$ of the closed subscheme Z_i in X along p as the schematic closure of $(Z_i - Z_i \times_S C) \times_S \tilde{S}$ in $X \times_S \tilde{S}$. We define the strict transform of α along p as the cycle over $\tilde{S}$

$$\tilde{\alpha} = \sum_{i \in I} n_i.\langle \tilde{Z}_i \rangle_{X \times_S \tilde{S}}.$$

As in [**SV00b**], we remark that a corollary of the platification theorem of Gruson-Raynaud is the following :

LEMMA 8.1.18. *Let S be a reduced scheme and α be a pseudo-dominant cycle over S.*

Then there exists a dominant blow-up $p : \tilde{S} \to S$ such that the strict transform $\tilde{\alpha}$ of α along p is a Hilbert cycle over $\tilde{S}$.

We conclude this part by recalling an elementary lemma about cycles and Galois descent which will be used extensively in the next sections :

LEMMA 8.1.19. *Let L/K be an extension of fields and X be a K-scheme. We put $X_L = X \times_K \operatorname{Spec}(L)$ and consider the faithfully flat morphism $f : X_L \to X$.*

Denote by $\operatorname{Cycl}(X)$ (resp. $\operatorname{Cycl}(X_L)$) the cycles with domain X (resp. X_L).

(1) *The morphism $f^* : \operatorname{Cycl}(X) \to \operatorname{Cycl}(X_L), \beta \mapsto \beta \otimes^\flat_K L$ is a monomorphism.*
(2) *Suppose L/K is finite. For any K-cycle $\beta \in \operatorname{Cycl}(X)$,* $f_*(\beta \otimes^\flat_K L) = [L : K].\beta$.
(3) *Suppose L/K is finite normal with Galois group G.*
The cycles in the image of f^* are invariant under the action of G. For any cycle $\beta \in \operatorname{Cycl}(X_L)^G$, there exists a unique cycle $\beta_K \in \operatorname{Cycl}(X)$ such that
$$\beta_K \otimes^\flat_K L = [L : K]_i.\beta$$
where $[L : K]_i$ is the inseparable degree *of L/K.*

8.1.c. *Specialization.* The aim of this section is to give conditions on cycles so that one can define a *relative tensor product* on them.

DEFINITION 8.1.20. Consider two cycles $\alpha = \sum_{i \in I} n_i.s_i$ and $\beta = \sum_{j \in J} m_i.x_j$. Let S be the support of α.

A morphism $\beta \xrightarrow{f} \alpha$ of cycles is said to be *pre-special* if it is of finite type and for any $j \in J$, there exists $i \in I$ such that $f(x_j) = s_i$ and $n_i | m_j$ in Λ. We define the reduction of β/α as the cycle over S

$$\beta_0 = \sum_{j \in J, f(x_j) = s_i} \frac{m_j}{n_i}.x_j.$$

EXAMPLE 8.1.21. Let S be a scheme and α a Hilbert S-cycle. Then the canonical morphism of cycles $\alpha \to \langle S \rangle$ is pre-special. If S is the spectrum of a discrete valuation ring, an S-cycle α is pre-special if and only if it is a Hilbert S-cycle.

DEFINITION 8.1.22. Let α be a cycle.

A *point* (resp. *trait*) of α will be a morphism $\operatorname{Spec}(k) \xrightarrow{x} \alpha$ (resp. $\operatorname{Spec}(R) \xrightarrow{\tau} \alpha$) such that k is a field (resp. R is a discrete valuation ring). We simply say that x (resp. τ) is *dominant* if the image of the generic point in the domain of α is a generic point of α.

Let $x : \operatorname{Spec}(k_0) \to \alpha$ be a point. An extension of x will be a point y on α of the form $\operatorname{Spec}(k) \to \operatorname{Spec}(k_0) \xrightarrow{x} \alpha$.

A *fat point* of α will be morphisms

$$\operatorname{Spec}(k) \xrightarrow{s} \operatorname{Spec}(R) \xrightarrow{\tau} \alpha$$

such that τ is a dominant trait and the image of s is the closed point of $\mathrm{Spec}\,(R)$.
Given a point $x : \mathrm{Spec}\,(k) \to \alpha$, a fat point over x is a factorization of x through a dominant trait as above.

In the situation of the last definition, we denote simply by (R, k) a fat point over x, without indicating in the notation the morphisms s and τ.

Remark 8.1.23. With our choice of terminology, a point of α is in general an extension of a specialization of a generic point of α. As a further example, a dominant point of α is an extension of a generic point of α.

Lemma 8.1.24. *For any cycle α and any non dominant point $x : \mathrm{Spec}\,(k_0) \to \alpha$, there exists an extension $y : \mathrm{Spec}\,(k) \to \alpha$ of x and a fat point (R, k) over y.*

Proof. Replacing α by its support S, we can assume $\alpha = \langle S \rangle$. Let s be the image of x in S, κ its residue field. We can assume S is reduced, irreducible by taking one irreducible component containing s, and local with closed point s. Let $S = \mathrm{Spec}\,(A)$, $K = \mathrm{Frac}(A)$. According to [**EGA2**, 7.1.7], there exists a discrete valuation ring R such that $A \subset R \subset K$, and R/A is an extension of local rings. Then any composite extension k/κ of k_0 and the residue field of R over κ gives the desired fat point (R, k). □

Definition 8.1.25. Let $\beta \to \alpha$ be a pre-special morphism of cycles. Consider S the support of α and X the domain of β. Let $\beta_0 = \sum_{j \in J} m_j.\langle Z_j \rangle_X$ be the reduction of β/α written in standard form.

(1) Let $\mathrm{Spec}\,(K) \to \alpha$ be a dominant point. We define the following cycle over $\mathrm{Spec}\,(K)$ with domain $X_K = X \times_S \mathrm{Spec}\,(K)$:
$$\beta_K = \sum_{j \in J} m_j.\langle Z_j \times_S \mathrm{Spec}\,(K) \rangle_{X_K}.$$
(2) Let $\mathrm{Spec}\,(R) \xrightarrow{\tau} S$ be a dominant trait, K be the fraction field of R and $j : X_K \to X_R$ be the canonical open immersion. We define the following cycle over R with domain X_R :
$$\beta_R = j_*(\beta_K).$$
According to example 8.1.11, β_R is a Hilbert cycle over R.
(3) Let $x : \mathrm{Spec}\,(k) \to \alpha$ be a point on α and (R, k) be a fat point over x.
We define the *specialization of β along the fat point* (R, k) as the cycle
$$\beta_{R,k} := \beta_R \otimes^{\flat}_R k$$
using the above notation and definition 8.1.14. It is a cycle over $\mathrm{Spec}\,(k)$ with domain $X_k = X \times_S \mathrm{Spec}\,(k)$.

Remark 8.1.26. Let $\beta \subset X$ be an S-cycle, $x : \mathrm{Spec}\,(K) \to S$ be a dominant point and U be an open neighborhood of x in S.
Then if β is pre-special over S, $\beta|_U$ is pre-special over U and $\beta_K = (\beta|_U)_K$.
If $\tau : \mathrm{Spec}\,(R) \to S$ (resp. (R, k)) is a trait (resp. fat point) with generic point x, we also get $\beta_R = (\beta|_U)_R$ (resp. $\beta_{R,k} = (\beta|_U)_{R,k}$).

8.1.27. Let S be a reduced scheme, and $\beta = \sum_{i \in I} n_i.x_i$ be an S-cycle with domain X. For any index $i \in I$, let κ_i be the residue field of x_i.

Consider a dominant point $x : \mathrm{Spec}\,(K) \to S$. Let η be its image in S and F be the residue field of η. We put $I' = \{i \in I \mid f(x_i) = \eta\}$ where $f : X \to S$ is the structural morphism. With these notations, we get
$$\beta_K = \sum_{i \in I'} n_i.\langle \mathrm{Spec}\,(\kappa_i \otimes_F K) \rangle_{X_K},$$
and for a dominant trait $\mathrm{Spec}\,(R) \to S$ with generic point x,
$$\beta_R = \sum_{i \in I'} n_i.\langle \mathrm{Spec}\,(\kappa_i \otimes_F K) \rangle_{X_R}, \tag{8.1.27.1}$$

where Spec $(\kappa_i \otimes_F K)$ is seen as a subscheme of X_K (resp. X_R).

Consider a fat point (R, k) with generic point x and write $\beta = \sum_{i \in I} n_i.\langle Z_i \rangle_X$ in standard form (*i.e.* Z_i is the closure of $\{x_i\}$ in X). Then according to (8.1.6.1), we obtain[75]

$$\beta_{R,k} = \sum_{i \in I'} n_i.\left\langle \overline{Z_{i,K}} \times_R \mathrm{Spec}\,(k) \right\rangle_{X_k}$$

where $Z_{i,K} = Z_i \times_S \mathrm{Spec}\,(K)$ is considered as a subscheme of X_K and the schematic closure is taken in X_R.

Considering the description of the schematic closure for the generic fiber of an R-scheme (*cf.* [**EGA4**, 2.8.5]), we obtain the following way to compute $\beta_{R,k}$. By definition, R is an F-algebra. For $i \in I'$, let A_i be the image of the canonical morphism

$$\kappa_i \otimes_F R \to \kappa_i \otimes_F K.$$

It is an R-algebra without R-torsion. Moreover, the factorization

$$\mathrm{Spec}\,(\kappa_i \otimes_F K) \to \mathrm{Spec}\,(A_i) \to \mathrm{Spec}\,(\kappa_i \otimes_F R)$$

defines Spec (A_i) as the schematic closure of the left hand side in the right hand side (*cf.* [**EGA4**, 2.8.5]). In particular, we get an immersion Spec $(A_i \otimes_R k) \to X_k$ and the nice formula :

$$\beta_{R,k} = \sum_{i \in I'} n_i.\left\langle \mathrm{Spec}\,(A_i \otimes_R k) \right\rangle_{X_k}.$$

DEFINITION 8.1.28. Consider a morphism of cycles $f : \beta \to \alpha$ and a point $x : \mathrm{Spec}\,(k_0) \to \alpha$. We say that f is special at x if it is pre-special and for any extension $y : \mathrm{Spec}\,(k) \to \alpha$ of x, for any fat points (R, k) and (R', k) over y, the equality $\beta_{R,k} = \beta_{R',k}$ holds in X_k. Equivalently, we say that β/α is special at x.
We say that f is special (or that β is special over α) if it is special at every point of α.

REMARK 8.1.29. (1) Trivially, f is special at every dominant point of α.
(2) Given an extension y of x, it is equivalent for f to be special at x or at y (use Lemma 8.1.19(1)). Thus, in the case where $\alpha = \langle S \rangle$, we can restrict our attention to the points $s \in S$.
(3) According to 8.1.26, the property that β/S is special at $s \in S$ depends only on an open neighbourhood U of s in S. More precisely, the following conditions are equivalent :
   (i) β is special at s over S.
   (ii) $\beta|_U$ is special at s over U.

EXAMPLE 8.1.30. Let S be a scheme and β be a Hilbert cycle over S. We have already seen that $\beta \to \langle S \rangle$ is pre-special. The next lemma shows this morphism is in fact special.

LEMMA 8.1.31. *Let S be a scheme and β be a Hilbert cycle over S. Consider a point x : Spec $(k) \to S$ and a fat point (R, k) over x.*
Then $\beta_{R,k} = \beta \otimes^\flat_S k$.

PROOF. According to the preceding definition and Lemma 8.1.15 it is sufficient to prove $\beta_R = \beta \otimes^\flat_S R$. As the two sides of this equation are unchanged when replacing β by the reduction β_0 of β/S, we can assume that S is reduced. By additivity, we are reduced to the case where $\beta = \langle X \rangle$ is the fundamental cycle associated with a flat S-scheme X. According to 8.1.6.1, $\beta_R = \left\langle \overline{X_K} \right\rangle_{X_R}$. Applying now [**EGA4**, 2.8.5], $\overline{X_K}$ is the unique closed subscheme Z of X_R such that Z is flat over Spec (R) and $Z \times_R \mathrm{Spec}\,(K) = X_K$. Thus, as X_R is flat over Spec (R), we get $\overline{X_K} = X_R$ and this concludes. □

LEMMA 8.1.32. *Let $p : \tilde{S} \to S$ be a birational morphism and consider a commutative diagram*

$$\begin{array}{ccccc} & & & \nearrow & \tilde{S} \\ \mathrm{Spec}\,(k) & \longrightarrow & \mathrm{Spec}\,(R) & & \downarrow p \\ & & & \searrow & S \end{array}$$

[75]This shows that our definition coincide with the one given in [**SV00b**] (p. 23, paragraph preceding 3.1.3) in the case where $\alpha = \langle S \rangle$, S reduced.

such that (R, k) is a fat point of $\tilde{S}$ and S.

Consider a pre-special cycle β over S and $\tilde{\beta}$ its strict transform along p. Then, $\tilde{\beta}$ is pre-special and $\tilde{\beta}_{R,k} = \beta_{R,k}$.

Proof. Using 8.1.26, we reduce to the case where p is an isomorphism which is trivial. □

Lemma 8.1.33. *Let S be a reduced scheme, $x : \operatorname{Spec}(k_0) \to S$ be a point and α be a pre-special cycle over S. Let $p : \tilde{S} \to S$ be a dominant blow-up such that the strict transform $\tilde{\alpha}$ of α along p is a Hilbert cycle over $\tilde{S}$. Then the following conditions are equivalent :*

- *(i) α is special at x.*
- *(ii) for every points $x_1, x_2 : \operatorname{Spec}(k) \to \tilde{S}$ such that $p \circ x_1 = p \circ x_2$ and $p \circ x_1$ is an extension of x, $\tilde{\alpha} \otimes^{\flat}_{\tilde{S}} x_1 = \tilde{\alpha} \otimes^{\flat}_{\tilde{S}} x_2$.*

Proof. The case where x is a dominant point follows from the definitions and the fact p is an isomorphism at the generic point. We thus assume x is non dominant.
(i) $\Rightarrow$ *(ii)* : Applying Lemma 8.1.24 to x_i, $i = 1, 2$, we can find an extension $x'_i : \operatorname{Spec}(k_i) \to \tilde{S}$ of x_i and a fat point (R_i, k_i) over x'_i. Taking a composite extension L of k_1 and k_2 over k, we can further assume $L = k_1 = k_2$ and $p \circ x'_1 = p \circ x'_2$. Then for $i = 1, 2$, we get

$$\big(\tilde{\alpha} \otimes^{\flat}_{\tilde{S}} x_i\big) \otimes^{\flat}_k L \overset{8.1.15}{=\!=} \tilde{\alpha} \otimes^{\flat}_{\tilde{S}} x'_i \overset{8.1.31}{=\!=} \tilde{\alpha}_{R_i,L} \overset{8.1.32}{=\!=} \alpha_{R_i,L},$$

and this concludes according to 8.1.19(1).
(ii) $\Rightarrow$ *(i)* : Consider an extension $y : \operatorname{Spec}(k) \to \alpha$ over x and two fat point (R_1, k), (R_2, k) over y. Fix $i \in \{1, 2\}$. As p is proper birational, the trait $\operatorname{Spec}(R_i)$ on S can be extended (uniquely) to $\tilde{S}$. Let $x_i : \operatorname{Spec}(k) \to \operatorname{Spec}(R_i) \to \tilde{S}$ be the induced point. Then the following computation allows to conclude :

$$\alpha_{R_i,k} \overset{8.1.32}{=\!=} \tilde{\alpha}_{R_i,k} \overset{8.1.31}{=\!=} \tilde{\alpha} \otimes^{\flat} x_i$$ □

8.1.d. *Pullback.*

8.1.34. In this part, we construct a *pullback* which extends the pullback defined by Suslin et Voevodsky in [**SV00b**, 3.3.1] to the case of morphism of cycles. Consider the situation of a diagram of cycles

$$\begin{array}{ccc} & & \beta \\ & & \downarrow{\scriptstyle f} \\ \alpha' & \longrightarrow & \alpha \end{array} \quad \subset \quad \begin{array}{ccc} & & X \\ & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

where the diagram on the right is the domain of the one on the left. Let n be exponential characteristic of $\operatorname{Supp}(\alpha')$.

The pullback of β, considered as an α-cycle, over α' will be a $\Lambda[1/n]$-cycle denoted by $\beta \otimes_\alpha \alpha'$. It will fits into the following commutative diagram of cycles

$$\begin{array}{ccc} \beta \otimes_\alpha \alpha' & \longrightarrow & \beta \\ \downarrow & & \downarrow \\ \alpha' & \longrightarrow & \alpha \end{array} \quad \subset \quad \begin{array}{ccc} X \times_S S' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

where the right commutative square is again the support of the left one.

It will be defined under an assumption on β/α and is therefore non symmetric[76]. This assumption will imply that β/α is pre-special, and the first property of $\beta \otimes_\alpha \alpha'$ is that it is pre-special over α'.
We define this product in three steps in which the following properties[77] will be a guideline :

(P1) Let S_0 be the support of α and β_0 be the reduction of β/α as an S_0-cycle. Consider the canonical factorization $\alpha' \to S_0 \to \alpha$.
Then, $\beta \otimes_\alpha \alpha' = \beta_0 \otimes_{S_0} \alpha'$.

[76] See further 8.2.3 for this question.

[77] All these properties except (P3) will be particular cases of the associativity of the pullback.

(P2) Consider a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}\,(E) & \longrightarrow & \mathrm{Spec}\,(R') & \longrightarrow & \mathrm{Spec}\,(R) \\ & & \downarrow & (*) & \downarrow \\ & & \alpha' & \longrightarrow & \alpha \end{array}$$

such that (R, E) (resp. (R', E)) is a fat point on α (resp. α').
Then, $(\beta \otimes_\alpha \alpha')_{R',E} = \beta_{R,E}$.

Assume $\alpha' \to \alpha = \langle S' \to S \rangle$.

(P3) If β is a Hilbert cycle over S, $\beta \otimes_S S' = \beta \otimes^\flat_S S'$.

(P4) Consider a factorization $S' \to U \xrightarrow{j} S$ such that j is an open immersion. Then $\beta \otimes_S S' = \beta|_U \otimes_U S'$.

(P5) Consider a factorization $S' \to \tilde{S} \xrightarrow{p} S$ such that p is a birational morphism. Then $\beta \otimes_S S' = \tilde{\beta} \otimes_{\tilde{S}} S'$.

LEMMA 8.1.35. *Consider the hypothesis of 8.1.34 in the case where $\alpha' = \mathrm{Spec}\,(k)$ is a point x of α.*

We suppose that f is special at x.

Then the pre-special $\Lambda[1/n]$-cycle $\beta \otimes_\alpha k$ exists and is uniquely determined by property (P2) above. We also put $\beta_k := \beta \otimes_\alpha k$.

The properties (P1) to (P5) are fulfilled and in addition :
(P6) For any extension fields L/k, $\beta_L = \beta_k \otimes^\flat_k L$.

PROOF. According to Lemma 8.1.24 there always exists a fat point (R, E) over an extension of x. Thus the unicity statement follows from 8.1.19(1).

For the existence, we first consider the case where $\alpha = \langle S \rangle$ is a reduced scheme. Applying Lemma 8.1.18, there exists a blow-up $p : \tilde{S} \to S$ such that the strict transform $\tilde{\beta}$ of β along p is a Hilbert cycle over $\tilde{S}$.

As p is surjective, the fiber $\tilde{S}_k$ is a non empty algebraic k-scheme. Thus, it admits a closed point given by a finite extension k'_0 of k. Let k'/k be a normal closure of k'_0/k and G be its Galois group. As β/S is special at x by hypothesis, Lemma 8.1.33 implies that $\tilde{\beta} \otimes^\flat_{\tilde{S}} k'$ is G-invariant. Thus, applying Lemma 8.1.19, there exists a unique cycle $\beta_k \subset X_k$ with coefficients in $\Lambda[1/n]$ such that $\beta_k \otimes^\flat_k k' = \tilde{\beta} \otimes^\flat_{\tilde{S}} k'$.

We prove (P2). Given a diagram $(*)$ with $\alpha' = \mathrm{Spec}\,(k)$, we first remark that $(\beta_k)_{R',E} = \beta_k \otimes^\flat_k E$. As p is proper birational, the dominant trait $\mathrm{Spec}\,(R) \to S$ lifts to a dominant trait $\mathrm{Spec}\,(R) \to \tilde{S}$. Let E'/k be a composite extension of k'/k and E/k. With these notations, we get the following computation :

$$\beta_{R,E} \otimes^\flat_E E' \overset{8.1.32}{=\!=} \tilde{\beta}_{R,E} \otimes^\flat_E E' \overset{8.1.31}{=\!=} \tilde{\beta} \otimes^\flat_{\tilde{S}} E' \overset{8.1.15}{=\!=} (\tilde{\beta} \otimes^\flat_{\tilde{S}} k') \otimes^\flat_E E' =\!= \beta_k \otimes^\flat_k E',$$

so that we can conclude by applying 8.1.19(1).

In the general case, we consider he support S of α abd β_0/S the reduction of β/α. According to (P1), we are led to put $\beta_k := (\beta_0)_k$ with the help of the preceding case. Considering the definition of specialization along fat points, we easily check this cycle satisfies property (P2).

Finally, property (P6) (resp. (P3), (P5)) follows from the unicity statement applying lemmas 8.1.24, 8.1.19(1) (resp. and moreover Lemma 8.1.31, 8.1.32). □

REMARK 8.1.36. In the case where x is a dominant point, the cycle β_k defined in the previous proposition agrees with the one defined in 8.1.25(1).

LEMMA 8.1.37. *Consider the hypothesis of 8.1.34 in the case where $\alpha' = \mathrm{Spec}\,(O)$ is a trait of α. Let K be the fraction field of O and x the corresponding point on α.*

We suppose that f is special at x.

Then the pre-special $\Lambda[1/n]$-cycle $\beta \otimes_\alpha O$ exists and is uniquely defined by the property $(\beta \otimes_\alpha O) \otimes^\flat_O K = \beta_K$ with the notations of the preceding lemma. We also put $\beta_O := \beta \otimes_\alpha O$.

The properties (P1) to (P5) are fulfilled and in addition :
(P6') For any extension O'/O of discrete valuation rings, $\beta_{O'} = \beta_O \otimes^\flat_O O'$.

Proof. Remark that, with the notation of definition 8.1.7, $\beta_O \otimes^\flat_O K = \beta_O|_{\mathrm{Spec}(K)}$. For the first statement, we simply apply Lemma 8.1.8 and put $\beta_O = j_*(\beta_K)$ where $j : X_K \to X_O$ is the canonical open immersion.

Then properties (P1), (P3), (P4), (P5) and (P6') of the case considered in this lemma follows easily from the uniqueness statement and the corresponding properties in the preceding lemma (applying again 8.1.8).

It remains to prove (P2). According to (P1), we reduce to the case $\alpha = \langle S \rangle$ for a reduced scheme S. We choose a birational morphism $p : \tilde{S} \to S$ such that the proper transform $\tilde{\beta}$ is a Hilbert $\tilde{S}$-cycles. Consider a diagram of the form $(*)$ in this case. According to property (P3), we can assume $R' = O$.
Remark the trait $\mathrm{Spec}\,(R) \to S$ admits an extension $\mathrm{Spec}\,(R) \to \tilde{S}$ as p is proper. The point x admits an extension K'/K which lifts to a point $x' : \mathrm{Spec}\,(K') \to \tilde{S}$ – again $\tilde{S}_K$ is a non empty algebraic scheme. The discrete valuation corresponding to $O \subset K$ extends to a discrete valuation on K' as K'/K is finite. Let $O' \subset K'$ be the corresponding valuation ring. The corresponding trait $\mathrm{Spec}\,(O') \to S$ thus admits a lifting to $\tilde{S}$ corresponding to the point x' as p is proper. Considering a composite extension E'/K of K'/K and E/K, we have obtained a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}\,(E') & \longrightarrow & \mathrm{Spec}\,(O') & \longrightarrow & \mathrm{Spec}\,(R) \\ & & \| & & \downarrow \\ & & \mathrm{Spec}\,(O') & \longrightarrow & \tilde{S} \end{array}$$

which lifts our original diagram $(*)$. Let x_1 (resp. x_2) be the point $\mathrm{Spec}\,(E)' \to \tilde{S}$ corresponding to the the composite through the upper way (resp. lower way) in the preceding diagram.

Then, $\beta_{R,E} \otimes^\flat_E E' = \tilde{\beta}_{x_1}$. Moreover, we get

$$(\beta \otimes_S O)_{O,E} \otimes^\flat_E E' \overset{8.1.31}{=\!=} (\beta \otimes_S O) \otimes^\flat_O E' \overset{(P5)+(P6')}{=\!=\!=} (\tilde{\beta} \otimes_{\tilde{S}} O') \otimes^\flat_{O'} E' \overset{(P3)}{=\!=} \tilde{\beta}_{x_2}.$$

By hypothesis, β/α is special at $\mathrm{Spec}\,(K') \to S$. Thus Lemma 8.1.33 concludes. □

Theorem 8.1.38. *Consider the hypothesis of 8.1.34.*
Assume f is special at the generic points of α'.
Then the pre-special $\Lambda[1/n]$-cycle $\beta \otimes_\alpha \alpha'$ exists and is uniquely determined by property (P2).
It satisfies all the properties (P1) to (P5).

Proof. According to Lemma 8.1.24, for any point s of S' with residue field κ, there exists an extension E/κ and a fat point (R, E) (resp. (R', E)) of α (resp. α') over $\mathrm{Spec}\,(E) \to \alpha$ (resp. $\mathrm{Spec}\,(E) \to \alpha'$). The uniqueness statement follows by applying Lemma 8.1.19(1).

For the existence, we write $\alpha' = \sum_{i \in I} n_i.\langle Z_i \rangle_{S'}$ in standard form.

For any $i \in I$, let K_i be the function field of Z_i and consider the canonical morphism $\mathrm{Spec}\,(K_i) \to \alpha$. Let $\beta_{K_i} \subset X_{K_i}$ be the $\Lambda[1/n]$-cycle defined in lemma 8.1.35. Let $j_i : X_{K_i} \to X'$ be the canonical immersion and put :

$$\beta \otimes_\alpha \alpha' = \sum_{i \in I} n_i.j_{i*}(\beta_{K_i}). \tag{8.1.38.1}$$

Then properties (P1), (P3), (P4) and (P5) are direct consequences of this definition and of the corresponding properties of Lemma 8.1.35.

We check property (P2). Given a diagram of the form $(*)$, there exists a unique $i \in I$ such that $\mathrm{Spec}\,(R')$ dominates Z_i. Thus we get for this choice of $i \in I$ that $(\beta \otimes_\alpha \alpha')_{R',E} = \big(j_{i*}(\beta_{K_i})\big)_{R',E}$. Let K' be the fraction field of R' and consider the open immersion $j' : X_{K'} \to X_{R'}$. The following computation then concludes :

$$\big(j_{i*}(\beta_{K_i})\big)_{R',E} = j'_*\big(j_{i*}(\beta_{K_i})_{K'}\big) \otimes^\flat_{R'} E \overset{8.1.26}{=\!=} j'_*(\beta_{K'}) \otimes^\flat_{R'} E \overset{8.1.37}{=\!=} \beta_{R'} \otimes^\flat_{R'} E \overset{8.1.37(P2)}{=\!=\!=} \beta_{R,E}.$$

□

DEFINITION 8.1.39. In the situation of the previous theorem, we call the $\Lambda[1/n]$-cycle $\beta \otimes_\alpha \alpha'$ the pullback of β/α by α'.

8.1.40. By construction, the cycle $\beta \otimes_\alpha \alpha'$ is bilinear with respect to addition of cycles in the following sense:

(P7) Consider the hypothesis of 8.1.34. Let α'_1, α'_2 be cycles with domain S' such that $\alpha = \alpha'_1 + \alpha'_2$. If β/α is special at the generic points of α_1 and α_2, then the following cycles are equal in $X \times_S S'$:

$$\beta \otimes_\alpha (\alpha'_1 + \alpha'_2) = \beta \otimes_\alpha \alpha'_1 + \beta \otimes_\alpha \alpha'_2.$$

(P7') Consider the hypothesis of 8.1.34. Let β_1, β_2 be cycles with domain X such that $\beta = \beta_1 + \beta_2$. If β_1 and β_2 are special over α at the generic points of α', then β/α is special at the generic points of α' and the following cycles are equal in $X \times_S S'$:

$$(\beta_1 + \beta_2) \otimes_\alpha \alpha' = \beta_1 \otimes_\alpha \alpha' + \beta_2 \otimes_\alpha \alpha'.$$

In the theorem above, we can assume that X (resp. S, S') is the support of β (resp. α, α'). Thus the support of $\beta \otimes_\alpha \alpha'$ is included in $X \times_S S'$. More precisely:

LEMMA 8.1.41. *Consider the hypothesis of 8.1.34 and assume that X (resp. S, S') is the support of β (resp. α, α'). Then, if β/α is special at the generic points of α', we obtain:*

(i) Let $(X \times_S S')^{(0)}$ be the generic points of $X \times_S S'$. Then, we can write

$$\beta \otimes_\alpha \alpha' = \sum_{x \in (X \times_S S')^{(0)}} m_x . x$$

(ii) For any generic point x of $X \times_S S'$, if $m_x \neq 0$, the image of x in S' is a generic point s' and the multiplicity of s' in α' divides m_x in $\Lambda[1/n]$.

PROOF. Point (ii) is just a traduction that $\beta \otimes_\alpha \alpha'$ is pre-special over α'. For point (i), we reduce easily to the case where α is the scheme S and S is reduced. We can also assume that α' is the spectrum of a field k. It is sufficient to check point (i) after an extension of k. Thus we can apply Lemma 8.1.18 to reduce to that case where β is a Hilbert cycle over S. This case is obvious. □

DEFINITION 8.1.42. In the situation of the previous lemma, we put

$$m^{SV}(x; \beta \otimes_\alpha \alpha') := m_x \in \Lambda[1/n]$$

and we call them the Suslin-Voevodsky multiplicities (in the operation of pullback).

REMARK 8.1.43. Consider the notations of the previous lemma:

(1) Assume that α is the spectrum of a field k. Then the product $\beta \otimes_k \alpha'$ is always defined and agrees with the classical *exterior product* (according to (P3)).
(2) According to the previous lemma, the irreducible components of $X \times_S S'$ which does not dominate an irreducible component of S' have multiplicity 0: they correspond to the "non proper components" with respect to the operation $\beta \otimes_\alpha \alpha'$.
(3) Assume $\alpha' \to \alpha = \langle S' \xrightarrow{p} S \rangle$, $\beta = \sum_{i \in I} n_i . x_i$. Let y be a generic point of $X \times_S S'$ lying over a generic point s' of S'. Let S'_0 be the irreductible component of S' corresponding to s'. Consider *any* irreductible component S_0 of S which contains $p(s')$ and let $\beta_0 = \sum_i n_i . x_i$ where the sums runs over the indexes i such that x_i lies over S_0. Then, according to (8.1.38.1),
$$m^{SV}(y; \beta \otimes_S \langle S' \rangle) = m^{SV}(y; \beta_0 \otimes_{S_0} \langle S'_0 \rangle).$$
This is a key property of the Suslin-Voevodsky multiplicities which explains why we have to consider the property that β/α is special at s' (see 8.3.25 for a refined statement).

LEMMA 8.1.44. *Consider a morphism of cycles $\alpha' \to \alpha$ and a pre-special morphism $f : \beta \to \alpha$ which is special at the generic points of α. Consider a commutative square*

$$\begin{array}{ccc} \mathrm{Spec}\,(k') & \xrightarrow{x'} & \alpha' \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,(k) & \xrightarrow{x} & \alpha \end{array}$$

such that k and k' are fields. Then the following conditions are equivalent :

(i) *f is special at x.*
(ii) *$\beta \otimes_\alpha \alpha' \to \alpha'$ is special at x'.*

PROOF. This follows easily from Lemma 8.1.24 and property (P2). □

COROLLARY 8.1.45. *Let $f : \beta \to \alpha$ be a special morphism.*
Then for any morphism $\alpha' \to \alpha$, $\beta \otimes_\alpha \alpha' \to \alpha'$ is special.

DEFINITION 8.1.46. Let $f : \beta \to \alpha$ be a morphism of cycles and $x : \mathrm{Spec}\,(k) \to \alpha$ be a point. We say that f is Λ-universal at x if it is special at x and the cycle $\beta \otimes_\alpha k$ has coefficients in Λ.

In the situation of this definition, let s be the image of x in the support of α, and κ_s be its residue field. Then according to (P6), $\beta_k = \beta_{\kappa_s} \otimes^\flat_{\kappa_s} k$. Thus f is Λ-universal at x if and only if it is Λ-universal at s. Furthermore, the following lemma follows easily :

LEMMA 8.1.47. *Let $f : \beta \to \alpha$ be a morphism of cycles. The following conditions are equivalent :*

(i) *For any point $s \in \overline{\alpha}$, f is Λ-universal at s.*
(ii) *For any point $x : \mathrm{Spec}\,(k) \to \alpha$, f is Λ-universal at x.*
(iii) *For any morphism of cycles $\alpha' \to \alpha$, $\beta \otimes_\alpha \alpha'$ has coefficients in Λ.*

DEFINITION 8.1.48. We say that a morphism of cycles f is Λ-universal if it satisfies the equivalent properties of the preceding lemma.

Of course, Λ-universal morphisms are stable by base change. These definitions will be applied similarly to morphisms of schemes by considering the associated morphism of cycles.

EXAMPLE 8.1.49. According to property (P3) of the pullback, a flat equidimensional morphism of schemes is Λ-universal.

8.2. Intersection theoretic properties.

8.2.a. *Commutativity.*

LEMMA 8.2.1. *Consider morphisms of cycles with support in the left diagram*

$$\begin{array}{ccc} & & \beta \\ & & \downarrow \\ \gamma & \longrightarrow & \alpha \end{array} \quad \subset \quad \begin{array}{ccc} & & X \\ & & \downarrow f \\ T & \xrightarrow{g} & S \end{array}$$

such that β/α is pre-special and γ/α is pseudo-dominant.

Assume

$$\alpha = \sum_{i \in I} n_i.s_i, \quad \beta = \sum_{j \in J} m_j.x_j, \quad \gamma = \sum_{l \in H} p_l.t_l$$

and denote by κ_{s_i} (resp. κ_{x_j}, κ_{t_l}) the residue field of s_i (resp. x_j, t_l) in S (resp. X, T). Considering $(i,j,l) \in I \times J \times H$ such that $f(x_j) = g(t_l) = s_i$, we denote by $\nu_{j,l} : \mathrm{Spec}\left(\kappa_{x_j} \otimes_{\kappa_{s_i}} \kappa_{t_l}\right) \to X \times_S T$ the canonical immersion.

Then the following assertions hold :

(i) *β is special at the generic points of γ.*
(ii) *The cycle $\beta \otimes_\alpha \gamma$ has coefficients in Λ.*

(iii) The following equality of cycles holds

$$\beta \otimes_\alpha \gamma = \sum_{i,j,l} \frac{m_j}{n_i} p_l.\nu_{j,l*}\big(\langle \mathrm{Spec}\,(\kappa_{y_j} \otimes_{\kappa_{x_i}} \kappa_{z_l})\rangle\big)$$

where the sum runs over $(i,j,l) \in I \times J \times H$ such that $f(x_j) = g(t_j) = s_i$.

Proof. Assertion (i) is in fact the first point of 8.1.29. Assertion (ii) follows from assertion (iii), which is a consequence of the defining formula (8.1.38.1) and remark 8.1.36. □

Corollary 8.2.2. *Let $g : T \to S$ be a flat morphism and $\beta = \sum_{j\in J} m_j.\langle Z_j\rangle_X$ be a pre-special S-cycle written in standard form.*

Then β/S is pre-special at the generic points of T and

$$\beta \otimes_S \langle T\rangle = \sum_{j\in J} m_j.\langle Z_j \times_S T\rangle.$$

The pullback $\beta \otimes_\alpha \gamma$, at it is defined only when β/α is special, is in general non symmetric in β and γ. However the previous lemma implies it is symmetric whenever it makes sense :

Corollary 8.2.3. *Consider pre-special morphisms of cycles $\beta \to \alpha$ and $\gamma \to \alpha$.*

Then β (resp. γ) is special at the generic points of γ (resp. β) and the following equality holds: $\beta \otimes_\alpha \gamma = \gamma \otimes_\alpha \beta$.

8.2.b. *Associativity.*

Proposition 8.2.4. *Consider morphism of cycles $\beta \xrightarrow{f} \alpha$, $\alpha'' \to \alpha' \to \alpha$ such that f is special at the generic points of α' and of α''. Let n be the exponential characteristic of α''.*

Then the following assertions hold:

(i) The relative cycle $(\beta \otimes_\alpha \alpha')/\alpha'$ is special at the generic points of α''.
(ii) The cycle $(\beta \otimes_\alpha \alpha') \otimes_{\alpha'} \alpha''$ has coefficients in $\Lambda[1/n]$.
(iii) $(\beta \otimes_\alpha \alpha') \otimes_{\alpha'} \alpha'' = \beta \otimes_\alpha \alpha''$.

Proof. Assertion (i) is a corollary of Lemma 8.1.44. Assertion (ii) is in fact a corollary of assertion (iii), which in turn follows easily from the uniqueness statement in theorem 8.1.38. □

Lemma 8.2.5. *Let $\gamma \xrightarrow{g} \beta \xrightarrow{f} \alpha$ be two pre-special morphisms of cycles with domains $Y \to X \to S$. Consider a fat point (R, k) over α such that γ/β is special at the generic points of $\beta_{R,k}$.*

Then γ/α is pre-special and the following equality of cycles holds in Y_k:

$$\gamma_{R,k} = \gamma \otimes_\beta (\beta_{R,k}).$$

Proof. The first statement is obvious.

We first prove: $\gamma_R = \gamma \otimes_\beta \beta_R$.

Remark that $\beta_R \to \beta$ is pseudo-dominant. Thus γ/β is special at the generic points of β_R and the right hand side of the preceding equality is well defined. Moreover, according to Lemma 8.2.1, we can restrict to the case where $\alpha = s$, $\beta = x$ and $\gamma = y$, with multiplicity 1. Let κ_s, κ_x, κ_y be the corresponding respective residue fields, and K be the fraction field of R.

Then, according to (8.1.27.1), $\gamma_R = \langle \kappa_y \otimes_{\kappa_s} K\rangle_{Y_R}$ and $\beta_R = \langle \kappa_x \otimes_{\kappa_s} K\rangle_{X_R}$. But Lemma 8.2.1 implies that $\gamma \otimes_\beta \beta_R = \langle \kappa_y \otimes_{\kappa_x} (\kappa_x \otimes_{\kappa_s} K)\rangle_{X_R}$. Thus the associativity of the tensor product of fields allows to conclude.

From this equality and Proposition 8.2.4, we deduce that:

$$\gamma_R \otimes_{\beta_R} \beta_{R,k} = (\gamma \otimes_\beta \beta_R) \otimes_{\beta_R} \beta_{R,k} = \gamma \otimes_\beta \beta_{R,k}.$$

Thus, the equality we have to prove can be written $\gamma_R \otimes^\flat_R k = \gamma_R \otimes_{\beta_R} (\beta_R \otimes^\flat_R k)$ and we are reduced to the case $\alpha = \mathrm{Spec}\,(R)$.

In this case, we can assume $\beta = \langle X\rangle$ with X integral. Let us consider a blow-up $\tilde{X} \xrightarrow{p} X$ such that the proper transform $\tilde{\gamma}$ of γ along p is a Hilbert cycle over $\tilde{X}$ (8.1.18). We easily get (from (P3) and 8.1.15) that

$$\tilde{\gamma}_k = \tilde{\gamma} \otimes_{\tilde{X}} \langle \tilde{X}_k\rangle.$$

Let Y (resp. $\tilde{Y}$) be the support of γ (resp. $\tilde{\gamma}$), $q : \tilde{Y} \to Y$ the canonical projection. We consider the cartesian square obtained by pullback along $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(R)$:

$$\begin{array}{ccc} \tilde{Y}_k & \xrightarrow{q_k} & Y_k \\ \downarrow & & \downarrow \\ \tilde{X}_k & \xrightarrow{p_k} & X_k. \end{array}$$

As $X_k \subset X$ (resp. $Y_k \subset Y$) is purely of codimension 1, the proper morphism p_k (resp. q_k) is still birational. As a consequence, $q_{k*}(\tilde{\gamma}) = \gamma$. Let y be a point in $\tilde{Y}_k^{(0)} \simeq Y_k^{(0)}$ which lies above a point x in $\tilde{X}_k^{(0)} \simeq X_k^{(0)}$ Then, according to (P5) and using the notations of 8.1.42, we get

$$m^{SV}(y; \tilde{\gamma} \otimes_{\tilde{X}} \langle \tilde{X}_k \rangle) = m^{SV}(y; \gamma \otimes_X \langle X_k \rangle).$$

This readily implies $q_{k*}(\tilde{\gamma} \otimes_{\tilde{X}} \langle \tilde{X}_k \rangle) = \gamma \otimes_X \langle X_k \rangle$ and allows us to conclude. □

As a corollary of this lemma using the uniqueness statement in Theorem 8.1.38, we obtained :

Corollary 8.2.6. *Let $\gamma \xrightarrow{g} \beta \xrightarrow{f} \alpha$ be pre-special morphisms of cycles.*

Let $x : \mathrm{Spec}\,(k) \to \alpha$ be a point. If β/α is special (resp. Λ-universal) at x and γ/β is special (resp. Λ-universal) at the generic points of β_k, then γ/α is special at x.

Let $\alpha' \to \alpha$ be any morphism of cycles with domain $S' \to S$ and n be the exponential characteristic of α'. Then, whenever it is well defined, the following equality of $\Lambda[1/n]$-cycles holds:

$$\gamma \otimes_\beta (\beta \otimes_\alpha \alpha') = \gamma \otimes_\alpha \alpha'.$$

A consequence of the transitivity formulas is the associativity of the pullback :

Corollary 8.2.7. *Suppose given the following morphisms of cycles*

$$\begin{array}{ccccccc} \alpha & & & \beta & & & \gamma \\ & \searrow & \swarrow_{f} & & \searrow & \swarrow_{g} & \\ & & \delta & & \sigma & & \end{array}$$

such that f and g are pre-specials.

Then, whenever it is well defined, the following equality of cycles hold:

$$\gamma \otimes_\sigma (\beta \otimes_\delta \alpha) = (\gamma \otimes_\sigma \beta) \otimes_\delta \alpha$$

Proof. Indeed, by the transitivity formulas 8.2.4 and 8.2.6, both members of the equation are equal to $(\gamma \otimes_\sigma \beta) \otimes_\beta (\beta \otimes_\delta \alpha)$. □

8.2.c. *Projection formulas.*

Proposition 8.2.8. *Consider morphisms of cycles with support in the left diagram*

$$\begin{array}{ccc} & & \beta \\ & & \downarrow \\ \alpha' & \longrightarrow & \alpha \end{array} \quad \subset \quad \begin{array}{ccc} & & X \\ & & \downarrow \\ S' & \xrightarrow{q} & S \end{array}$$

such that β/α is special at the generic points of α'.
Consider a factorization $S' \xrightarrow{g} T \to S$.

Then β/α is special at the generic points of $g_*(\alpha)$ and the following equality of cycles holds in $X \times_S T$:

$$\beta \otimes_\alpha g_*(\alpha') = (1_X \times_S g)_*(\beta \otimes_\alpha \alpha').$$

Proof. The first assuption is obvious. By linearity, we can assume S' is integral and α' is the generic point s of S' with multiplicity 1. Let L (resp. E) be the residue field of s (resp. $g(s)$).

Consider the pullback square $\begin{array}{ccc} X_L & \xrightarrow{g_0} & X_E \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ X \times_S S' & \xrightarrow{g_X} & X \times_S T \end{array}$ where i and j are the natural immersions.

Let d be the degree of L/E if it is finite and 0 otherwise. We are reduced to prove the equality $g_{X*}(j_*(\beta_L)) = d.i_*(\beta_E)$. Using the functoriality of pushforward and property (P6), it is sufficient to prove the equality $g_{0*}(\beta_E \otimes^\flat_E L) = d.\beta_E$. If $d = 0$, the morphism g_0 induces an infinite extension of fields on any point of X_L which concludes. If L/E is finite, g_0 is finite flat and $\beta_E \otimes^\flat_E L$ is the usual pullback by g_0. Then the needed equality follows easily (see [**Ful98**, 1.7.4]). □

LEMMA 8.2.9. *Let $\beta \to \alpha$ be a pre-special morphism of cycles with domain $X \xrightarrow{p} S$. Let (R, k) a fat point over α and $X \xrightarrow{f} Y \to S$ be a factorization of p. Let f_k be the pullback of f over* $\operatorname{Spec}(k)$.

Suppose that the support of β is proper with respect to f. Then $f_*(\beta)$ is pre-special over α and the equality of cycles $\big(f_*(\beta)\big)_{R,k} = f_{k*}(\beta_{R,k})$ holds in Y_k.

PROOF. As usual, considering the support S of α, we reduce to the case where $\alpha = \langle S \rangle$. Let K be the fraction field of R. As $\operatorname{Spec}(K)$ maps to a generic point of S, we can assume S is integral. Let F be its function field. We can assume by linearity that β is a point x in X with multiplicity 1.

Let L (resp. E) be the residue field of x (resp. $y = f(x)$). Let d be the degree of L/E if it is finite and 0 otherwise. Consider the following pullback square

$$\begin{array}{ccc} \operatorname{Spec}(L \otimes_F K) & \xrightarrow{j} & X \times_S \operatorname{Spec}(R) = X_R \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_R} \\ \operatorname{Spec}(E \otimes_F K) & \xrightarrow{i} & Y \times_S \operatorname{Spec}(R) = Y_R. \end{array}$$

According to the formula (8.1.27.1), we obtain:

$$\begin{aligned} f_{R*}(\beta_R) &= f_{R*}j_*(\langle L \otimes_F K \rangle) = i_* f_{0*}(\langle L \otimes_F K \rangle) \\ &= i_* f_{0*}(f_0^*(\langle E \otimes_F K \rangle) = i_*(d.\langle E \otimes_F K \rangle) = \langle f_*(\beta) \rangle_R. \end{aligned}$$

We are finally reduced to the case $S = \operatorname{Spec}(R)$ and β is a Hilbert cycle over $\operatorname{Spec}(R)$. Note that $f_*(\beta)$ is still a Hilbert cycle over $\operatorname{Spec}(R)$. As $\beta_{R,k} = \beta \otimes^\flat_R k$, the result follows now from Proposition 8.1.16. □

COROLLARY 8.2.10. *Consider morphisms of cycles with support in the left diagram*

$$\begin{array}{ccc} & & \beta \\ & & \downarrow \\ \alpha' & \longrightarrow & \alpha \end{array} \quad \subset \quad \begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle p} \\ S' & \longrightarrow & S \end{array}$$

such that β/α is special at the generic points of α' (resp. Λ-universal).

Consider a factorization $X \xrightarrow{f} Y \to S$ of p.

Suppose that the support of β is proper with respect to f. Then $f_*(\beta)/\alpha$ is special at the generic points of α' (resp. Λ-universal) and the following equality of cycles holds in $X \times_S S'$:

$$(f \times_S 1_{S'})_*(\beta \otimes_\alpha \alpha') = \big(f_*(\beta)\big) \otimes_\alpha \alpha'.$$

8.3. Geometric properties.

8.3.1. We introduce a notation which will come often in the next section. Let S be a scheme and $\alpha = \sum_{i \in I} n_i.\langle Z_i \rangle_X$ an S-cycle written in standard form.

Let s be a point of S and $\operatorname{Spec}(k) \xrightarrow{\bar{s}} S$ be a geometric point of S with k separably closed. Let S' be one of the following local schemes: the localization of S at s, the Hensel localization of S at s, the strict localization of S at $\bar{s}$.

We then define the cycle with coefficients in Λ and domain $X \times_S S'$ as:

$$\alpha|_{S'} = \sum_{i \in I} n_i \langle Z_i \times_S S' \rangle_{X \times_S S'}.$$

REMARK 8.3.2. The canonical morphism $S' \to S$ is flat. In particular, α/S is special at the generic points of S' and we easily get: $\alpha|_{S'} = \alpha \otimes_S S'$.

8.3.a. *Constructibility.*

Definition 8.3.3. Let S be a scheme and $s \in S$ a point. We say that a pre-special S-cycle α is emphtrivial at s if it is special at s and $\alpha \otimes_S s = 0$.

Naturally, we say that α is trivial if it is zero. Thus α is trivial if and only if it is trivial at the generic points of S.

Recall from [**EGA4**, 1.9.6] that an ind-constructible subset of a noetherian scheme X is a union of locally closed subset of X.

Lemma 8.3.4. *Let S be a noetherian scheme, and α/S be a pre-special cycle. Then the set*

$$T = \big\{s \in S \mid \alpha/S \text{ is special (resp. trivial, } \Lambda\text{-universal) at } s\big\}$$

is ind-constructible in S.

Proof. Let s be a point of T, and Z be its closure in S with its reduced subscheme structure. Put $\alpha_Z = \alpha \otimes_S Z$, defined because α is special at the generic point of Z. Given any point t of Z, we know that α/S is special at t if and only if α_Z/Z is special at t (*cf.* 8.1.44). But there exists a dense open subset U_s of Z such that $\alpha_Z|_{U_Z}$ is a Hilbert cycle over U_Z. Thus, α/S is special at each point of U_s and $U_s \subset T$. This concludes and the same argument proves the respective statements. □

8.3.5. Let I be a left filtering category and $(S_i)_{i\in I}$ be a projective system of noetherian schemes with affine transition morphisms. We let S be the projective limit of (S_i) and we assume the followings:

(1) S is noetherian.
(2) There exists an index $i \in I$ such that the canonical projection $S \xrightarrow{p_i} S_i$ is dominant.

In this case, there exists an index j/i such that for any k/j, the map p_k induces an isomorphism $S^{(0)} \to S_k^{(0)}$ on the generic points (*cf.* [**EGA4**, 8.4.1]). Thus, replacing I by I/j, we can assume that this property is satisfied for all index $i \in I$. As a consequence, the following properties are consequences of the previous ones:

(3) For any $i \in I$, $p_i : S \to S_i$ is pseudo-dominant and p_i induces an isomorphism $S^{(0)} \to S_i^{(0)}$.
(4) For any arrow $j \to i$ of I, $p_{ji} : S_j \to S_i$ is pseudo-dominant and p_{ji} induces an isomorphism $S_j^{(0)} \to S_i^{(0)}$.

Proposition 8.3.6. *Consider the notations and hypothesis above. Assume we are given a projective system of cycles $(\alpha_i)_{i\in I}$ such that α_i is a pre-special cycle over S_i and for any $j \to i$, $\alpha_j = \alpha_i \otimes_{S_i} S_j$. Put $\alpha = \alpha_i \otimes_{S_i} S$ for an index $i \in I$.*[78]

The following conditions are equivalent:

(i) α/S is special (resp. Λ-universal).
(ii) There exists $i \in I$ such that α_i/S_i is special (resp. Λ-universal).
(iii) There exists $i \in I$ such that for all j/i, α_j/S_j is special (resp. Λ-universal).

Let s be point of S and s_i its image in S_i. Then the following conditions are equivalent:

(i) α/S is special (resp. Λ-universal) at s.
(ii) There exists $i \in I$ such that α_i/S_i is special (resp. Λ-universal) at s_i.
(iii) There exists $i \in I$ such that for all j/i, α_j/S_j is special (resp. Λ-universal) at s_j.

Proof. Let P be one of the respective properties: "special", "trivial", "Λ-universal". Using the fact that being P at s is an ind-constructible property (from Lemma 8.3.4), it is sufficient to apply [**EGA4**, th. 8.3.2] to the following family of sets:

$$F_i = \{s_i \in S_i \mid \ \alpha_i \text{ satisfies } \mathsf{P} \text{ at } s_i\}, \quad F = \{s \in S \mid \ \alpha \text{ satisfies } \mathsf{P} \text{ at } s\}.$$

[78]The pullback is well defined because of point (3) and (4) of the hypothesis above.

To get the two sets of equivalent conditions of the statement from *op. cit.* we have to prove the following relations:

$$(1) : \forall (j \to i) \in \mathrm{Fl}(I), p_{ji}^{-1}(F_i) \subset F_j,$$
$$(2) : F = \cup_{i \in I} p_i^{-1}(F_i).$$

We consider the case where P is the property "special". For relation (1), we apply 8.1.44 which implies the stronger relation $p_{ji}^{-1}(F_i) = F_j$. For relation (2), another application of 8.1.44 gives in fact the stronger relation $F = p_i^{-1}(F_i)$ for any $i \in I$.

Consider a point $s_j \in S$ and put $s_i = p_{ji}(s_j)$. Assume α_i is special at s_i. Then, applying 8.2.4 and (P3), we get:

$$\alpha_j \otimes_{S_j} s_j = (\alpha_i \otimes_{S_i} s_i) \otimes^{\flat}_{\kappa(s_i)} \kappa(s_j). \tag{8.3.6.1}$$

Similarly, given $s \in S_j$, $s_i = p_i(s)$, and assuming α_i is special at s_i, we get:

$$\alpha \otimes_S s = (\alpha_i \otimes_{S_i} s_i) \otimes^{\flat}_{\kappa(s_i)} \kappa(s). \tag{8.3.6.2}$$

We consider now the case where P is the property "trivial". Then relation (1) follows from (8.3.6.1). Relation (2) follows from (8.3.6.1) and 8.1.19(1).

We finally consider the case P is the property "Λ-universal". Relation (1) in this case is again a consequence of (8.3.6.1). According to (8.3.6.2), we get the inclusion $\cup_{i \in I} f_i^{-1}(F_i) \subset F$. We have to prove the reciprocal inclusion.

Consider a point $s \in S$ with residue field k such that α/S is Λ-universal at s. For any $i \in I$, we put $s_i = p_i(s)$ and denote by k_i its residue field. It is sufficient to find an index $i \in I$ such that $\alpha_i \otimes_{S_i} s_i$ has coefficients in Λ. Thus we are reduced to the following lemma:

Lemma 8.3.7. *Let $(k_i)_{i \in I^{op}}$ be an ind-field and put: $k = \varinjlim_{i \in I^{op}} k_i$.*
Consider a family $(\beta_i)_{i \in I}$ such that β_i is a k_i-cycle of finite type with coefficients in $\mathbf{Q}$ and for any j/i, $\beta_j = \beta_i \otimes^{\flat}_{k_i} k_j$. We put $\beta = \beta_i \otimes^{\flat}_{k_i} k$.

If for an index $i \in I$, $\beta_i \otimes^{\flat}_{k_i} k$ has coefficients in Λ, then there exists j/i such that β_j has coefficients in Λ.

We can assume that for any j/i, β_j has positive coefficients. Let X_j (resp. X) be the support of β_j (resp. β). We obtain a pro-scheme $(X_j)_{j/i}$ such that $X = \varprojlim_{i \in I} X_i$. The transition maps of $(X_j)_{j/i}$ are dominant. Thus, by enlarging i, we can assume that for any j/i, the induced map $\pi_0(X_i) \to \pi_0(X_j)$ is a bijection. Thus we can consider each element of $\pi_0(X)$ separately and assume that all the X_i are integrals: for any j/i, $\beta_j = n_j.\langle X_j \rangle$ for a positive element $n_j \in \mathbf{Q}$. Arguing generically, we can further assume $X_j = \mathrm{Spec}\,(L_j)$ for a field extension of finite type L_j of k_j. By assumption now, for any j/i, $L_i \otimes_{k_i} k_j$ is an Artinian ring whose reduction is the field L_j. Moreover, $n_j = n_i.\mathrm{lg}(L_i \otimes_{k_i} k_j)$ and we know that $n := n_i.\mathrm{lg}(L_i \otimes_{k_i} k)$ belongs to Λ.

Let p be a prime not invertible in Λ such that $v_p(n_i) < 0$ where v_p denotes the p-adic valuation on $\mathbf{Q}$. It is sufficient to find an index j/i such that $v_p(n_j) \geq 0$. Let $L = (L_i \otimes_{k_i} k)_{red}$. Remark that $L = \varinjlim_{i \in I^{op}} L_i$. It is a field extension of finite type of k. Consider elements $a_1, ..., a_n$ algebraically independant over k such that L is a finite extension of $k(a_1, ..., a_n)$. By enlarging i, we can assume that $a_1, ..., a_n$ belongs to L_i. Thus L_i is a finite extension of $k_i(a_1, ..., a_n)$: replacing k_i by $k_i(a_1, ..., a_n)$, we can assume that L_i/k_i is finite.

Let L' be the subextension of L over k generated by the p-th roots of elements of k. As L/k is finite, L'/k is finite, generated by elements $b_1, ..., b_r \in L$. consider an index j/i such that $b_1, ..., b_r$ belongs to L_j. It follows that $v_p(\mathrm{lg}(L_i \otimes_{k_i} k_j)) = v_p(\mathrm{lg}(L_i \otimes_{k_i} k))$. Thus $v_p(n_j) = v_p(n) \geq 0$ and we are done. □

Corollary 8.3.8. *Let S be a scheme and α be a pre-special S-cycle.*
Let $\bar{s}$ be a geometric point of S, with image s in S, and S' be the strict localization of S at $\bar{s}$.

Then the following conditions are equivalent:

(i) α/S is special at s.
(i') α/S is special at $\bar{s}$.

(ii) $\big(\alpha|_{S'}\big)/S'$ *is special at* $\bar{s}$ *(notation of 8.3.1).*

(iii) There exists an étale neighbourhood V *of* $\bar{s}$ *in* S *such that* $(\alpha \otimes_S V)/V$ *is special at* $\bar{s}$*.*

Proof. The equivalence of (i) and (i') follows trivially from definition (*cf.* 8.1.29). Recall from 8.3.1 that $\alpha|_{S'} = \alpha \otimes_S S'$. Thus $(i') \Rightarrow (ii)$ is easy (see 8.1.44). Moreover, $(ii) \Rightarrow (iii)$ is a consequence of the previous proposition applied to the pro-scheme of étale neighbourhood of $\bar{s}$. Finally, $(iii) \Rightarrow (i)$ follows from Lemma 8.1.44. □

Proposition 8.3.9. *Consider the notations and hypothesis of 8.3.5. Assume that* S *and* S_i *are reduced for any* $i \in I$*.*

Suppose given a projective system $(X_i)_{i \in I}$ *of* S_i*-schemes of finite type such that for any* j/i*,* $X_j = X_i \times_{S_i} S_j$*. We let* X *be the projective limit of* (X_i)*.*

Then for any pre-special (resp. special, Λ*-universal)* S*-cycle* $\alpha \subset X$*, there exists* $i \in I$ *and a pre-special (resp. special,* Λ*-universal)* S_i*-cycle* $\alpha_i \subset X_i$ *such that* $\alpha = \alpha_i \otimes_{S_i} S$*.*[79]

Proof. Using Proposition 8.3.6, we are reduced to consider the first of the respective cases of the proposition. Write $\alpha = \sum_{r \in \Theta} n_r.\langle Z_r \rangle_X$ in standard form.
Consider $r \in \Theta$. As X is noetherian, there exists an index $i \in I$ and a closed subscheme $Z_{r,i} \subset X_i$ such that $Z_r = Z_{r,i} \times_{S_i} S$. Moreover, replacing $Z_{r,i}$ by the reduced closure of the image of the canonical map $Z_r \xrightarrow{(*)} Z_{r,i}$, we can assume that the map $(*)$ is dominant. For any $j \in I/i$, we put $Z_{r,j} = Z_{r,i} \times_{S_i} S_j$. The limit of the pro-scheme $(Z_{r,j})_{j \in I/i^{op}}$ is the integral scheme Z_r. Thus, applying [**EGA4**, 8.2.2], we see that by enlarging i, we can assume that for any $j \in I/i$, $Z_{r,j}$ is irreducible (but not necessarily reduced).
We repeat this construction for every $r \in \Theta$, enlarging i at each step. Fix now an element $j \in I/i$. The scheme $Z_{r,j}$ may not be reduced. However, its reduction $Z'_{r,j}$ is an integral scheme such that $Z'_{r,j} \times_{S_j} S = Z_r$. We put

$$\alpha_j = \sum_{r \in \Theta} n_r \langle Z'_{r,j} \rangle_{X_j}.$$

Let $z_{r,j}$ be the generic point of $Z'_{r,j}$, and $s_{r,j}$ be its image in S_j. It is a generic point and corresponds uniquely to a generic point s_r of S according to the point (3) of the hypothesis 8.3.5. Thus α_j/S_j is pre-special. Moreover, we get from the above that $\kappa(z_{r,j}) \otimes_{\kappa(s_{r,j})} \kappa(s_r) = \kappa(z_r)$ where z_r is the generic point of Z_r. Thus the relation $\alpha_j \otimes_{S_j} S = \alpha$ follows from lemma 8.2.1. □

8.3.b. *Samuel multiplicities.*

8.3.10. We give some recall on Samuel multiplicities, following as a general reference [**Bou93**, VIII.§7].
Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $M \neq 0$ be a A-module of finite type and $\mathfrak{q} \subset \mathfrak{m}$ an ideal of A such that $M/\mathfrak{q}M$ has finite length. Let d be the dimension of the support of M. Recall from *loc. cit.* that *Samuel multiplicity* of M at $\mathfrak{q}$ is defined as the integer:

$$e^A_{\mathfrak{q}}(M) := \lim_{n \to \infty} \left(\frac{d!}{n^d} \lg_A(M/\mathfrak{q}^n M) \right)$$

In the case $M = A$, we simply put $e_{\mathfrak{q}}(A) := e^A_{\mathfrak{q}}(A)$ and $e(A) := e^A_{\mathfrak{m}}(A)$.

We will use the following properties of these multiplicities that we recall for the convenience of the reader; let A be a local noetherian ring with maximal ideal $\mathfrak{m}$:

Let Φ be the generic points $\mathfrak{p}$ of Spec (A) such that $\dim(A/\mathfrak{p}A) = \dim A$. Then according to proposition 3 of *loc. cit.*:

$$e_{\mathfrak{q}}(A) = \sum_{\mathfrak{p} \in \Phi} \lg(A_{\mathfrak{p}}).e_{\mathfrak{q}}(A/\mathfrak{p}). \qquad (\mathcal{S}1)$$

Let B be a local flat A-algebra such that $B/\mathfrak{m}B$ has finite length over B. Then according to proposition 4 of *loc. cit.*:

$$\frac{e_{\mathfrak{m}B}(B)}{e(A)} = \lg_B(B/\mathfrak{m}B). \qquad (\mathcal{S}2)$$

[79] This pullback is defined in any case because of point (3) of the hypothesis above.

Let B be a local flat A-algebra such that $\mathfrak{m}B$ is the maximal ideal of B. Let $\mathfrak{q} \subset A$ be an ideal such that $A/\mathfrak{q}A$ has finite length. Then according to the corollary of proposition 4 in *loc. cit.*:

$$(\mathcal{S}3) \qquad e_{\mathfrak{q}B}(B) = e_{\mathfrak{q}}(A).$$

Assume A is integral with fraction field K. Let B be a finite local A-algebra such that $B \supset A$. Let k_B/k_A be the extension of the residue fields of B/A. Then, according to proposition 5 and point b) of the corollary of proposition 4 in *loc. cit.*,

$$(\mathcal{S}4) \qquad \frac{e_{\mathfrak{m}B}(B)}{e(A)} = \frac{\dim_K(B \otimes_A K)}{[k_B : k_A]}.$$

DEFINITION 8.3.11. (i) Let $S = \operatorname{Spec}(A)$ be a local scheme, $s = \mathfrak{m}$ the closed point of S. Let Z be an S-scheme of finite type with special fiber Z_s. For any generic point z of Z_s, denoting by B the local ring of Z at z, we define the *Samuel multiplicity of Z at z over S* as the rational integer:

$$m^{\mathcal{S}}(z, Z/S) = \frac{e_{\mathfrak{m}B}(B)}{e(A)}.$$

In the case where Z is integral, we define the *Samuel specialization of the S-cycle $\langle Z\rangle$ at s* as the cycle with rational coefficients and domain Z_s:

$$\langle Z\rangle \otimes_S^{\mathcal{S}} s = \sum_{z \in Z_s^{(0)}} m^{\mathcal{S}}(z, Z/S).z.$$

Consider an S-cycle of finite type $\alpha = \sum_{i \in I} n_i.\langle Z_i\rangle_X$ written in standard form. We define the *Samuel specialization of the S-cycle α at s* as the cycle with domain X_s:

$$\alpha \otimes_S^{\mathcal{S}} s = \sum_{i \in I} n_i.\langle Z_i\rangle \otimes_S^{\mathcal{S}} s.$$

(ii) Let S be a scheme. For any point s of S, we let $S_{(s)}$ be the localized scheme of S at s. Let $f : Z \to S$ be an S-scheme of finite type, and z a point of Z which is generic in its fiber. Put $s = f(z)$. We define the *Samuel multiplicity of Z/S at z* as the integer

$$m^{\mathcal{S}}(z, Z/S) := m^{\mathcal{S}}(z, Z \times_S S_{(s)}/S_{(s)}).$$

Consider an S-cycle of finite type α with domain X and a point s of S. We define the *Samuel specialization of the S-cycle α at s* as the cycle with rational coefficients:

$$\alpha \otimes_S^{\mathcal{S}} s = \big(\alpha|_{S_{(s)}}\big) \otimes_{S_{(s)}}^{\mathcal{S}} s.$$

LEMMA 8.3.12. *Let S be a scheme, and $p : Z' \to Z$ an S-morphism which is a birational universal homeomorphism. Then for any point $s \in S$,*

$$\langle Z'\rangle \otimes_S^{\mathcal{S}} s = \langle Z\rangle \otimes_S^{\mathcal{S}} s$$

in $(Z'_s)_{red} = (Z_s)_{red}$.

PROOF. By hypothesis, p induces an isomorphism $Z'^{(0)} \simeq Z^{(0)}$ between the generic points. Given any irreducible component T' of Z' corresponding to the irreducible component T of Z, we get by hypothesis:

$$T'_{red} \simeq T_{red} \text{ (as schemes)}, \ \lg\big(\mathcal{O}_{Z',T'}\big) = \lg\big(\mathcal{O}_{Z,T}\big).$$

Thus, we easily concludes from the definition. □

8.3.13. Let $Z \xrightarrow{f} S$ be a morphism of finite type and a z a point of Z, $s = f(z)$. Assume z is a generic point of Z_s. We introduce the following condition:

$$\mathcal{D}(z, Z/S) : \left\{ \begin{array}{c} \text{For any irreducible component } T \text{ of } Z_{(z)}, \\ T_s = \emptyset \text{ or } \dim(T) = \dim(Z_{(z)}). \end{array} \right.$$

REMARK 8.3.14. This condition is in particular satisfied if $Z_{(z)}$ is absolutely equidimensional (and a fortiori if Z is absolutely equidimensional).

An immediate translation of $(\mathcal{S}1)$ gives:

LEMMA 8.3.15. *Let S be a local scheme with closed point s and Z be an S-scheme of finite type such that Z_s is irreducible with generic point z.*

If the condition $\mathcal{D}(z, Z/S)$ is satisfied, then $\langle Z\rangle \otimes_S^{\mathcal{S}} s = m^{\mathcal{S}}(z, Z/S).z$.

We get directly from ($\mathcal{S}2$) the following lemma:

LEMMA 8.3.16. *Let S be a scheme, s be a point of S, and $\alpha = \sum_{i\in I} n_i.\langle Z_i\rangle_X$ be an S-cycle in standard form such that Z_i is a flat S-scheme of finite type.*

Then α is a Hilbert S-cycle and $\alpha \otimes_S^{\mathcal{S}} s = \alpha \otimes_S^{\flat} s$.

With the notations of 8.3.1, we get from ($\mathcal{S}3$):

LEMMA 8.3.17. *Let S be a scheme, s a point of S with residue field k and α an S-cycle of finite type.*

(i) Let S' be the Hensel localization of S at s. Then, $\alpha \otimes_S^{\mathcal{S}} s = \big(\alpha|_{S'}\big) \otimes_{S'}^{\mathcal{S}} s$.

(ii) Let $\bar{k}$ a separable closure corresponding and $\bar{s}$ the corresponding geometric point of S. Let $S_{(\bar{s})}$ be the strict localization of S at $\bar{s}$. Then,

$$\big(\alpha \otimes_S^{\mathcal{S}} s\big) \otimes_k^{\flat} \bar{k} = \big(\alpha|_{S_{(\bar{s})}}\big) \otimes_{S_{(\bar{s})}}^{\mathcal{S}} \bar{s}.$$

Let us recall from [**EGA4**, 13.3.2] the following definition:

DEFINITION 8.3.18. Let $f : X \to S$ be a morphism of finite type between noetherian schemes, and x a point of X.

We say f is equidimensional at x if there exists an open neighbourhood U of x in X and a quasi-finite pseudo-dominant S-morphism $U \to \mathbf{A}_S^d$ for $d \in \mathbf{N}$. The integer d is independant of the choice of U: it is called the relative dimension of f at x.

We say f is equidimensional if it is equidimensional at every point of X.

REMARK 8.3.19. A quasi-finite morphism is equidimensional if and only if it is pseudo-dominant. According to [**EGA4**, 12.1.1.5], this definition agrees with the convention stated in paragraph 8.1.9 in the case of flat morphisms.

Note that a direct translation of ($\mathcal{S}4$) gives:

LEMMA 8.3.20. *Let $S = \operatorname{Spec}(A)$ be an integral local scheme with closed point s and fraction field K. Let Z be a finite equidimensional S-scheme and z a generic point of Z_s. Let B be the local ring of Z at z.*

Then,

$$m^{\mathcal{S}}(z, Z/S) = \frac{\dim_K(B \otimes_A K)}{[\kappa(x) : \kappa(s)]}.$$

8.3.21. Recall that a scheme S is said to be *unibranch (* resp. *geometrically unibranch) at a point $s \in S$* if the henselisation (resp. strict henselisation) of the local ring $\mathcal{O}_{S,s}$ is irreducible (see [**EGA4**, 6.15.1, 18.8.16]). The scheme S is said to be *unibranch* (resp. *geometrically unibranch*) if it is so at any point $s \in S$.

The following result is the key point of this subsection.

PROPOSITION 8.3.22. *Consider a cartesian square*

$$\begin{array}{ccc} Z' & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

and a point s' of S', $s = g(s')$. Let k (resp. k') be the residue field of s (resp. s'). We assume the following conditions:

(1) S (resp. S') is geometrically unibranch at s (resp. s').
(2) f and f' are equidimensional of dimension n.
(3) For any generic point z of Z_s (resp. z' of $Z_{s'}$) the condition $\mathcal{D}(z, Z/S)$ (resp. $\mathcal{D}(z', Z'/S')$) is satisfied.

Then, the following equality holds in $Z_{s'}$:

$$\langle Z'\rangle \otimes^{\mathcal{S}}_{S'} s' = (\langle Z\rangle \otimes^{\mathcal{S}}_{S} s) \otimes^{\flat}_{k} k'.$$

Proof. According to Lemma 8.3.15, we have to prove the equality:

$$\sum_{z' \in Z_{s'}^{(0)}} m^{\mathcal{S}}(z', Z'/S').z' = \sum_{z \in Z_s^{(0)}} m^{\mathcal{S}}(z, Z/S).\langle \mathrm{Spec}\,(\kappa(z) \otimes_k k')\rangle_{Z_{s'}}. \tag{8.3.22.1}$$

As f is equidimensional of dimension n, we can assume according to 8.3.18 that there exists a quasi-finite pseudo-dominant S-morphism $p : Z \to \mathbf{A}^n_S$. For any generic point z of Z_s, $t = p(z)$ is the generic point of $\mathbf{A}^n_s$. Thus applying ($\mathcal{S}3$), we get:

$$m^{\mathcal{S}}(z, Z/S) = m^{\mathcal{S}}(z, Z/\mathbf{A}^n_S).$$

Consider the S' morphism $p' : Z' \to \mathbf{A}^n_{Z'}$ obtained by base change. It is quasi-finite. As Z'/S' is equidimensional of dimension n, p' must be pseudo-dominant. For any generic point z' of $Z_{s'}$, $t' = p'(z')$ is the generic point of $\mathbf{A}^n_{s'}$ and as in the preceding paragraph, we get

$$m^{\mathcal{S}}(z', Z'/S') = m^{\mathcal{S}}(z', Z'/\mathbf{A}^n_{S'}).$$

Moreover, the residue field κ_t of t (resp. $\kappa_{t'}$ of t') is $k(t_1, ..., t_n)$ (resp. $k'(t_1, ..., t_n)$) and this implies $\mathrm{Spec}\,(\kappa(z) \otimes_{\kappa_t} \kappa_{t'})$ is homeomorphic to $\mathrm{Spec}\,(\kappa(z) \otimes_k k')$ and has the same geometric multiplicities. Putting this and the two preceding relations in (8.3.22.1), we get reduced to the case $n = 0$ – indeed, according to [**EGA4**, 14.4.1.1], $\mathbf{A}^n_S$ (resp. $\mathbf{A}^n_{S'}$) is geometrically unibranch at t (resp. t').

Assume now $n = 0$, so that f and f' are quasi-finite pseudo-dominant.

Let $\bar{k}$ be a separable closure of k and $\bar{k}'$ a separable closure of a composite of $\bar{k}$ and k'. It is sufficient to prove relation (8.3.22.1) after extension to $\bar{k}'$ (Lemma 8.1.19). Thus according to 8.3.17 and hypothesis (3), we can assume S and S' are integral strictly local schemes.

For any $z \in Z_s^{(0)}$, the extension $\kappa(z)/k$ is totally inseparable. Moreover, z corresponds to a unique point $z' \in Z_{s'}^{(0)}$ and we have to prove for any $z \in Z_s^{(0)}$:

$$m^{\mathcal{S}}(z', Z'/S') = m^{\mathcal{S}}(z, Z/S).\, \mathrm{lg}(\kappa(z) \otimes_k k').$$

Let $S = \mathrm{Spec}\,(A)$, $K = \mathrm{Frac}(A)$ and $B = \mathcal{O}_{Z,z}$ (resp. $S' = \mathrm{Spec}\,(A')$, $K' = \mathrm{Frac}(A')$ and $B' = \mathcal{O}_{Z',z'}$). As B is quasi-finite dominant over A and A is henselian, B/A is necessarily finite dominant. The same is true for B'/A' and ($\mathcal{S}4$) gives the formulas:

$$m^{\mathcal{S}}(z, Z/S) = \frac{\dim_K(B \otimes_A K)}{[\kappa(z) : k]}, \qquad m^{\mathcal{S}}(z', Z'/S') = \frac{\dim_{K'}(B' \otimes_{A'} K')}{[\kappa(z') : k']}.$$

As $B' \otimes_{A'} K' = (B \otimes_A K) \otimes_K K'$, the numerator of these two rationals are the same. To conclude, we are reduced to the easy relation

$$[\kappa(z') : k'].\, \mathrm{lg}(\kappa(z) \otimes_k k') = [\kappa(z) : k].$$

□

Definition 8.3.23. Let S be a scheme and $\alpha = \sum_{i \in I} n_i.\langle Z_i\rangle_X$ be an S-cycle in standard form.

We say α/S is *pseudo-equidimensional over s* if it is pre-special and for any $i \in I$, the structual map $Z_i \to S$ is equidimensional at the generic points of the fiber $Z_{i,s}$.

Proposition 8.3.24. *Let S be a strictly local integral scheme with closed point s and residue field k and α be an S-cycle pseudo-equidimensional at s.*

Then for any extension $\mathrm{Spec}\,(k') \xrightarrow{s'} S$ *of s and any fat point (R, k') of S over s', the following relation holds:*

$$\alpha_{R,k'} = \left(\alpha \otimes^{\mathcal{S}}_S s\right) \otimes^{\flat}_k k'.$$

Proof. We put $S' = \mathrm{Spec}\,(R)$ and denote by s' its closed point.

Reductions.– By additivity, we reduce to the case $\alpha = \langle Z\rangle$, Z is integral and the structural morphism $f : Z \to S$ is equidimensional at the generic points of Z_s. Any generic points of $S'_{s'}$ dominantes a generic point of Z_s so that we can argue locally at each generic point x of Z_s. Thus we can assume Z_s is irreducible with generic point x. Moreover, as Z is equidimensional at x, we can assume according to 8.3.18 there exists a quasi-finite pseudo-dominant S-morphism

$$Z \xrightarrow{p} \mathbf{A}^n_S. \tag{8.3.24.1}$$

Note that S is geometrically unibranch at s. Thus, applying [**EGA4**, 14.4.1] ("critère de Chevalley"), f is universally open at x. As S' is a trait whose close point goes to s in S, it follows from [**EGA4**, 14.3.7] that the base change $f' : Z' \to S'$ of f along S'/S is pseudo-dominant.

Let T be an irreducible component of Z', with special fiber $T_{s'}$ and generic fiber $T_{K'}$ over S'. Then $T \to S'$ is a dominant morphism of finite type. Thus, according to [**EGA4**, 14.3.10], either $T_{s'} = \emptyset$ or $\dim(T_{s'}) = \dim(T_{K'})$. Moreover, the dimension of T_η is equal to the transcendantal degree of the function field of T over K', which is equal to the transcendental degree of Z over K. This is n according to (8.3.24.1). Thus, in any case, T is equidimensional of dimension n over S' and this implies Z' is equidimensional of dimension n over S'. Moreover, either $T_{s'} = \emptyset$ or $\dim(T) = n + 1 = \dim(Z')$. Note this implies that for any generic point z' of $Z_{s'}$, the condition $\mathcal{D}(z', Z'/S')$ is satisfied.

Middle step.– We prove: $\alpha_{R,k} = \langle Z'\rangle \otimes^{\mathcal{S}}_{S'} s'$.
According to Lemma 8.3.16,

$$\alpha_{R,k} = \langle \overline{Z'_K}\rangle \otimes^\flat_R k' = \langle \overline{Z'_K}\rangle \otimes^{\mathcal{S}}_{S'} s'.$$

But the canonical map $\overline{Z'_K} \to Z'$ is a birational universal homeomorphism so that we conclude this step by Lemma 8.3.12.

Final step.– We have only to point out that the conditions of Proposition 8.3.22 are fulfilled for the obvious square; this is precisely what we need. □

Corollary 8.3.25. *Let S be a reduced scheme, s a point of S and α an S-cycle which is pseudo-equidimensional over s.*

Let $\bar{s}$ be a geometric point of S with image s in S and S' be the strict localization of S at $\bar{s}$. We let $S' = \cup_{i\in I} S'_i$ be the irreducible components of S' and α_i be the cycle made by the part of the cycle $\alpha \otimes^\flat_S S'$ whose points dominate S'_i.

Then the following conditions are equivalent:

(i) α/S is special at s.

(ii) the cycle $\alpha_\lambda \otimes^{\mathcal{S}}_{S'_i} \bar{s}$ does not depend on $i \in I$.

Moreover, when these conditions are fulfilled, $\alpha \otimes_S \bar{s} = \alpha_\lambda \otimes^{\mathcal{S}}_{S'_i} \bar{s}$.

Proof. According to Corollary 8.3.8, we reduce to the case $S = S'$. Then this follows directly from the preceding proposition. □

Corollary 8.3.26. *Let S be a reduced scheme, geometrically unibranch at a point $s \in S$, and α an S-cycle. The following conditions are equivalent:*

(i) α/S is pseudo-equidimensional over s.

(ii) α/S is special at s.

Under these conditions, $\alpha \otimes_S s = \alpha \otimes^{\mathcal{S}}_S s$.

Remark 8.3.27. In particular, over a reduced geometrically unibranch scheme S, every cycle whose support is equidimensional over S is special.

Corollary 8.3.28. *Let S be a reduced scheme and $s \in S$ a point such that S is geometrically unibranch at s and $e(\mathcal{O}_{S,s}) = 1$. Then for any S-cycle α, the following conditions are equivalent:*

(i) α/S is pseudo-equidimensional over s.

(ii) α/S is Λ-universal at s.

Remark 8.3.29. In particular, over a regular scheme S, every cycle whose support is equidimensional over S is Λ-universal. Remark also the following theorem:

Theorem 8.3.30. *Let S be an excellent scheme, $s \in S$ a point. The following conditions are equivalent:*

(i) *S is regular at s.*
(ii) *S is geometrically unibranch at s and $e(\mathcal{O}_{S,s}) = 1$.*
(iii) *S is unibranch at s and $e(\mathcal{O}_{S,s}) = 1$.*

Bibliographical references for the proof. We can assume S is the spectrum of an excellent local ring A with closed point s. The implication $(i) \Rightarrow (ii)$ follows from the fact that a normal local ring is geometrically unibranch (at its closed point) and from [**Bou93**, AC.VIII.§7, prop. 2]. $(ii) \Rightarrow (iii)$ is trivial. Concerning the implication $(iii) \Rightarrow (i)$, let $\hat{A}$ be the completion of the local ring A. We know from [**Bou93**, AC.VIII.108, ex. 24] that when $e(A) = 1$ and $\hat{A}$ is integral, A is regular. Note $e(A) = 1$ implies A is reduced. To conclude, we refer to [**EGA4**, 7.8.3, (vii)] which established that if A is local excellent reduced, $\hat{A}$ is integral if and only if A is unibranch.

Finally, we get the following theorem already proved by Suslin and Voevodsky ([**SV00b**, 3.5.9]):

Theorem 8.3.31. *Let S be a scheme and s a point with residue field κ_s such that the local ring A of S at s is regular. Then for any equidimensional S-scheme Z and any generic point z of Z_s,*

$$m^{SV}(z, \langle Z \rangle \otimes_S s) = \sum_i (-1)^i \lg_A \operatorname{Tor}_i^A(\mathcal{O}_{Z,z}, \kappa_s).$$

Proof. We reduce to the case $S = \operatorname{Spec}(A)$. Then Z is absolutely equidimensional and we can apply Lemma 8.3.15 together with Corollary 8.3.26 to get that $m^{SV}(z, \langle Z \rangle \otimes_S s) = m^S(z, Z/S)$. Then the result follows from a theorem of Serre [**Ser75**, IV.12, th. 1]. □

Remark 8.3.32. Let S be a regular scheme, X a smooth S-scheme and $\alpha \subset X$ an S-cycle whose support is equidimensional over S. Let s be a point of S and $i : X_s \to X$ the closed immersion of the fiber of X at s. Then the cycle $i^*(\alpha)$ of [**Ser75**, V-28, par. 7] is well defined and we get:

$$\alpha \otimes_S s = i^*(\alpha).$$

9. Finite correspondences

9.0. In this section, $\mathscr{S}$ is the category of all noetherian schemes. We fix an admissible class $\mathscr{P}$ of morphisms in $\mathscr{S}$ and assume in addition that $\mathscr{P}$ is contained in the class of separated morphisms of finite type.

Consider two S-schemes X and Y. To clarify certain formulas, we will denote $X \times_S Y$ simply by XY and let $p^X_{XY} : XY \to X$ be the canonical projection morphism.

We fix a ring of coefficients $\Lambda \subset \mathbf{Q}$.

9.1. Definition and composition.

9.1.1. Let S be a base scheme. For any $\mathscr{P}$-scheme X/S, we let $c_0(X/S, \Lambda)$ be the Λ-module made of the finite and Λ-universal S-cycles with domain X.[80] Consider a morphism $f : Y \to X$ of $\mathscr{P}$-schemes over S. Then the pushforward of cycles induces a well defined morphism:

$$f_* : c_0(Y/S, \Lambda) \to c_0(X/S, \Lambda).$$

Indeed, consider a cycle $\alpha \in c_0(Y/S)$. Let us denote by Z its support in Y and by $f(Z) \subset X$ image of the latter by f. We consider these subsets as reduced subschemes. Note that $f(Z)$ is separated and of finite type over S because X/S is noetherian, separated, and of finite type, by assumption 9.0. Because Z/S is proper, [**EGA2**, 5.4.3(ii)] shows that $f(Z)$ is indeed proper over S. Thus, the cycle $f_*(\alpha)$ is Λ-universal according to Corollary 8.2.10. Finally, Z/S is finite, we deduce that $f(Z)$ is quasi-finite, thus finite, over S. This implies the result.

[80] With the notations of [**SV00b**], $c_0(X/S, \mathbf{Z}) = c_{equi}(X/S, 0)$ when S is reduced.

Definition 9.1.2. Let X and Y be two $\mathscr{P}$-schemes over S.

A finite S-correspondence from X to Y with coefficients in Λ is an element of

$$c_S\left(X,Y\right)_\Lambda := c_0(X \times_S Y/X).$$

We denote such a correspondence by the symbol $X \bullet\!\xrightarrow{\alpha} Y$.

In the case $\Lambda = \mathbf{Z}$, we simply put $c_S\left(X,Y\right) := c_S\left(X,Y\right)_{\mathbf{Z}}$. Through the rest of this section, unless explicitely stated, any cycle and any finite S-correspondence are assumed to have coefficients in Λ.

Remark 9.1.3.

1. According to properties (P7) and (P7') (*cf.* 8.1.40) of the pullback, $c_S\left(X,Y\right)_\Lambda$ commutes with finite sums in X and Y.
2. Consider $\alpha \in c_S\left(X,Y\right)_\Lambda$. Let Z be the support of α. Then, Z is finite pseudo-dominant over X (by definition 8.1.20). This means that Z is finite equidimensional over X.

   When X is regular (resp. X is reduced geometrically unibranch and $\mathrm{char}(X) \subset \Lambda^\times$), a cycle $\alpha \subset X \times_S Y$ written in standard form:

   $$\alpha = \sum_i n_i \langle Z_i \rangle_{X \times_S Y}$$

   defines a finite S-correspondence from X to Y if and only if for any index $i \in I$, the scheme Z_i is finite equidimensional over X (*i.e.* finite and dominant over an irreducible component of X) – *cf.* 8.3.29 (resp. 8.3.27).

   Moreover, in each respective case, $c_S(X,Y)_\Lambda$ is the free Λ-module generated by the closed integral subschemes Z of $X \times_S Y$ which are finite equidimensional over X.
3. Recall that in general, there is only an inclusion

   $$c_S\left(X,Y\right) \otimes_{\mathbf{Z}} \Lambda \subset c_S\left(X,Y\right)_\Lambda .$$

   This inclusion is an equality if S is regular (*cf.* 8.3.29) or $\mathrm{char}(S) \subset \Lambda^\times$.[81]

   Given more generally inclusions of rings $\Lambda \subset \Lambda' \subset \mathbf{Q}$, we get an inclusion of groups

   (9.1.3.1) $$c_S\left(X,Y\right)_\Lambda \otimes_\Lambda \Lambda' \subset c_S\left(X,Y\right)_{\Lambda'}$$

   which, for the same reasons, is an equality when S is regular or $\mathrm{char}(S) \subset \Lambda^\times$.

Example 9.1.4.

1. Let $f : X \to Y$ be a morphism in $\mathscr{P}/S$.

   Because X/S is separated (assumption 9.0), the graph Γ_f of f is a closed subscheme of $X \times_S Y$. The canonical projection $\Gamma_f \to X$ is an isomorphism. Thus $\langle \Gamma_f \rangle_{XY}$ is a Hilbert cycle over X. In particular, it is Λ-universal and also finite over X, thus it defines a finite S-correspondence from X to Y.
2. Let $f : Y \to X$ be a finite S-morphism which is Λ-universal (as a morphism of the associated cycles). Then the graph Γ_f of f is closed in $X \times_S Y$ and the projection $\Gamma_f \to X$ is isomorphic to f. Thus the cycle $\langle \Gamma_f \rangle_{XY}$ is a finite Λ-universal cycle over X which therefore define a finite S-correspondence ${}^t f : X \bullet\!\!\longrightarrow Y$ called the *transpose* of the finite Λ-universal morphism f.

Suppose we are given finite S-correspondences $X \bullet\!\xrightarrow{\alpha} Y \bullet\!\xrightarrow{\beta} Z$. Consider the following diagram of cycles :

(9.1.4.1)
$$\begin{array}{ccccc}
\beta \otimes_Y \alpha & \rightarrow & \beta & \rightarrow & Z. \\
\downarrow & & \downarrow & & \\
\alpha & \longrightarrow & Y & & \\
\downarrow & & & & \\
X & & & &
\end{array}$$

The pullback cycle is well defined and has coefficients in Λ as β is Λ-universal over Y. Moreover, according to the definition of pullback (*cf.* 8.1.38) and Corollary 8.2.6, $\beta \otimes_Y \alpha$ is a finite Λ-universal cycle over X with domain XYZ. Note finally that according to 9.1.1, the pushforward of this latter cycle by p^{XZ}_{XYZ} is an element of $c_S\left(X,Z\right)_\Lambda$.

[81] Indeed Suslin-Voeodsky's multiplicities of a cycle over a scheme X can only have denominators whose prime factors divide the residue characteristics of X according to 8.1.38.

Definition 9.1.5. Using the preceding notations, we define the *composition product* of β and α as the finite S-correspondence

$$\beta \circ \alpha = p^{XZ}_{XYZ*}(\beta \otimes_Y \alpha) : X \bullet\!\!\longrightarrow Z.$$

Remark 9.1.6. In the case where S is regular and X, Y, Z are smooth over S, the composition product defined above agree with the one defined in [**Dég07**, 4.1.16] in terms of the Tor-formula of Serre. In fact, this is a direct consequence of 8.3.31 after reduction to the case where α and β are represented by closed integral subschemes (see also point (2) of remark 9.1.3).

We sum up the main properties of the composition for finite correspondences in the following proposition :

Proposition 9.1.7. *Let X, Y, Z be $\mathscr{P}$-schemes over S.*

(1) For any finite S-correspondences $X \bullet\!\!\xrightarrow{\alpha} Y \bullet\!\!\xrightarrow{\beta} Z \bullet\!\!\xrightarrow{\gamma} T$, we have $(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$.

(2) For any $X \bullet\!\!\xrightarrow{\alpha} Y \xrightarrow{g} Z$, $\langle \Gamma_g \rangle_{YZ} \circ \alpha = (1_X \times_S g)_*(\alpha)$.

(3) For any $X \xrightarrow{f} Y \bullet\!\!\xrightarrow{\beta} Z$, $\beta \circ \langle \Gamma_f \rangle_{XY} = \beta \otimes_Y \langle X \rangle$.
Moreover, if f is flat, $\beta \circ \langle \Gamma_f \rangle_{XY} = (f \times_S 1_Z)^*(\beta)$ considering the flat pullback of cycles in the classical sense.

(4) For any $X \xleftarrow{f} Y \bullet\!\!\xrightarrow{\beta} Z$ such that f is finite Λ-universal, $\beta \circ {}^t f = (f \times_S 1_Z)_*(\beta)$.

(5) For any $X \bullet\!\!\xrightarrow{\alpha} Y \xleftarrow{g} Z$ such that g is finite Λ-universal, ${}^t g \circ \alpha = \langle Z \rangle \otimes_Y \alpha$.
If we suppose that g is finite flat, then ${}^t g \circ \alpha = (1_X \times_S g)^*(\alpha)$.

Proof. (1) Using respectively the projection formulas 8.2.10 and 8.2.8, we obtain

$$(\gamma \circ \beta) \circ \alpha = p^{XT}_{XYZT*}\big((\gamma \otimes_Z \beta) \otimes_Y \alpha\big)$$
$$\gamma \circ (\beta \circ \alpha) = p^{XT}_{XYZT*}\big(\gamma \otimes_Z (\beta \otimes_Y \alpha)\big).$$

Thus this formula is a direct consequence of the associativity 8.2.7.

(2) Let $\epsilon : \Gamma_g \to Y$ and $p^{XZ}_{X\Gamma_g} : X\Gamma_g \to XZ$ be the canonical projections. As ϵ is an isomorphism, we have tautologically $\langle Y \rangle = \epsilon_*(\langle \Gamma_g \rangle)$. We conclude by the following computation :

$$\begin{aligned}(1_X \times_S g)_*(\alpha) &= (1_X \times_S g)_*(\langle Y \rangle \otimes_Y \alpha) = (1_X \times_S g)_*(\epsilon_* \langle \Gamma_g \rangle \otimes_Y \alpha) \\ &\overset{(*)}{=} (1_X \times_S g)_*(1_X \times_S \epsilon)_*(\langle \Gamma_g \rangle \otimes_Y \alpha) = p^{XZ}_{X\Gamma_g *}(\langle \Gamma_g \rangle \otimes_Y \alpha) \\ &\overset{(*)}{=} p^{XZ}_{XYZ*}(\langle \Gamma_g \rangle_{YZ} \otimes_Y \alpha)\end{aligned}$$

The equalities labeled $(*)$ follow from the projection formula of 8.2.10.

(3) The first assertion follows from projection formula of 8.2.8 and the fact that Γ_f is isomorphic to X :

$$\beta \circ \langle \Gamma_f \rangle_{XY} = p^{XZ}_{XYZ*}(\beta \otimes_Y \langle \Gamma_f \rangle_{XY}) = \beta \otimes_Y p^{X}_{XY*}(\langle \Gamma_f \rangle_{XY}) = \beta \otimes_Y \langle X \rangle$$

The second assertion follows from Corollary 8.2.2.

(4) and (5): The proof of these assertions is strictly similar to that of (2) and (3) instead that we use the projection formula of 8.2.8 (and do not need the commutativity 8.2.3). ☐

As a corollary, we obtain that the composition of S-morphisms coincide with the composition of the associated graph considered as finite S-correspondences. For any S-morphism $f : X \to Y$, we will still denote by $f : X \bullet\!\!\longrightarrow Y$ the finite S-correspondence equal to $\langle \Gamma_f \rangle_{XY}$. Note moreover that for any $\mathscr{P}$-scheme X/S, the identity morphism of X is the neutral element for the composition of finite S-correspondences.

Definition 9.1.8. We let $\mathscr{P}^{cor}_{\Lambda,S}$ be the category of $\mathscr{P}$-schemes over S with morphisms the finite S-correspondences and the composition product of definition 9.1.5.

An object of $\mathscr{P}^{cor}_{\Lambda,S}$ will be denoted by $[X]$. The category $\mathscr{P}^{cor}_{\Lambda,S}$ is additive, and the direct sum is given by the disjoint union of $\mathscr{P}$-schemes over S. We have a canonical faithful functor

$$\gamma : \mathscr{P}/S \to \mathscr{P}^{cor}_{\Lambda,S} \tag{9.1.8.1}$$

which is the identity on objects and the graph on morphisms. We call it the *graph functor*.

9.1.9. Given extension of rings $\Lambda \subset \Lambda' \subset \mathbf{Q}$, we get according to Remark 9.1.3(3) and the definition of composition of finite correspondences a functor of Λ'-linear categories:

$$\mathscr{P}^{cor}_{\Lambda,S} \otimes_\Lambda \Lambda' \to \mathscr{P}^{cor}_{\Lambda',S} \tag{9.1.9.1}$$

which is the identity on objects and the inclusions of the form (9.1.3.1) on morphisms.

Proposition 9.1.10. *Consider the above notations. If S is regular or* $\mathrm{char}(S) \subset \Lambda^\times$ *then the functor* (9.1.9.1) *is an equality of categories.*

Indeed according to point (3) of Proposition 9.1.3, the inclusions of groups of correspondences used to define the above functors are all equalities in each respective cases.

9.1.11. Given two S-morphisms $f : Y \to X$ and $g : X' \to X$ such that g is finite Λ-universal, we get from the previous proposition the equality of cycles in YX':

$${}^t g \circ f = \langle X' \rangle \otimes_X \langle Y \rangle_{YX}$$

where Y is seen as a closed subscheme of YX through the graph of f.

In particular, when either f or g is flat, we get (use property (P3) of 8.1.34 or Corollary 8.2.2):

$${}^t g \circ f = \langle X' \times_X Y \rangle_{YX'}.$$

To state the next formulas (the generalized degree formulas), we introduce the following notion:

Definition 9.1.12. Let $f : X' \to X$ be a finite equidimensional morphism.

For any generic point x of X, we define the degree of f at x as the integer:

$$\deg_x(f) = \sum_{x'/x} [\kappa_{x'} : \kappa_x]$$

where the sum runs over the generic points of X' lying above x.

Proposition 9.1.13. *Let X be a connected S-scheme and $f : X' \to X$ be a finite S-morphism.*

If f is special then there exists an integer $d \in \mathbf{N}^*$ such that for any generic point x of X, $\deg_x(f) = d$.

Moreover, $f \circ {}^t f = d.1_X$.

We simply call d the *degree* of the finite special morphism f.

Proof. Let Δ' be the diagonal of X'/S. For any generic point x of X, we let Δ_x be the diagonal of the corresponding irreducible component of X, seen as a closed subscheme of X. According to Proposition 9.1.7, and the definition of pushforwards, we get

$$\alpha := f \circ {}^t f = (f \times_S f)_*(\langle \Delta' \rangle_{X'X'}) = \sum_{x \in X^{(0)}} \deg_x(f).\langle \Delta_x \rangle_{XX}.$$

Considering generic points x, y of X, we prove $\deg_x(f) = \deg_y(f)$. By induction, we can reduce to the case where x and y have a common specialisation s in X because X is connected and noetherian. Then, as α/X is special, we get by definition of the pullback (see more precisely 8.1.43)

$$\alpha \otimes_S s = \deg_x(f).s = \deg_y(f).s$$

as required. The remaining assertion then follows. □

Proposition 9.1.14. *Let $f : X' \to X$ be an S-morphism which is finite, radicial and Λ-universal.*

Assume X is connected, and let d be the degree of f.

Then ${}^tf \circ f = d.1_{X'}$. In particular, if d is invertible in Λ, f is an isomorphism in $\mathscr{P}^{cor}_{\Lambda,S}$.

Proof. According to 9.1.11, ${}^tf \circ f = \langle X' \rangle \otimes_X \langle X' \rangle$ as cycles in $X'X'$. Let x be the generic point of X and k be its residue field. Let $\{x'_i, i \in I\}$ be the set of generic points of X, and for any $i \in I$, k'_i be the residue field of x'_i. According to 8.2.1, we thus obtain:

$$ {}^tf \circ f = \sum_{(i,j)\in I^2} \langle \mathrm{Spec}\left(k'_i \otimes_k k'_j\right) \rangle_{X'X'}. $$

The result now follows by the definition of the degree and the fact that for any $i \in I$, k'_i/k is radicial. □

9.2. Monoidal structure. Fix a base scheme S. Let X, X', Y, Y' be $\mathscr{P}$-schemes over S.

Consider finite S-correspondences $\alpha : X \bullet\!\!\longrightarrow Y$ and $\alpha' : X' \bullet\!\!\longrightarrow Y'$. Then $\alpha X' := \alpha \otimes_X \langle XX' \rangle$ and $\alpha' X := \alpha' \otimes_{X'} \langle XX' \rangle$ are both finite Λ-universal cycles over XX'. Using stability by composition of finite Λ-universal morphisms (*cf.* Corollary 8.2.6), the cycle $(\alpha X') \otimes_{XX'} (\alpha' X)$ is finite Λ-universal over XX'.

Definition 9.2.1. Using the above notation, we define the *tensor product* of α and α' over S as the finite S-correspondence

$$ \alpha \otimes^{tr}_S \alpha' = (\alpha X') \otimes_{XX'} (\alpha' X) : XX' \bullet\!\!\longrightarrow YY'. $$

Let us first remark that this tensor product is commutative (use commutativity of the pullback 8.2.3) and associative (use associativity of the pullback 8.2.7). Moreover, it is compatible with composition :

Lemma 9.2.2. *Suppose given finite S-correspondences :*
$\alpha : X \to Y,\ \beta : Y \to Z,\ \alpha' : X' \to Y',\ \beta' : Y' \to Z'$. Then

$$ (\beta \circ \alpha) \otimes^{tr}_S (\beta' \circ \alpha') = (\beta \otimes^{tr}_S \beta') \circ (\alpha \otimes^{tr}_S \alpha'). $$

Proof. We put $\alpha X' = \alpha \otimes_X \langle XX' \rangle$, $\alpha' X = \alpha' \otimes_X \langle XX' \rangle$ and $\beta Y' = \beta \otimes_Y \langle YY' \rangle$, $\beta' Y = \beta' \otimes_Y \langle YY' \rangle$. We can compute the right hand side of the above equation as follows :

$$
\begin{aligned}
& p^{XX'ZZ'}_{XX'YY'ZZ'*}\Big((\beta Y' \otimes_{YY'} \beta' Y) \otimes_{YY'} (\alpha X' \otimes_{XX'} \alpha' X) \Big) \\
& \overset{(1)}{=} p^{XX'ZZ'}_{XX'YY'ZZ'*}\Big((\beta Y' \otimes_{YY'} \beta' Y) \otimes_{YY'} (\alpha' X \otimes_{XX'} \alpha X') \Big) \\
& \overset{(2)}{=} p^{XX'ZZ'}_{XX'YY'ZZ'*}\Big(\beta Y' \otimes_{YY'} ((\beta' Y \otimes_{YY'} \alpha' X) \otimes_{XX'} \alpha X') \Big) \\
& \overset{(3)}{=} p^{XX'ZZ'}_{XX'YY'ZZ'*}\Big((\beta Y' \otimes_{YY'} \alpha X') \otimes_{XX'} (\beta' Y \otimes_{YY'} \alpha' X) \Big).
\end{aligned}
$$

Equality (1) follows from commutativity 8.2.3, equality (2) from associativity 8.2.7 and equality (3) by both commutativity and associativity.

For the left hand side, we note that using the projection formula 8.2.10, the left hand side is equal to

$$ p^{XX'ZZ'}_{XX'YY'ZZ'*}\Big(\big((\beta \otimes_Y \alpha) \otimes_X \langle XX' \rangle \big) \otimes_{XX'} \big((\beta' \otimes_{Y'} \alpha') \otimes_{X'} \langle XX' \rangle \big) \Big). $$

We are left to remark that

$$ (\beta \otimes_Y \alpha) \otimes_X \langle XX' \rangle = \big((\beta Y') \otimes_{YY'} \alpha \big) \otimes_X \langle XX' \rangle = \beta Y' \otimes_{YY'} \alpha X', $$

using transitivity 8.2.4 and associativity 8.2.7. We thus conclude by symmetry of the other part in the left hand side. □

Definition 9.2.3. We define a symmetric monoidal structure on the category $\mathscr{P}^{cor}_{\Lambda,S}$ by putting $[X] \otimes^{tr}_S [Y] = [X \times_S Y]$ on objects and using the tensor product of the previous definition for morphisms.

9.2.4. Note that the functor $\gamma : \mathscr{P}/S \to \mathscr{P}^{cor}_{\Lambda,S}$ is monoidal for the cartesian structure on the source category. Indeed, this is a consequence of property (P3) of the relative product (see 8.1.34) and the remark that for any morphisms $f : X \to Y$ and $f' : X' \to Y'$, $(\Gamma_f \times_S X') \times_{XX'} (\Gamma'_f \times_S X) = \Gamma_{f \times_S f'}$.

9.3. Functoriality. Fix a morphism of schemes $f : T \to S$. For any $\mathscr{P}$-scheme X/S, we put $X_T = X \times_S T$. For a pair of $\mathscr{P}$-schemes over S (resp. T-schemes) (X, Y), we put $XY = X \times_S Y$ (resp. $XY_T = X \times_T Y$).

9.3.a. *Base change.* Consider a finite S-correspondence $\alpha : X \bullet\!\!\longrightarrow Y$. The cycle $\alpha \otimes_X \langle X_T \rangle$ defines a finite T-correspondence from X_T to Y_T denoted by α_T.

Lemma 9.3.1. *Consider finite S-correspondences $X \overset{\alpha}{\bullet\!\!\to} Y \overset{\beta}{\bullet\!\!\to} Y$.*
Then $(\beta \circ \alpha)_T = \beta_T \circ \alpha_T$.

Proof. This follows easily using the projection formula 8.2.10, the associativity formula 8.2.7 and the transitivity formula 8.2.4 :

$$\begin{aligned} &p^{XZ}_{XYZ*}(\beta \otimes_Y \alpha) \otimes_X \langle X_T \rangle = p^{XZ_T}_{XYZ_T*}\big((\beta \otimes_Y \alpha) \otimes_X \langle X_T \rangle\big) \\ &= p^{XZ_T}_{XYZ_T*}\big(\beta \otimes_Y (\alpha \otimes_X \langle X_T \rangle)\big) = p^{XZ_T}_{XYZ_T*}\big((\beta \otimes_Y \langle Y_T \rangle) \otimes_{Y_T} (\alpha \otimes_X \langle X_T \rangle)\big). \end{aligned}$$

□

Definition 9.3.2. Let $f : T \to S$ be a morphism of schemes. Using the preceding lemma, we define the base change functor

$$\begin{array}{rcl} f^* : \mathscr{P}^{cor}_{\Lambda,S} & \to & \mathscr{P}^{cor}_{\Lambda,T} \\ {[X/S]} & \mapsto & [X_T/T] \\ c_S\,(X,Y)_\Lambda \ni \alpha & \mapsto & \alpha_T. \end{array}$$

We sum up the basic properties of the base change for correspondences in the following lemma.

Lemma 9.3.3. *Take the notation and hypothesis of the previous definition.*

(1) *The functor f^* is symmetric monoidal.*
(2) *Let $f^*_0 : \mathscr{P}/S \to \mathscr{P}/T$ be the classical base change functor on $\mathscr{P}$-schemes over S. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathscr{P}/S & \xrightarrow{\gamma_S} & \mathscr{P}^{cor}_{\Lambda,S} \\ {\scriptstyle f^*_0}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathscr{P}/T & \xrightarrow{\gamma_T} & \mathscr{P}^{cor}_{\Lambda,T}. \end{array}$$

(3) *Let $\sigma : T' \to T$ be a morphism of schemes. Through the canonical isomorphisms $(X_T)_{T'} \simeq X_{T'}$, equality $(f \circ \sigma)^* = \sigma^* \circ f^*$ holds.*

Proof. (1) This point follows easily using the associativity formula 8.2.7 and the transitivity formulas 8.2.4, 8.2.6.
(2) This point follows from the fact that for any S-morphism $f : X \to Y$, there is a canonical isomorphism $\Gamma_{f_T} \to \Gamma_f \times_S T$.
(3) This point is a direct application of the transitivity 8.2.4. □

Lemma 9.3.4. *Let $f : T \to S$ be a universal homeomorphism.*
Then $f^* : \mathscr{P}^{cor}_{\Lambda,S} \to \mathscr{P}^{cor}_{\Lambda,T}$ is fully faithful.

Proof. Let X and Y be $\mathscr{P}$-schemes over S. Then $X_T \to X$ is a universal homeomorphism. Any generic point x of X corresponds uniquely to a generic point of X_T. Let m_x (resp. m'_x) be the geometric multiplicity of x in X (resp. X_T). Consider a finite S-correspondence $\alpha = \sum_{i\in I} n_i.z_i$. For each $i \in I$, let x_i be the generic point of X dominated by z_i. Then we get by definition:

$$f^*(\alpha) = \sum_{i\in I} m'_{x_i} \frac{n_i}{m_{x_i}}.z_i$$

and the lemma is clear. □

9.3.b. *Restriction.* Consider a $\mathscr{P}$-morphism $p : T \to S$. For any pair of T-schemes (X, Y), we denote by $\delta_{XY} : X \times_T Y \to X \times_S Y$ the canonical closed immersion deduced by base change from the diagonal immersion of T/S.

Consider a finite T-correspondence $\alpha : X \bullet\!\!\longrightarrow Y$. The cycle $\delta_{XY*}(\alpha)$ is the cycle α considered as a cycle in $X \times_S Y$. It defines a finite S-correspondence from X to Y.

Lemma 9.3.5. *Let X, Y and Z be T-schemes. The following relations are true :*

(1) For any T-morphism $f : X \to Y$, $\delta_{XY*}\big(\langle\Gamma_f\rangle_{XY_T}\big) = \langle\Gamma_f\rangle_{XY}$.
(2) For all $\alpha \in c_T(X,Y)_\Lambda$ and $\beta \in c_T(Y,Z)_\Lambda$,

$$\delta_{XZ*}(\beta\circ\alpha) = (\delta_{YZ*}(\beta)) \circ (\delta_{XY*}(\alpha)).$$

Proof. The first assertion is obvious.

The second assertion is a consequence of the projection formulas 8.2.8 and 8.2.10, and the functoriality of pushforwards :

$$\begin{aligned}(\delta_{YZ*}(\beta)) \circ (\delta_{XY*}(\alpha)) &= p^{XZ}_{XYZ*}\big(\delta_{YZ*}(\beta) \otimes_Y \delta_{XY*}(\alpha)\big)\\ &= p^{XZ}_{XYZ*}\delta_{XYZ*}(\beta \otimes_Y \alpha) = \delta_{XZ*}p^{XZ_T}_{XYZ_T*}(\beta \otimes_Y \alpha).\end{aligned}$$

□

Definition 9.3.6. Let $p : T \to S$ be a $\mathscr{P}$-morphism.

Using the preceding lemma, we define a functor

$$\begin{aligned} p_\sharp : \mathscr{P}^{cor}_{\Lambda,T} &\to \mathscr{P}^{cor}_{\Lambda,S}\\ [X \to T] &\mapsto [X \to T \xrightarrow{p} S]\\ c_T(X,Y)_\Lambda \ni \alpha &\mapsto \delta_{XY*}(\alpha).\end{aligned}$$

This functor enjoys the following properties:

Lemma 9.3.7. *Let $p : T \to S$ be a $\mathscr{P}$-morphism.*

(1) The functor $p_\sharp$ is left adjoint to the functor p^*.
(2) For any composable $\mathscr{P}$-morphisms $Z \xrightarrow{q} T \xrightarrow{p} S$, $(pq)_\sharp = p_\sharp q_\sharp$.
(3) Let $p^0_\sharp : \mathscr{P}/T \to \mathscr{P}/S$ be the functor induced by composition with p. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{P}/T & \xrightarrow{\gamma_T} & \mathscr{P}^{cor}_{\Lambda,T}\\ {\scriptstyle p^0_\sharp}\downarrow & & \downarrow{\scriptstyle p_\sharp}\\ \mathscr{P}/S & \xrightarrow{\gamma_S} & \mathscr{P}^{cor}_{\Lambda,S}.\end{array}$$

Proof. For point (1), we have to construct for schemes X/T and Y/S a natural isomorphism $c_S(p_\sharp X, Y)_\Lambda \simeq c_T(X, p^*Y)_\Lambda$. It is induced by the canonical isomorphism of schemes $(p_\sharp X)\times_S Y \simeq X \times_T (p^*Y)$.

Point (2) follows from the associativity of the pushforward functor on cycles. Note also that this identification is compatible with the transposition of the identification of 9.3.3(3) according to the adjunction property just obtained.

Point (3) is a reformulation of 9.3.5(2). □

9.3.c. *A finiteness property.*

9.3.8. We assume here that $\mathscr{P}$ is the class of all separated morphisms of finite type in $\mathscr{S}$.
Let I be a left filtering category and $(X_i)_{i\in I}$ be a projective system of separated S-schemes of finite type with affine dominant transition morphisms. We let $\mathcal{X}$ be the projective limit of $(X_i)_i$ and assume that $\mathcal{X}$ is Noetherian over S.

Proposition 9.3.9. *Let Y be a $\mathscr{P}$-scheme of finite type over S. Then the canonical morphism*

$$\varphi : \varinjlim_{i\in I^{op}} c_S\left(X_i, Y\right)_\Lambda \to c_0(\mathcal{X}\times_S Y/\mathcal{X},\Lambda).$$

is an isomorphism.

Proof. Note that according to [**SGA4**, IV, 8.3.8(i)], we can assume the conditions (2) of 8.3.5 is verified for $(X_i)_{i\in I}$. Thus conditions (1) to (4) of *loc. cit.* are verified. Then the surjectivity of φ follows from 8.3.9 and the injectivity from 8.3.6. □

9.4. The fibred category of correspondences. We can summarize the preceding constructions:

Proposition 9.4.1. *The* 2*-functor*

$$\mathscr{P}^{cor}_\Lambda : S \mapsto \mathscr{P}^{cor}_{\Lambda,S}$$

equipped with the pullback defined in 9.3.2 and with the tensor product of 9.2.3 is a monoidal $\mathscr{P}$-fibred category such that the functor

$$\gamma : \mathscr{P} \to \mathscr{P}^{cor}_\Lambda$$

(see (9.1.8.1)*) is a morphism of monoidal $\mathscr{P}$-fibred category.*

Proof. According to Lemma 9.3.7, for any $\mathscr{P}$ morphisms p, p^* admits a left adjoint $p_\sharp$. We have checked that γ is symmetric monoidal and commutes with f^* and $p_\sharp$ (see respectively 9.2.4, 9.3.3 and 9.3.7). But γ is essentially surjective. Thus, to prove the properties ($\mathscr{P}$-BC) and ($\mathscr{P}$-PF) for the fibred category $\mathscr{P}^{cor}_\Lambda$, we are reduced to the case of $\mathscr{P}$ which is easy (see example 1.1.28). This concludes. □

Remark 9.4.2. Consider the definition above.

(1) The category $\mathscr{P}^{cor}_\Lambda$ is Λ-linear. For any scheme S, $\mathscr{P}^{cor}_{\Lambda,S}$ is additive. For any finite family of schemes $(S_i)_{i\in I}$ which admits a sum S in $\mathscr{S}$, the canonical map

$$\mathscr{P}^{cor}_{\Lambda,S} \to \bigoplus_{i\in I} \mathscr{P}^{cor}_{\Lambda,S_i}$$

is an isomorphism.

(2) The functor $\gamma : \mathscr{P} \to \mathscr{P}^{cor}_\Lambda$ is nothing else than the canonical geometric sections of $\mathscr{P}^{cor}_\Lambda$ (see definition 1.1.35).

We will apply these definitions in the particular cases $\mathscr{P} = Sm$ (resp. $\mathscr{P} = \mathscr{S}^{ft}$) the class of smooth separated (resp. separated) morphisms of finite type. Note that we get a commutative square

$$\begin{array}{ccc} Sm & \xrightarrow{\gamma} & \mathscr{S}m^{cor}_\Lambda \\ \downarrow & & \downarrow \\ \mathscr{S}^{ft} & \xrightarrow{\gamma} & \mathscr{S}^{ft,cor}_\Lambda \end{array}$$

where the vertical maps are the obvious embeddings of monoidal Sm-fibred categories.

9.4.3. Consider extensions of rings $\Lambda \subset \Lambda' \subset \mathbf{Q}$. The functors (9.1.9.1) for various schemes S in $\mathscr{S}$ are compatible with the operations of a $\mathscr{P}$-fibred category because it is just concerned with adding denominators in the coefficients of the finite correspondences considered. Thus they induce a morphism of monoidal $\mathscr{P}$-fibred categories over $\mathscr{S}$:

$$\mathscr{P}^{cor}_\Lambda \otimes_\Lambda \Lambda' \to \mathscr{P}^{cor}_{\Lambda'}. \tag{9.4.3.1}$$

According to Proposition 9.1.10, we get the following result:

PROPOSITION 9.4.4. *Consider the above notations. Then the above morphism of monoidal $\mathscr{P}$-fibred categories is an equality whenever it is restricted to one of the following subcategories of $\mathscr{S}$:*

- *The category of regular schemes.*
- *The category of noetherian finite dimensional schemes S such that* $\mathrm{char}(S) \subset \Lambda^\times$.

REMARK 9.4.5. The restriction of the category $\mathscr{P}^{cor}_{\mathbf{Z}}$ to the category of regular schemes was already defined in [**Dég07**]. Indeed, one can check using the comparison of Suslin-Voevodsky's multiplicites with Serre's intersection multiplicities (using Tor-formulas ; *cf.* 8.3.31), that the operations τ^*, $\tau_\sharp$, and $\otimes^{tr}$ defined here coincide with that of [**Dég07**]. This remark extends 9.1.6.

10. Sheaves with transfers

10.0. The category $\mathscr{S}$ is the category of noetherian schemes of finite dimension. We fix an admissible class $\mathscr{P}$ of morphisms in $\mathscr{S}$ satisfying the following assumptions:

(a) Any morphism in $\mathscr{P}$ is separated of finite type.
(b) Any étale separated morphism of finite type is in $\mathscr{P}$.

We fix a topology t on $\mathscr{S}$ which is $\mathscr{P}$-admissible and such that:

(c) For any scheme S, there is a class of covers $\mathcal{E}$ of the form $(p : W \to S)$ with p a $\mathscr{P}$-morphism such that t is the topology generated by $\mathcal{E}$ and the covers of the form $(U \to U \sqcup V, V \to U \sqcup V)$ for any schemes U and V in $\mathscr{S}$.

We fix a ring of coefficients Λ. Whenever we speak of Λ-cycles (or the premotivic category $\mathscr{P}^{cor}_\Lambda$, etc...), we mean cycles with coefficients in the localization of $\mathbf{Z}$ with respect to invertible primes in Λ.

Note that in sections 10.4 and 10.5, we will apply the conventions of section 1.4 by replacing the class of smooth morphisms of finite type (resp. morphisms of finite type) there by the class of smooth separated morphisms of finite type (resp. separated morphisms of finite type).

10.1. Presheaves with transfers.

We consider the additive category $\mathscr{P}^{cor}_{\Lambda,S}$ of definition 9.1.8 and the graph functor $\gamma : \mathscr{P}/S \to \mathscr{P}^{cor}_{\Lambda,S}$ of (9.1.8.1).

DEFINITION 10.1.1. A *presheaf with transfers* F over S is an additive presheaf of Λ-modules over $\mathscr{P}^{cor}_{\Lambda,S}$. We denote by $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ the corresponding category.

If X is a $\mathscr{P}$-scheme over S, we denote by $\Lambda^{tr}_S(X)$ the presheaf with transfers represented by X.

We denote by $\hat{\gamma}_*$ the functor

$$\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right) \to \mathrm{PSh}(\mathscr{P}/S, \Lambda), F \mapsto F \circ \gamma. \tag{10.1.1.1}$$

Note that $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ is obviously a Grothendieck abelian category generated by the objects $\Lambda^{tr}_S(X)$ for a $\mathscr{P}$-scheme X/S. Moreover, the following proposition is straightforward:

PROPOSITION 10.1.2. *There is an essentially unique Grothendieck abelian $\mathscr{P}$-premotivic category* $\mathrm{PSh}\left(\mathscr{P}^{cor}_\Lambda\right)$ *which is geometrically generated (cf. 1.1.41), whose fiber over a scheme S is* $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ *and such that the functor Λ^{tr}_S induces a morphism of additive monoidal $\mathscr{P}$-fibred categories.*

$$\mathscr{P}^{cor}_\Lambda \to \mathrm{PSh}\left(\mathscr{P}^{cor}_\Lambda\right). \tag{10.1.2.1}$$

Moreover, the functor (10.1.1.1) *induces a morphism of abelian $\mathscr{P}$-premotivic categories*

$$\hat{\gamma}^* : \mathrm{PSh}(\mathscr{P}, \Lambda) \rightleftarrows \mathrm{PSh}\left(\mathscr{P}^{cor}_\Lambda\right) : \hat{\gamma}_*.$$

PROOF. To help the reader, we recall the following consequence of Yoneda's lemma:

LEMMA 10.1.3. *Let* $F : (\mathscr{P}^{cor}_{\Lambda,S})^{op} \to \Lambda\text{-}\mathrm{mod}$ *be a presheaf with transfers. Let* $\mathcal{I}$ *be the category of representables preshaves with transfers over* F*. Then the canonical map*

$$\varinjlim_{\Lambda^{tr}_S(X)\to F} \Lambda^{tr}_S(X) \to F$$

is an isomorphism. The limit is taken in $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ *and runs over* $\mathcal{I}$*.*

This lemma allows us to define the structural left adjoint of $\mathrm{PSh}\,(\mathscr{P}^{cor}_{\Lambda})$ (recall f^*, $p_\sharp$ for p a $\mathscr{P}$-morphism and the tensor product) because they are indeed determined by (10.1.2.1). The existence of the structural right adjoints is formal.

The same lemma allows to get the adjunction $(\hat{\gamma}^*, \hat{\gamma}_*)$. □

REMARK 10.1.4. Note that for any presheaf with transfers F over S, and any morphism $f : T \to S$ (resp. $\mathscr{P}$-morphism $p : S \to S'$), we get as usual $f_*F = F \circ f^*$ (resp. $p^*F = F \circ p_\sharp$) where the functor f^* (resp. $p_\sharp$) on the right hand side is taken with respect to the $\mathscr{P}$-fibred category $\mathscr{P}^{cor}_{\Lambda}$.

Let us state the following lemma for future use.

LEMMA 10.1.5. *Let Let* $(S_\alpha)_{\alpha\in A}$ *be a projective system of schemes in* $\mathscr{S}$*, with dominant affine transition maps, and such that* $S = \varprojlim_{\alpha\in A} S_\alpha$ *is representable in* $\mathscr{S}$*.*

Consider an index $\alpha_0 \in A$ *and a presheaf with transfers* F *over* S_{α_0}*. For any index* α/α_0*, we denote by* F_α *(resp.* F*) the pullback of* F_{α_0} *over* S_α *(resp.* S*) in the sense of the premotivic structure on* $\mathrm{PSh}\,(\mathscr{P}^{cor}_{\Lambda})$*.*

Then the canonical map:

$$\varinjlim_{\alpha\in A/\alpha_0} F_\alpha(S_\alpha) \longrightarrow F(S)$$

is an isomorphism.

PROOF. The presheaf F_{α_0} can be written as an inductive limit of representable sheaves of the form $\Lambda^{tr}_{S_{\alpha_0}}(X_{\alpha_0})$ of a $\mathscr{P}$-scheme $X_{\alpha_0}/S_{\alpha_0}$. As the global section functor on presheaves with transfers commute with inductive limit, we are reduced to the case where $F = \Lambda^{tr}_{S_{\alpha_0}}(X_{\alpha_0})$. In this case, the lemma follows directly from Proposition 9.3.9. □

10.2. Sheaves with transfers.

DEFINITION 10.2.1. A t-sheaf with transfers over S is a presheaf with transfers F such that the functor $F \circ \gamma_S$ is a t-sheaf. We denote by $\mathrm{Sh}_t\big(\mathscr{P}^{cor}_{\Lambda,S}\big)$ the full subcategory of $\mathrm{PSh}(\mathscr{P}^{cor}_{\Lambda,S}, \Lambda)$ of sheaves with transfers.

According to this definition, we get a canonical faithful functor

$$\gamma_* : \mathrm{Sh}_t\big(\mathscr{P}^{cor}_{\Lambda,S}\big) \to \mathrm{Sh}_t(\mathscr{P}/S, \Lambda)\,, F \mapsto F \circ \gamma.$$

EXAMPLE 10.2.2. A particularly important case for us is the case when $t = \mathrm{Nis}$ is the Nisnevich topology. According to the original definition of Voevodsky, a Nisnevich sheaf with transfers will be called simply a *sheaf with transfers*.

REMARK 10.2.3. Later on, in the case $\mathscr{P} = \mathscr{S}^{ft}$, we will use the notation $\underline{\Lambda}^{tr}_S(X)$ to denote the presheaf on the big site $\mathscr{S}^{ft,cor}_{\Lambda,S}$ represented by a separated S-scheme of finite type.

PROPOSITION 10.2.4. *Let* X *be an* $\mathscr{P}$*-scheme over* S*.*

1. *The presheaf* $\Lambda^{tr}_S(X)$ *is an étale sheaf with transfers.*
2. *If* $\mathrm{char}(X) \subset \Lambda^\times$*,* $\Lambda^{tr}_S(X)$ *is a* qfh*-sheaf with transfers.*

PROOF. For point (1), we follow the proof of [**Dég07**, 4.2.4]: the computation of the pullback by an étale map is given in our context by point (3) of Proposition 9.1.7. Moreover, the property for a cycle α/Y to be Λ-universal is étale-local on Y according to 8.3.8. For point (2), we refer to [**SV00b**, 4.2.7]. □

We can actually describe explicitely representable presheaves with transfers in the following case:

Proposition 10.2.5. *Let S be a scheme and X be a finite étale S-scheme. Then for any $\mathscr{P}$-scheme Y over S,*

$$\Gamma(Y, \Lambda_S^{tr}(X)) = \pi_0(Y \times_S X).\Lambda.$$

This readily follows from the following lemma:

Lemma 10.2.6. *Let $f : X \to S$ be an étale separated morphism of finite type. Let $\pi_0^{finite}(X/S)$ be the set of connected components V of X such that $f(V)$ is equal to a connected component of S (*i.e.* f is finite over V).*

Then $c_0(X/S, \Lambda) = \pi_0^{finite}(X/S).\Lambda$.

Proof. We can assume that S is reduced and connected.

We first treat the case where $X = S$. Consider a finite Λ-universal S-cycle α with domain S. Write $\alpha = \sum_{i \in I} n_i.\langle Z_i \rangle_S$ in standard form. By definition, Z_i dominates an irreducible component of S thus Z_i is equal to that irreducible component.

Consider S_0 an irreducible component of S and an index $i \in I$ such that $S_0 \cap Z_i$ is not empty. Consider a point $s \in S_0 \cap Z_i$. We have obviously $\alpha_s = n_i.\langle \mathrm{Spec}\,(\kappa(s)) \rangle \neq 0$. Thus there exists a component of α which dominates S_0 *i.e.* $\exists j \in I$ such that $Z_j = S_0$. Moreover, computing α_s using alternatively Z_i and Z_j gives $n_i = n_j$.

As S is noetherian, we see inductively $\{Z_i | i \in I\}$ is the set of irreducible components of S and for any $i, j \in I$, $n_i = n_j$. Thus $c_0(S/S, \Lambda) = \mathbf{Z}$.

Consider now the case of an étale S-scheme X. By additivity of c_0, we can assume that X is connected. Consider the following canonical map:

$$c_0(X/S, \Lambda) \to c_0(X \times_S X/X, \Lambda), \alpha \mapsto \alpha \otimes_S^{\flat} X.$$

Note that considering the projection $p : X \times_S X \to X$, by definition, $\alpha \otimes_S^{\flat} X = p^*(\alpha)$.

Consider the diagonal $\delta : X \to X \times_S X$ of X/S. Because X/S is étale and separated, δ is a direct factor of $X \times_S X$ and we can write $X \times_S X = X \sqcup U$. Because c_0 is additive,

$$c_0(X \times_S X/X, \Lambda) = c_0(X/X, \Lambda) \oplus c_0(U/X, \Lambda).$$

Moreover, the projection on the first factor is induced by the map δ^* on cycles. Because $\delta^* p^* = 1$, we deduce that a cycke in $c_0(X/S, \Lambda)$ corresponds uniquely to a cycle in $c_0(X/X, \Lambda)$. According to the preceding case, this latter group is the free group generated by the cycle $\langle X \rangle$. This latter cycle is Λ-universal over S, because X/S is flat. Thus, if X/S is finite, it is an element of $c_0(X/S, \Lambda)$ so that $c_0(X/S, \Lambda) = \Lambda$. Otherwise, not any of the Λ-linear combination of $\langle X \rangle$ belongs to $c_0(X/S, \Lambda)$ so that $c_0(X/S, \Lambda) = 0$. ☐

10.3. Associated sheaf with transfers.

10.3.1. Recall from 3.2.1 that we denote by $(\mathscr{P}/S)^{\amalg}$ the category of I-diagrams of objects in $\mathscr{P}/S$ indexed by a discrete category I. Given any simplicial object $\mathcal{X}$ of $(\mathscr{P}/S)^{\amalg}$, we will consider the complex $\Lambda_S^{tr}(\mathcal{X})$ of $\mathrm{PSh}\left(\mathscr{P}_{\Lambda,S}^{cor}\right)$ applying the definition of 5.1.8 to the Grothendieck $\mathscr{P}$-fibred category $\mathrm{PSh}\,(\mathscr{P})$.

Consider a t-cover $p : W \to X$ in $\mathscr{P}/X$. We denote by W_X^n the n-fold product of W over X (in the category $\mathscr{P}/X$). We denote by $\check{S}(W/X)$ the Čech simplicial object of $\mathscr{P}_{\Lambda,S}^{cor}$ such that $\check{S}_n(W/X) = W_X^{n+1}$. The canonical morphism $\check{S}(W/X) \to X$ is a t-hypercover according to definition 3.2.1. We will call these particular type of t-hypercovers the *Čech t-hypercovers* of X.

Definition 10.3.2. We will say that the admissible topology t on $\mathscr{P}$ is *compatible with transfers* (resp. *weakly compatible with transfers*) if for any scheme S and any t-hypercover (resp. any Čech t-hypercover) $\mathcal{X} \to X$ in the site $\mathscr{P}/S$, the canonical morphism of complexes

$$\Lambda_S^{tr}(\mathcal{X}) \to \Lambda_S^{tr}(X) \qquad (10.3.2.1)$$

induces a quasi-isomorphism of the associated t-sheaves on $\mathscr{P}/S$.

Obviously, if t is compatible with transfers then it is weakly compatible with transfers.

Recall from 10.2.4 that, in the cases $t = \mathrm{Nis}, \text{ét}$, (10.3.2.1) is actually a morphism of complexes of t-sheaves with transfers. The following proposition is a generalisation of [**Voe96**, 3.1.3] but its proof is in fact the same.

PROPOSITION 10.3.3. *The Nisnevich (resp étale) topology t on $\mathscr{P}$ is weakly compatible with transfers.*

PROOF. We consider a t-cover $p : W \to X$, the associated Čech hypercover $\mathcal{X} = \check{S}(W/X)$ of X and we prove that the map (10.3.2.1) is a quasi-isomorphism of t-sheaves. Recall that a point of $\mathscr{P}/S$ for the topology t is given by an essentially affine pro-object $(V_i)_{i\in I}$ of $\mathscr{P}/S$. Moreover, its projective limit $\mathcal{V}$ in the category of schemes is in particular a local henselian noetherian scheme. It will be sufficient to check that the fiber of (10.3.2.1) at the point $(V_i)_{i\in I}$ is a quasi-isomorphism. Thus, according to Proposition 9.3.9, we can assume that $S = \mathcal{V}$ is a local henselian scheme and we are to reduce to prove that the complex of abelian groups

$$\ldots \to c_0(W \times_X W/S, \Lambda) \to c_0(W/S, \Lambda) \xrightarrow{p_*} c_0(X/S, \Lambda) \to 0$$

is acyclic. We denote this complex by C.

Recall that the abelian group $c_0(X/S)$ is covariantly functorial in X with respect to separated morphisms of finite type $f : X' \to X$ (*cf.* paragraph 9.1.1). Moreover, if f is an immersion, f_* is obviously injective.

Let $\mathcal{F}_0$ be the set of closed subschemes Z of X such that Z/S is finite. Given a closed subscheme Z in $\mathcal{F}_0$, we let C_Z be the complex of abelian groups

$$\ldots \to c_0(W_Z \times_Z W_Z/S, \Lambda) \to c_0(W_Z/S, \Lambda) \xrightarrow{p_{Z*}} c_0(Z/S, \Lambda) \to 0 \qquad (10.3.3.1)$$

where p_Z is the t-cover obtained by pullback along $Z \to X$. From what we have just recalled, we can identify C_Z with a subcomplex of C. The set $\mathcal{F}_0$ can be ordered by inclusion, and C is the union of its subcomplexes C_Z. If $\mathcal{F}_0$ is empty, then $C = 0$ and the proposition is clear. Otherwise, $\mathcal{F}_0$ is filtered and we can write:

$$C = \varinjlim_{Z\in\mathcal{F}_0} C_Z.$$

Thus, it will be sufficient to prove that C_Z is acyclic for any $Z \in \mathcal{F}_0$. Because S is henselian and Z is finite over S, Z is indeed a finite sum of local henselian schemes. This implies that the t-cover p_Z, which is in particular étale surjective, admits a splitting $s : Z \to W_Z$. Then the complex (10.3.3.1) is contractible with contracting homotopy defined by the family

$$(s \times_Z 1_{W_Z^n})_* : c_0(W_Z^n/S, \Lambda) \to c_0(W_Z^{n+1}/S, \Lambda).$$

□

10.3.4. Considering an additive abelian presheaf G on $\mathscr{P}/S$, the natural transformation

$$X \mapsto \mathrm{Hom}_{\mathrm{PSh}(\mathscr{P}/S)}(\hat{\gamma}_* \Lambda_S^{tr}(X), G)$$

defines a presheaf with transfers over S.[82] We will denote by G_τ its restriction to the site $\mathscr{P}/S$. Note that this definition can be applied in the case where G is a t-sheaf on $\mathscr{P}/S$, because under the assumption 10.0 on t, it is in particular an additive presheaf.

DEFINITION 10.3.5. We will say that t is mildly compatible with transfers if for any scheme S and any t-sheaf F on $\mathscr{P}/S$, F_τ is a t-sheaf on $\mathscr{P}/S$.

If t is weakly compatible with transfers then is it mildly compatible with transfers.

REMARK 10.3.6. Assume t is mildly compatible with transfers. Then for any scheme S, any t-cover $p : W \to X$ in $\mathscr{P}/S$ induces a morphism

$$p_* : \Lambda_S^{tr}(W) \to \Lambda_S^{tr}(X)$$

[82] Actually, this defines a right adjoint to the functor $\hat{\gamma}_*$.

which is an epimorphism of the associated t-sheaves on $\mathscr{P}/S$. This means that for any correspondence $\alpha \in c_S(Y, X)$, there exists a t-cover $q : W' \to Y$ and a correspondence $\alpha' \in c_S(W', W)$ making the following diagram commutative:

(10.3.6.1)
$$\begin{array}{ccc} W' & \overset{\hat{\alpha}}{\bullet\!\!\longrightarrow} & W \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \overset{\alpha}{\bullet\!\!\longrightarrow} & X \end{array}$$

Lemma 10.3.7. *Assume t is mildly compatible with transfers.*

Let S be a scheme and P^{tr} be a presheaf with transfers over S. We put $P = P^{tr} \circ \gamma$ as a presheaf on $\mathscr{P}/S$. We denote by F the t-sheaf associated with P and by $\eta : P \to F$ the canonical natural transformation.

Then there exists a unique pair (F^{tr}, η^{tr}) such that:

(1) F^{tr} is a sheaf with transfers over S such that $F^{tr} \circ \gamma = F$.
(2) $\eta^{tr} : P^{tr} \to F^{tr}$ is a natural transformation of presheaves with transfers such that the induced transformation
$$P = (P^{tr} \circ \gamma) \to (F^{tr} \circ \gamma) = F$$
coincides with η.

Proof. As a preliminary observation, we note that given a presheaf G on $\mathscr{P}/S$, the data of a presheaf with transfers G^{tr} such that $G^{tr} \circ \gamma = G$ is equivalent to the data for any $\mathscr{P}$-schemes X and Y of a bilinear product

(10.3.7.1)
$$G(X) \otimes_{\mathbf{Z}} c_S(Y, X) \to G(Y), \rho \otimes \alpha \mapsto \langle \rho, \alpha \rangle$$

such that:

(a) For any morphism $f : Y' \to Y$ in $\mathscr{P}/S$, $f^*\langle \rho, \alpha \rangle = \langle \rho, \alpha \circ f \rangle$.
(b) For any morphism $f : X \to X'$ in $\mathscr{P}/S$, if $\rho = f^*(\rho')$, $\langle \rho, \alpha \rangle = \langle \rho', f \circ \alpha \rangle$.
(c) When $X = Y$, for any $\rho \in F(X)$, $\langle \rho, 1_X \rangle = \rho$.
(d) For any finite S-correspondence $\beta \in c_S(Z, Y)$, $\langle \langle \rho, \alpha \rangle, \beta \rangle = \langle \rho, \alpha \circ \beta \rangle$.

On the other hand, the data of products of the form (10.3.7.1) for any $\mathscr{P}$-schemes X and Y over S which satisfy the conditions (a) and (b) above is equivalent to the data of a natural transformation
$$\phi : G \to G_\tau$$
by putting $\langle \rho, \alpha \rangle_\phi = [\phi_X(\rho)]_Y.\alpha$.

Applying this to the presheaf with transfers P^{tr}, we obtain a canonical natural transformation
$$\psi : P \to P_\tau.$$
By assumption on t, F_τ is a t-sheaf. Thus there existe a unique natural transformation ψ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\psi} & P_\tau \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta_\tau} \\ F & \xrightarrow{\phi} & F_\tau \end{array}$$

Thus we get products of the form 10.3.7.1 associated with ϕ which satisfies (a) and (b). The commutativity of the above diagram asserts they are compatible with the ones corresponding to P^{tr} and the unicity of the natural transformation ϕ implies the uniqueness statement of the lemma.

To finish the proof of the existence, we must show (c) and (d) for the product $\langle ., \rangle_\phi$. Consider a couple $(\rho, \alpha) \in F(X) \times c_S(Y, X)$. Because F is the t-sheaf associated with P, there exists a t-cover $p : W \to X$ and a section $\hat{\rho} \in P(W)$ such that $p^*(\rho) = \eta_W(\hat{\rho})$. Moreover, according to remark 10.3.6, we get a t-cover $q : W' \to Y$ and a correspondence $\hat{\alpha} \in c_S(W', W)$ making the diagram (10.3.6.1) commutative. Then we get using (a) and (b):
$$q^*\langle \rho, \alpha \rangle_\phi = \langle \rho, \alpha \circ q \rangle_\phi = \langle \rho, p \circ \hat{\alpha} \rangle_\phi = \langle p^*\rho, \hat{\alpha} \rangle_\phi = \langle \eta_W(\hat{\rho}), \hat{\alpha} \rangle_\phi = \langle \hat{\rho}, \hat{\alpha} \rangle_\psi.$$

Because $q^* : F(X) \to F(W)$ is injective, we deduce easily from this principle the properties (c) and (d) and this concludes. □

10.3.8. Let us consider the canonical adjunction

$$a_t^* : \mathrm{PSh}(\mathscr{P}/S, \Lambda) \rightleftarrows \mathrm{Sh}_t(\mathscr{P}/S, \Lambda) : \mathcal{O}_t$$

where $\mathcal{O}_t$ is the canonical forgetful functor.

We also denote by $\mathcal{O}_t^{tr} : \mathrm{Sh}_t\big(\mathscr{P}^{cor}_{\Lambda,S}\big) \to \mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ the obvious forgetful functor. Trivially, the following relation holds:

$$\hat{\gamma}_*\, a_{t,*} = a_{t,*}\, \gamma_*. \tag{10.3.8.1}$$

Proposition 10.3.9. *Using the notations above, the following condition on the admissible topology t are equivalent:*

(i) t is mildly compatible with transfers.

(ii) For any scheme S, the functor $\mathcal{O}_t^{tr}$ admits a left adjoint $a_t^* : \mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right) \to \mathrm{Sh}_t\big(\mathscr{P}^{cor}_{\Lambda,S}\big)$ which is exact and such that the exchange transformation

$$a_t^*\, \hat{\gamma}_* \to \gamma_*\, a_t^* \tag{10.3.9.1}$$

induced by the identification (10.3.8.1) *is an isomorphism.*

Under these conditions, the following properties hold for any scheme S:

(iii) The category $\mathrm{Sh}_t\big(\mathscr{P}^{cor}_{\Lambda,S}\big)$ is a Grothendieck abelian category.

(iv) The functor γ_* commutes with every limits and colimits.

Proof. The fact (i) implies (ii) follows from the preceding lemma as we can put $a_t^{tr}(F) = F^{tr}$ with the notation of the lemma. The fact this defines a functor, as well as the properties stated in (ii), follows from the uniqueness statement of *loc. cit.*

Let us assume (ii). Then (iii) follows formally because from (ii), from the existence, adjunction property and exactness of a_t^*, because $\mathrm{PSh}\left(\mathscr{P}^{cor}_{\Lambda,S}\right)$ is a Grothendieck abelian category. Moreover, we deduce from the isomorphism (10.3.9.1) that γ_* is exact: indeed, a_t^* and $\hat{\gamma}_*$ are exact. As γ_* commutes with arbitrary direct sums, we get (iv).
From this point, we deduce the existence of a right adjoint $\gamma^!$ to the functor γ_*. Using again the isomorphism (10.3.9.1), we obtain for any t-sheaves F on $\mathscr{P}/S$ and any $\mathscr{P}$-scheme X/S a canonical isomorphism $F_\tau(X) = \gamma^! F(X)$. This proves (i). □

10.3.10. Under the assumption of the previous proposition, given any $\mathscr{P}$-scheme X/S, we will put $\Lambda_S^{tr}(X)_t = a_t^* \Lambda_S^{tr}(X)$. The above proposition shows that the family $\Lambda_S^{tr}(X)_t$ for $\mathscr{P}$-schemes X/S is a generating family in $\mathrm{Sh}_t\big(\mathscr{P}^{cor}_{\Lambda,S}\big)$. Moreover, we get easily the following corollary of the preceding proposition and Proposition 10.1.2:

Corollary 10.3.11. *Assume that t is mildly compatible with transfers.*

Then there exists an essentially unique Grothendieck abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda)$ which is geometrically generated, whose fiber over a scheme S is $\mathrm{Sh}_t\big(\mathscr{P}^{cor}_{\Lambda,S}\big)$ and such that the t-sheafification functor induces an adjunction of abelian $\mathscr{P}$-premotivic categories:

$$a_t^* : \mathrm{PSh}\left(\mathscr{P}^{cor}_\Lambda\right) \rightleftarrows \mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda) : \mathcal{O}_t^{tr}.$$

Moreover, the functor γ_* induces an adjunction of abelian $\mathscr{P}$-premotivic categories:

$$\gamma^* : \mathrm{Sh}_t(\mathscr{P}, \Lambda) \rightleftarrows \mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda) : \gamma_*. \tag{10.3.11.1}$$

Remark 10.3.12. Notice moreover that $\gamma^*\, a_t^* = a_t^*\, \hat{\gamma}^*$.

Proof. In fact, using the exactness of a_t^*, given any sheaf F with transfers F over S, we get a canonical isomorphism

$$F = \varinjlim_{\Lambda_S^{tr}(X)_t \to F} \Lambda_S^{tr}(X)_t$$

where the limit is taken in $\mathrm{Sh}_t\big(\mathscr{P}^{cor}_{\Lambda,S}\big)$ and runs over the representable t-sheaves with transfers over F. As in the proof of 10.1.2, this allows to define uniquely the structural left adjoints of

$\mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda)$. The existence (and uniqueness) of the structural right adjoints then follows formally. The same remark allows to construct the functor γ^*. □

REMARK 10.3.13. Let us explicit the meaning of the preceding Corollary for a topology t which is compatible with transfers. Given a complex C with coefficients in the category $\mathrm{Sh}_t\big(\mathscr{P}^{cor}_{\Lambda,S}\big)$, the following conditions are equivalent:

(i) C is local (Definition 5.1.9),
(i') $\gamma_*(C)$ is local,
(i") given any $\mathscr{P}$-scheme X/S and any integer $n \in \mathbf{Z}$, the canonical map

$$H^n(C(X)) \to H^n_t(X, \gamma_*(C))$$

is an isomorphism,
(ii) C is t-flasque (Definition 5.1.9),
(ii') $\gamma_*(C)$ is t-flasque,
(ii") given any t-hypercover $p : \mathcal{X} \to X$ in $\mathscr{P}/S$ and any integer $n \in \mathbf{Z}$, the canonical map

$$p^* : H^n(C(X)) \to H^n(C(\mathcal{X}))$$

is an isomorphism.

More precisely, the equivalence of (i) and (ii) is the preceding corollary, while the equivalence of (i) and (i') (resp. (ii) and (ii')) follows from the existence of the adjunction (10.3.11.1) and the fact γ_* is exact. The equivalence between (i') and (i") (resp. (ii') and (ii")) is a simple translation of Definition 5.1.9.

10.3.14. Recall from Definition 5.1.9 we say that the abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda)$ satisfies cohomological t-descent if for any scheme S, and any t-hypercover $\mathcal{X} \to X$ in $\mathscr{P}/S$, the induced morphism of complexes in $\mathrm{Sh}_t\big(\mathscr{P}^{cor}_{\Lambda,S}\big)$

$$\Lambda^{tr}_S(\mathcal{X})_t \to \Lambda^{tr}_S(X)_t$$

is a quasi-isomorphism. The preceding corollary thus gives the following one:

COROLLARY 10.3.15. *Assume t is mildly compatible with transfers.*
Then the following conditions are equivalent:

(i) The topology t is compatible with transfers.
(ii) The abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda)$ satisfies cohomological t-descent.
(iii) The abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda)$ is compatible with t (see 5.1.9).

PROOF. The equivalence of (i) and (ii) follows easily from the isomorphism (10.3.9.1). The equivalence of (ii) and (iii) is Proposition 5.1.26 applied to the adjunction (10.3.11.1), in view of 10.3.9(iv). □

10.3.16. Recall from Paragraph 2.1.10 that a cd-structure P on $\mathscr{S}$ is the data of a family of commutative squares, called P-distinguished, of the form

$$\begin{array}{ccc} B & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & Q & \downarrow{\scriptstyle f} \\ A & \xrightarrow[i]{} & X \end{array} \tag{10.3.16.1}$$

which is stable by isomorphisms. Further, we will consider the following assumptions on P:

(a) P is complete, regular and bounded in the sense of [**Voe10c**].
(b) Any P-distinguished square as above is made of $\mathscr{P}$-morphisms and k is an immersion.
(c) Any square as above which is cartesian and such that $X = A \sqcup Y$, i and f being the obvious immersions, is P-distinguished.

Then the topology t_P associated with P (see 2.1.10) is $\mathscr{P}$-admissible and satisfy assumption 10.0(c). Moreover, according to [**Voe10c**, 2.9], we obtain the following properties:

(d) A presheaf F on $\mathscr{P}/S$ is a t_P-sheaf if and only if $F(\varnothing) = 0$ and for any P-distinguished square (10.3.16.1) in $\mathscr{P}/S$, the sequence

$$0 \to F(X) \xrightarrow{f^*+e^*} F(Y) \oplus F(A) \xrightarrow{k^*-g^*} F(B)$$

is exact.

(e) For any P-distinguished square (10.3.16.1) the sequence of representable pre-sheaves on $\mathscr{P}/S$

$$0 \to \Lambda_S(B) \xrightarrow{k_*-g_*} \Lambda_S(Y) \oplus \Lambda_S(A) \xrightarrow{f_*+e_*} \Lambda_S(X) \to 0$$

becomes exact on the associated t_P-sheaves.

Proposition 10.3.17. *Consider a cd-structure P satisfying properties (a) and (b) above and assume that $t = t_P$ is the topology associated with P. Then the following conditions are equivalent:*

(i) The topology t is compatible with transfers.

(ii) The topology t is mildly compatible with transfers.

(iii) For any scheme S and any P-distinguished square (10.3.16.1) *in $\mathscr{P}/S$, the short sequence of presheaves with transfers over S*

$$0 \to \Lambda^{tr}_S(B) \xrightarrow{k_*-g_*} \Lambda^{tr}_S(Y) \oplus \Lambda^{tr}_S(A) \xrightarrow{f_*+e_*} \Lambda^{tr}_S(X) \to 0$$

becomes exact on the associated t-sheaves on $\mathscr{P}/S$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious.

The implication (ii) $\Rightarrow$ (iii) follows from point (e) above and the following facts: γ^* is right exact (Corollary 10.3.11), $\gamma^* a_t = a^{tr}_t \hat{\gamma}^*$ (remark 10.3.12), $k_* : \Lambda^{tr}_S(B) \to \Lambda^{tr}_S(Y)$ is a monomorphism of presheaves with transfers (use 9.1.7(2) and the fact k is an immersion from assumption (b)).

Assume (iii). Then we obtain (ii) as a direct consequence of the point (d) above. Thus, to prove (i), we have only to prove that the abelian $\mathscr{P}$-premotivic category $\mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda)$ satisfies cohomological t-descent according to 10.3.15.

Let S be a scheme. Recall that the category $\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S, \Lambda)$ has a canonical DG-structure (see for example 5.0.27). The cohomological t-descent for $\mathrm{Sh}_t(\mathscr{P}^{cor}_{\Lambda,S})$ can be reformulated by saying that for any complex K of t-sheaves on $\mathscr{P}/S$, and any t-hypercover $\mathcal{X} \to X$, the canonical map of $\mathrm{D}(\Lambda\text{-}\mathrm{mod})$

$$\mathbf{R}\mathrm{Hom}^\bullet_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\gamma_* \Lambda^{tr}_S(X)_t, K) \to \mathbf{R}\mathrm{Hom}^\bullet_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\gamma_* \Lambda^{tr}_S(\mathcal{X})_t, K)$$

is an isomorphism. Recall also there is the injective model structure on $\mathrm{C}(\mathrm{Sh}_t(\mathscr{P}/S, \Lambda))$ for which every object is cofibrant and with quasi-isomorphisms as weak equivalences (see [**CD09**, 2.1] for more details). Replacing K by a fibrant resolution for the injective model structure, we get for any simplicial objects $\mathcal{X}$ of $\mathscr{P}/S^{\amalg}$ that:

$$\mathbf{R}\mathrm{Hom}^\bullet_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\gamma_* \Lambda^{tr}_S(\mathcal{X})_t, K) = \mathrm{Hom}^\bullet_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\gamma_* \Lambda^{tr}_S(\mathcal{X})_t, K).$$

Thus it is sufficient to prove that the presheaf

$$E : \mathscr{P}/S^{op} \to \mathrm{C}(\Lambda\text{-}\mathrm{mod}), X \mapsto \mathrm{Hom}^\bullet_{\mathrm{D}(\mathrm{Sh}_t(\mathscr{P}/S,\Lambda))}(\gamma_* \Lambda^{tr}_S(X)_t, K)$$

satisfies t-descent in the sense of 3.2.5.

We derive from (iii) that E sends a P-distinguished square to a homotopy cartesian square in $\mathrm{D}(\Lambda\text{-}\mathrm{mod})$. Thus the assertion follows from the arguments on t-descent from [**Voe10b, Voe10c**]. □

Remark 10.3.18. It follows from Remark 10.3.13 that under the equivalent conditions (i), (ii), (iii) of the above corollary, the admissible topology $t = t_P$ is bounded in $\mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda)$ in the sense of Definition 5.1.28. Over a scheme S, a bounded generating family is given by the following complexes of $\mathrm{Sh}_t(\mathscr{P}^{cor}_{\Lambda,S})$:

$$\ldots \to 0 \to \Lambda^{tr}_S(B) \xrightarrow{k_*-g_*} \Lambda^{tr}_S(Y) \oplus \Lambda^{tr}_S(A) \xrightarrow{f_*+e_*} \Lambda^{tr}_S(X) \to 0 \to \ldots$$

induced by a P-distignuished square of the form (10.3.16.1) – see also Example 5.1.29.

We end-up this section with a compatibility of certain sheaves with transfers with projective limits of schemes. This will be the key point to establish continuity for motivic complexes.

PROPOSITION 10.3.19. *Let t be one of the topologies* Nis, ét, cdh.

Let $(S_\alpha)_{\alpha\in A}$ be a projective system of schemes in $\mathscr{S}$, with dominant affine transition maps, and such that $S = \varprojlim_{\alpha\in A} S_\alpha$ is representable in $\mathscr{S}$.

Consider an index $\alpha_0 \in A$ and a t-sheaf with transfers F over $\mathscr{S}^{ft,cor}_{\Lambda,S_0}$. For any index α/α_0, we denote by F_α (resp. F) the pullback of F_{α_0} over S_α (resp. S) in the sense of the premotivic structure on $\mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda)$ (obtained in Corollary 10.3.11).

Then the canonical map:

$$\varinjlim_{\alpha\in A/\alpha_0} F_\alpha(S_\alpha) \longrightarrow F(S)$$

is an isomorphism.

PROOF. We consider the forgetful functor: $\mathcal{O}^{tr}_t : \mathrm{Sh}_t\left(\mathscr{S}^{ft,cor}_\Lambda\right) \to \mathrm{PSh}\left(\mathscr{S}^{ft,cor}_\Lambda\right)$. It is fully faithful and it commutes with the global section functor. We want to prove the proposition by using Lemma 10.1.5. Thus it is sufficient to prove that, for any morphism $f : X \to S$ in $\mathscr{S}$, the functor $\mathcal{O}^{tr}_t$ commutes with f^*. In other words, the pullback functor $\hat{f}^*$ for presheaves with transfers on $\mathscr{S}^{ft,cor}_\Lambda$ preserves t-sheaves with transfers: for any t-sheaf with transfers F over S, $\hat{f}^*(F)$ is a t-sheaf with transfers.

Let us first treat the case where f is separated of finite type. Then $\hat{f}^*$ admits a left adjoint $\hat{f}_\sharp$ which preserves t-covers. Thus the property is clear.

In the general case, we write f as a projective limit of morphisms of schemes $(f_\alpha : X_\alpha \to S)_{\alpha\in A}$ such that the transition morphisms of the projective scheme $(X_\alpha)_{\alpha\in A}$ are affine and dominant and each f_α is separated of finite type.[83] To check that $\hat{f}^*(F)$ is a t-sheaf, we consider a t-cover $p : W \to X$ of an S-scheme separated of finite type. Because of our choice of topology t, there exists an index α_1/α_0 such that $p : W \to X$ can be lifted as a t-cover $p_1 : W_{\alpha_1} \to X_{\alpha_1}$ over S_{α_1}. Using Lemma 10.1.5 again, we now are reduced to prove that for any α/α_1, $\hat{f}^*_{\alpha_1}(F)$ satisfies the t-sheaf property with respect to the pullback of p_1 over S_α/S_{α_1}. This follows from the first case treated. □

REMARK 10.3.20. The previous proposition generalizes [**Dég07**, Prop. 2.19].

10.4. Examples.

10.4.1. Assume that t is the Nisnevich topology. According to Lemma 10.3.3 and Proposition 10.3.17, t is then compatible with transfers. With the notation of Corollary 10.3.11, we get the following definition:

DEFINITION 10.4.2. We denote by

$$\mathrm{Sh}^{tr}(-,\Lambda), \qquad \underline{\mathrm{Sh}}^{tr}(-,\Lambda)$$

the respective abelian premotivic and generalized abelian premotivic categories defined in Corollary 10.3.11 in the respective cases $\mathscr{P} = \mathscr{S}m$, $\mathscr{P} = \mathscr{S}^{ft}$.

From now on, the objects of $\mathrm{Sh}^{tr}(S,\Lambda)$ (resp. $\underline{\mathrm{Sh}}^{tr}(S,\Lambda)$) are called *sheaves with transfers* over S (resp. *generalized sheaves with transfers* over S).

Let X be a separated S-scheme of finite type. We let $\underline{\Lambda}^{tr}_S(X)$ be the generalized sheaf with transfers represented by X (*cf.* 10.2.4). If X is S-smooth, we denote by $\Lambda^{tr}_S(X)$ its restriction to $\mathscr{S}m^{cor}_{\Lambda,S}$ – *i.e.* the sheaf with transfers over S represented by X.

An important property of sheaves with transfers is that the abelian premotivic category $\mathrm{Sh}^{tr}(-,\Lambda)$ (resp. $\underline{\mathrm{Sh}}^{tr}(-,\Lambda)$) is compatible with the Nisnevich topology on $\mathscr{S}m$ (resp. $\mathscr{S}^{ft}$) according to Proposition 10.3.17. Note moreover that it is compactly geometrically generated.

[83] Write the $\mathcal{O}_S$-algebra $f_*(\mathcal{O}_X)$ as the filtered union of its finite type sub-$\mathcal{O}_S$-algebras, ordered by inclusion.

10.4.3. We also obtained an adjunction (resp. generalized adjunction) of premotivic abelian categories

$$\gamma^* : \mathrm{Sh}(\mathscr{S}m, \Lambda) \rightleftarrows \mathrm{Sh}^{tr}(-, \Lambda) : \gamma_*$$
$$\gamma^* : \mathrm{Sh}\big(\mathscr{S}^{ft}, \Lambda\big) \rightleftarrows \underline{\mathrm{Sh}}^{tr}(-, \Lambda) : \gamma_*.$$

Note that in each case γ_* is conservative and exact according to 10.3.9(iv).

Remark 10.4.4. An important application of the existence of the pair of adjoint functors (γ^*, γ_*) is the following computation: given any complex K of sheaves with transfers over S and any smooth S-scheme X,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{tr}(S,\Lambda))}(\Lambda_S^{tr}(X), K[n]) &= \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{tr}(S,\Lambda))}(\mathbf{L}\gamma^* \Lambda_S(X), K[n]) \\ &= \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}(\mathscr{S}m,\Lambda))}(\Lambda_S(X), \gamma_*(K)[n]) = H^n_{\mathrm{Nis}}(X, \gamma_*(K)).\end{aligned}$$

This is a generalization of [**VSF00**, chap. 5, 3.1.9] to unbounded complexes and arbitrary bases.

10.4.5. Let S be a scheme. Consider the inclusion functor $\varphi : \mathscr{S}m^{cor}_{\Lambda,S} \to \mathscr{S}^{ft,cor}_{\Lambda,S}$. It induces a functor

$$\varphi^* : \underline{\mathrm{Sh}}^{tr}(S, \Lambda) \to \mathrm{Sh}^{tr}(S, \Lambda), F \mapsto F \circ \varphi$$

which is obviously exact and commute with arbitrary direct sums. In particular, it commutes with arbitrary colimits.

Lemma 10.4.6. *With the notations above, the functor φ^* admits a left adjoint $\varphi_!$ such that for any smooth S-scheme X, $\varphi_!(\Lambda_S^{tr}(X)) = \underline{\Lambda}_S^{tr}(X)$. The functor $\varphi_!$ is fully faithful.*

In other words, we have defined an enlargement of premotivic abelian categories (*cf.* definition 1.4.13)

(10.4.6.1) $$\varphi_! : \mathrm{Sh}^{tr}(-, \Lambda) \to \underline{\mathrm{Sh}}^{tr}(-, \Lambda) : \varphi^*.$$

Proof. Let F be a sheaf with transfers. Let $\{X/F\}$ be the category of representable sheaf $\Lambda_S^{tr}(X)$ over F for a smooth S-scheme X. We put

$$\varphi_!(F) = \varinjlim_{\{X/F\}} \underline{\Lambda}_S^{tr}(X).$$

The adjunction property of $\varphi_!$ is immediate from the Yoneda lemma. We prove that for any sheaf with transfers F, the unit adjunction morphism $F \to \varphi^*\varphi_!(F)$ is an isomorphism. As already remarked, φ^* commutes with colimits so that we are restricted to the case where $F = \Lambda_S^{tr}(X)$ which follows by definition. □

10.4.7. Assume now that $t = \mathrm{cdh}$ is the cdh-topology, and $\mathscr{P} = \mathscr{S}^{ft}$ is the class of separated morphisms of finite type. Recall the topology t is associated with the *lower cd-structure* – see Example 2.1.11. Then the assumptions of Proposition 10.3.17 with respect to the lower cd-structure are fulfilled according to [**SV00b**, 4.3.3] combined with [**SV00b**, 4.2.9]. Thus we get the following result:

Proposition 10.4.8. *The admissible topology* cdh *on $\mathscr{S}^{ft}$ is compatible with transfers.*

As a corollary, we get a generalized premotivic abelian category whose fiber over a scheme S is the category $\underline{\mathrm{Sh}}^{tr}_{\mathrm{cdh}}(S, \Lambda)$ of cdh-sheaves with transfers on $\mathscr{S}^{ft}$. It is compatible with the cdh-topology. Moreover, the restriction of a_{cdh} to $\underline{\mathrm{Sh}}^{tr}(S, \Lambda)$ induces a morphism of generalized premotivic categories; we get the following commutative diagram of such morphisms:

$$\begin{array}{ccc} \underline{\mathrm{Sh}}(-, \Lambda) & \xrightarrow{a^*_{\mathrm{cdh}}} & \underline{\mathrm{Sh}}_{\mathrm{cdh}}(-, \Lambda) \\ {\scriptstyle \gamma^*}\downarrow & & \downarrow{\scriptstyle \gamma^*_{\mathrm{cdh}}} \\ \underline{\mathrm{Sh}}^{tr}(-, \Lambda) & \xrightarrow{a^*_{\mathrm{cdh}}} & \underline{\mathrm{Sh}}^{tr}_{\mathrm{cdh}}(-, \Lambda) \end{array}$$

10.5. Comparison results.

10.5.a. *Change of coefficients.*

10.5.1. Assume the topology t is mildly compatible with transfers and consider a localization Λ' of Λ.

Then the morphism (9.4.3.1) of $\mathscr{P}$-premotivic categories extends to an adjunction of abelian $\mathscr{P}$-premotivic categories:

$$\mathrm{Sh}_t(\mathscr{P}^{cor}_\Lambda) \otimes_\Lambda \Lambda' \leftrightarrows \mathrm{Sh}_t(\mathscr{P}^{cor}_{\Lambda'}) \tag{10.5.1.1}$$

Proposition 9.4.4 immediately yields the following result:

PROPOSITION 10.5.2. *Consider the above notations. Then the above $\mathscr{P}$-premotivic adjunction is an equality whenever it is restricted to one of the following subcategories of $\mathscr{S}$:*

- *The category of regular schemes.*
- *The category of noetherian finite dimensional schemes S such that $\mathrm{char}(S) \subset \Lambda^\times$.*

REMARK 10.5.3. Remark 9.4.5 can be extended to sheaves with transfers: for any regular scheme S, the category $\mathrm{Sh}^{tr}(S, \mathbf{Z}) = \mathrm{Sh}_{\mathrm{Nis}}(\mathscr{S}m^{cor}_{\mathbf{Z},S})$ defined here coincides with that defined in [**Dég07**], as well as its operations of a $\mathscr{P}$-premotivic category when restricted to regular schemes.

10.5.b. *Representable* qfh*-sheaves.*

10.5.4. Let us denote by $\mathrm{Sh}_{\mathrm{qfh}}(S, \Lambda)$ the category of qfh-sheaves of Λ-modules over $\mathscr{S}^{ft}/S$. Remark that for an S-scheme X, the Λ-presheaf represented by X is not a sheaf for the qfh-topology. We denote the associated sheaf by $\Lambda_S^{\mathrm{qfh}}(X)$. We let a_{qfh} be the associated qfh-sheaf functor. Recall that for any S-scheme X, the graph functor (10.4.3) induces a morphism of sheaves

$$\underline{\Lambda}_S(X) \xrightarrow{\gamma_{X/S}} \underline{\Lambda}_S^{tr}(X).$$

We recall the following theorem of Suslin and Voevodsky (see [**SV00b**, 4.2.7+4.2.12]):

THEOREM 10.5.5. *Let S be a scheme such that $\mathrm{char}(S) \subset \Lambda^\times$. Then, for any S-scheme X, the application of a_{qfh} to the map $\gamma_{X/S}$ gives an isomorphism in $\mathrm{Sh}_{\mathrm{qfh}}(S, \Lambda)$:*

$$\Lambda_S^{\mathrm{qfh}}(X) \xrightarrow{\gamma^{\mathrm{qfh}}_{X/S}} \underline{\Lambda}_S^{tr}(X).$$

10.5.6. Assume $\mathrm{char}(S) \subset \Lambda^\times$. Using the previous theorem, we associate to any qfh-sheaf $F \in \mathrm{Sh}_{\mathrm{qfh}}(S, \Lambda)$ a presheaf with transfers

$$\rho(F) : X \mapsto \mathrm{Hom}_{\mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)}(\underline{\Lambda}_S^{tr}(X), F).$$

We obviously get $\gamma^*\rho(F) = F$ as a presheaf over $\mathscr{S}^{ft}/S$ so that $\rho(F)$ is a sheaf with transfers and we have defined a functor

$$\rho : \mathrm{Sh}_{\mathrm{qfh}}(S, \Lambda) \to \underline{\mathrm{Sh}}^{tr}(S, \Lambda).$$

For any S-scheme X, $\rho(\Lambda_S^{\mathrm{qfh}}(X)) = \underline{\Lambda}_S^{tr}(X)$ according to the previous proposition.

COROLLARY 10.5.7. *Assume $\mathrm{char}(S) \subset \Lambda^\times$. Let $f : X' \to X$ be a morphism of S-schemes. If f is a universal homeomorphism, then the map $f_* : \underline{\Lambda}_S^{tr}(X') \to \underline{\Lambda}_S^{tr}(X)$ is an isomorphism in $\underline{\mathrm{Sh}}^{tr}(S, \Lambda)$.*

PROOF. Indeed, according to [**Voe96**, 3.2.5], $\Lambda_S^{\mathrm{qfh}}(X') \to \Lambda_S^{\mathrm{qfh}}(X)$ is an isomorphism in $\mathrm{Sh}_{\mathrm{qfh}}(S, \Lambda)$ and we conclude by applying the functor ρ. □

10.5.c. qfh*-sheaves and transfers.*

PROPOSITION 10.5.8. *Assume $\mathrm{char}(S) \subset \Lambda^\times$. Any* qfh*-sheaf of Λ-modules over the category of S-schemes of finite type is naturally endowed with a unique structure of a sheaf with transfers, and any morphism of such* qfh*-sheaves is a morphism of sheaves with transfers.*

In particular, the qfh*-sheafification functor defines a left exact functor left adjoint to the forgetful functor $\rho : \mathrm{Sh}_{\mathrm{qfh}}(S, \Lambda) \to \underline{\mathrm{Sh}}^{tr}(S, \Lambda)$ introduced in 10.5.6.*

Proof. It follows from Theorem 10.5.5 that the category of Λ-linear finite correspondences is canonically equivalent to the full subcategory of the category of qfh-sheaves of Λ-modules spanned by the objects of shape $\Lambda_S^{\mathrm{qfh}}(X)$ for X separated of finite type over S. The first assertion is thus an immediate consequence of Theorem 10.5.5 and of the (additive) Yoneda lemma. The fact that the qfh-sheafification functor defines a left adjoint to the restriction functor ρ is then obvious, while its left exactness is a consequence of the facts that it is left exact (at the level of sheaves without transfers) and that forgetting transfers defines a conservative and exact functor from the category of Nisnevich sheaves with transfers to the category of Nisnevich sheaves. □

Recall the following theorem:

Theorem 10.5.9. *Assume Λ is a $\mathbf{Q}$-algebra. Let F be an étale Λ-sheaf on $\mathscr{S}^{ft}/S$. Then for any S-scheme X, and any integer i, the canonical map*

$$H^i_{\mathrm{Nis}}(X,F) \to H^i_{\mathrm{\acute{e}t}}(X,F)$$

is an isomorphism.

Proof. Using the compatibility of étale cohomology with projective limits of schemes, we are reduced to prove that $H^i_{\mathrm{\acute{e}t}}(X,F)=0$ whenever X is henselian local and $i>0$. Let k be the residue field of X, G its absolute Galois group and F_0 the restriction of F to Spec (k). Then F_0 is a G-module and according to [**SGA4**, 8.6], $H^i_{\mathrm{\acute{e}t}}(X,F)=H^i(G,F_0)$. As G is profinite, this group must be torsion so that it vanishes by assumption. □

Remark 10.5.10. The preceding theorem also follows formally from Theorem 3.3.23.

Proposition 10.5.11. *Assume Λ is a $\mathbf{Q}$-algebra. Let S be an excellent scheme and F be a* qfh*-sheaf of Λ-modules on $\mathscr{S}^{ft}/S$. Then for any geometrically unibranch S-scheme X of finite type, and any integer i, the canonical map*

$$H^i_{\mathrm{Nis}}(X,F) \to H^i_{\mathrm{qfh}}(X,F)$$

is an isomorphism.

Proof. According to 10.5.9, $H^i_{\mathrm{Nis}}(X,F)=H^i_{\mathrm{\acute{e}t}}(X,F)$. Let $p:X'\to X$ be the normalization of X. As X is an excellent geometrically unibranch scheme, p is a finite universal homeomorphism. It follows from [**SGA4**, VII, 1.1] that $H^i_{\mathrm{\acute{e}t}}(X,F)=H^i_{\mathrm{\acute{e}t}}(X',F)$ and from [**Voe96**, 3.2.5] that $H^i_{\mathrm{qfh}}(X,F)=H^i_{\mathrm{qfh}}(X',F)$. Thus we can assume that X is normal, and the result is now exactly [**Voe96**, 3.4.1]. □

Corollary 10.5.12. *Assume Λ is a $\mathbf{Q}$-algebra. Let S be an excellent scheme.*

(1) *Let X be a geometrically unibranch S-scheme of finite type. For any point x of X, the local henselian scheme X^h_x is a point for the category of sheaves $\mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)$ (*i.e.* evaluating at X^h_x defines an exact functor).*
(2) *The family of points X^h_x of the previous type is a conservative family for $\mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)$.*

Proof. The first point follows from the previous proposition. For any excellent scheme X, the normalization morphism $X'\to X$ is a qfh-cover. Thus the category $\mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)$ is equivalent to the category of qfh-sheaves on the site made of geometrically unibranch S-schemes of finite type. This implies the second assertion. □

10.5.13. Given any scheme S, we introduce the following composite functor using the notations of 10.5.6 and 10.4.5:

$$\psi^*:\mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)\xrightarrow{\rho}\underline{\mathrm{Sh}}^{tr}(S,\Lambda)\xrightarrow{\varphi^*}\mathrm{Sh}^{tr}(S,\Lambda).$$

Theorem 10.5.14. *Assume Λ is a $\mathbf{Q}$-algebra and let S be a geometrically unibranch excellent scheme. Considering the above notation, the following conditions are true :*

(i) *For any S-scheme X of finite type, $\psi^*\big(\Lambda_S^{\mathrm{qfh}}(X)\big)=\Lambda_S^{tr}(X)$.*
(ii) *The functor ψ^* admits a left adjoint $\psi_!$.*
(iii) *For any smooth S-scheme X, $\psi_!\big(\Lambda_S^{tr}(X)\big)=\Lambda_S^{\mathrm{qfh}}(X)$.*
(iv) *The functor ψ^* is exact and preserves colimits.*

(v) *The functor $\psi_!$ is fully faithful.*

According to property (iii), the functor $\psi_!$ commutes with pullbacks. In other words, we have defined an enlargement of abelian premotivic categories (*cf.* definition 1.4.13) over the category of (noetherian) geometrically unibranch schemes:

$$\psi_! : \mathrm{Sh}^{tr}(-,\Lambda) \rightleftarrows \mathrm{Sh}_{\mathrm{qfh}}(-,\Lambda) : \psi^* \tag{10.5.14.1}$$

Proof. Point (i) follows from the fact $\Lambda_S^{tr}(X) = \Lambda_S^{\mathrm{qfh}}(X)$. Recall the enlargement of (10.4.6.1):

$$\varphi_! : \mathrm{Sh}^{tr}(-,\Lambda) \to \underline{\mathrm{Sh}}^{tr}(-,\Lambda) : \varphi^*.$$

We define the functor $\psi_!$ as the composite :

$$\mathrm{Sh}^{tr}(S,\Lambda) \xrightarrow{\varphi_!} \underline{\mathrm{Sh}}^{tr}(S,\Lambda) \xrightarrow{\gamma^*} \underline{\mathrm{Sh}}(S,\Lambda) \xrightarrow{a_{\mathrm{qfh}}} \mathrm{Sh}_{\mathrm{qfh}}(S,\Lambda)\,.$$

According to the properties of the functors in this composite, $\psi_!$ is exact and preserves colimits. Moreover, for any smooth S-scheme X, as $\Lambda_S^{tr}(X)$ is a qfh-sheaf over $\mathscr{S}^{ft}/S$ according to 10.2.4, $\psi_!(\Lambda_S^{tr}(X)) = \Lambda_S^{\mathrm{qfh}}(X)$ which proves (iii). Property (ii) follows from (iii) and the fact $\psi_!$ commutes with colimits, while the sheaves $\Lambda_S^{tr}(X)$ for X/S smooth generate $\mathrm{Sh}^{tr}(S,\Lambda)$.

For any smooth S-scheme X, $\Gamma(X;\psi^*(F)) = F(X)$. Thus the exactness of ψ^* follows from corollary 10.5.12. As ψ^* obviously preserves direct sums, we get (iv).

To check that for any sheaf with transfers F the unit map $F \to \psi^*\psi_!(F)$ is an isomorphism, we thus are reduced to the case where $F = \Lambda_S^{tr}(X)$ for a smooth S-scheme X which follows from (i) and (iii). □

11. Motivic complexes

11.0. In this section, $\mathscr{S}$ is the category of noetherian finite dimensional schemes. It is adequate in the sense of 2.0. Given a scheme S, we denote by $\mathscr{S}m_S$ the category smooth separated S-schemes of finite type. It is admissible in the sense of 1.0.

We fix a ring of coefficients Λ.

11.1. Definition and basic properties.

11.1.a. *Premotivic categories.* According to Proposition 10.3.17 and Corollary 10.3.15, the abelian premotivic category $\mathrm{Sh}^{tr}(-,\Lambda)$ constructed in 10.4.2 is compatible with Nisnevich topology. Thus we can apply to it the general definitions of section 5. This gives the following definition:

Definition 11.1.1. We define the (Λ-linear) category of *motivic complexes* (resp. *stable motivic complexes* or simply *motives*) following definition 5.3.22 (resp. definition 5.2.16) as

$$\mathrm{DM}_\Lambda^{eff} = \mathrm{D}_{\mathbf{A}^1}^{eff}\left(\mathrm{Sh}^{tr}(-,\Lambda)\right)$$
$$\text{resp. } \mathrm{DM}_\Lambda = \mathrm{D}_{\mathbf{A}^1}\left(\mathrm{Sh}^{tr}(-,\Lambda)\right).$$

Given a scheme S, we will put: $\mathrm{DM}^{eff}(S,\Lambda) = \mathrm{DM}_\Lambda^{eff}(S)$, $\mathrm{DM}(S,\Lambda) = \mathrm{DM}_\Lambda(S)$.

11.1.2. Let us unfold the preceding definition. Given a scheme S in $\mathscr{S}$, the triangulated category $\mathrm{DM}^{eff}(S,\Lambda)$ is equal to the $\mathbf{A}^1$-localization of the derived category $\mathrm{D}(\mathrm{Sh}^{tr}(S,\Lambda))$ of the category of sheaves with transfers over S.

Given a smooth scheme S-scheme X of finite type, we have denoted by $\Lambda_S^{tr}(X)$ the sheaf with transfers represented by X over S. We will see this sheaf as an object of $\mathrm{DM}^{eff}(S,\Lambda)$, as a complex concentrated in degree 0, and call it the effective motivic complex associated with X/S.

Recall the following operations as part of the premotivic structure:

- Given any morphism $f : T \to S$ in $\mathscr{S}$, there exists an adjunction of the form:
$$\mathbf{L}f^* : \mathrm{DM}^{eff}(S,\Lambda) \leftrightarrows \mathrm{DM}^{eff}(T,\Lambda) : \mathbf{R}f_*.$$
- Given a separated smooth morphism of finite type $f : T \to S$ in $\mathscr{S}$, there exists an adjunction of the form:
$$\mathbf{L}f_\sharp : \mathrm{DM}^{eff}(S,\Lambda) \leftrightarrows \mathrm{DM}^{eff}(T,\Lambda) : f^*.$$

- Given any noetherian finite dimensional scheme S, the category $\mathrm{DM}^{\mathit{eff}}(S,\Lambda)$ is symmetric closed monoidal.

These operations are subject to the properties of a premotivic category: functoriality, smooth base change formula, smooth projection formula – see section 1 for more details. By construction, the triangulated premotivic category $\mathrm{DM}^{\mathit{eff}}_\Lambda$ satisfies the homotopy property and the Nisnevich descent properties.

By construction (cf. (5.3.23.2)), we get an adjunction of triangulated premotivic categories

$$\Sigma^\infty : \mathrm{DM}^{\mathit{eff}}_\Lambda \rightleftarrows \mathrm{DM}_\Lambda : \Omega^\infty. \tag{11.1.2.1}$$

Considering the *Tate motivic complex*

$$\Lambda^{tr}_S(1) := \Lambda^{tr}_S(\mathbf{P}^1_S/\{1\}), \tag{11.1.2.2}$$

the object $\Sigma^\infty(\Lambda^{tr}_S(1))$ is $\otimes$-invertible in $\mathrm{DM}(S,\Lambda)$ and this property characterizes uniquely the homotopy category $\mathrm{DM}(S,\Lambda)$ – see Remark 5.3.29. Given a smooth separated S-scheme X of finite type, we put:

$$M_S(X) := \Sigma^\infty\Lambda^{tr}_S(X)$$

and simply call it the motive associated with X/S. Usually we denote by $\mathbb{1}_S$ the unit of the monoidal category $\mathrm{DM}(S,\Lambda)$.

By construction, the premotivic category DM_Λ satisfies the homotopy, stability and Nisnevich descent properties (see Paragraph 5.3.23).

Example 11.1.3.
- Let k be a perfect field. Then $\mathrm{DM}^{\mathit{eff}}(k,\mathbf{Z})$ contains as a full subcategory the category $\mathrm{DM}^{\mathit{eff}}_{-}(k)$ defined by Voevodsky (cf [**VSF00**, Chap. 5]). This is the content of the proof of [**VSF00**, Chap. 5, Prop. 3.2.3]. Indeed, recall from Paragraph 5.2.18 that $\mathrm{DM}^{\mathit{eff}}(k,\mathbf{Z})$ is equivalent to the full subcategory of $\mathrm{D}(\mathrm{Sh}^{tr}(k,\mathbf{Z}))$ made by the complexes which are $\mathbf{A}^1$-local. Over a perfect field, Theorem 3.1.12 of [**VSF00**, Chap. 5] implies that a complex of sheaves with transfers is $\mathbf{A}^1$-local if and only if its homotopy sheaves are $\mathbf{A}^1$-invariant.
- Let S be a regular scheme. The triangulated categories $\mathrm{DM}^{\mathit{eff}}(S,\mathbf{Z})$ and $\mathrm{DM}(S,\mathbf{Z})$ introduced here coincide with that constructed in [**CD09**]. The same is true concerning the operations of premotivic triangulated categories (see Remark 10.5.3).

11.1.4. Let Λ' be a localization of Λ. The premotivic adjunction

$$\mathrm{Sh}^{tr}(-,\Lambda)\otimes_\Lambda\Lambda' \rightleftarrows \mathrm{Sh}^{tr}(-,\Lambda') \tag{11.1.4.1}$$

obtained as a particular case of (10.5.1.1) gives the following adjunctions of triangulated premotivic categories:

$$\begin{aligned} \mathrm{DM}_\Lambda\otimes_\Lambda\Lambda' &\rightleftarrows \mathrm{DM}_{\Lambda'}, \\ \mathrm{DM}^{\mathit{eff}}_\Lambda\otimes_\Lambda\Lambda' &\rightleftarrows \mathrm{DM}^{\mathit{eff}}_{\Lambda'}. \end{aligned} \tag{11.1.4.2}$$

Proposition 10.5.2 gives the following result:

Proposition 11.1.5. *The above premotivic adjunctions are equalities whenever it is restricted to one of the following subcategories of $\mathscr{S}$:*

- *The category of regular schemes.*
- *The category of noetherian finite dimensional schemes S such that* $\mathrm{char}(S)\subset\Lambda^\times$.

In other words, when S is a scheme of one of the categories listed above, the triangulated monoidal category $\mathrm{DM}(S,\Lambda')$ (resp. $\mathrm{DM}^{\mathit{eff}}(S,\Lambda')$) is the naive localization of the category $\mathrm{DM}(S,\Lambda)$ (resp. $\mathrm{DM}^{\mathit{eff}}(S,\Lambda)$) with respect to integers invertible in Λ'.

11.1.b. *Constructible and geometric motives.*

11.1.6. The premotivic triangulated category $\mathrm{DM}^{\mathit{eff}}_\Lambda$ is geometrically generated: given any scheme S, the essentially small set $\mathcal{G}^{eff}_S$ of motivic complexes of the form $\Lambda^{tr}_S(X)$ for a smooth separated S-scheme X of finite type form a set of generators in the triangulated category $\mathrm{DM}^{\mathit{eff}}(S,\Lambda)$.

Similarly, the premotivic triangulated category DM_Λ is $\mathbf{Z}$-generated where $\mathbf{Z}$ is the set of twists corresponding to the Tate twist: given any scheme S, the essentially small set $\mathcal{G}_S$ of motives of the form $M_S(X)(n)$ for a smooth separated S-scheme X of finite type and an integer $n \in \mathbf{Z}$ form a set of generators in the triangulated category $\mathrm{DM}(S,\Lambda)$.

Following the general conventions about premotivic triangulated category (Definition 1.4.9), we define the notion of constructibility for motives as follows:

Definition 11.1.7. Given any scheme S, we define the category of *constructible motives* (resp. *constructible motivic complexes*) over S as the thick triangulated subcategory of $\mathrm{DM}(S,\Lambda)$ (resp. $\mathrm{DM}^{\mathit{eff}}(S,\Lambda)$) generated by $\mathcal{G}_S$ (resp. $\mathcal{G}^{eff}_S$). We denote it by $\mathrm{DM}_c(S,\Lambda)$ (resp. $\mathrm{DM}^{\mathit{eff}}_c(S,\Lambda)$).

Remark 11.1.8. Recall that $\mathrm{DM}_{c,\Lambda}$ (resp. $\mathrm{DM}^{\mathit{eff}}_{c,\Lambda}$) is Sm-fibred monoidal subcategory of DM_Λ (resp. $\mathrm{DM}^{\mathit{eff}}_\Lambda$) over $\mathscr{S}$. In other words, constructible motives (resp. motivic complexes) are stable by the operations f^*, $p_\sharp$ for p smooth and tensor product. This is obvious from definitions.

11.1.9. Let S be a scheme. Consider the triangulated subcategory $\mathscr{V}_S$ of $\mathrm{K}^b(\mathscr{S}m^{cor}_{\Lambda,S})$ generated by complexes of one the following forms :

(1) for any distinguished square $\begin{array}{ccc} W & \xrightarrow{k} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$ of smooth S-schemes,

$$[W] \xrightarrow{g_*-k_*} [U] \oplus [V] \xrightarrow{j^*+f^*} [X]$$

(2) for any smooth S-scheme X, $p : \mathbf{A}^1_X \to X$ the canonical projection.

$$[\mathbf{A}^1_X] \xrightarrow{p_*} [X].$$

Definition 11.1.10. We define the category $\mathrm{DM}^{\mathit{eff}}_{gm}(S,\Lambda)$ of *geometric effective motives* over S as the pseudo-abelian envelope of the triangulated category

$$\mathrm{K}^b(\mathscr{S}m^{cor}_{\Lambda,S})/\mathscr{V}_S.$$

We define the category $\mathrm{DM}_{gm,\Lambda}(S)$ of *geometric motives* over S as the triangulated category obtained from $\mathrm{DM}^{\mathit{eff}}_{gm}(S,\Lambda)$ by formally inverting the Tate complex

$$[\mathbf{P}^1_S] \to [S].$$

Remark 11.1.11. The categories of geometric motives (resp. effective geometric motives) over an arbitrary base, as defined here, already appears in the work of Ivorra [**Ivo07**, sec. 1.3].

11.1.12. According to this definition, we can construct for any scheme S a commutative diagram of functors:

$$(11.1.12.1)\qquad \begin{array}{ccc} \mathrm{DM}^{\mathit{eff}}_{gm}(S,\Lambda) & \longrightarrow & \mathrm{DM}^{\mathit{eff}}(S,\Lambda) \\ \downarrow & & \downarrow{\scriptstyle \Sigma^\infty} \\ \mathrm{DM}_{gm}(S,\Lambda) & \longrightarrow & \mathrm{DM}(S,\Lambda) \end{array}$$

where the right vertical map is the left adjoint of (11.1.2.1).

Recall from Remark 10.3.18 that the Nisnevich topology is bounded in $\mathrm{Sh}^{tr}(-,\Lambda)$. Thus, as a corollary of Proposition 5.2.38, Corollary 5.2.39 and Corollary 5.3.42 we get the following result:

Theorem 11.1.13. *The horizontal functors of the square* (11.1.12.1) *are fully faithful and their essential images consist of constructible objects in the sense of Definition 11.1.7.*

Given any motive (resp. motivic complex) $\mathcal{M}$ over S, the following conditions are equivalent:

*(i) $\mathcal{M}$ is geometric (*i.e. *in the image of the horizontal map of diagram* (11.1.12.1)*),*

(ii) $\mathcal{M}$ *is constructible,*

(iii) $\mathcal{M}$ *is compact.*

The triangulated category $\mathrm{DM}(S,\Lambda)$ *(resp.* $\mathrm{DM}^{\mathit{eff}}(S,\Lambda)$*) is compactly generated. More precisely, the objects of the set of generators* $\mathcal{G}_S$ *(resp.* $\mathcal{G}_S^{eff}$*) defined in Paragraph 11.1.6 are compact.*

Remark 11.1.14. If $S = \mathrm{Spec}\,(k)$ is the spectrum of a perfect field, then the categories $\mathrm{DM}_{gm}(S,\Lambda)$ and $\mathrm{DM}^{\mathit{eff}}_{gm}(S,\Lambda)$ coincide with the categories introduced by Voevodsky in [**VSF00**, chap. 5, Sec. 2.1]. The above theorem is a generalization of [**VSF00**, chap. 5, Th. 3.2.6] to an arbitrary base (and the non effective case).

11.1.c. *Enlargement, descent and continuity.*

11.1.15. We can apply the definitions of section 5 to the generalized abelian premotivic category $\underline{\mathrm{Sh}}^{tr}(-,\Lambda)$ constructed in 10.4.2

Definition 11.1.16. We define the (Λ-linear) category of *generalized motivic complexes* (resp. *generalized motives*) following definition 5.3.22 (resp. definition 5.2.16) as

$$\underline{\mathrm{DM}}^{\mathit{eff}}_{\Lambda} = \mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}\left(\underline{\mathrm{Sh}}^{tr}(-,\Lambda)\right)$$
$$\text{resp. } \underline{\mathrm{DM}}_{\Lambda} = \mathrm{D}_{\mathbf{A}^1}\left(\underline{\mathrm{Sh}}^{tr}(-,\Lambda)\right).$$

11.1.17. The advantage of this definition is that any separated S-scheme X of finite type defines a generalized motivic complex, given by the sheaf with transfers $\underline{\Lambda}^{tr}_S(X)$ seen as a complex concentrated in degree 0 (see Definition 10.4.2).

The category $\underline{\mathrm{DM}}^{\mathit{eff}}_{\Lambda}$, as a generalized premotivic category, admits the following operations:

- Given any morphism $f : T \to S$ in $\mathscr{S}$, there exists an adjunction of the form:
$$\mathbf{L}f^* : \underline{\mathrm{DM}}^{\mathit{eff}}(S,\Lambda) \rightleftarrows \underline{\mathrm{DM}}^{\mathit{eff}}(T,\Lambda) : \mathbf{R}f_*.$$
- Given a separated morphism $f : T \to S$ of finite type in $\mathscr{S}$ (non necessarily smooth), there exists an adjunction of the form:
$$\mathbf{L}f_\sharp : \underline{\mathrm{DM}}^{\mathit{eff}}(S,\Lambda) \rightleftarrows \underline{\mathrm{DM}}^{\mathit{eff}}(T,\Lambda) : f^*.$$
- Given any noetherian finite dimensional scheme S, the category $\underline{\mathrm{DM}}^{\mathit{eff}}(S,\Lambda)$ is symmetric closed monoidal.

These operations satisfies the properties of a genaralized premotivic category for which we refer the reader to section 1.4.

As in the non generalized case, we get from the general construction (see (5.3.23.2)) an adjunction of triangulated generalized premotivic categories

$$\Sigma^\infty : \underline{\mathrm{DM}}^{\mathit{eff}}_{\Lambda} \rightleftarrows \underline{\mathrm{DM}}_{\Lambda} : \Omega^\infty. \tag{11.1.17.1}$$

To any separated S-scheme X of finite type, we associate a generalized motive as:

$$\underline{M}_S(X) := \Sigma^\infty \underline{\Lambda}^{tr}_S(X).$$

By construction, the generalized premotivic category $\underline{\mathrm{DM}}^{\mathit{eff}}_{\Lambda}$ (resp. $\underline{\mathrm{DM}}_{\Lambda}$) satisfies the homotopy property, Nisnevich descent property (resp. and stability property).

11.1.18. According to Remark 10.3.18, the Nisnevich topology is bounded in $\underline{\mathrm{Sh}}^{tr}(-,\Lambda)$. Thus, as a corollary of Proposition 5.2.38 (resp. Corollary 5.2.39), we obtain in particular that $\underline{\mathrm{DM}}^{\mathit{eff}}(S,\Lambda)$ (resp. $\underline{\mathrm{DM}}(S,\Lambda)$) is compactly generated, with the essentially small family of objects $\underline{\Lambda}^{tr}_S(X)$ (resp. $\underline{M}_S(X)(n)$) for a separated S-scheme of finite type X (resp. and an integer $n \in \mathbf{Z}$) as compact generators.

Recall that for any scheme S, the obvious restriction functor $\varphi^* : \mathrm{Sh}^{tr}(S,\Lambda) \to \underline{\mathrm{Sh}}^{tr}(S,\Lambda)$ admits a left adjoint $\varphi_!$ which is fully faithful (Lemma 10.4.6). Moreover, the adjoint pair $(\varphi_!, \varphi^*)$ satisfies the assumption of Proposition 6.1.4 so that applying Corollary 6.1.9 gives the following proposition:

PROPOSITION 11.1.19. *Given any scheme S, the adjoint pair $(\varphi_!, \varphi^*)$ can be derived and induces the following pair of adjoint functors*

$$\varphi_! : \mathrm{DM}(S,\Lambda) \rightleftarrows \underline{\mathrm{DM}}(S,\Lambda) : \varphi^*, \qquad (11.1.19.1)$$
$$\textit{resp. } \varphi_! : \mathrm{DM}^{\mathit{eff}}(S,\Lambda) \rightleftarrows \underline{\mathrm{DM}}^{\mathit{eff}}(S,\Lambda) : \varphi^*,$$

such that $\varphi_!$ is fully faithful.

More generally, the family of these adjunctions for a noetherian finite dimensional scheme S defines an enlargement of premotivic categories (Definition 1.4.13).

The abuse of notations is justified because of the following essentially commutative diagram of functors:

$$\begin{array}{ccc} \mathrm{DM}^{\mathit{eff}}_\Lambda & \xrightarrow{\Sigma^\infty} & \mathrm{DM}_\Lambda \\ {\scriptstyle\varphi_!}\downarrow & & \downarrow{\scriptstyle\varphi_!} \\ \underline{\mathrm{DM}}^{\mathit{eff}}_\Lambda & \xrightarrow{\Sigma^\infty} & \underline{\mathrm{DM}}_\Lambda \end{array} \qquad (11.1.19.2)$$

Recall that, given a smooth separated S-scheme X, we have the relation:

$$\varphi_!(M_S(X)) = \underline{M}_S(X).$$

REMARK 11.1.20. Beware that the functor φ^* is far from being conservative. The following example was suggested by V.Vologodsky: let Z be a nowhere dense closed subscheme of S. Then $\varphi^*(\underline{M}_S(Z)) = 0$. In fact, one can see that $\mathrm{DM}(S,\Lambda)$ is a localization of the category $\underline{\mathrm{DM}}(S,\Lambda)$ with respect to the objects $\underline{\mathscr{M}}$ such that $\varphi^*(\underline{\mathscr{M}}) = 0$.

11.1.21. With rational coefficients, the preceding proposition can be refined. Recall that the qfh-sheafification functor (10.5.8) induces by 5.3.28 a premotivic adjunction

$$\underline{\alpha}^* : \underline{\mathrm{DM}}_{\mathbf{Q}} \rightleftarrows \underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}} : \underline{\alpha}_* .$$

THEOREM 11.1.22. *If S is a geometrically unibranch excellent noetherian scheme of finite dimension then the following composite functor*

$$\underline{\alpha}^*\varphi_! : \mathrm{DM}(S,\mathbf{Q}) \to \underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}(S)$$

is fully faithful.

PROOF. Note that $\mathrm{DM}^{\mathit{eff}}(S,\mathbf{Q})$ and $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1}(\mathrm{Sh}_{\mathrm{qfh}}(S,\mathbf{Q}))$ are compactly generated; see example 5.1.29 and Proposition 5.2.38. Hence this corollary follows from Theorem 10.5.14 and Proposition 6.1.8. □

REMARK 11.1.23. Recall this theorem can be rephrased by saying that motives over S satisfies qfh-descent – see Remark 5.2.11 and more generally Section 3. In the next section, we will give applications of this fact to motivic cohomology.

THEOREM 11.1.24. *The following assertions hold:*

(1) *The triangulated premotivic categories $\mathrm{DM}^{\mathit{eff}}_\Lambda$ and DM_Λ are continuous when we restrict ourself to pro-schemes whose transition maps are affine and dominant.*
(2) *The generalized triangulated premotivic categories $\underline{\mathrm{DM}}^{\mathit{eff}}_\Lambda$ and $\underline{\mathrm{DM}}_\Lambda$ are continuous with the same restriction on pro-schemes.*

PROOF. Note that Proposition 10.3.19 shows precisely that the generalized premotivic abelian category $\underline{\mathrm{Sh}}^{tr}(-,\Lambda)$ satisfies Property (C) of Paragraph 5.1.35. Therefore the assertion (2) follows from Propositions 5.2.41 and 5.3.44.[84]

Moreover, the assertion (1) follows from Corollary 6.1.12 given the enlargement obtained in Proposition 11.1.19. □

[84]These propositions are also true with the restriction on pro-schemes considered in the statement of the Theorem.

Example 11.1.25. From the previous theorem and Proposition 4.3.4, we obtain in particular that for any pro-scheme $(S_\alpha)_{\alpha\in A}$ with affine and dominant transition map such that $S = \varprojlim_{\alpha\in A} S_\alpha$ is noetherian finite dimensional, there exists canonical equivalences of categories:

$$2\text{-}\varinjlim_{\alpha} \big(\mathrm{DM}^{\mathit{eff}}_{gm,\Lambda}(S_\alpha)\big) \to \mathrm{DM}^{\mathit{eff}}_{gm,\Lambda}(S),$$

$$2\text{-}\varinjlim_{\alpha} \big(\mathrm{DM}_{gm,\Lambda}(S_\alpha)\big) \to \mathrm{DM}_{gm,\Lambda}(S).$$

This result generalizes [**Ivo07**, 4.16].

11.2. Motivic cohomology.

11.2.a. *Definition and functoriality.*

Definition 11.2.1. Let S be a scheme and $(n, m) \in \mathbf{Z}^2$ a couple of integer. We define the *motivic cohomology* of S in degree n and twist m with coefficients in Λ as the Λ-module

$$H^{n,m}_{\mathcal{M}}(S,\Lambda) = \mathrm{Hom}_{\mathrm{DM}(S,\Lambda)}\big(\mathbb{1}_S, \mathbb{1}_S(m)[n]\big).$$

Assuming $m \geq 0$, we define the *effective motivic cohomology* of S in degree n and twist m with coefficients in Λ as the Λ-module

$$H^{n,m}_{\mathcal{M},\mathit{eff}}(S,\Lambda) = \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S,\Lambda)}\big(\Lambda^{tr}_S, \Lambda^{tr}_S(m)[n]\big).$$

Motivic cohomology (resp. effective motivic cohomology) is contravariant with respect to morphisms of schemes and the monoidal structure on DM_Λ (resp. $\mathrm{DM}^{\mathit{eff}}_\Lambda$) defines a ring structure compatible with pullbacks: given two cohomology classes:

$$\alpha : \mathbb{1}_S \to \mathbb{1}_S(m)[n], \alpha' : \mathbb{1}_S \to \mathbb{1}_S(m')[n'],$$

one simply put:

$$\alpha.\alpha' = \alpha \otimes_S \alpha'.$$

The link between motivic cohomology and effective motivic cohomology is provided by Proposition 5.3.39. Given any scheme S and any couple of integers $(n, m) \in \mathbf{Z}^2$, one has a canonical identification:

$$H^{n,m}_{\mathcal{M}}(S,\Lambda) = \varinjlim_{r>>0} \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S,\Lambda)}\big(\Lambda^{tr}_S(r), \Lambda^{tr}_S(m+r)[n]\big).$$

11.2.2. Let Λ' be a localization of Λ. Then using the left adjoint of the premotivic adjunction (11.1.4.2), we get a canonical morphism

$$H^{n,m}_{\mathcal{M}}(S,\Lambda) \otimes_\Lambda \Lambda' \to H^{n,m}_{\mathcal{M}}(S,\Lambda').$$

It is obviously compatible with pullbacks and the product structure. According to Proposition 11.1.5, this map is an isomorphism (even an identity) when one the following conditions are fulfilled:

- S is a regular schemes.
- $\mathrm{char}(S) \subset \Lambda^\times$.

Example 11.2.3. Let k be a perfect field. Given any smooth separated k-scheme S of finite type, with structural morphism f, and any pair of integers $(n, m) \in \mathbf{Z}^2$, motivic cohomology as defined in the previous definition coincide with motivic cohomology as defined by Voevodsky in [**VSF00**, chap. 5] according to the following computation and Remark 11.1.14:

$$\begin{aligned} H^{n,m}_{\mathcal{M}}(X,\mathbf{Z}) &= \mathrm{Hom}_{\mathrm{DM}(X,\mathbf{Z})}(\mathbb{1}_X, \mathbb{1}_X(m)[n]) = \mathrm{Hom}_{\mathrm{DM}(X,\mathbf{Z})}(\mathbb{1}_X, f^*(\mathbb{1}_k)(m)[n]) \\ &= \mathrm{Hom}_{\mathrm{DM}(k,\mathbf{Z})}(\mathbf{L}f_\sharp(\mathbb{1}_X), \mathbb{1}_k(m)[n]) = \mathrm{Hom}_{\mathrm{DM}(k,\mathbf{Z})}(M_k(X), \mathbb{1}_k(m)[n]) \\ &= \mathrm{Hom}_{\mathrm{DM}_{gm}(k,\mathbf{Z})}(M_k(X), \mathbb{1}_k(m)[n]). \end{aligned}$$

In particular, it coincides with higher Chow groups (cf [**Voe02a**]) according to the following formula:

$$H^{n,m}_{\mathcal{M}}(X,\mathbf{Z}) = CH^m(X, 2m-n).$$

Recall in particular the following computations:

$$H^{n,m}_{\mathcal{M}}(X,\mathbf{Z})=\begin{cases}\mathbf{Z}^{\pi_0(X)} & \text{if } n=m=0,\\ \mathbf{G}_m(X) & \text{if } n=m=1,\\ CH^m(X) & \text{if } n=2m,\\ 0 & \text{if } m<0, n>\min(m+\dim(X),2m)\end{cases}$$

where $CH^m(X)$ is the usual Chow group of m-codimensional cycles in X.

Note we will extend the identification of motivic cohomology as defined in the previous definition with the general version defined by Voevodsky – [**Voe98**] – in section 11.2.c.

11.2.4. Consider a separated morphism $p: X\to S$ of finite type. Recall from the $\mathscr{S}^{ft}$-fibred structure of $\underline{\mathrm{DM}}_\Lambda$ that $\underline{M}_S(X)=\mathbf{L}p_\sharp p^*(\mathbb{1}_S)$. Using the adjunction property of the pair $(\mathbf{L}p_\sharp, p^*)$, we easily get:

$$\begin{aligned}H^{n,m}_{\mathcal{M}}(X,\Lambda) &= \mathrm{Hom}_{\mathrm{DM}(X,\Lambda)}\big(\mathbb{1}_X,\mathbb{1}_X(m)[n]\big)=\mathrm{Hom}_{\underline{\mathrm{DM}}(X,\Lambda)}\big(\mathbb{1}_X,\mathbb{1}_X(m)[n]\big)\\ &= \mathrm{Hom}_{\underline{\mathrm{DM}}(S,\Lambda)}\big(\underline{M}_S(X),\mathbb{1}_S(m)[n]\big).\end{aligned}\tag{11.2.4.1}$$

In particular, given any finite S-correspondence $\alpha: X\bullet\!\!\longrightarrow Y$ between separated S-schemes of finite type, we obtain a pullback

$$\alpha^*: H^{n,m}_{\mathcal{M}}(Y,\Lambda)\to H^{n,m}_{\mathcal{M}}(X,\Lambda)$$

which is, among other properties, functorial with respect to composition of finite S-correspondences and extends the natural contravariant functoriality of motivic cohomology.

In particular, given any finite Λ-universal morphism $f: Y\to X$, we obtain a pushout

$$f_*: H^{n,m}_{\mathcal{M}}(Y,\Lambda)\to H^{n,m}_{\mathcal{M}}(X,\Lambda)$$

by considering the transpose of the graph of f.

PROPOSITION 11.2.5. *Let $f: Y\to X$ be a finite Λ-universal morphism of schemes. Assume X is connected and let $d>0$ be the degree of f (cf. 9.1.12). Then for any element $x\in H^{n,m}_{\mathcal{M}}(X,\Lambda)$, $f_*f^*(x)=d.x$.*

This is a simple application of Proposition 9.1.13. We left to the reader the exercice to state projection and base change formulas for this pushout.

EXAMPLE 11.2.6. Let $f: Y\to X$ be a finite morphism. Recall that f is Λ-universal in the following particular cases:

- f is flat (see Example 8.1.49);
- X is regular and f sends the generic points of Y to generic points of X (see Corollary 8.3.28).

In particular, motivic cohomology is covariant with respect to this kind of finite morphisms.

Another important application of the generalized motives is obtained using the Corollary 10.5.7:

PROPOSITION 11.2.7. *Let $f: X'\to X$ be a separated universal homeomorphism of finite type. Assume that* $\mathrm{char}(X)\subset\Lambda^\times$. *Then the pullback functor*

$$H^{n,m}_{\mathcal{M}}(X,\Lambda)\to H^{n,m}_{\mathcal{M}}(X',\Lambda)$$

is an isomorphism.

REMARK 11.2.8. The preceding considerations hold similarly for the effective motivic cohomology.

EXAMPLE 11.2.9. In characteristic 0, motivic cohomology (effective and non effective) is invariant under seminormalization ([**Swa80**]).

When restricted to excellent geometrically unibranch scheme X, motivic cohomology (effective and non effective) is invariant under normalization. Indeed, the normalization $X'\to X$ of such a scheme is a universal homeomorphism ([**EGA4**, IV_0, 23.2.2]) of finite type.

11.2.b. *Effective motivic cohomology in weight* 0 *and* 1.

11.2.10. Let S be a scheme and X a smooth S-scheme. For any subscheme Y of X, we denote by $\Lambda_S^{tr}(X/Y)$ the cokernel of the canonical morphism of sheaf with transfers $\Lambda_S^{tr}(Y) \to \Lambda_S^{tr}(X)$. As this morphism is a monomorphism, we obtain a canonical distinguished triangle in $\mathrm{DM}^{\mathit{eff}}(S,\Lambda)$

$$\Lambda_S^{tr}(Y) \to \Lambda_S^{tr}(X) \to \Lambda_S^{tr}(X/Y) \to \Lambda_S^{tr}(X)[1].$$

Using this notation and according to Definition 2.4.17, the Tate motivic complex is defined as: $\Lambda_S^{tr}(1) = \Lambda_S^{tr}(\mathbf{P}^1_S/\{\infty\})[-2]$.

The following computation is classical:

$$\Lambda_S^{tr}(1) = \Lambda_S^{tr}(\mathbf{P}^1_S/\mathbf{A}^1_S)[-2] = \Lambda_S^{tr}(\mathbf{A}^1_S/\mathbf{G}_m)[-2];$$

the first identification follows from homotopy invariance and the second one by Nisnevich descent (cf. Prop. 5.2.13).

Proposition 11.2.11. *Suppose S is a normal scheme.*

Then the sheaf on Sm_S represented by $\mathbf{G}_m$ admits a canonical structure of a sheaf with transfers and there is a canonical isomorphism in $\mathrm{DM}^{\mathit{eff}}(S,\Lambda)$:

$$\mathbf{G}_m \otimes_{\mathbf{Z}} \Lambda \xrightarrow{\simeq} \Lambda_S^{tr}(1)[1].$$

Proof. Let U be an open subscheme of $\mathbf{A}^1_S$ and X be a smooth S-scheme. Note that X is normal according to [**EGA4**, 18.10.7]. Consider a cycle

$$\alpha = \sum_i n_i.\langle Z_i \rangle$$

of $X \times_S U$ with $n_i \in \Lambda$ and Z_i irreducible finite and dominant over an irreducible component of X. Then Z_i is a divisor in $X \times_S U$ and according to [**EGA4**, 21.14.3], it is flat over X. In other words, α is a Hilbert bert cycle which implies it is Λ-universal (Example 8.1.49). As a consequence, we obtain the equality

$$H^i\Gamma(X; \underline{C}^*\Lambda_S^{tr}(U)) = H_{-i}^{sing}(X \times_S U/X) \otimes_{\mathbf{Z}} \Lambda$$

where the functor $\underline{C}^*$ is the associated Suslin singular complex (see (5.2.32.1)) and the right hand side is the Suslin homology of $X \times_S U/X$ (*cf.* [**SV00b**]).

Suppose in addition that X and U are affine and let $Z = \mathbf{P}^1_S - U$. According to a theorem of Suslin and Voevodsky (*cf.* [**SV00b**, th. 3.1]),

$$H_{-i}^{sing}(X \times_S U/X) = \begin{cases} \mathrm{Pic}(X \times_S \mathbf{P}^1_S, X \times_S Z) & \text{if } i = 0 \\ 0 & \text{otherwise;} \end{cases}$$

the group on the left hand side is the *relative Picard group*. In particular, the complex $\underline{C}^*\Lambda_S^{tr}(U)$, seen as a complex of presheaves with transfers, is concentrated in cohomological degree 0 and its 0-th cohomology is the presheaf $X \mapsto \mathrm{Pic}(X \times_S \mathbf{P}^1_S, X \times_S Z) \otimes_{\mathbf{Z}} \Lambda$.

Consider the following exact sequence of presheaves with transfers:

$$0 \to \Lambda_S^{tr}(\mathbf{G}_m) \to \Lambda_S^{tr}(\mathbf{A}^1_S) \to \tilde{\Lambda}_S^{tr}(\mathbf{A}^1_S/\mathbf{G}_m) \to 0.$$

Applying the functor $\underline{C}^*$ to it, relatively to the category of complexes of presheaves with transfers, we obtain a distinguished triangle in $\mathrm{D}(\mathrm{PSh}^{tr}(S,\Lambda))$:

$$\underline{C}^*\Lambda_S^{tr}(\mathbf{G}_m) \to \underline{C}^*\Lambda_S^{tr}(\mathbf{A}^1_S) \to \underline{C}^*\tilde{\Lambda}_S^{tr}(\mathbf{A}^1_S/\mathbf{G}_m) \xrightarrow{+1} \underline{C}^*\Lambda_S^{tr}(\mathbf{G}_m).$$

Taking the associated long exact sequence of cohomology presheaves, we obtain that the complex of presheaves with transfers $\underline{C}^*\tilde{\Lambda}_S^{tr}(\mathbf{A}^1_S/\mathbf{G}_m)$ is concentrated in cohomological degree 0 and -1, and we get an exact sequence of presheaves:

$$0 \to \hat{H}^{-1}[\underline{C}^*\tilde{\Lambda}_S^{tr}(\mathbf{A}^1_S/\mathbf{G}_m)] \to \hat{H}^0[\underline{C}^*\Lambda_S^{tr}(\mathbf{G}_m)] \to \hat{H}^0[\underline{C}^*\Lambda_S^{tr}(\mathbf{A}^1_S)] \to \hat{H}^0[\underline{C}^*\tilde{\Lambda}_S^{tr}(\mathbf{A}^1_S/\mathbf{G}_m)] \to 0.$$

By definition of the relative Picard group, given any smooth (affine) scheme X, we get an exact sequence of abelian groups:

$$0 \to \mathbf{G}_m(X) \to \mathrm{Pic}(X \times_S \mathbf{P}^1_S, X_0 \sqcup X_\infty) \to \mathrm{Pic}(X \times_S \mathbf{P}^1_S, X_0) \to 0. \tag{11.2.11.1}$$

Thus we deduce that:

$$\hat{H}^0[\underline{C}^*\tilde{\Lambda}^{tr}_S(\mathbf{A}^1_S/\mathbf{G}_m)] = 0,$$
$$\hat{H}^{-1}[\underline{C}^*\tilde{\Lambda}^{tr}_S(\mathbf{A}^1_S/\mathbf{G}_m)] = \mathbf{G}_m \otimes_{\mathbf{Z}} \Lambda.$$

This gives in particular a canonical isomorphism:

$$\underline{C}^*\tilde{\Lambda}^{tr}_S(\mathbf{A}^1_S/\mathbf{G}_m)[-1] \simeq \mathbf{G}_m \otimes_{\mathbf{Z}} \Lambda$$

in $\mathrm{D}(\mathrm{PSh}^{tr}(S,\Lambda))$. Taking its image in $\mathrm{DM}^{eff}(S,\Lambda)$ we obtain a canonical isomorphism which can be written as:

$$\underline{C}^*\Lambda^{tr}_S(\mathbf{A}^1_S/\mathbf{G}_m)[-1] \simeq \mathbf{G}_m \otimes_{\mathbf{Z}} \Lambda.$$

Thus we can conclude because, according to Lemma 5.2.35, the canonical map

$$\Lambda^{tr}_S(\mathbf{A}^1_S/\mathbf{G}_m) \to \underline{C}^*\Lambda^{tr}_S(\mathbf{A}^1_S/\mathbf{G}_m)$$

is an isomorphism in $\mathrm{DM}^{eff}(S,\Lambda)$. □

Remark 11.2.12. In the course of the proof, a canonical structure of a sheaf with transfers over S on $\mathbf{G}_m$ has naturally appeared – described by the exact sequence (11.2.11.1). This structure is classical (see [**MVW06**, Ex. 2.4]). One can describe it as follows.

Let X and Y be smooth S-schemes. Assume X is connected (thus irreducible as it is normal). Let Z be a closed integral subscheme Z of $X \times_S Y$ which is finite surjective over X. Then Z/X corresponds to an extension of function fields L/K. The norm map of L/K induces a morphism of abelian groups: $N_{Z/X} : \mathbf{G}_m(Z) \to \mathbf{G}_m(X)$. Then we associate with Z, seen as a finite correspondence from X to Y, the following morphism:

$$\mathbf{G}_m(Y) \xrightarrow{p^*} \mathbf{G}_m(Z) \xrightarrow{N_{Z/X}} \mathbf{G}_m(X)$$

where $p : Y \to Z$ is the natural projection.

The following proposition is well known to the expert. We include a proof for completeness.

Proposition 11.2.13. *For any regular scheme X and any interger $i \geq 0$,*

$$H^i_{\mathrm{Nis}}(X,\mathbf{G}_m) = \begin{cases} \mathcal{O}_X(X)^\times & \text{if } i=0, \\ \mathrm{Pic}(X) & \text{if } i=1, \\ 0 & \text{otherwise} \end{cases}$$

where $\mathrm{Pic}(X)$ *is the Picard group of* X*.*

Proof. Let Y be any étale scheme over X. We let $C^0(V)$ be the abelian group made by the invertible rational functions on V and $C^1(V)$ be the group of 1-codimensional cycles in V. Classically, one associates with any rational function f on V its Weil divisor $\mathrm{div}(f) \in C^1(V)$. Recall, when V is integral with function field K, $f \in K$, one puts:

$$\mathrm{div}_V(f) = \sum_{x \in V^{(1)}} v_x(f).x;$$

the sum runs over the points of codimension 1 in V and $v_x(f)$ is the valuation of f corresponding to the valuation ring $\mathcal{O}_{X,x}$.

According to this definition, we get a complex:

$$0 \to \mathbf{G}_m(V) \to C^0(V) \xrightarrow{\mathrm{div}_V} C^1(V).$$

This sequence is functorial with respect to pullback of étale X-schemes. Thus we have defined a morphism of presheaves on $X_{\text{ét}}$:

$$\pi : \mathbf{G}_m \to C^*.$$

Given any Nisnevich distinguished square Q (Example 2.1.11), one can check easily that the image of Q by C^0 (resp. C^1) is cocartesian. As a consequence C^* is a complex of Nisnevich sheaves which satisfies the Brown-Gersten property – *i.e.* it is Nisnevich flasque in the sense of Definition 5.1.9 according to Proposition 5.2.13 applied to the derived category of Nisnevich sheaves over X.

On the other hand, π is a quasi-isomorphism of Nisnevich sheaves over S: indeed it is well known that for any regular local ring A, the sequence

$$0 \to A^\times \to \mathrm{Frac}(A)^\times \xrightarrow{\mathrm{div}_A} Z^1(A) \to 0$$

is exact. This is an easy consequence of the fact A is a unique factorization domain – the classical Auslander-Buchsbaum theorem, (e.g. [**Mat70**, 20.3]).

In particular, we get $H^i(X, \mathbf{G}_m) = H^i(C^*(X))$ and this concludes. □

The following theorem is a generalization of a well-known computation of Voevodsky for smooth schemes over a perfect field. The second case is a corollary of the two preceding propositions.

THEOREM 11.2.14. *Let S be a scheme and $n \in \mathbf{Z}$ an integer. The following computation holds:*

(1)

$$H^{n,0}_{\mathcal{M},\mathit{eff}}(S,\Lambda) = \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\Lambda_S^{tr}, \Lambda_S^{tr}[n]) = \begin{cases} \Lambda^{\pi_0(S)} & \text{if } n=0 \\ 0 & \text{otherwise;} \end{cases}$$

(2) if S is regular,

$$H^{n,1}_{\mathcal{M},\mathit{eff}}(S,\Lambda) = \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\Lambda_S^{tr}, \Lambda_S^{tr}(1)[n]) = \begin{cases} \mathcal{O}_S(S)^\times \otimes_{\mathbf{Z}} \Lambda & \text{if } n=1 \\ \mathrm{Pic}(S) \otimes_{\mathbf{Z}} \Lambda & \text{if } n=2 \\ 0 & \text{otherwise} \end{cases}$$

PROOF. For the first case, according to Proposition 10.2.5, the sheaf Λ_S^{tr} is Nisnevich local and $\mathbf{A}^1$-local as a complex of sheaves. It is obviously acyclic for the Nisnevich topology. Thus, we conclude using again 10.2.5 in the case $n = 0$.

Consider the second case. According to Proposition 11.2.13, the sheaf $\mathbf{G}_m$ on Sm_S is $\mathbf{A}^1$-local. Thus according to Proposition 11.2.11 $\mathbf{G}_m \otimes \Lambda[-1]$ is an $\mathbf{A}^1$-resolution of $\Lambda_S^{tr}(1)$. In particular,

$$\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\Lambda_S^{tr}, \Lambda_S^{tr}(1)[n]) = \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{tr}(S,\Lambda))}(\Lambda_S^{tr}, \mathbf{G}_m \otimes \Lambda[n-1]) = H^{n-1}_{\mathrm{Nis}}(S, \mathbf{G}_m) \otimes \Lambda$$

where the second identification follows from Remark 10.4.4. The conclusion follows from another application of Proposition 11.2.13. □

The following corollary is a (very) weak cancellation result in $\mathrm{DM}^{\mathit{eff}}(S)$:

COROLLARY 11.2.15. *Let S be a regular scheme. Then*

$$\mathbf{R}\mathit{Hom}(\Lambda_S^{tr}(1), \Lambda_S^{tr}(1)) = \Lambda_S^{tr}.$$

Moreover, if $m = 0$ or $m = 1$, for any integer $n > m$,

$$\mathbf{R}\mathit{Hom}(\Lambda_S^{tr}(n), \Lambda_S^{tr}(m)) = 0.$$

PROOF. We consider the first assertion. Any smooth S-scheme is regular. Hence it is sufficient to prove that for any connected regular scheme S, for any integer $n \in \mathbf{Z}$,

$$\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\Lambda_S^{tr}(1), \Lambda_S^{tr}(1)[n]) = \begin{cases} \Lambda & \text{if } n=0 \\ 0 & \text{otherwise.} \end{cases}$$

Using the exact triangle

$$\Lambda_S^{tr}(\mathbf{G}_m) \to \Lambda_S^{tr}(\mathbf{A}^1) \to \Lambda_S^{tr}(1)[2] \xrightarrow{+1} \qquad (11.2.15.1)$$

and the second case of the previous theorem, we obtain the following long exact sequence

$$\cdots \to \mathrm{Hom}(\Lambda_S^{tr}(\mathbf{A}^1), \Lambda_S^{tr}(1)[n]) \to \mathrm{Hom}(\Lambda_S^{tr}(\mathbf{G}_m), \Lambda_S^{tr}(1)[n])$$
$$\to \mathrm{Hom}(\Lambda_S^{tr}(1), \Lambda_S^{tr}(1)[n-1]) \to \mathrm{Hom}(\Lambda_S^{tr}(\mathbf{A}^1), \Lambda_S^{tr}(1)[n+1]) \to \cdots$$

Then we conclude using the previous theorem and the fact

$$\mathrm{Pic}(\mathbf{A}^1 \times S) = \mathrm{Pic}(\mathbf{G}_m \times S)$$

whenever S is regular.

For the last assertion, we are reduced to prove that if S is a regular scheme, for any integers $n > 0$ and i,

$$\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S)}(\Lambda_S^{tr}(n), \Lambda_S^{tr}[i]) = 0.$$

This is obviously implied by the case $n = 1$.

Consider the distinguished triangle (11.2.15.1) again. Then the long exact sequence attached to the cohomological functor $\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S,\Lambda)}(-, \Lambda_S^{tr})$ and applied to this triangle together with the first case of the previous theorem allows us to conclude. □

11.2.c. *The motivic cohomology ring spectrum.*

11.2.16. According to definition 10.4.2 and paragraph 10.4.3, we have an adjunction of abelian premotivic categories

$$\gamma^* : \mathrm{Sh}(-, \Lambda) \rightleftarrows \mathrm{Sh}^{tr}(-, \Lambda) : \gamma_*$$

such that γ_* is conservative and exact. According to Paragraph 5.3.28, it induces an adjunction of triangulated premotivic categories

$$\mathbf{L}\gamma^* : \mathrm{D}_{\mathbf{A}^1,\Lambda} \rightleftarrows \mathrm{DM}_\Lambda : \mathbf{R}\gamma_*. \tag{11.2.16.1}$$

Composing with the premotivic adjunction between the stable homotopy category and the $\mathscr{A}^1$-derived homtopoy category (5.3.35.1), we finally get a canonical premotivic adjunction:

$$\varphi^* : \mathrm{SH} \rightleftarrows \mathrm{DM}_\Lambda : \varphi_*. \tag{11.2.16.2}$$

Recall that, because φ^* is monoidal, φ_* is weakly monoidal. In particular, for any scheme S, one gets canonical morphisms

$$\mathbb{1}_S \to \varphi_*(\mathbb{1}_S), \varphi_*(\mathbb{1}_S) \wedge \varphi_*(\mathbb{1}_S) \to \varphi_*(\mathbb{1}_S)$$

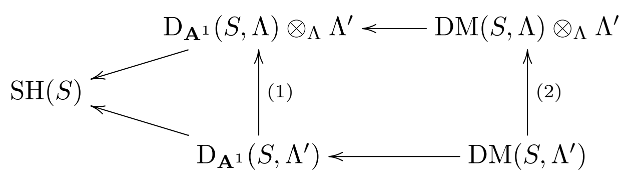
which gives a structure of a commutative monoid to the spectrum $\varphi_*(\mathbb{1}_S)$ *i.e.* a ring spectrum.

Definition 11.2.17. Given any scheme S, one defines the *motivic cohomology ring spectrum* over S with coefficients in Λ as the commutative ring spectrum:

$$\mathrm{H}^\Lambda_{\mathcal{M},S} := \varphi_*(\mathbb{1}_S).$$

The properties of the functor φ_* immediately implies that the ring spectrum $\mathrm{H}^\Lambda_{\mathcal{M},S}$ represents motivic cohomology. One easily checks now that this ring specutrm coincides with the original one of Voevodsky (see [**Voe98**, sec. 6.1]) – in the case $\Lambda = \mathbf{Z}$. Therefore, our definition of motivic cohomology (with $\mathbf{Z}$-coefficients) agrees with that given by Voevodsky in *loc. cit.*

11.2.18. Consider a localization Λ' of Λ. Then one gets an essentially commutative diagram of premotivic adjunctions:



where the map (1) is the canonical equivalence and the map (2) is the left adjoint of (11.1.4.2) (in the stable case). Note that (2) is weakly monoidal. Thus this essentially commutative diagram defines a canonical morphism of ring spectra:

$$\mathrm{H}^\Lambda_{\mathcal{M},S} \otimes_\Lambda \Lambda' \to \mathrm{H}^{\Lambda'}_{\mathcal{M},S}. \tag{11.2.18.1}$$

As a corollary of Proposition 11.1.5, we get the following result:

Proposition 11.2.19. *If S ir regular or* $\mathrm{char}(S) \subset \Lambda^\times$ *then the map* (11.2.18.1) *is an isomorphism.*

Remark 11.2.20. We do not know what is the nature of the map (11.2.18.1) in the case of a general scheme S.

11.2.21. Let $f : T \to S$ be a morphism of schemes. Recall from the structure of the premotivic adjunction (φ^*, φ_*) defined above that we get an exchange morphism:

$$f^*\varphi_* \to \varphi_* f^*$$

Applying this natural transformation to the unit object $\mathbb{1}_S$ of $\mathrm{DM}(S, \Lambda)$, one gets a canonical morphism of ring spectra:

$$\tau_f : f^*(\mathrm{H}^\Lambda_{\mathcal{M},S}) \to \mathrm{H}^\Lambda_{\mathcal{M},T}.$$

Remark that this shows the collection $(\mathrm{H}^\Lambda_{\mathcal{M},S})$ is a section of the fibred category SH. Recall also the following conjecture of Voevodsky ([**Voe02b**, conj. 17]):

Conjecture. *For any morphism f as above, the map τ_f is an isomorphism.*

At least, Voevodsky formulated this conjecture in the case where $\Lambda = \mathbf{Z}$. As we have warn the reader in remark 11.2.20, the case $\Lambda = \mathbf{Z}$ does not necessarily implies the case of an arbitrary ring $\Lambda \subset \mathbf{Q}$.

Remark 11.2.22. We will solve affirmatively a particular case of this conjecture in 16.1.7.

11.3. Orientation and purity.

11.3.1. For any scheme S, we let $\mathbf{P}^\infty_S$ be the ind-scheme

$$S \to \mathbf{P}^1_S \to \cdots \to \mathbf{P}^n_S \to \mathbf{P}^{n+1}_S \to$$

made of the obvious closed immersions. This ind-scheme has a comultiplication given by the Segre embeddings

$$\mathbf{P}^\infty_S \times_S \mathbf{P}^\infty_S \to \mathbf{P}^\infty_S$$

Define $\Lambda^{tr}_S(\mathbf{P}^\infty) = \varinjlim \Lambda^{tr}_S(\mathbf{P}^n)$. Applying Theorem 11.2.14 in the case $S = \mathrm{Spec}\,(\mathbf{Z})$, we obtain a canonical isomorphism:

$$\mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(\mathrm{Spec}(\mathbf{Z}),\Lambda)}(\Lambda^{tr}(\mathbf{P}^\infty), \Lambda^{tr}(1)[2]) = \mathrm{Pic}(\mathbf{P}^\infty) \otimes_{\mathbf{Z}} \Lambda,$$

whose aim is a free Λ-algebra of power series in one variable. Considering the canonical dual invertible sheaf on $\mathbf{P}^\infty$, we obtain a canonical formal generator of this Λ-algebra and thus a morphism $\mathrm{DM}^{\mathit{eff}}(\mathrm{Spec}\,(\mathbf{Z})\,, \Lambda)$:

$$\mathfrak{c}_1 : \Lambda^{tr}(\mathbf{P}^\infty) \to \Lambda^{tr}(1)[2].$$

For any scheme S, considering the canonical projection $f : S \to \mathrm{Spec}\,(\mathbf{Z})$, we obtain by pullback along f a morphism of $\mathrm{DM}^{\mathit{eff}}(S, \Lambda)$

$$\mathfrak{c}_{1,S} : \Lambda^{tr}_S(\mathbf{P}^\infty_S) \to \Lambda^{tr}_S(1)[2].$$

Consider $\mathbf{G}_m$ as a sheaf of groups over Sm_S. Following [**MV99**, part 4], we introduce its classifying space $B\mathbf{G}_m$ as a simplicial sheaf over Sm_S. From proposition 1.16 of *loc. cit.*, we get $\mathrm{Hom}_{\mathscr{H}^s_\bullet(S)}(S_+, B\mathbf{G}_m) = \mathrm{Pic}(S)$. Moreover, in the homotopy category of pointed simplicial sheaves $\mathscr{H}_\bullet(S)$, we have a canonical isomorphism $B\mathbf{G}_m \simeq \mathbf{P}^\infty_S$ (*cf. loc. cit.*, prop. 3.7). Thus finally, we obtain a canonical map of pointed sets

$$\begin{aligned}\mathrm{Pic}(S) = \mathrm{Hom}_{\mathscr{H}^s_\bullet(S)}(S_+, B\mathbf{G}_m) &\to \mathrm{Hom}_{\mathscr{H}_\bullet(S)}(S_+, \mathbf{P}^\infty)\\ &\to \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S,\Lambda)}(\Lambda^{tr}_S, \Lambda^{tr}_S(\mathbf{P}^\infty/*)) \to \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S,\Lambda)}(\Lambda^{tr}_S, \Lambda^{tr}_S(\mathbf{P}^\infty)).\end{aligned}$$

Definition 11.3.2. Consider the above notations. We define the first motivic Chern class as the following composite morphism

$$\begin{aligned}c_1 : \mathrm{Pic}(S) &\longrightarrow \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S,\Lambda)}(\Lambda^{tr}_S, \Lambda^{tr}_S(\mathbf{P}^\infty_S)) \xrightarrow{(\mathfrak{c}_{1,S})_*} \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S,\Lambda)}(\Lambda^{tr}_S, \Lambda^{tr}_S(1)[2])\\ &\longrightarrow \mathrm{Hom}_{\mathrm{DM}(S,\Lambda)}(\mathbb{1}_S, \mathbb{1}_S(1)[2]) = H^{2,1}_{\mathcal{M}}(S, \Lambda)\end{aligned}$$

The first motivic Chern class is evidently compatible with pullback.

REMARK 11.3.3. Beware that the map

$$\mathrm{Pic}(S) \to \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S,\Lambda)}(\Lambda_S^{tr}, \Lambda_S^{tr}(\mathbf{P}_S^\infty))$$

defined above is not necessarily a morphism of abelian groups. However, the composite:

$$\mathrm{Pic}(S) \longrightarrow \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S,\Lambda)}(\Lambda_S^{tr}, \Lambda_S^{tr}(\mathbf{P}_S^\infty)) \xrightarrow{(\mathfrak{c}_{1,S})_*} \mathrm{Hom}_{\mathrm{DM}^{\mathit{eff}}(S,\Lambda)}(\Lambda_S^{tr}, \Lambda_S^{tr}(1)[2])$$

is the isomorphism of Theorem 11.2.14 when S is normal. In particular, it is a morphism of abelian groups in this case. We will give an argument below for the general case.

11.3.4. The triangulated category $\mathrm{DM}(S,\Lambda)$ thus satisfies all the axioms of [**Dég08**, 2.1] (see also 2.3.1 of *loc. cit.* in the regular case). In particular, we derive from the main results of *loc. cit.* the following facts:

(1) Let $p : P \to S$ be a projective bundle of rank n. Let $c : \mathbb{1}_S \to \mathbb{1}_S(1)[2]$ be the first Chern class of the canonical line bundle on P. Then the map

$$M_S(P) \xrightarrow{\sum_i p\otimes c^i} \bigoplus_{i=0}^{n} \mathbb{1}_S(i)[2i]$$

is an isomorphism. This gives the projective bundle theorem in motivic cohomology for any base scheme.

One deduces using the method of Grothendieck that motivic cohomology possesses Chern classes of vector bundles which satisfies all the usual properties (see remark below for additivity).

(2) Let $i : Z \to X$ be a closed immersion between smooth separated S-schemes of finite type. Assume i has pure codimension c and let j be the complementary open immersion. Then there is a canonical *purity isomorphism*:

$$\mathfrak{p}_{X,Z} : M_S(X/X-Z) \to M_S(Z)(c)[2c].$$

This defines in particular the *Gysin triangle*

$$M_S(X-Z) \xrightarrow{j_*} M_S(X) \xrightarrow{i^*} M_S(Z)(c)[2c] \xrightarrow{\partial_{X,Z}} M_S(X-Z)[1].$$

(3) Let $f : Y \to X$ be a projective morphism between smooth separated S-schemes of finite type. Assume f has pure relative dimension d. Then there is an associated *Gysin morphism*

$$f^* : M_S(X) \to M_S(Y)(d)[2d]$$

functorial in f. We refer the reader to *loc. cit* for various formulas involving the Gysin morphism (projection formula, excess intersection,...)

Note in particular that we deduce from that Gysin morphism the following map in motivic cohomology:

$$f_* : H_{\mathcal{M}}^{n,i}(Y,\Lambda) \to H_{\mathcal{M}}^{n+2d,i+d}(X,\Lambda).$$

(4) For any smooth projective S-scheme X, the premotive $M_S(X)$ admits a *strong dual*. If X has pure relative dimension d over S, the strong dual of $M_S(X)$ is $M_S(X)(-d)[-2d]$.

REMARK 11.3.5. According to *loc. cit.*, there exists for any scheme S a formal group law $F_S(x,y)$ with coefficients in the graded ring $H_{\mathcal{M}}^{2*,*}(S,\Lambda)$. If one consider the Segre embedding

$$\Sigma : \mathbf{P}_S^\infty \to \mathbf{P}_S^\infty \times_S \mathbf{P}_S^\infty$$

one has: $F_S(x,y) = \sigma^*(1)$ through the isomorphism:

$$H_{\mathcal{M}}^{2*,*}(\mathbf{P}_S^\infty \times_S \mathbf{P}_S^\infty, \Lambda) \simeq H_{\mathcal{M}}^{2*,*}(S,\Lambda)[[x,y]]$$

which results from the projective bundle formula in motivic cohomology.

According to Remark 11.3.3, whenever S is normal, one gets $F_S(x,y) = x + y$. In particular, $F_{\mathrm{Spec}(\mathbf{Z})}(x,y) = x + y$. On the other hand, according to the above definition of $F_S(x,y)$, $F_S(x,y)$ is compatible with pullback. Thus one deduces that $F_S(x,y) = x + y$ for any scheme S.

11.3.6. According to the properties that we have previously proved, motivic cohomology, and in particular the bigraded part $H^{2n,n}_{\mathcal{M}}(X, \mathbf{Z})$, possesses many of the desired property of a generalized Chow theory for regular schemes (see [**SGA6**, XIV, §8]).

Note in particular that the existence of Chern classes allows to define a Chern character:

$$K_0(X) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{ch} H^{2*,*}_{\mathcal{M}}(X, \mathbf{Z}) \otimes \mathbf{Q} \simeq \mathrm{H}^{2*,*}_{\mathcal{M}}(X, \mathbf{Q})$$

where the final isomorphism follows from Paragraph 11.2.2. In particular we will prove in the next section (Corollary 16.1.7) that, when X is regular, this map is an isomorphism as expected.

Remark 11.3.7. Among the good properties of motivic cohomology is the fact it is defined, with its ring structure and natural functoriality, other arbitrary schemes. On the other hand, even when X is regular, one cannot describe at the moment the cohomology group $H^{2n,n}_{\mathcal{M}}(X, \mathbf{Z})$ in terms of classes of n-codimensional cycles in X modulo an appropriate equivalence relation.

Let us however mention the two following interesting facts:

(1) Let X be a scheme of finite type over Spec ($\mathbf{Z}$) and X_p be its fiber over a primer p. Then one has a pullback map:

$$H^{2n,n}_{\mathcal{M}}(X, \mathbf{Z}) \to H^{2n,n}_{\mathcal{M}}(X_p, \mathbf{Z}), \sigma \mapsto \sigma_p.$$

When X is an arithmetic scheme (regular and flat over $\mathbf{Z}$) with good reduction at p, the target is the Chow group of n-codimensional cycles (see Example 11.2.3). Then σ_p should be thought as the specialization of its generic fiber (which lies in $H^{2n,n}_{\mathcal{M}}(X_{\mathbf{Q}}, \mathbf{Z}) = CH^n(X_{\mathbf{Q}})$ according to the Example 11.2.3). This construction should coincide with other specialization maps in the arithmetic case (see for example [**Ful98**, §20.3]).

(2) Let X be a smooth S-scheme. Then any n-codimensional closed subscheme Z of X which is smooth over S defines using the Gysin morphism an element

$$[Z] = i_*(1) \in H^{2n,n}_{\mathcal{M}}(X, \mathbf{Z})$$

which should be called the fundamental class of X. One can extract from [**Dég08**] some of the expected properties of these fundamental classes (relation to Chern classes, pullback properties such as compatibility with base change).

In particular, any S-point of X defines an element of $H^{2d,d}_{\mathcal{M}}(X, \mathbf{Z})$ where d is the dimension of X (assumed of pure dimension). In particular, the group $H^{2d,d}_{\mathcal{M}}(X, \mathbf{Z})$ is close to a group of cycles in X of relative dimension 0 over S.

11.3.8. We end up this series of remarks on motivic cohomology with the following construction that the reader might enjoy.

Let S be any scheme and $\mathscr{P}_S$ be the category of smooth projective S-schemes. Given any scheme X and Y in $\mathscr{P}_S$, one can use the group

$$H^{2d,d}_{\mathcal{M}}(X \times_S Y, \Lambda)$$

where d is the relative dimension of Y as a group of correspondences using the properties obtained so far from motivic cohomology. In particular, one can mimic the construction of the category of Chow motives over a field k using the category $\mathscr{P}_S$ and these correspondences. One obtains an additive monoidal category Chow$'(S, \Lambda)$ of *strong Chow motives*.

According to the duality property of motives (Paragraph 11.3.4, point 4) one also obtains a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{DM}(S,\Lambda)}(M_S(X), M_S(Y)) = H^{2d,d}_{\mathcal{M}}(X \times_S Y, \Lambda).$$

Thus one deduces a canonical full embedding of monoidal categories:

$$\mathrm{Chow}'(S, \Lambda) \to \mathrm{DM}_{gm}(S, \Lambda)$$

which extends the well known case when S is a perfect field.

Remark 11.3.9. Beware that, with rational coefficients, a sharper notion of Chow motives – in more precise terms, these are motives *of weight zero* – have been introduced recently (see [**Héb11**], [**Bon10**]).

11.4. The six functors.

11.4.1. Recall that according to Definition 10.4.2 and Paragraph 10.4.3, we have an adjunction of abelian premotivic categories

$$\gamma^* : \mathrm{Sh}(-,\Lambda) \rightleftarrows \mathrm{Sh}^{tr}(-,\Lambda) : \gamma_*$$

such that γ_* is exact and conservative. Moreover, for any scheme S, any smooth S-schemes X, Y and any open immersion $j : U \to X$, the canonical map:

$$j_* : c_S(Y,U) \to c_S(Y,X)$$

is obviously a monomorphism. Thus the abelian premotivic category $\mathrm{Sh}^{tr}(-,\Lambda)$ satisfies the assumptions (i)-(iv) of Paragraph 6.3.1. In particular, we deduce from Corollaries 6.3.12 and 6.3.15 the following theorem:

Proposition 11.4.2. *The premotivic triangulated category* DM_Λ *satisfies the support property. Moreover, for any scheme S and any closed immersion $i : Z \to X$ between smooth S-schemes,* DM_Λ *satisfies the localization property with respect to i, (Loc_i).*

An important corollary of this proposition is that given any separated morphism $f : Y \to X$ of finite type, one can construct an adjunction of triangulated categories:

$$f_! : \mathrm{DM}(Y,\Lambda) \leftrightarrows \mathrm{DM}(X,\Lambda) : f^!$$

such that $f_! = f_*$ when f is proper (see Section 2.2). We will elaborate on this fact at the end of this section.

11.4.3. Note that in particular, the premotivic category DM_Λ satisfies the weak localization property (wLoc). According to the premotivic adjunction (11.2.16.2) and the existence of the first Chern class in motivic cohomology (Definition 11.3.2), one can apply Example 2.4.40 to the premotivic triangulated category DM_Λ (which satisfies the Nisnevich separation property by construction). This implies in particular that DM_Λ is oriented as a premotivic triangulated category (Definition 2.4.38).

In particular, one can apply Corollary 2.4.43 to DM_Λ and get the following result:

Proposition 11.4.4. *Any smooth projective morphism f is* DM_Λ*-pure: the canonical purity map* (2.4.39.3)

$$f_\sharp \to f_!(d)[2d],$$

is an isomorphism where d is the relative dimension of f.

In particular, DM_Λ is weakly pure. The only property of the premotivic triangulated category DM_Λ that we cannot prove is the localization property for general closed immersions. However, the properties we have seen so far allows to construct the 6 operations and establish some of its properties that are already of interest. Let us summarize this formalism, from Theorem 2.2.14 together with Lemma 2.4.23:

Theorem 11.4.5. *For any separated morphism of finite type $f : Y \to X$, there exists an essentially unique pair of adjoint functors*

$$f_! : \mathrm{DM}(Y,\Lambda) \rightleftarrows \mathrm{DM}(X,\Lambda) : f^!$$

such that:

1. *There exists a structure of a covariant (resp. contravariant)* 2*-functor on $f \mapsto f_!$ (resp. $f \mapsto f^!$).*
2. *There exists a natural transformation $\alpha_f : f_! \to f_*$ which is an isomorphism when f is proper. Moreover, α is a morphism of* 2*-functors.*
3. *For any smooth projective morphism $f : X \to S$ of relative dimension d, there are canonical natural isomorphisms*

$$\mathfrak{p}_f^{\mathfrak{t}} : f_\sharp \longrightarrow f_!(d)[2d]$$
$$\mathfrak{p}_f^{\prime\mathfrak{t}} : f^* \longrightarrow f^!(-d)[-2d]$$

which are dual to each other.

(4) *For any cartesian square:*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

such that f *is separated of finite type, there exist natural transformations*

$$g^* f_! \xrightarrow{\sim} f'_! g'^* \,,$$
$$g'_* f'^! \xrightarrow{\sim} f^! g_* \,,$$

which are isomorphisms in the following cases:

- g *is smooth;*
- f *is projective and smooth.*

(5) *For any smooth projective morphism* $f : Y \to X$*, there exist natural isomorphisms*

$$Ex(f_!^*, \otimes) : (f_! K) \otimes_X L \xrightarrow{\sim} f_!(K \otimes_Y f^* L)\,,$$
$$Hom_X(f_!(L), K) \xrightarrow{\sim} f_* Hom_Y(L, f^!(K))\,,$$
$$f^! Hom_X(L, M) \xrightarrow{\sim} Hom_Y(f^*(L), f^!(M))\,.$$

Remark 11.4.6. As an example of application, let us recall the construction of the general trace map (from [**SGA4**]) in the case of a smooth projective morphism $f : Y \to X$ of relative dimension d. It is the following composite map:

$$f_* f^* \xrightarrow{\alpha_f^{-1}} f_! f^* \xrightarrow{\mathfrak{p}'^{\mathfrak{t}}_f} f_! f^!(d)[2d] \xrightarrow{ad'(f_!, f^!)} 1(d)[2d].$$

This allows one to recover the Gysin map associated with f, already constructed in Paragraph 11.3.4, as well as the duality property for the motive $M_X(Y)$.

Part 4

Beilinson motives and algebraic K-theory

12.0. In all this part, $\mathscr{S}$ is assumed to be the category of noetherian schemes of finite dimension.

12. Stable homotopy theory of schemes

12.1. Ring spectra. Consider a base scheme S.

Recall that a ring spectrum E over S is a monoid object in the monoidal category $\mathrm{SH}(S)$. We say that E is commutative if it is commutative as a monoid in the symmetric monoidal category $\mathrm{SH}(S)$. In what follows, we will assume that all our ring spectra are commutative without mentioning it.
The premotivic category is $\mathbf{Z}^2$-graded where the first index refers to the simplicial sphere and the second one to the Tate twist. According to our general convention, a cohomology theory representable in SH is $\mathbf{Z}^2$-graded accordingly: given such a ring spectrum E, for any smooth S-scheme X, and any integer $(i,n)\in\mathbf{Z}^2$, we get a bigraded ring:

$$E^{n,i}(X)=\mathrm{Hom}_{\mathrm{SH}(S)}\big(\Sigma^\infty X_+,E(i)[n]\big).$$

When X is a pointed smooth S-scheme, it defines a pointed sheaf of sets still denoted by X and we denote by $\tilde{E}^{n,i}(X)$ for the corresponding cohomology ring.
The *coefficient ring* associated with E is the cohomology of the base $E^{**}:=E^{**}(S)$. The ring $E^{**}(X)$ (resp. $\tilde{E}^{**}(X)$) is in fact an E^{**}-algebra.

12.1.1. We say E is a *strict ring spectrum* if there exists a monoid in the category of symmetric Tate spectra E' and an isomorphism of ring spectra $E\simeq E'$ in $\mathrm{SH}(S)$. In this case, a module M over the monoid E in the monoidal category $\mathrm{SH}(S)$ will be said to be *strict* if there exists an E'-module M' in the category of symmetric Tate spectra, as well as an isomorphism of E-modules $M\simeq M'$ in $\mathrm{SH}(S)$.

12.2. Orientation.

12.2.1. Consider the infinite tower

$$\mathbf{P}^1_S\to\mathbf{P}^2_S\to\cdots\to\mathbf{P}^n_S\to\cdots$$

of schemes pointed by the infinity. We denote by $\mathbf{P}^\infty_S$ the limit of this tower as a pointed Nisnevich sheaf of sets and by $\iota:\mathbf{P}^1_S\to\mathbf{P}^\infty_S$ the induced inclusion. Recall the following definition, classically translated from topology:

Definition 12.2.2. Let E be a ring spectrum over S. An *orientation* of E is a cohomology class c in $\tilde{E}^{2,1}(\mathbf{P}^\infty_S)$ such that $\iota^*(c)$ is sent to the unit of the coefficient ring of E by the canonical isomorphism $\tilde{E}^{2,1}(\mathbf{P}^1_S)=E^{0,0}$.

We then say that (E,c) is an *oriented ring spectrum*. We shall say also that E is *orientable* if there exists an orientation c.

According to [**MV99**, 1.16 and 3.7], we get a canonical map for any smooth S-scheme X

$$\mathrm{Pic}(X)=H^1(X,\mathbf{G}_m)\to\mathrm{Hom}_{\mathscr{H}_\bullet(S)}(X_+,\mathbf{P}^\infty)\to\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty X_+,\Sigma^\infty\mathbf{P}^\infty)$$

(the first map is an isomorphism whenever S is regular (or even geometrically unibranch)). Given this map, an orientation c of a ring spectrum E defines a map of sets

$$c_{1,X}:\mathrm{Pic}(X)\to E^{2,1}(X)$$

which is natural in X (and from its construction in [**MV99**], one can check that $c=c_{1,\mathbf{P}^\infty_S}(\mathcal{O}(1))$). Usually, we put $c_1=c_{1,X}$.

Example 12.2.3.
1. The original example of an oriented ring spectrum is the algebraic cobordism spectrum MGL introduced by Voevodsky (*cf.* [**Voe98**]).
2. According to Definition 11.3.2, the motivic cohomology ring spectrum $\mathrm{H}^\Lambda_{\mathcal{M},S}$ defined in 11.2.17 is an oriented ring spectrum.

(3) Consider a triangulated premotivic category $\mathscr{T}$ which satisfies the weak localization property (wLoc) and such that there exists an adjunction of triangulated premotivic categories:

$$\varphi^* : \mathrm{SH} \rightleftarrows \mathscr{T} : \varphi_*.$$

Recall that φ^* is symmetric monoidal. Thus, its right adjoint is weakly symmetric monoidal and for any the spectrum

$$\mathrm{H}_{\mathscr{T},S} := \varphi_*(\mathbb{1}_S)$$

admits a (commutative) ring structure.

Then $\mathscr{T}$ is oriented in the sense of Definition 2.4.38 if and only if the ring spectrum $\mathrm{H}_{\mathscr{T},S}$ is oriented in the sense of Definition 12.2.2 – see Example 2.4.40. Moreover, an orientation of $\mathscr{T}$ is equivalent to the data of orientations $\mathrm{H}_{\mathscr{T},S}$ for any scheme S which are stable by pullbacks (on cohomology).

Remark 12.2.4. When E is a strict ring spectrum, the category E-mod satisfies the axioms of [**Dég08**, 2.1] (see example 2.12 of *loc.cit.*).

Recall the following result, which first appeared in [**Vez01**]:

Proposition 12.2.5 (Morel). *Let (E,c) be an oriented ring spectrum. Then:*

$$E^{**}(\mathbf{P}^\infty_S) = E^{**}[[c]]$$
$$E^{**}(\mathbf{P}^\infty_S \times \mathbf{P}^\infty_S) = E^{**}[[x,y]]$$

where x (resp. y) is the pullback of c along the first (resp. second) projection.

Remark 12.2.6. When E is a strict ring spectrum, this is [**Dég08**, 3.2] according to remark 12.2.4. The proof follows an argument of Morel ([**Dég08**, lemma 3.3]) and the arguments of *op.cit.*, p. 634, can be easily used to obtain the proposition arguing directly for the cohomology functor $X \mapsto E^{*,*}(X)$.

12.2.7. Recall that the Segre embeddings

$$\mathbf{P}^n_S \times \mathbf{P}^m_S \to \mathbf{P}^{n+m+nm}_S$$

define a map

$$\sigma : \mathbf{P}^\infty_S \times \mathbf{P}^\infty_S \to \mathbf{P}^\infty_S.$$

It gives the structure of an H-group to $\mathbf{P}^\infty_S$ in the homotopy category $\mathscr{H}(S)$. Consider the hypothesis of the previous proposition. Then the pullack along σ in E-cohomology induces a map

$$E^{**}[[c]] \xrightarrow{\sigma^*} E^{**}[[x,y]]$$

and following Quillen, we check that the formal power series $\sigma^*(c)$ defines a formal group law over the ring E^{**}.

Definition 12.2.8. Let (E,c) be an oriented ring spectrum and consider the previous notation.

The formal group law $F_E(x,y) := \sigma^*(c)$ will be called the formal group law associated to (E,c).

Recall the formal group law has the form:

$$F_E(x,y) = x + y + \sum_{i+j>0} a_{ij}.x^i y^j$$

with $a_{ij} = a_{ji}$ in $E^{-2i-2j,-i-j}$.
The coefficients a_{ij} describe the failure of additivity of the first Chern class c_1. Indeed, assuming the previous definition, we get the following result:

Proposition 12.2.9. *Let X be a smooth S-scheme.*

(1) For any line bundle L/X, the class $c_1(L)$ is nilpotent in $E^{**}(X)$.

(2) Suppose X admits an ample line bundle. For any line bundles L, L' over X,

$$c_1(L_1 \otimes L_2) = F_E(c_1(L_1), c_1(L_2)) \in E^{2,1}(X).$$

We refer to [**Dég08**, 3.8] in the case where E is strict; the proof is the same in the general case.

Recall the following theorem of Vezzosi (*cf.* [**Vez01**, 4.3]):

Theorem 12.2.10 (Vezzosi). *Let (E, c) be an oriented spectra over S, with formal group law F_E. Then there exists a bijection between the following sets:*

(i) Orientation classes c' of E.
(ii) Morphisms of ring spectra $MGL \to E$ in $\mathrm{SH}(S)$.
(iii) Couples (F, φ) where F is a formal group law over E^{**} and φ is a power series over E^{**} which defines an isomorphism of formal group law: φ is invertible as a power series and $F_E(\varphi(x), \varphi(y)) = F(x, y)$.

12.3. Rational category. In what follows, we shall use frequently the equivalence of premotivic categories (see 5.3.35)

$$\mathrm{SH}_{\mathbf{Q}} \rightleftarrows \mathrm{D}_{\mathbf{A}^1, \mathbf{Q}}\,,$$

and shall identify freely any rational spectrum over a scheme S with an object of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$.

13. Algebraic K-theory

13.1. The K-theory spectrum. We consider the spectrum KGL_S which represents homotopy invariant K-theory in $\mathrm{SH}(S)$ according to Voevodsky (see [**Cis13**], [**Voe98**, 6.2], [**Rio10**, 5.2] and [**PPR09**]). It is characterized by the following properties:

(K1) For any morphism of schemes $f : T \to S$, there is an isomorphism $f^*KGL_S \simeq KGL_T$ in $\mathrm{SH}(T)$.

(K2) For any regular scheme S and any integer n, there exists an isomorphism

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{1}_S[n], KGL_S) \to K_n(S)$$

(where the right hand side is Quillen algebraic K-theory) such that, for any morphism $f : T \to S$ of regular schemes, the following diagram is commutative:

$$\begin{array}{ccccc} \mathrm{Hom}(\mathbb{1}_S[n], KGL_S) & \longrightarrow & \mathrm{Hom}(f^*\mathbb{1}_S[n], f^*KGL_S) & = & \mathrm{Hom}(\mathbb{1}_T[n], KGL_T) \\ \downarrow & & & & \downarrow \\ K_n(S) & & \xrightarrow{\quad f^* \quad} & & K_n(T) \end{array}$$

(where the lower horizontal map is the pullback in Quillen algebraic K-theory along the morphism f and the upper horizontal map is obtained by using the functor $f^* : \mathrm{SH}(S) \to \mathrm{SH}(T)$ and the identification (K1)).

(K3) For any scheme S, there exists a unique structure of a commutative monoid on KGL_S which is compatible with base change – using the identification (K1) – and induces the canonical ring structure on $K_0(S)$.

Thus, according to (K1) and (K3), the collection of the ring spectrum KGL_S for any scheme S form an absolute ring spectrum. As usual, when no confusion can arise, we will not indicate the base in the notation KGL.

Note that (K1) implies formally that the isomorphism of (K2) can be extended for any smooth S-scheme X (with S regular), giving a natural isomorphism:

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty X_+[n], KGL) \to K_n(X)\,.$$

13.2. Periodicity.

13.2.1. Recall from the construction the following property of the spectrum KGL:

(K4) the spectrum KGL is a $\mathbf{P}^1$-periodic spectrum in the sense that there exists a canonical isomorphism

$$KGL \xrightarrow{\sim} \mathbf{R}Hom\left(\Sigma^\infty \mathbf{P}^1_S, KGL\right) = KGL(-1)[-2].$$

As usual, $\mathbf{P}^1_S$ is pointed by the infinite point.

This isomorphism, classically called the Bott isomorphism, is characterized uniquely by the fact that its adjoint isomorphism (obtained by tensoring with $\mathbb{1}_S(1)[2]$) is equal to the composite

$$\gamma_u : KGL(1)[2] \xrightarrow{1\otimes u} KGL \wedge KGL \xrightarrow{\mu} KGL. \tag{13.2.1.1}$$

where $u : \Sigma^\infty \mathbf{P}^1 \to KGL$ corresponds to the class $[\mathcal{O}(1)] - 1$ in $\tilde{K}_0(\mathbf{P}^1)$ through the isomorphism (K2) and μ is the structural map of monoid from (K3).

Using the isomorphism of (K4), the property (K1) can be extended as follows: for any smooth S-scheme X and any integers $(i, n) \in \mathbf{Z}^2$, there is a canonical isomorphism:

$$KGL^{n,i}(X) \xrightarrow{\sim} K_{2i-n}(X). \tag{13.2.1.2}$$

Remark 13.2.2. The element u is invertible in the ring $KGL^{*,*}(S)$. Its inverse is the *Bott element* $\beta \in KGL^{2,1}(S)$. If we chose as an orientation of the ring spectrum KGL (*cf.* 12.2.2) the class

$$\beta.([\mathcal{O}(1)] - 1) \in KGL^{2,1}(\mathbf{P}^\infty),$$

the corresponding formal group law is the multiplicative formal group law:

$$F(x, y) = x + y + \beta^{-1}.xy.$$

13.3. Modules over algebraic K-theory.

Theorem 13.3.1 (Röndigs, Spitzweck, Østvær). *The spectrum KGL can be represented canonically by a cartesian monoid KGL', as well as by a homotopy cartesian commutative monoid KGL^β in the fibred model category of symmetric $\mathbf{P}^1$-spectra, in such a way that there exists a morphism of monoids $KGL' \to KGL^\beta$ which is a termwise stable $\mathbf{A}^1$-equivalence.*

Proof. For any noetherian scheme of finite dimension S, one has a strict commutative ring spectrum KGL^β_S which is canonically isomorphic to KGL_S in $\mathrm{SH}(S)$ as ring spectra; see [**RSØ10**]. One can check that the objects KGL^β_S do form a commutative monoid over the diagram of all noetherian schemes of finite dimension (i.e. a commutative monoid in the category of sections of the fibred category of $\mathbf{P}^1$-spectra over the category of noetherian schemes of finite dimension), either by hand, by following the explicit construction of *loc. cit.*, either by modifying its construction very slightly as follows: one can perform *mutatis mutandis* the construction of *loc. cit.* in the $\mathbf{P}^1$-stabilization of the $\mathbf{A}^1$-localization of the model category of Nisnevich simplicial sheaves over (any essentially small adequate subcategory of) the category of all noetherian schemes of finite dimension, and get an object KGL^β, whose restriction to each of the categories Sm/S is the object KGL^β_S. From this point of view, we clearly have canonical maps $f^*(KGL^\beta_S) \to KGL^\beta_T$ for any morphism of schemes $f : T \to S$. The object KGL^β is homotopy cartesian, as the composed map

$$\mathbf{L}f^*(KGL_S) \simeq \mathbf{L}f^*(KGL^\beta_S) \to f^*(KGL^\beta_S) \to KGL^\beta_T \simeq KGL_T$$

is an isomorphism in $\mathrm{SH}(T)$. Consider now a cofibrant resolution

$$KGL'_{\mathrm{Spec}(\mathbf{Z})} \to KGL^\beta_{\mathrm{Spec}(\mathbf{Z})}$$

in the model category of monoids of the category of symmetric $\mathbf{P}^1$-spectra over $\mathrm{Spec}\,(\mathbf{Z})$; see Theorem 7.1.3. Then, we define, for each noetherian scheme of finite dimension S, the $\mathbf{P}^1$-spectrum KGL'_S as the pullback of $KGL'_{\mathrm{Spec}(\mathbf{Z})}$ along the map $f : S \to \mathrm{Spec}\,(\mathbf{Z})$. As the functor f^* is a left Quillen functor, the object KGL'_S is cofibrant (both as a monoid and as a $\mathbf{P}^1$-spectrum), so that we get, by construction, a termwise cofibrant cartesian strict $\mathbf{P}^1$-ring spectrum KGL', as well as a morphism $KGL' \to KGL^\beta$ which is a termwise stable $\mathbf{A}^1$-equivalence. □

13.3.2. For each noetherian scheme of finite dimension S, one can consider the model categories of modules over KGL'_S and KGL^β_S respectively; see 7.2.2. The change of scalars functor along the stable $\mathbf{A}^1$-equivalence $KGL'_S \to KGL^\beta_S$ defines a left Quillen equivalence, whence an equivalence of homotopy categories:

$$\mathrm{Ho}(KGL'_S\text{- mod}) \simeq \mathrm{Ho}(KGL^\beta_S\text{- mod}).$$

We put

$$\mathrm{Ho}(KGL\text{- mod})(S) = \mathrm{Ho}(KGL^\beta_S\text{- mod})\,,$$

and call this category the *homotopy category of KGL-modules over S*. By definition, for any smooth S-scheme X, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(X_+), KGL[n]) \simeq \mathrm{Hom}_{KGL}(KGL_S(X), KGL[n])$$

(where $KGL_S(X) = KGL_S \wedge^{\mathbf{L}}_S \Sigma^\infty(X_+)$, while Hom_{KGL} stands for $\mathrm{Hom}_{\mathrm{Ho}(KGL\text{- mod})(S)}$).

According to (K1) and (K3), for any regular scheme X, we thus get a canonical isomorphism:

(13.3.2.1) $$\epsilon_S : \mathrm{Hom}_{KGL}(KGL_S[n], KGL_S) \xrightarrow{\sim} K_n(S).$$

Using Bott periodicity (K4), and the compatibility with base change, this isomorphism can be extended for any smooth S-scheme X and any pair $(n, m) \in \mathbf{Z}^2$:

(13.3.2.2) $$\epsilon_{X/S} : \mathrm{Hom}_{KGL}(KGL_S(X), KGL_S(m)[n]) \xrightarrow{\sim} K_{2m-n}(X).$$

Corollary 13.3.3. *The categories* $\mathrm{Ho}(KGL\text{- mod})(S)$ *form a motivic category, and the functors*

$$\mathrm{SH}(S) \to \mathrm{Ho}(KGL\text{- mod})(S)\ , \quad M \mapsto KGL_S \wedge^{\mathbf{L}}_S M$$

define a morphism of motivic categories

$$\mathrm{SH} \to \mathrm{Ho}(KGL\text{- mod})$$

over the category of noetherian schemes of finite dimension.

Proof. This follows from the preceding theorem and from 7.2.13 and 7.2.18. □

13.4. K-theory with support.

13.4.1. Consider a closed immersion $i : Z \to S$ with complementary open immersion $j : U \to S$. Assume S is regular.
We use the definition of [**Gil81**, 2.13] for the K-theory of S with support in Z denoted by $K^Z_*(S)$. In other words, we define $K^Z(S)$ as the homotopy fiber of the restriction map

$$\mathbf{R}\Gamma(S, KGL_S) = K(S) \to K(U) = \mathbf{R}\Gamma(U, KGL_U)\,,$$

and put: $K^Z_n(S) = \pi_n(K^Z(S))$.

Applying the derived global section functor $\mathbf{R}\Gamma(S, -)$ to the homotopy fiber sequence

(13.4.1.1) $$i_!\, i^! KGL_S \to KGL_S \to j_*\, j^* KGL_S\,,$$

we get a homotopy fiber sequence

(13.4.1.2) $$\mathbf{R}\Gamma(S, i_!\, i^! KGL_S) \to \mathbf{R}\Gamma(S, KGL_S) \to \mathbf{R}\Gamma(U, KGL_S)$$

from which we deduce an isomorphism in the stable homotopy category of S^1-spectra:

(13.4.1.3) $$\mathbf{R}\Gamma(Z, i^! KGL_S) = \mathbf{R}\Gamma(S, i_!\, i^! KGL_S) \simeq K^Z(S)\,.$$

We thus get the following property:

(K6) There is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{SH}(S)}\left(\mathbb{1}_S[n], i_! i^! KGL_S\right) \to K^Z_n(S)$$

which satisfies the following compatiblities:

(K6a) the following diagram is commutative:

$$\begin{array}{ccccc}
\mathrm{Hom}(\mathbb{1}[n+1], j_*j^*KGL_S) & \longrightarrow & \mathrm{Hom}\big(\mathbb{1}[n], i_!i^!KGL_S\big) & \longrightarrow & \mathrm{Hom}(\mathbb{1}[n], KGL_S) \\
\downarrow & & \downarrow & & \downarrow \\
K_{n+1}(U) & \longrightarrow & K_n^Z(S) & \longrightarrow & K_n(S)
\end{array}$$

where the upper horizontal arrows are induced by the localization sequence (13.4.1.1), and the lower one is the canonical sequence of K-theory with support. The extreme left and right vertical maps are the isomorphisms of (K2);

(K6b) for any morphism $f : Y \to S$ of regular schemes, $k : T \to Y$ the pullback of i along f, the following diagram is commutative:

$$\begin{array}{ccccc}
\mathrm{Hom}\left(\mathbb{1}[n], i_!i^!KGL_S\right) & \longrightarrow & \mathrm{Hom}\left(f^*\mathbb{1}[n], f^*i_!i^!KGL_S\right) & \longrightarrow & \mathrm{Hom}\left(\mathbb{1}[n], k_!k^!KGL_Y\right) \\
\downarrow & & & & \downarrow \\
K_n^Z(S) & & \xrightarrow{\qquad f^* \qquad} & & K_n^T(Y)
\end{array}$$

where the lower horizontal map is given by the functoriality of relative K-theory (induced by the funtoriality of K-theory) and the left one is obtained using the functor f^* of SH, the canonical exchange morphism $f^*i_!i^! \to k_!k^!f^*$ and the identification (K1).

This property can be extended to the motivic category Ho(KGL-mod) and we get a canonical isomorphism

$$\epsilon_i : \mathrm{Hom}_{KGL}(KGL_S[n], i_!i^!KGL_S) \xrightarrow{\sim} K_n^Z(S) \tag{13.4.1.4}$$

satisfying the analog of (K6a) and (K6b).

13.5. Fundamental class.

13.5.1. Consider a cartesian square of regular schemes

$$\begin{array}{ccc}
Z' & \xrightarrow{k} & S' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{i} & S
\end{array}$$

with i a closed immersion. We will say that this square is *Tor-independant* if Z and S' are Tor-independant over S in the sense of [**SGA6**, III, 1.5]: for any $i > 0$, $\mathrm{Tor}_i^S(\mathcal{O}_Z, \mathcal{O}_{S'}) = 0$.[85]

In this case, when we assume in addition that all the schemes in the previous square are regular and that i is a closed immersion we get from [**TT90**, 3.18][86] the formula

$$f^*i_* = k_*g^* : K_*(Z) \to K_*(S')$$

in Quillen K-theory. An important point for us is that there exists a *canonical homotopy* between these morphisms at the level of the Waldhausen spectra.[87] According to the localization theorem of Quillen [**Qui73**, 3.1], we get:

THEOREM 13.5.2 (Quillen). *For any closed immersion $i : Z \to S$ between regular schemes, there exists a canonical isomorphism*

$$\mathfrak{q}_i : K_n^Z(S) \to K_n(Z).$$

Moreover, this isomorphism is functorial with respect to the Tor-independant squares as above, with i a closed immersion and all the schemes regular.

---

[85]For example, when i is a regular closed immersion of codimension 1, this happens if and only if the above square is transversal.

[86]When all the schemes in the square admit ample line bundles, we can refer to [**Qui73**, 2.11].

[87]In the proof of Quillen, one can also trace back a canonical homotopy with the restriction mentioned in the preceding footnote.

Remark 13.5.3. In the condition of this theorem, the following diagram is commutative by construction:

$$\begin{array}{ccc} K_n^Z(S) & \longrightarrow & \\ {\scriptstyle \mathfrak{q}_i}\downarrow & & K_n(S) \\ K_n(Z) & \xrightarrow{i_*} & \end{array}$$

where the non labelled map is the canonical one.

Definition 13.5.4. Let $i : Z \to S$ be a closed immersion between regular schemes.

We define the *fundamental class* associated with i as the morphism of KGL-modules:

$$\eta_i : i_*KGL_Z \to KGL_S$$

defined by the image of the unit element 1 through the following morphism:

$$K_0(Z) \xrightarrow{\mathfrak{q}_i^{-1}} K_0^Z(S) \xrightarrow{\epsilon_i^{-1}} \mathrm{Hom}_{KGL}(KGL_S, i_!i^!KGL_S) = \mathrm{Hom}_{KGL}(i_*KGL_Z, KGL_S).$$

We also denote by $\eta'_i : KGL_Z \to i^!KGL_S$ the morphism obtained by adjunction.

Remark 13.5.5. The fundamental class has the following functoriality properties.

(1) By definition, and applying remark 13.5.3, the composite map
$$KGL_S \to i_*i^*(KGL_S) = i_*KGL_Z \xrightarrow{\eta_i} KGL_S$$
corresponds via the isomorphism ϵ_S to $i_*(1) \in K_0(S)$. According to [**SGA6**, Exp. VII, 2.7], this class is equal to $\lambda_{-1}(N_i)$ where N_i is the conormal sheaf of the regular immersion i.

(2) In the situation of a Tor-independant square as in 13.5.1, remark that $f^*\eta_i = \eta_k$ through the canonical exchange isomorphism $f^*i_* = k_*g^*$ — apply the functoriality of ϵ_i from (K6b) and the one of $\mathfrak{q}_i$.

(3) Using the identification $i^!i_* = 1$, we get $\eta'_i = i^!\eta_i$. Consider a cartesian square as in 13.5.1 and assume f is smooth. Then the square is Tor-independant and we get $g^*\eta'_i = \eta'_k$ using the exchange isomorphism $g^*i^! = k^!f^*$.

13.6. Absolute purity for K-theory.

Proposition 13.6.1. *For any closed immersion $i : Z \to S$ between regular schemes, the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}_{KGL}(KGL_Z[n], KGL_Z) & \xrightarrow{\eta'_i} & \mathrm{Hom}_{KGL}(KGL_Z[n], i^!KGL_S) \\ {\scriptstyle \epsilon_Z}\downarrow & (*) & \downarrow{\scriptstyle \epsilon_i} \\ K_n(Z) & \xrightarrow{\mathfrak{q}_i^{-1}} & K_n^Z(S) \end{array}$$

Proof. In this proof, we denote by $[-,-]$ the bifunctor $\mathrm{Hom}_{KGL}(-,-)$.

Step 1: We assume that $i : Z \to S$ admits a retraction $p : S \to Z$.

Consider a KGL-linear map $\alpha : KGL_Z[n] \to KGL_Z$. Then, $\eta'_i(\alpha)$ corresponds by adjunction to the composition

$$i_*KGL_Z[n] \xrightarrow{i_*(\alpha)} i_*KGL_Z \xrightarrow{\eta_i} KGL_S.$$

Applying the projection formula for the motivic category $\mathrm{Ho}(KGL\text{-}\mathrm{mod})$, we get:

$$i_*(\alpha) = i_*(1 \otimes i^*p^*(\alpha)) = i_*(1) \otimes p^*(\alpha).$$

Here 1 stands for the identity morphism of the KGL-module KGL_Z. This shows that $\eta'_i(\alpha)$ corresponds by adjunction to the composite map:

$$\eta_i \otimes p^*(\alpha) : i_*KGL_Z[n] = i_*KGL_Z[n] \otimes KGL_S \to KGL_S \otimes KGL_S = KGL_S$$

(the tensor product is the KGL-linear one). By assumption, $i_* : K_*(Z) \to K_*(S)$ admits a retraction which implies the canonical map $\mathcal{O}_i : K_*^Z(S) \to K_*(S)$ admits a retraction (*cf.* remark 13.5.3). To check that the diagram $(*)$ is commutative, we can thus compose with $\mathcal{O}_i$.

Recall the first point of remark 13.5.5: applying property (K6a) and the fact the isomorphism $\epsilon_S : [KGL_S[n], KGL_S] \to K_n(S)$ is compatible with the algebra structures, we are finally reduced to prove that

$$i_*(\alpha) = i_*(1).p^*(\alpha) \in K_n(S).$$

This follows from the projection formula in K-theory (see [**Qui73**, 2.10] and [**TT90**, 3.17]).
Step 2: We shall reduce the general case to Step 1. We consider the following deformation to the normal cone diagram: let D be the blow-up of $\mathbf{A}^1_S$ in the closed subscheme $\{0\} \times Z$, P be the projective completion of the normal bundle of Z in S and s be the canonical section of P/Z; we get the following diagram of regular schemes:

(13.6.1.1)
$$\begin{array}{ccccc} Z & \xrightarrow{s_1} & \mathbf{A}^1_Z & \xleftarrow{s_0} & Z \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow{\scriptstyle s} \\ S & \longrightarrow & D & \longleftarrow & P \end{array}$$

where s_0 (resp. s_1) is the zero (resp. unit) section of $\mathbf{A}^1_Z$ over Z. These squares are cartesian and Tor-independant in the sense of 13.5.1. The maps s_0 and s_1 induce isomorphisms in K-theory because Z is regular. Thus, the second point of remark 13.5.5 allows to reduce to the case of the immersion s which was done in Step 1. □

13.6.2. Consider a cartesian square

$$\begin{array}{ccc} T & \xrightarrow{k} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & S \end{array}$$

such that S and Z are regular, i is a closed immersion and f is smooth. In this case, the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}_{KGL}(KGL_Z(T)[n], KGL_Z) & \xrightarrow{\eta'_i} & \mathrm{Hom}_{KGL}(KGL_Z(T)[n], i^! KGL_S) \\ \| & & \| \\ \mathrm{Hom}_{KGL}(KGL_T[n], KGL_T) & \xrightarrow{\eta'_k} & \mathrm{Hom}_{KGL}(KGL_T[n], k^! KGL_X) \end{array}$$

using the adjunction $(g_\sharp, g_*)$, the exchange isomorphism $g^* i^! \simeq k^! f^*$ (which uses relative purity for smooth morphisms) and the third point of remark 13.5.5. In particular, the preceding proposition has the following consequences:

Theorem 13.6.3 (Absolute purity). *For any closed immersion $i : Z \to S$ between regular schemes, the map*

$$\eta'_i : KGL_Z \to i^! KGL_S$$

is an isomorphism in the category $\mathrm{Ho}(KGL\text{-}\mathrm{mod})(Z)$ *(or in* $\mathrm{SH}(Z)$*).*

Corollary 13.6.4. *Given a cartesian square as above, for any pair $(n, m) \in \mathbf{Z}^2$, the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}(KGL_S(X), i_* KGL_Z(m)[n]) & \xrightarrow{\eta_i} & \mathrm{Hom}(KGL_S(X), KGL_S(m)[n]) \\ \| & & \\ \mathrm{Hom}(KGL_Z(T), KGL_Z(m)[n]) & & \sim \big\downarrow {\scriptstyle \epsilon_{X/S}} \\ {\scriptstyle \epsilon_{T/Z}} \big\downarrow \sim & & \\ K_{2m-n}(T) & \xrightarrow{k_*} & K_{2m-n}(X) \end{array}$$

where the vertical maps are the isomorphisms (13.3.2.2).

13.7. Trace maps.

13.7.1. Let S be a regular scheme. Let Y be a smooth S-scheme. The canonical map

$$\mathrm{Pic}(Y) \to K_0(Y) \xrightarrow{\sim} \mathrm{Hom}_{KGL}(KGL_S(Y), KGL_S) \xrightarrow{\beta_*} \mathrm{Hom}_{KGL}(KGL_S(Y), KGL_S(1)[2])$$

defines Chern classes in the category $\mathrm{Ho}(KGL\text{-}\mathrm{mod})(S)$; they corresponds to the orientation defined in remark 13.2.2.
Let $p : P \to S$ be a projective bundle of rank n. Let $v = [\mathcal{O}(1)] - 1$ in $K_0(P)$. It corresponds to a map $\mathfrak{v} : KGL_S(P) \to KGL_S$. According to [**Dég08**, 3.2] and our choice of Chern classes, the following map

$$KGL_S(P) \xrightarrow{\sum_i \beta^i.\mathfrak{v}^i \boxtimes p_*} \bigoplus_{0 \leq i \leq n} KGL_S(i)[2i]$$

is an isomorphism. As β is invertible, it follows that the map

$$\varphi_{P/S} : KGL_S(P) \xrightarrow{\sum_i \mathfrak{v}^i \boxtimes p_*} \bigoplus_{0 \leq i \leq n} KGL_S \tag{13.7.1.1}$$

is an isomorphism as well. Using this formula, the map $\mathrm{Hom}(\varphi_{P/S}, KGL_S)$ is equal to the isomorphism of Quillen's projective bundle theorem in K-theory (*cf.* [**Qui73**, 4.3]):

$$f_{P/S} : \bigoplus_{i=0}^{n} K_*(S) \to K_*(P), (S_0, ..., S_n) \mapsto \sum_i p^*(S_i).v^i.$$

Let $p_* : K_*(P) \to K_*(S)$ be the pushout by the projective morphism p. According to the projection formula, it is $K_*(S)$-linear. In particular, it is determined by the $n+1$-uple $(a_0, ..., a_n)$ where $a_i = p_*(v^i) \in K_0(S)$ through the isomorphism $f_{P/S}$. Let $\mathfrak{a}_i : KGL_S \to KGL_S$ be the map corresponding to a_i.

Definition 13.7.2. Consider the previous notations. We define the *trace map* associated with the projection $p : P \to S$ as the morphism of KGL-modules

$$\mathrm{Tr}_p^{KGL} : p_*(KGL_P) = \mathbf{R}Hom(KGL_S(P), KGL_S) \xrightarrow{(\varphi^*_{P/S})^{-1}} \bigoplus_{i=0}^{n} KGL_S \xrightarrow{(\mathfrak{a}_0, ..., \mathfrak{a}_n)} KGL_S.$$

From this definition, it follows that Tr_p represents the pushout by p in K-theory:

$$\begin{array}{ccc} \mathrm{Hom}_{KGL}(KGL_S[n], p_*KGL_P) & \xrightarrow{\mathrm{Tr}_{p*}^{KGL}} & \mathrm{Hom}_{KGL}(KGL_S[n], KGL_S) \\ \| & & \\ \mathrm{Hom}_{KGL}(KGL_P[n], KGL_P) & & \downarrow{\scriptstyle \epsilon_S} \\ {\scriptstyle \epsilon_P}\downarrow & & \\ K_n(P) & \xrightarrow{p_*} & K_n(S) \end{array}$$

Consider moreover a cartesian square:

$$\begin{array}{ccc} Q & \xrightarrow{q} & P \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & S \end{array}$$

such that f is smooth. From the projective base change theorem, we get $f^*p_*p^* = q_*q^*g^*$. Using this identification, we easily obtain that $f^* \mathrm{Tr}_p^{KGL} = \mathrm{Tr}_q^{KGL}$. Thus, we conclude that the map

$$\mathrm{Hom}_{KGL}(KGL_S(Y)[n], p_*KGL_P) \xrightarrow{\mathrm{Tr}_p^{KGL}} \mathrm{Hom}_{KGL}(KGL_S(Y)[n], KGL_S)$$

represents the usual pushout map

$$q_* : K_n(Q) \to K_n(Y)$$

through the canonical isomorphisms (13.3.2.2).

13.7.3. Consider a projective morphism $f : T \to S$ between regular schemes and choose a factorization

$$T \xrightarrow{i} P \xrightarrow{p} S$$

where i is a closed immersion and p is the projection of a projective bundle. Let us define a morphism

$$\mathrm{Tr}_{(p,i)}^{KGL} : f_*KGL_T = p_*i_*KGL_T \xrightarrow{p_*\eta_i} p_*KGL_P \xrightarrow{\mathrm{Tr}_p^{KGL}} KGL_S.$$

According to 13.6.4 and the previous paragraph, for any cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ T & \xrightarrow{f} & S \end{array}$$

such that a is smooth, the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Hom}_{KGL}(KGL_S(X), f_*KGL_T(m)[n]) & \xrightarrow{\mathrm{Tr}_{(p,i)*}^{KGL}} & \mathrm{Hom}_{KGL}(KGL_S(X), KGL_S(m)[n]) \\ \| & & \\ \mathrm{Hom}_{KGL}(KGL_T(Y), KGL_Z(m)[n]) & & \simeq \downarrow \epsilon_{X/S} \\ \epsilon_{Y/T} \downarrow \simeq & & \\ K_{2m-n}(Y) & \xrightarrow{g_*} & K_{2m-n}(X) \end{array} \tag{13.7.3.1}$$

Definition 13.7.4. Considering the above notations, we define the *trace map* associated to f as the morphism

$$\mathrm{Tr}_f^{KGL} = \mathrm{Tr}_{(p,i)}^{KGL} : f_*f^*KGL_S \to KGL_S.$$

Remark 13.7.5. By definition, the trace map Tr_f^{KGL} is a morphism of KGL-modules. As a consequence, the map obtained by adjuntion

$$\eta'_f : KGL_T \simeq f^*KGL_S \to f^!KGL_S$$

is also a morphism of KGL-module. This implies that the morphism η'_f (and thus also Tr_f^{KGL}) is completely determined by the element

$$\eta'_f \in \mathrm{Hom}_{KGL}(KGL_T, f^!KGL_S) \simeq \mathrm{Hom}_{\mathrm{SH}(T)}(\mathbb{1}_T, f^!KGL_S)\,.$$

Moreover, as p is smooth, there is a canonical isomorphism $p^!KGL_S \simeq KGL_P$ (by relative purity for p and by periodicity; see [**Rio10**, lemma 6.1.3.3]). From there, we deduce from Theorem 13.6.3 that we have a canonical isomorphism

$$f^!KGL_S \simeq i^!KGL_P \simeq KGL_T\,.$$

This implies that we have an isomorphism:

$$\mathrm{Hom}_{\mathrm{SH}(T)}(\mathbb{1}_T, f^!KGL_S) \simeq K_0(T)\,.$$

Hence the map η'_f is completely determined by a class in $K_0(T)$. The problem of the functoriality of trace maps in the motivic category $\mathrm{Ho}(KGL\text{-}\mathrm{mod})$ is thus a matter of functoriality of this element η'_f in K_0, which can be translated faithfully to the problem of the functoriality of pushforwards for K_0.

However, the only property of trace maps we shall use here is the following.

Proposition 13.7.6. *Let $f : T \to S$ be a finite flat morphism of regular schemes such that the $\mathcal{O}_S$-module $f_*\mathcal{O}_T$ is (globally) free of rank d. Then the following composite map*

$$KGL_S \to f_*f^*KGL_S \xrightarrow{\mathrm{Tr}_f^{KGL}} KGL_S$$

is equal to $d.1_{KGL_S}$ in $\mathrm{Ho}(KGL\text{-}\mathrm{mod})(S)$ (whence in $\mathrm{SH}(S)$).

Proof. Let φ be the composite map of Ho(KGL-mod)(S)

$$KGL_S \to f_*f^*KGL_S \xrightarrow{\mathrm{Tr}_f} KGL_S\,.$$

As φ is KGL_S-linear by construction, it corresponds to an element

$$\varphi \in \mathrm{Hom}_{KGL}(KGL_S, KGL_S) \simeq \mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{1}_S, KGL_S) \simeq K_0(S)\,.$$

According to the commutative diagram (13.7.3.1), if we apply the functor $\mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{1}_S, -)$ to φ, we obtain through the evident canonical isomorphisms the composition of the usual pullback and pushforward by f in K-theory:

$$K_0(S) \xrightarrow{f^*} K_0(T) \xrightarrow{f_*} K_0(S).$$

With these notations, the element of $K_0(S)$ corresponding to φ is the pushforward of $1_T = f^*(1_S)$ by f, while the element corresponding to the identity of KGL_S is of course 1_S. Under our assumptions on f, it is obvious that we have the identity $f_*(1_T) = d.1_S \in K_0(S)$. This means that φ is d times the identity of KGL_S. □

14. Beilinson motives

14.1. The γ-filtration.

14.1.1. We denote by $KGL_{\mathbf{Q}}$ the $\mathbf{Q}$-localization of the absolute ring spectrum KGL, considered as a cartesian section of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$. From [**Rio10**, 5.3.10], this spectrum has the following property:

(K5) For any scheme S, there exists a canonical decomposition, called the *Adams decomposition*

$$KGL_{\mathbf{Q},S} \simeq \bigoplus_{i\in\mathbf{Z}} KGL_S^{(i)}$$

compatible with base change and such that for any regular scheme S, the isomorphism of (K2) induces an isomorphism:

$$\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})}\Big(\mathbf{Q}_S(X)[n], KGL_S^{(i)}\Big) \simeq K_n^{(i)}(S) := Gr_\gamma^i K_n(S)_{\mathbf{Q}}$$

where the right hand side is the i-th graded piece of the γ-filtration on K-theory groups.

We will denote by

$$\pi_i : KGL_{\mathbf{Q},S} \to KGL_S^{(i)},$$
$$\text{resp. } \iota_i : KGL_S^{(i)} \to KGL_{\mathbf{Q},S}$$

the projection (resp. inclusion) defined by the decomposition (K3) and we put $p_i = \iota_i\pi_i$ for the corresponding projector on $KGL_{\mathbf{Q},S}$.

Definition 14.1.2 (Riou). We define the *Beilinson motivic cohomology spectrum* as the rational Tate spectrum $H_{\mathrm{B},S} = KGL_S^{(0)}$.

Remark 14.1.3. Note that, by definition, for any morphism of schemes $f : T \to S$, we have $f^*H_{\mathrm{B},S} \simeq H_{\mathrm{B},T}$.

Lemma 14.1.4. *The isomorphism γ_u of* (13.2.1.1) *is homogenous of degree* $+1$ *with respect to the graduation (K5). In other words, for any integer $i \in \mathbf{Z}$, the following composite map is an isomorphism*

$$KGL^{(i)}(1)[2] \xrightarrow{\iota_i} KGL_{\mathbf{Q}}(1)[2] \xrightarrow{\gamma_u} KGL_{\mathbf{Q}} \xrightarrow{\pi_i} KGL^{(i+1)}.$$

For any integer $i \in \mathbf{Z}$, we thus get a canonical isomorphism

(14.1.4.1) $$H_{\mathrm{B}}(i)[2i] \xrightarrow{\sim} KGL^{(i)}.$$

Proof. It is sufficient to check that, for $j \neq i+1$,

$$\begin{cases} p_j \circ \gamma_u \circ p_i = 0, \\ p_j \circ \gamma_u^{-1} \circ p_i = 0 \end{cases}$$

in $\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})}(KGL_{\mathbf{Q}}, KGL_{\mathbf{Q}})$. But according to [**Rio10**, 5.3.1 and 5.3.6], we have only to check these equalities for the induced endomorphism of K_0 (seen as a presheaf on the category of smooth schemes over Spec $(\mathbf{Z})$). This follows then from the compatibility of the projective bundle isomorphism with the γ-filtration; see [**SGA6**, Exp. VI, 5.6]. □

14.1.5. Recall from [**NSØ09**] that $KGL_{\mathbf{Q}}$ is canonically isomorphic (with respect to the orientation 13.2.2) to the universal oriented rational ring spectrum with multiplicative formal group law introduced in [**NSØ09**]. The isomorphism of the preceding corollary shows in particular that $H_{Б}$ is obtained from $KGL_{\mathbf{Q}}$ by killing the elements β^n for $n \neq 0$. In particular, this shows that $H_{Б}$ is canonically isomorphic to the spectrum denoted by $\mathsf{L}\mathbf{Q}$ in [**NSØ09**], which corresponds to the universal additive formal group law over $\mathbf{Q}$. This implies that $H_{Б}$ has a natural structure of a (commutative) ring spectrum.

Proposition 14.1.6. *The multiplication map*

$$\mu : H_{Б} \otimes H_{Б} \to H_{Б}$$

is an isomorphism.

This trivially implies that the following map is an isomorphism:

(14.1.6.1) $$1 \otimes \eta : H_{Б} \to H_{Б} \otimes H_{Б}.$$

Proof. It is enough to treat the case $S = \mathrm{Spec}\,(\mathbf{Z})$. We will proove that the projector

$$\psi : H_{Б} \otimes H_{Б} \xrightarrow{\mu} H_{Б} \xrightarrow{1\otimes\eta} H_{Б} \otimes H_{Б}$$

is an isomorphism (in which case it is in fact the identity). We do that for the isomorphic ring spectrum $\mathsf{L}\mathbf{Q}$.

Let $\mathsf{H}^{top}\mathbf{Q}$ be the topological spectrum representing rational singular cohomology. In the terminology of [**NSØ09**], $\mathsf{L}\mathbf{Q}$ is a Tate spectrum representing the Landweber exact cohomology which corresponds to the Adams graded MU_*-algebra $\mathbf{Q}$ obtained by killing every generators of the Lazard ring MU_*. The corresponding topological spectrum is of course $\mathsf{H}^{top}\mathbf{Q}$.
According to [**NSØ09**, 9.2], the spectrum $E = \mathsf{L}\mathbf{Q} \otimes \mathsf{L}\mathbf{Q}$ is a Landweber exact spectrum corresponding to the MU_*-algebra $\mathbf{Q} \otimes_{MU_*} \mathbf{Q} = \mathbf{Q}$. In particular, the corresponding topological spectrum is simply $\mathsf{H}^{top}\mathbf{Q}$. Thus, according to [**NSØ09**, 9.7], applied with $F = E = \mathsf{L}\mathbf{Q} \otimes \mathsf{L}\mathbf{Q}$, we get an isomorphism of $\mathbf{Q}$-vector spaces

$$\mathrm{End}(\mathsf{L}\mathbf{Q} \otimes \mathsf{L}\mathbf{Q}) = \mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}, E_{**}) = \mathbf{Q}.$$

Thus $\psi = \lambda.Id$ for $\lambda \in \mathbf{Q}$. But $\lambda = 0$ is excluded because ψ is a projector on a non trivial factor, so that we can conclude. □

14.2. Definition.

Definition 14.2.1. Let S be any scheme.

We say that an object E of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ is $H_{Б}$-*acyclic* if $H_{Б} \otimes E = 0$ in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$. A morphism of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ is an $H_{Б}$-*equivalence* if its cone is $H_{Б}$-acyclic (or, equivalently, if its tensor product with $H_{Б}$ is an isomorphism).

An object M of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ is $H_{Б}$-*local* if, for any $H_{Б}$-acyclic object E, the group $\mathrm{Hom}(E, M)$ vanishes.

We denote by $\mathrm{DM}_{Б}(S)$ the Verdier quotient of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ by the localizing subcategory made of $H_{Б}$-acyclic objects (i.e. the localization of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ by the class of $H_{Б}$-equivalences).

The objects of $\mathrm{DM}_{Б}(S)$ are called the *Beilinson motives*.

Proposition 14.2.2. *An object E of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ is $H_{Б}$-acyclic if and only if we have $KGL_{\mathbf{Q}} \otimes E = 0$.*

Proof. This follows immediately from property (K5) (see 14.1.1) and Lemma 14.1.4. $\square$

Proposition 14.2.3. *The localization functor* $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q}) \to \mathrm{DM}_{\mathrm{B}}(S)$ *admits a fully faithful right adjoint whose essential image in* $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ *is the full subcategory spanned by* H_{B}*-local objects. More precisely, there is a left Bousfield localization of the stable model category of symmetric Tate spectra* $\mathrm{Sp}(S,\mathbf{Q})$ *by a small set of maps whose homotopy category is precisely* $\mathrm{DM}_{\mathrm{B}}(S)$.

Proof. For each smooth S-scheme X and any integers $n, i \in \mathbf{Z}$, we have a functor with values in the category of $\mathbf{Q}$-vector spaces

$$F_{X,n,i} = \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})}(\Sigma^\infty \mathbf{Q}_S(X), H_{\mathrm{B}} \otimes (-)(i)[n]) : \mathrm{Sp}(S,\mathbf{Q}) \to \mathbf{Q}\text{-}\mathrm{mod}$$

which preserves filtered colimits. We define the class of H_{B}-weak equivalences as the class of maps of $\mathrm{Sp}(S,\mathbf{Q})$ whose image by $F_{X,n,i}$ is an isomorphism for all X and n, i. By virtue of [**Bek00**, Prop. 1.15 and 1.18], we can apply Smith's theorem [**Bek00**, Theorem 1.7] (with the class of cofibrations of $\mathrm{Sp}(S,\mathbf{Q})$), which implies the proposition. $\square$

Remark 14.2.4. We shall often make the abuse of considering $\mathrm{DM}_{\mathrm{B}}(S)$ as a full subcategory in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, with an implicit reference to the preceding proposition.

Note that H_{B}-acyclic objects are stable by the operations f^*, $f_\sharp$ and $\otimes$, so that applying Corollary 5.2.5, we obtain a premotivic category DM_{B} together with a premotivic adjunction:

$$\beta^* : \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \rightleftarrows \mathrm{DM}_{\mathrm{B}} : \beta_*. \tag{14.2.4.1}$$

Proposition 14.2.5. *The spectrum* $H_{\mathrm{B},S}$ *is* H_{B}*-local and the unit map* $\eta_{H_{\mathrm{B}}} : \mathbb{1} \to H_{\mathrm{B},S}$ *is an* H_{B}*-equivalence in* $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$.

Proof. The unit map $\eta : \mathbb{1}_S \to H_{\mathrm{B},S}$ is an H_{B}-equivalence by 14.1.6.

Consider a rational spectrum E over S such that $E \otimes H_{\mathrm{B}} = 0$ and a map $f : E \to H_{\mathrm{B}}$. It follows trivially from the commutative diagram

$$\begin{array}{ccccc}
E & \xrightarrow{f} & H_{\mathrm{B},S} & & \\
{\scriptstyle 1\otimes\eta}\downarrow & & {\scriptstyle 1\otimes\eta}\downarrow & \searrow{=} & \\
E \otimes H_{\mathrm{B},S} & \xrightarrow{f\otimes 1} & H_{\mathrm{B},S} \otimes H_{\mathrm{B},S} & \xrightarrow{\mu} & H_{\mathrm{B},S}
\end{array}$$

that $f = 0$, which shows that $H_{\mathrm{B},S}$ is H_{B}-local. $\square$

Corollary 14.2.6. *The family of ring spectra* $H_{\mathrm{B},S}$ *comes from a cofibrant cartesian commutative monoid (7.2.10) of the symmetric monoidal fibred model category of Tate spectra over the category of schemes.*

Proof. By virtue of Proposition 14.2.5 and of Corollary 7.1.9, there exists a cofibrant commutative monoid in the model category of symmetric Tate spectra over $\mathrm{Spec}(\mathbf{Z})$ which is canonically isomorphic to $H_{\mathrm{B},\mathbf{Z}}$ in $\mathrm{D}_{\mathbf{A}^1}(\mathrm{Spec}(\mathbf{Z}), \mathbf{Q})$ (as commutative ring spectrum). For a morphism of schemes $f : S \to \mathrm{Spec}(\mathbf{Z})$, we can then define $H_{\mathrm{B},S}$ as the pullback of $H_{\mathrm{B},\mathbf{Z}}$ (at the level of the model categories); using Proposition 7.1.11, we see that this defines a cofibrant cartesian commutative monoid on the fibred category of spectra which is isomorphic to $H_{\mathrm{B},S}$ as commutative ring spectra in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$. $\square$

14.2.7. From now on, we shall assume that H_{B} is given by a cofibrant cartesian commutative monoid of the symmetric monoidal fibred model category of Tate spectra over the category of schemes. By virtue of propositions 7.2.11 and 7.2.18), we get the motivic category $\mathrm{Ho}(H_{\mathrm{B}}\text{-}\mathrm{mod})$ of H_{B}-modules, together with a commutative diagram of morphisms of premotivic categories

$$\begin{array}{ccc}
\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} & \xrightarrow{H_{\mathrm{B}}\otimes(-)} & \mathrm{Ho}(H_{\mathrm{B}}\text{-}\mathrm{mod}) \\
& {\scriptstyle\beta}\searrow \quad \nearrow{\scriptstyle\varphi} & \\
& \mathrm{DM}_{\mathrm{B}} &
\end{array}$$

(any H_{B}-acyclic object becomes null in the homotopy category of H_{B}-modules by definition, so that $H_{\mathrm{B}} \otimes (-)$ factors uniquely through DM_{B} by the universal property of localization).

PROPOSITION 14.2.8. *The forgetful functor* $U : \mathrm{Ho}(H_{\text{Б}}\text{-mod})(S) \to \mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ *is fully faithful.*

PROOF. We have to prove that, for any $H_{\text{Б},S}$-module M, the map

$$H_{\text{Б},S} \otimes M \to M$$

is an isomorphism in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$. As this is a natural transformation between exact functors which commute with small sums, and as $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ is a compactly generated triangulated category, it is sufficient to check this for $M = H_{\text{Б},S} \otimes E$, with E a (compact) object of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ (see 7.2.7). In this case, this follows immediately from the isomorphism (14.1.6.1). □

THEOREM 14.2.9. *The functor* $\mathrm{DM}_{\text{Б}}(S) \to \mathrm{Ho}(H_{\text{Б},S}\text{-mod})$ *is an equivalence of triangulated monoidal categories.*

PROOF. This follows formally from the preceding proposition by definition of $\mathrm{DM}_{\text{Б}}$ (see for instance [**GZ67**, Chap. I, Prop. 1.3]). □

REMARK 14.2.10. The preceding theorem shows that the premotivic category $\mathrm{Ho}(H_{\text{Б}}\text{-mod})$ as well as the morphism $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \to \mathrm{Ho}(H_{\text{Б}}\text{-mod})$ are completely independent of the choice of the strictification of the (commutative) monoid structure on $H_{\text{Б}}$ given by Corollary 14.2.6.

COROLLARY 14.2.11. *The premotivic category* $\mathrm{DM}_{\text{Б}} \simeq \mathrm{Ho}(H_{\text{Б}}\text{-mod})$ *is a* $\mathbf{Q}$*-linear motivic category.*

PROOF. It follows from Proposition 7.2.18 and Theorem 14.2.9 that $\mathrm{DM}_{\text{Б}}$ satisfies the homotopy, stability and localization properties (because this is true for $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ by 6.2.2). It is also well generated because it is a localization of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$. Thus we can apply Remark 2.4.47 to conclude. □

REMARK 14.2.12. One can also prove that $\mathrm{DM}_{\text{Б}}$ is motivic much more directly: this follows from the fact that $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ is motivic and that the six Grothendieck operations preserve $H_{\text{Б}}$-acyclic objects, so that all the properties of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ induce their analogs on $\mathrm{DM}_{\text{Б}}$ by the 2-universal property of localization (we leave this as an easy exercise for the reader).

DEFINITION 14.2.13. For a scheme X, we define its Beilinson motivic cohomology by the formula:

$$H^q_{\text{Б}}(X, \mathbf{Q}(p)) = \mathrm{Hom}_{\mathrm{DM}_{\text{Б}}(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q])\,.$$

In fact, according to the preceding corollary, the cohomology theory defined above is represented by the ring spectrum $H_{\text{Б}}$. In particular, we can now justify the terminology of Beilinson motives:

COROLLARY 14.2.14. *For any regular scheme* X*, we have a canonical isomorphism*

$$H^q_{\text{Б}}(X, \mathbf{Q}(p)) \simeq Gr^p_\gamma\, K_{2p-q}(X)_{\mathbf{Q}}\,.$$

14.2.15. Recall from Paragraph 14.1.5 that $H_{\text{Б},S}$ is canonically oriented for any scheme S. Moreover, these orientations are compatible with pullbacks with respect to S. This means in particular that the motivic triangulated category $\mathrm{DM}_{\text{Б}}$ is oriented (see Example 12.2.3).

In particular, the fibred category $\mathrm{DM}_{\text{Б}}$ satisfies the usual Grothendieck 6 functors formalism. We refer the reader to Theorem 2.4.50 for the precise statement.

It was remarked in Paragraph 14.1.5 that $H_{\text{Б},S}$ is the universal oriented ring spectrum with additive formal group law over S. This property can be expressed by the following nice description of Beilinson motives:

COROLLARY 14.2.16. *Let* E *be a rational spectrum over* S*. The following conditions are equivalent:*

(i) E *is a Beilinson motive (i.e. is in the essential image of the right adjoint of the localization functor* $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \to \mathrm{DM}_{\text{Б}}$*);*
(ii) E *is* $H_{\text{Б}}$*-local;*
(iii) the map $\eta \otimes 1_E : E \to H_{\text{Б}} \otimes E$ *is an isomorphism;*

(iv) E *is an* H_{B}*-module in* $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$*;*

(v) E *is admits a strict* H_{B}*-module structure.*

If, in addition, E *is a commutative ring spectrum, these conditions are equivalent to the following ones:*

(Ri) E *is orientable;*

(Rii) E *is an* H_{B}*-algebra;*

(Riii) E *admits a unique structure of* H_{B}*-algebra;*

And, if E *is a strict commutative ring spectrum, these conditions are equivalent to the following conditions:*

(Riv) there exists a morphism of commutative monoids $H_{\mathrm{B}} \to E$ *in the stable model category of Tate spectra;*

(Rv) there exists a unique morphism $H_{\mathrm{B}} \to E$ *in the homotopy category of commutative monoids of the category of Tate spectra.*

Proof. The equivalence between statements (i)–(v) follows immediately from 14.2.9. If E is a ring spectrum, the equivalence with (Ri), (Rii) and R(iii) is a consequence of 12.2.10 and of the fact that $MGL_{\mathbf{Q}}$ is H_{B}-local; see [**NSØ09**, Cor. 10.6]. It remains to prove the equivalence with (Riv) and (Rv). Then, E is H_{B}-local if and only if the map $E \to H_{\mathrm{B}} \otimes E$ is an isomorphism. But this map can be seen as a morphism of strict commutative ring spectra (using the model structure of 7.1.8 applied to the model category of Tate spectra) whose target is clearly an H_{B}-algebra, so that (Riv) is equivalent to (ii). It remains to check that there is at most one strict H_{B}-algebra structure on E (up to homotopy), which follows from the fact that H_{B} is the initial object in the homotopy category of commutative monoids of the model category given by Theorem 7.1.8 applied to the model structure of Proposition 14.2.3. □

Corollary 14.2.17. *One has the following properties.*

(1) The ring structure on the spectrum H_{B} *is given by the following structural maps (with the notations of 14.1.1).*

$$H_{\mathrm{B}} \otimes H_{\mathrm{B}} \xrightarrow{\iota_0 \otimes \iota_0} KGL_{\mathbf{Q}} \otimes KGL_{\mathbf{Q}} \xrightarrow{\mu_{KGL}} KGL_{\mathbf{Q}} \xrightarrow{\pi_0} H_{\mathrm{B}},$$
$$\mathbf{Q} \xrightarrow{\eta_{KGL}} KGL_{\mathbf{Q}} \xrightarrow{\pi_0} H_{\mathrm{B}}.$$

(2) The map $\imath_0 : H_{\mathrm{B}} \to KGL_{\mathbf{Q}}$ *is compatible with the monoid structures.*

(3) Let $H_{\mathrm{B}}[t,t^{-1}] = \bigoplus_{i\in\mathbf{Z}} H_{\mathrm{B}}(i)[2i]$ *be the free* H_{B}*-algebra generated by one invertible generator* t *of bidegree* $(2,1)$*. Then the section* $u : \mathbf{Q}(1)[2] \to KGL_{\mathbf{Q}}$ *induces an isomorphism of* H_{B}*-algebras:*

$$\gamma'_u : H_{\mathrm{B}}[t,t^{-1}] \to KGL_{\mathbf{Q}}.$$

Proof. Property (1) follows from properties (2) and (3). Property (2) is a trivial consequence of the previous corollary. Using the isomorphisms (14.1.4.1) of Lemma 14.1.4, we get a canonical isomorphism

$$H_{\mathrm{B},S}[t,t^{-1}] \xrightarrow{\sim} \bigoplus_{i\in\mathbf{Z}} KGL^{(i)}.$$

Through this isomorphism, the map γ'_u corresponds to the Adams decomposition (i.e. to the isomorphism (K5) of 14.1.1) from which we deduce property (3). □

Remark 14.2.18. One deduces easily, from the preceding proposition and from 14.1.6, another proof of the fact that $KGL_{\mathbf{Q}}$ is a strict commutative ring spectrum.

The isomorphism (3) is in fact compatible with the gradings of each term: the factor $H_{\mathrm{B}}.t^i$ is sent to the factor $KGL^{(i)}$. Recall also the parameter t corresponds to the unit β^{-1} in $KGL^{*,*}$.

Corollary 14.2.19. *The Adams decomposition is compatible with the monoid structure on* $KGL_{\mathbf{Q}}$*: for any integer* i, j, l *such that* $l \neq i+j$*, the following map is zero.*

$$KGL^{(i)} \otimes KGL^{(j)} \xrightarrow{\iota_i \otimes \iota_j} KGL_{\mathbf{Q}} \otimes KGL_{\mathbf{Q}} \xrightarrow{\mu} KGL_{\mathbf{Q}} \xrightarrow{\pi_l} KGL^{(l)}$$

14.2.20. Let R be a **Q**-algebra with structural morphism φ. Recall from Paragraph 5.3.36 that we get an adjunction of premotivic triangulated categories:

$$\varphi^* : \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \to \mathrm{D}_{\mathbf{A}^1,R} : \varphi_*.$$

Moreover, for any object M and N of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$, the canonical map

$$\mathrm{Hom}(M,N) \otimes_{\mathbf{Q}} R \to \mathrm{Hom}(\varphi^*(M), \varphi^*(N)). \tag{14.2.20.1}$$

is an isomorphism provided M is compact or R is a finite **Q**-vector space.

In particular, the ring spectrum $KGL_R := \varphi^*(KGL_{\mathbf{Q}})$ represents Quillen algebraic K-theory with coefficients in R over regular schemes. We can repeat Definition 14.2.1 with R-coefficients and this gives the category $\mathrm{DM}_{\mathrm{B}}(S,R)$ of Beilinson motives with R-coefficients together with an adjuntion:

$$\varphi^* : \mathrm{DM}_{\mathrm{B}} \to \mathrm{DM}_{\mathrm{B}}(-,R) : \varphi_*.$$

Moreover, using the canonical map (14.2.20.1) and the fact it is an isomorphism when M is a constructible Beilinson motives, we immediately extend all the properties proved so far from **Q**-coefficients to R-coefficients.

14.3. Motivic proper descent. Recall from Definition 4.3.2 we have defined the notion of continuity for a triangulated premotivic category which is the homotopy category of a premotivic model category, such as the triangulated motivic category DM_{B} – in this case, the notion of continuity is relative to the Tate twist.

PROPOSITION 14.3.1. *The motivic triangulated category* DM_{B} *is continuous.*

PROOF. We consider the adjunction (14.2.4.1). According to Theorem 14.2.9, the functor β_* commutes with pullbacks by arbitrary morphisms. Thus the continuity property for DM_{B} follows from the continuity property for $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ which was established in Example 6.1.13. □

We will give the main applications of continuity in the section on constructible Beilinson motives. Recall from 4.3.9 the following corollary of the continuity property of the motivic category DM_{B}:

COROLLARY 14.3.2. *Let X be a scheme, and consider an X-scheme Y of finite type. Given a point $x \in X$, we denote by X_x^h the spectrum of the local henselian ring of X at the point x. Let $a_x : Y \times_X X_x^h \to Y$ be the canonical map. Then the family of functors*

$$\mathrm{DM}_{\mathrm{B}}(Y) \to \mathrm{DM}_{\mathrm{B}}(Y \times_X X_x^h)\ , \quad E \mapsto a_x^*(E)$$

is conservative.

As the reader might expect, this proposition is very useful to reduce global properties of the motivic category DM_{B} to local properties. This is in particular illustrated by the following proposition.

THEOREM 14.3.3. *The motivic category* DM_{B} *is separated (on the category of noetherian schemes of finite dimension).*

PROOF. According to Proposition 2.3.9, it is sufficient to check that, for any finite surjective morphism $f : T \to S$, the pullback functor

$$f^* : \mathrm{DM}_{\mathrm{B}}(S) \to \mathrm{DM}_{\mathrm{B}}(T)$$

is conservative.

We argue by induction on the dimension of S.

Let us first treat the case where $\dim(S) = 0$. Using the localization property, we can assume that S and T are reduced (*cf.* 2.3.6). Then S is a disjoint sum of spectra of fields. In particular, f is not only finite surjective but also flat. Moreover, it is also globally free. It will be sufficient to prove that, for any Beilinson motive E over S, the adjunction map

$$E \to f_* f^*(E)$$

is a monomorphism in DM_{B}. Using the projection formula in DM_{B} applied to the finite morphism f (point (5) of Theorem 2.4.50), this latter map is isomorphic to

$$\big(H_{\text{B}} \to f_* f^*(H_{\text{B}})\big) \otimes 1_E.$$

We are finally reduced to prove that the map $H_{\text{B},S} \to f_* f^* H_{\text{B},S}$ is a monomorphism in DM_{B} (any monomorphism of a triangulated category splits). As $H_{\text{B},S}$ is a direct factor of $KGL_{\mathbf{Q},S}$, it is sufficient to find a retraction of the adjunction map

$$KGL_{\mathbf{Q},S} \to f_* f^* KGL_{\mathbf{Q},S}\ ,$$

and this follows from Proposition 13.7.6.

Let us finally solve the induction process. Applying the preceding proposition, we can assume that S is local henselian. Let s be the closed point of S and U be the open complement. Let f_s (resp. f_U) be the pullback of f above s (resp. U). Using the localization property of DM_{B} and the base change isomorphisms (point (4) of Theorem 2.4.50), it is sufficient to treat the case of the finite morphisms f_U and f_s. The case of f_U follows by the induction hypothesis while the case of f_s follows from the case treated previously. This ends up the induction process. □

According respectively to Proposition 3.3.33 and Theorem 3.3.37, we deduce from the preceding proposition the following result:

THEOREM 14.3.4. *(1) The motivic category* DM_{B} *satisfies étale descent.*

(2) The motivic category DM_{B} *satisfies* h-*descent when restricted to quasi-excellent schemes.*

Recall this means that for any étale hypercover (resp. h-hypercover of a quasi-excellent scheme) $p : \mathscr{X} \to X$ and for any Beilinson motive E over X, the map

$$p^* : \mathbf{R}\Gamma(X, E) \to \mathbf{R}\Gamma(\mathscr{X}, E) = \mathbf{R}\varprojlim_{n} \mathbf{R}\Gamma(\mathscr{X}_n, E)$$

is an isomorphism in the derived category of the category of $\mathbf{Q}$-vector spaces (see Corollary 3.2.17 taking into account Definition 3.2.20).

14.4. Motivic absolute purity.

THEOREM 14.4.1 (Absolute purity). *Let $i : Z \to S$ be a closed immersion between regular schemes. Assume i has pure codimension n.*

Then, considering the notations of 14.1.1, definition 13.5.4, and the identification (14.1.4.1)*, the composed map*

$$H_{\text{B},Z} \xrightarrow{\iota_0} KGL_{\mathbf{Q},Z} \xrightarrow{\eta'_i} i^! KGL_{\mathbf{Q},S} \xrightarrow{\pi_n} i^! H_{\text{B},S}(n)[2n]$$

is an isomorphism.

This isomorphism, of equivalently the map obtained by adjunction:

$$i_*(H_{\text{B},Z}) \to H_{\text{B},S}(n)[2n]$$

is called the *fundamental class* associated with i. In fact, this is a canonical class in the Beilinson motivic cohomology of X with support in Z of bidegree $(2n, n)$.

REMARK 14.4.2. It follows from Remark 13.5.5 that the fundamental class in Beilinson motivic cohomology is compatible with pullback with respect to Tor-independant square.

PROOF. We have only to check that the above composition induces an isomorphism after applying the functor $\mathrm{Hom}(\mathbf{Q}_S(X), -(a)[b])$ for a smooth S-scheme X and a couple of integers $(a, b) \in \mathbf{Z}^2$. Using Remark 13.5.5(3), this composition is compatible with smooth base change and we can assume $X = S$. Let us consider the projector

$$p_a : K_r^Z(S)_{\mathbf{Q}} = K_r(S/S - Z)_{\mathbf{Q}} \to K_r(S/S - Z)_{\mathbf{Q}}$$

induced by $\pi_a \circ \iota_a : KGL_{\mathbf{Q}} \to KGL_{\mathbf{Q}}$, and denote by $K_r^{(a)}(S/S - Z)$ (with $r = 2a - b$) its image. By virtue of Propostion 13.6.1, we only have to check that the following composite is an isomorphism:

$$\rho_i : K_r^{(a)}(Z) \xrightarrow{\iota_a} K_r(Z)_{\mathbf{Q}} \xrightarrow{\mathfrak{q}_i^{-1}} K_r(S/S - Z)_{\mathbf{Q}} \xrightarrow{\pi_a} K_r^{(a+n)}(S/S - Z).$$

From 13.5.2, the morphism ρ_i is functorial with respect to Tor-independant cartesian squares of regular schemes (*cf.* 13.5.1). Thus, using again the deformation diagram (13.6.1.1), we get a commutative diagram

$$\begin{array}{ccccc} K_r^{(a)}(Z) & \longrightarrow & K_r^{(a)}(\mathbf{A}^1_Z) & \longleftarrow & K_r^{(a)}(Z) \\ {\scriptstyle \rho_i}\downarrow & & \downarrow & & \downarrow{\scriptstyle \rho_s} \\ K_r^{(a+n)}(S/S-Z) & \longrightarrow & K_r^{(a+n)}(D/D-\mathbf{A}^1_Z) & \longleftarrow & K_r^{(a+n)}(P/P-Z) \end{array}$$

in which any of the horizontal maps is an isomorphism (as a direct factor of an isomorphism). Thus, we are reduced to the case of the closed immersion $s : Z \to P$, canonical section of the projectivisation of a vector bundle E (where E is the normal bundle of the closed immersion i). Moreover, as the assertion is local on Z, we may assume E is a trivial vector bundle.

Let $p : P \to Z$ be the canonical projection, $j : P - Z \to P$ the obvious open immersion. Considering the element $v' := \big([\mathcal{O}(1)] - 1\big)$ of $K_0(P)$, we let v be its projection on the first graded part of the γ-filtration, $v \in K_0^{(1)}(P)$.
Recall that, according to the projective bundle formula, the horizontal lines in the following commutative diagram are split short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_r(P/P-Z)_{\mathbf{Q}} & \xrightarrow{\nu} & K_r(P)_{\mathbf{Q}} & \xrightarrow{j^*} & K_r(P-Z)_{\mathbf{Q}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_r^{(a+n)}(P/P-Z) & \xrightarrow{\nu'} & K_r^{(a+n)}(P) & \longrightarrow & K_r^{(a+n)}(P-Z) & \longrightarrow & 0. \end{array}$$

By assumption on E, v^n lies in the kernel of j^* and the diagram allows to identify the graded piece $K_r^{(a+n)}(P/P-Z)$ with the submodule of $K_r^{(a+n)}(P)$ of the form $K_r^{(a)}(Z).v^n$.

On the other hand, $j^*s_* = 0$: there exists a unique element $\epsilon \in K_0(Z)$ such that $s_*(1) = p^*(\epsilon).v^n$ in $K_0(P)$. From the relation $p_*s_*(1) = 1$, we obtain that ϵ is a unit in $K_0(Z)$, with inverse the element $p_*(v^n)$. By virtue of [**SGA6**, Exp. VI, Cor. 5.8], $p_*(v^n)$ belongs to the 0-th γ-graded part of $K_0(P)_{\mathbf{Q}}$ so that the same holds for its inverse ϵ. In the end, for any element $z \in K_r(Z)$, we get the following expression:

$$s_*(z) = s_*(1.s^*p^*(z)) = s_*(1).p^*(z) = p^*(\epsilon.z).v^n.$$

Thus, the commutative diagram

$$\begin{array}{ccccccc} K_r^{(a)}(Z) & \longrightarrow & K_r(Z)_{\mathbf{Q}} & \xrightarrow{\mathfrak{q}_s^{-1}} & K_r(P/P-Z)_{\mathbf{Q}} & \longrightarrow & K_r^{(a+n)}(P/P-Z) \\ & & & \searrow^{s_*} & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ & & & & K_r(P)_{\mathbf{Q}} & \longrightarrow & K_r^{(n)}(P) \end{array}$$

implies that the isomorphism $\mathfrak{q}_s^{-1}$ preserves the γ-filtrations (up to a shift by n). Hence it induces an isomorphism on the graded pieces by functoriality. □

15. Constructible Beilinson motives

15.1. Definition and basic properties. In this section, we apply the general results of Section 4 to the triangulated motivic category DM_{B}. Let us first recall the definition of constructibility (Def. 4.2.1) which corresponds to the Tate twist.

Definition 15.1.1. Given any scheme S, we define the category $\mathrm{DM}_{\mathrm{B},c}(S)$ of *constructible Beilinson motives* over S as the thick triangulated subcategory of $\mathrm{DM}_{\mathrm{B}}(S)$ generated by the motives of the form $M_S(X)(i)$ for a smooth S-scheme X and an integer $i \in \mathbf{Z}$.

Remark 15.1.2. Constructible Beilinson motives plays towards Beilinson motives the same role than complexes of étale sheaves with bounded cohomology and constructible cohomology sheaves plays agains complexes of étale sheaves (in the case of torsion coefficients prime to the residue characteristics). This fact will be even more striking after Theorems 15.2.1 and 15.2.4.

15.1.3. Recall from Corollary 6.2.2 that $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ is compactly generated by the Tate twist. According to Theorem 14.2.9, the same is true for the motivic category $\mathrm{DM}_{\text{Б}}$. Thus Proposition 1.4.11 gives the following criterion of constructibility for Beilinson motives:

Proposition 15.1.4. *Given any base scheme S, a Beilinson motive $\mathcal{M}$ over S is constructible if and only if it is compact.*

Remark 15.1.5. In the sequel, we will give several concrete descriptions of the category of constructible Beilinson motives (see Corollaries 16.1.6 and 16.2.16).

Recall from Proposition 14.3.1 that $\mathrm{DM}_{\text{Б}}$ is continuous (with respect to the Tate twist). Proposition 4.3.4 thus implies the following properties of constructibles Beilinson motives:

Proposition 15.1.6. *Let $(S_\alpha)_{\alpha\in A}$ be a pro-object of noetherian finite dimensional schemes with affine transition maps and such that the scheme $S = \varprojlim_{\alpha\in A} S_\alpha$ is noetherian of finite dimension.*

Then the canonical functor:

$$2\text{-}\varinjlim_{\alpha} \mathrm{DM}_{\text{Б},c}(S_\alpha) \to \mathrm{DM}_{\text{Б},c}(S) \tag{15.1.6.1}$$

is an equivalence of monoidal triangulated categories.

Example 15.1.7. Under the assumptions of the abobe proposition, for any couple of integers (p,q), the canonical map

$$\varinjlim_{\alpha} H^q_{\text{Б}}(S_\alpha, \mathbf{Q}(p)) \to H^q_{\text{Б}}(S, \mathbf{Q}(p))$$

is an isomorphism.[88]

15.2. Grothendieck 6 functors formalism and duality. The motivic triangulated category $\mathrm{DM}_{\text{Б}}$ is separated (14.3.3) and weakly pure (see Definition 4.2.20 ; this follows directly from Theorem 14.4.1). Thus the abstract Theorem 4.2.29 gives the finiteness theorem, which we state here in an explicit way to help the reader:

Theorem 15.2.1. *The triangulated subcategory $\mathrm{DM}_{\text{Б},c}$ of $\mathrm{DM}_{\text{Б}}$ is stable by the following operations:*

1. *f^* for any morphism of schemes f.*
2. *f_* for any morphism $f : Y \to X$ of finite type such that X is quasi-excellent (resp. any proper morphism f).*
3. *$f_!$ for any separated morphism of finite type f.*
4. *$f^!$ for any separated morphism of finite type f.*
5. *$\otimes_X$ for any scheme X.*
6. *Hom_X for any quasi-excellent scheme X.*

To be more precise, point (1) and (5) are obvious, the non respe condition of point (2) is the hardest fact and follows from Theorem 4.2.24, point (3) as well as the respe condition of point (2) is Corollary 4.2.12, point (4) is Corollary 4.2.28 and point (6) is Corollary 4.2.25.

15.2.2. Let B be an excellent scheme such that $\dim(B) \le 2$. Recall that B satisfies wide resolution of singularities up to quotient singularities (see Def. 4.1.9 and the result of De Jong recalled in 4.1.11). Thus according to Corollary 4.4.3, we get the following description of constructible Beilinson motives:

Proposition 15.2.3. *Let S be a separated B-scheme of finite type, and $T \subset S$ a closed subscheme. Then the triangulated category $\mathrm{DM}_{\text{Б},c}(S)$ is the smallest triangulated category of $\mathrm{DM}_{\text{Б}}(S)$ which contained motives of the form*

$$f_*(\mathbb{1}_X)(n)$$

where n is an integer and $f : X \to S$ is a projective morphism such that X is regular connected and $f^{-1}(T)_{red}$ is either empty, either X of the support of a strict normal crossing divisor.

[88] This result is to be compared with [**Qui73**, Sec. 7, 2.2] – it concerns homotopy invariant K-theory rather than K-theory.

The main motivation to introduce the notion of constructibility is Grothendieck duality. We obtain this duality from the theoretical result on motivic triangulated categories, more precisely Corollary 4.4.24:

THEOREM 15.2.4. *Let B be an excellent scheme such that* $\dim(B) \leq 2$ *and S be a regular separated B-scheme of finite type.*

Then for any separated morphism $f : X \to S$ of finite type, the premotive $f^!(\mathbb{1}_S)$ is a dualizing object of $\mathrm{DM}_{\text{Б},c}(X)$*. In fact, if we put $D_X(M) := \mathit{Hom}_X(M, f^!(\mathbb{1}_S))$ for any constructible Beilinson motives M, the following properties hold:*

(a) For any separated S-scheme of finite type X, the functor D_X preserves constructible objects.

(b) For any separated S-scheme of finite type X, the natural map

$$M \to D_X(D_X(M))$$

is an isomorphism for any constructible Beilinson motive M.

(c) For any separated S-scheme of finite type X, and for any Beilinson motive M and N over X, if N is constructible then we have a canonical isomorphism

$$D_X(M \otimes_X D_X(N)) \simeq \mathit{Hom}_X(M, N)\,.$$

(d) For any morphism between separated S-schemes of finite type $f : Y \to X$, we have natural isomorphisms

$$\begin{aligned} D_Y(f^*(M)) &\simeq f^!(D_X(M)) \\ f^*(D_X(M)) &\simeq D_Y(f^!(M)) \\ D_X(f_!(N)) &\simeq f_*(D_Y(N)) \\ f_!(D_Y(N)) &\simeq D_X(f_*(N)) \end{aligned}$$

where M (resp. N) is a constructible Beilinson motive over X (resp. Y).

15.2.5. Let R be a $\mathbf{Q}$-algebra.[89]
We define the premotivic triangulated category of constructible Beilinson motives with coefficients in R as the category of constructible objects of the category $\mathrm{DM}_{\text{Б}}(-, R)$ defined in Paragraph 14.2.20.

According to *loc. cit.*, for any constructible Beilinson motives with coefficients in $\mathbf{Q}$, we get an isomorphism:

$$\mathrm{Hom}_{\mathrm{DM}_{\text{Б},c}(S)}(M, N) \otimes_{\mathbf{Q}} R \longrightarrow \mathrm{Hom}_{\mathrm{DM}_{\text{Б},c}(S,R)}\big(\mathbf{L}\varphi^*(M), \mathbf{L}\varphi^*(N)\big).$$

It is straightforward to see that this isomorphism allows to extend all the results proved so far for Beilinson motives with coefficient in $\mathbf{Q}$ to the case of R-coefficients.

16. Comparison theorems

16.1. Comparison with Voevodsky motives.

16.1.1. We consider the premotivic adjunction of 11.4.1

$$\gamma^* : \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \rightleftarrows \mathrm{DM}_{\mathbf{Q}} : \gamma_*\,. \tag{16.1.1.1}$$

For a scheme S, $\gamma_*(\mathbb{1}_S)$ is a (strict) commutative ring spectrum, and, for any object M of $\mathrm{DM}_{\mathbf{Q}}(S)$, $\gamma_*(M)$ is naturally endowed with a structure of $\gamma_*(\mathbb{1}_S)$-module. On the other hand, as we have the projective bundle formula in $\mathrm{DM}_{\mathbf{Q}}(S)$ (11.3.4), $\gamma_*(\mathbb{1}_S)$ is orientable (12.2.10), which implies that, for any object M of $\mathrm{DM}_{\mathbf{Q}}(S)$, $\gamma_*(M)$ is an $H_{\text{Б},S}$-module, whence is $H_{\text{Б}}$-local (14.2.16). As consequence, we get a canonical factorization of (16.1.1.1):

$$\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} \xrightarrow{\beta^*} \mathrm{DM}_{\text{Б}} \xrightarrow{\varphi^*} \mathrm{DM}_{\mathbf{Q}}. \tag{16.1.1.2}$$

[89] The examples we have in mind are: $R = E$ is a number field, $R = \mathbf{C}$, $R = \mathbf{Q}_l, \bar{\mathbf{Q}}_l$ for a prime l.

Consider the commutative diagram of premotivic categories

(16.1.1.3)
$$\begin{array}{ccc} \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} & \xrightarrow{\gamma^*} & \mathrm{DM}_{\mathbf{Q}} \\ {\scriptstyle \rho_\sharp}\downarrow & & \downarrow{\scriptstyle \psi_\sharp} \\ \underline{\mathrm{D}}_{\mathbf{A}^1,\mathbf{Q}} & \xrightarrow{\underline{\gamma}^*} & \underline{\mathrm{DM}}_{\mathbf{Q}} \end{array}$$

in which the two vertical maps are the canonical enlargements, and, in particular, are fully faithful (see 6.1.8).

Let t denotes either the qfh-topology or the h-topology. We also have the following commutative triangle

(16.1.1.4)
$$\underline{\mathrm{D}}_{\mathbf{A}^1,\mathbf{Q}} \xrightarrow{\underline{\gamma}^*} \underline{\mathrm{DM}}_{\mathbf{Q}} \xrightarrow{\underline{\alpha}^*} \underline{\mathrm{DM}}_{t,\mathbf{Q}}, \qquad \underline{a}^* : \underline{\mathrm{D}}_{\mathbf{A}^1,\mathbf{Q}} \to \underline{\mathrm{DM}}_{t,\mathbf{Q}}$$

in which both $\underline{a}^*$ and $\underline{\alpha}^*$ are induced by the t-sheafification functor; see 5.3.31 and 11.1.21. We obtain from (16.1.1.2), (16.1.1.3), and (16.1.1.4) the commutative diagram of premotivic categories below, in which $\chi_\sharp = \varphi^* \underline{\alpha}^* \psi_\sharp$.

(16.1.1.5)
$$\begin{array}{ccc} \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}} & \xrightarrow{\beta^*} & \mathrm{DM}_{\text{Б}} \\ {\scriptstyle \rho_\sharp}\downarrow & & \downarrow{\scriptstyle \chi_\sharp} \\ \underline{\mathrm{D}}_{\mathbf{A}^1,\mathbf{Q}} & \xrightarrow{\underline{a}^*} & \underline{\mathrm{DM}}_{t,\mathbf{Q}} \end{array}$$

From now on, we shall fix an excellent noetherian scheme of finite dimension S.

Theorem 16.1.2. *We have canonical equivalences of categories*

$$\mathrm{DM}_{\text{Б}}(S) \simeq \mathrm{DM}_{\mathrm{qfh},\mathbf{Q}}(S) \simeq \mathrm{DM}_{\mathrm{h},\mathbf{Q}}(S)$$

(recall that, for $t = \mathrm{qfh}, \mathrm{h}$*,* $\mathrm{DM}_{t,\mathbf{Q}}(S)$ *stands for the localizing subcategory of* $\underline{\mathrm{DM}}_{t,\mathbf{Q}}(S)$*, spanned by the objects of shape* $\Sigma^\infty \mathbf{Q}_S(X)(n)$*, where* X *runs over the family of smooth* S*-schemes, and* $n \leq 0$ *is an integer; see 5.3.31).*

Proof. Let t denote the qfh-topology or the h-topology. We shall prove that the functor

$$\chi_\sharp : \mathrm{DM}_{\text{Б}}(S) \to \underline{\mathrm{DM}}_{t,\mathbf{Q}}(S)$$

is fully faithful, and that its essential image is precisely $\mathrm{DM}_{t,\mathbf{Q}}$. The functor

$$\beta_* : \mathrm{DM}_{\text{Б}} \to \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$$

is fully faithful, so that its composition with its left adjoint β^* is canonically isomorphic to the identity. In particular, we get isomorphisms of functors:

$$\chi_\sharp \simeq \chi_\sharp \, \beta^* \, \beta_* \simeq \underline{a}^* \, \rho_\sharp \, \beta_* \, .$$

The right adjoint of $\underline{a}^*$ is fully faithful, and its essential image consists of the objects of $\underline{\mathrm{D}}_{\mathbf{A}^1,\mathbf{Q}}(S)$ which satisfy t-descent (5.3.30). On the other hand, the functor $\rho_\sharp$ is fully faithful, and an object of $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)$ satisfies t-descent if and only if its image by $\rho_\sharp$ satisfies t-descent (6.1.11). By virtue of Theorem 14.3.4, this implies immediately that $\chi_\sharp$ is fully faithful. Let $\mathrm{DM}_{t,\mathbf{Q}}(S)$ be the localizing subcategory of $DM_{t,\mathbf{Q}}(S)$ spanned by the objects of shape $\Sigma^\infty \mathbf{Q}(X)(n)$, where X runs over the family of smooth S-schemes, and $n \leq 0$ is an integer (5.3.31). We know that $\mathrm{DM}_{t,\mathbf{Q}}(S)$ is compactly generated (see 5.1.29, 5.2.38 and 5.3.40), and that $\chi_\sharp$ is a fully faithful exact functor which preserves small sums as well as compact objects from $\mathrm{DM}_{\text{Б}}(S)$ to $\mathrm{DM}_{t,\mathbf{Q}}(S)$. As, by construction, there exists a generating family of compact objects of $\mathrm{DM}_{t,\mathbf{Q}}(S)$ in the essential image of $\chi_\sharp$, this implies that $\chi_\sharp$ induces an equivalence of triangulated categories $\mathrm{DM}_{\text{Б}}(S) \simeq \mathrm{DM}_{t,\mathbf{Q}}(S)$ (see 1.3.21). □

Let us underline the following result which completes Corollary 14.2.16:

Theorem 16.1.3. *Let* E *be an object of* $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$*. The following conditions are equivalent:*

(i) E is a Beilinson motive;
(ii) E satisfies h-*descent;*
(iii) E satisfies qfh-*descent;*

Proof. We already know that condition (i) implies condition (ii) (second point of Theorem 14.3.4), and condition (ii) implies obviously condition (iii). It is thus sufficient to prove that condition (iii) implies condition (i). If E satisfies qfh-descent, then $\rho_\sharp(E)$ satisfies qfh-descent in $\underline{\mathrm{DM}}(S,\mathbf{Q})$ as well. The commutativity of (16.1.1.4) implies then that $\rho_\sharp(E)$ belongs to the essential image of $\underline{\gamma}_*$ (the right adjoint of $\underline{\gamma}^*$). As $\rho_\sharp$ is fully faithful, the commutativity of (16.1.1.3) thus implies that E itself belongs to the essential image of γ_* (the right adjoint to γ^*). In particular, E is then a module over the ring spectrum $\gamma_*(\mathbb{1}_S)$, which is itself an H_{B}-algebra. We conclude by Corollary 14.2.16. □

Theorem 16.1.4. *If S is geometrically unibranch, then the comparison functor*

$$\varphi^* : \mathrm{DM}_{\mathrm{B}}(S) \to \mathrm{DM}_{\mathbf{Q}}(S)$$

is an equivalence of triangulated monoidal categories.

Proof. If S is geometrically unibranch, then we know that the composed functor

$$\mathrm{DM}_{\mathbf{Q}}(S) \xrightarrow{\psi_\sharp} \underline{\mathrm{DM}}_{\mathbf{Q}}(S) \xrightarrow{\underline{\alpha}^*} \underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}(S)$$

is fully faithful (11.1.22). The commutative diagram

$$\mathrm{DM}_{\mathrm{B}}(S) \xrightarrow{\varphi^*} \mathrm{DM}_{\mathbf{Q}}(S) \xrightarrow{\underline{\alpha}^*\psi_\sharp} \underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}(S), \qquad \chi_\sharp : \mathrm{DM}_{\mathrm{B}}(S) \to \underline{\mathrm{DM}}_{\mathrm{qfh},\mathbf{Q}}(S)$$

and Theorem 16.1.2 imply that φ^* is fully faithful. As φ^* is exact, preserves small sums as well as compact objects, and as $\mathrm{DM}_{\mathbf{Q}}(S)$ has a generating family of compact objects in the essential image of φ^*, the functor φ^* has to be an equivalence of categories (1.3.21). □

Remark 16.1.5. Some version of the preceding theorem (the one obtained by replacing DM_{B} by $\mathrm{Ho}(H_{\mathrm{B}}\text{-}\mathrm{mod})$) was already known in the case where S is the spectrum of a perfect field; see [**RØ08**, theorem 68]. The proof used de Jong's resolution of singularities by alterations and Poincaré duality in a crucial way. The proof of the preceding theorem we gave here relies on proper descent but does not use any kind of resolution of singularities.

The preceding theorem allows to give the following description of constructible Beilinson motives over geometrically unibranch schemes:

Corollary 16.1.6. *For any geometrically unibranch scheme S, the functor φ^* induces an equivalence of triangulated monoidal categories:*

$$\mathrm{DM}_{\mathrm{B},c}(S) \xrightarrow{\sim} \mathrm{DM}_{gm}(S,\mathbf{Q})$$

where the right hand side is the **Q***-linear version of the category of geometric (Voevodsky) motives (Definition 11.1.10).*

Note that we also applied Proposition 11.1.5 to get this corollary.

We finally point out the following important fact about Voevodsky's motivic cohomology spectrum $H_{\mathcal{M},S} = \gamma_*(\mathbb{1}_S)$ with rational coefficients:

Corollary 16.1.7. *(1) For any geometrically unibranch excellent scheme S, the canonical map*

$$H_{\mathrm{B},S} \to H^{\mathbf{Q}}_{\mathcal{M},S}$$

is an isomorphism of ring spectra.

(2) For any morphism $f : T \to S$ of excellent geometrically unibranch schemes, the canonical map

$$f^* H^{\mathbf{Q}}_{\mathcal{M},S} \to H^{\mathbf{Q}}_{\mathcal{M},T}$$

is an isomorphism of ring spectra.

The second part is the last conjecture of Voevodsky's paper [**Voe02b**] with rational coefficients (and geometrically unibranch schemes) – see also Paragraph 11.2.21.

PROOF. The first part is a trivial consequence of the previous theorem, and the second follows from the first, as the Beilinson motivic cohomology spectrum is stable by pullbacks. □

16.2. Comparison with Morel motives.

16.2.1. Let S be a scheme. The permutation isomorphism

$$\tau : \mathbf{Q}(1)[1] \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{Q}(1)[1] \to \mathbf{Q}(1)[1] \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{Q}(1)[1] \tag{16.2.1.1}$$

satisfies the equation $\tau^2 = 1$ in $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$. Hence it defines an element ϵ in $\mathrm{End}_{\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})}(\mathbf{Q})$ which also satisfies the relation $\epsilon^2 = 1$. We define two projectors

$$e_+ = \frac{\epsilon - 1}{2} \quad \text{and} \quad e_- = \frac{\epsilon + 1}{2}\,. \tag{16.2.1.2}$$

As the triangulated category $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ is pseudo abelian, we can define two objects by the formulæ:

$$\mathbf{Q}_+ = \mathrm{Im}\, e_+ \quad \text{and} \quad \mathbf{Q}_- = \mathrm{Im}\, e_-\,. \tag{16.2.1.3}$$

Then for an object M of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$, we set

$$M_+ = \mathbf{Q}_+ \otimes^{\mathbf{L}}_{\mathbf{Q}} M \quad \text{and} \quad M_- = \mathbf{Q}_- \otimes^{\mathbf{L}}_{\mathbf{Q}} M\,. \tag{16.2.1.4}$$

It is obvious that for any objects M and N of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$, one has

$$\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})}(M_i, N_j) = 0 \quad \text{for } i, j \in \{+, -\} \text{ with } i \neq j. \tag{16.2.1.5}$$

Denote by $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})_+$ (resp. $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})_-$) the full subcategory of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ made of objects which are isomorphic to some M_+ (resp. some M_-) for an object M in $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$. Then (16.2.1.5) implies that the direct sum functor $(M_+, M_-) \mapsto M_+ \oplus M_-$ induces an equivalence of triangulated categories

$$(\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})_+) \times (\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})_-) \simeq \mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})\,. \tag{16.2.1.6}$$

We shall call $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})_+$ the *category of Morel motives over* S. The aim of this section is to compare this category with $\mathrm{DM}_{\mathrm{B}}(S)$ (see Theorem 16.2.13). This will consists essentially of proving that $\mathbf{Q}_+$ is nothing else than Beilinson's motivic spectrum H_{B} (which was announced by Morel in [**Mor06**]). The main ingredients of the proof are the description of $\mathrm{DM}_{\mathrm{B}}(S)$ as full subcategory of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$, the homotopy t-structure on $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$, and Morel's computation of the endomorphism ring of the motivic sphere spectrum in terms of Milnor-Witt K-theory [**Mor03, Mor04a, Mor04b, Mor12**].

16.2.2. For a little while, we shall assume that S is the spectrum of a field k.

Recall that the *algebraic Hopf fibration* is the map

$$\mathbf{A}^2 - \{0\} \to \mathbf{P}^1\ , \quad (x, y) \mapsto [x, y]\,.$$

This defines, by desuspension, a morphism

$$\eta : \mathbf{Q}(1)[1] \to \mathbf{Q}$$

in $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$; see [**Mor03**, 6.2] (recall that we identify $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$ with $\mathrm{SH}_{\mathbf{Q}}(S)$ and that, under this identification, $\mathbf{Q}(1)[1]$ corresponds to $\Sigma^\infty(\mathbf{G}_m)$).

LEMMA 16.2.3. *We have* $\eta = \epsilon\eta$ *in* $\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})}(\mathbf{Q}(1)[1], \mathbf{Q})$.

PROOF. See [**Mor03**, 6.2.3]. □

16.2.4. Recall the *homotopy t-structure* on $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$; see [**Mor03**, 5.2]. To remain close to the conventions of *loc. cit.*, we shall adopt homological notations, so that, for any object M of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$, we have the following truncation triangle

$$\tau_{>0} M \to M \to \tau_{\leq 0} M \to \tau_{>0} M[1]\,.$$

We whall write H_0 for the zeroth homology functor in the sense of this t-structure. This t-structure can be described in terms of generators, as in [**Ayo07a**, definition 2.2.41]: the category $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$ is the smallest full subcategory of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ which contains the objects of shape $\mathbf{Q}_S(X)(m)[m]$ for X smooth over S, $m \in \mathbf{Z}$, and which satisfies the following stability conditions:

(a) $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$ is stable under suspension; i.e. for any object M in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$, $M[1]$ is in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$;

(b) $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$ is closed under extensions: for any distinguished triangle
$$M' \to M \to M'' \to M'[1]\,,$$
if M' and M'' are in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$, so is M;

(c) $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$ is closed under small sums.

With this description, it is easy to see that $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$ is also closed under tensor product (because the class of generators has this property). The category $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\leq 0}$ is the full subcategory of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ which consists of objects M such that
$$\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})}(\mathbf{Q}_S(X)(m)[m+n], M) \simeq 0$$
for X/S smooth, $m \in \mathbf{Z}$, and $n > 0$; see [**Ayo07a**, 2.1.72].

Note that the heart of the homotopy t-structure is symmetric monoidal, with tensor product $\otimes^h$ defined by the formula:
$$F \otimes^h G = H_0(F \otimes^{\mathbf{L}}_S G)$$
(the unit object is $H_0(\mathbf{Q})$).

We shall still write $\eta : H_0(\mathbf{Q}(1)[1]) \to H_0(\mathbf{Q})$ for the map induced by the algebraic Hopf fibration.

Proposition 16.2.5. *Tensoring by $\mathbf{Q}(n)[n]$ defines a t-exact endofunctor of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ for any integer n.*

Proof. As tensoring by $\mathbf{Q}(n)[n]$ is an equivalence of categories, it is sufficient to prove this for $n \geq 0$. This is then a particular case of [**Ayo07a**, 2.2.51]. □

Proposition 16.2.6. *For any smooth S-scheme X of dimension d, and for any object M of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$, the map*
$$\mathrm{Hom}(\mathbf{Q}_S(X), M) \to \mathrm{Hom}(\mathbf{Q}_S(X), M_{\leq n})$$
is an isomorphism for $n > d$.

Proof. Using [**Mor03**, lemma 5.2.5], it is sufficient to prove the analog for the homotopy t-structure on $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Q}}(S)$, which follows from [**Mor05**, lemma 3.3.3]. □

Proposition 16.2.7. *The homotopy t-structure is non degenerated. Even better, for any object M of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$, we have canonical isomorphisms*
$$\mathbf{L}\varinjlim_n \tau_{>n}M \simeq M \quad \textit{and} \quad \mathbf{R}\varprojlim_n \tau_{>n}M \simeq 0\,,$$
as well as isomorphisms
$$\mathbf{L}\varinjlim_n \tau_{\leq n}M \simeq 0 \quad \textit{and} \quad M \simeq \mathbf{R}\varprojlim_n \tau_{\leq n}M\,.$$

Proof. The first assertion is a direct consequence of propositions 16.2.5 and 16.2.6 (because the objects of shape $\mathbf{Q}_S(X)(m)[i]$, for X/S smooth, and $m, i \in \mathbf{Z}$, form a generating family). As the objects $\mathbf{Q}_S(X)(m)[m+n]$ are compact in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$, the category $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\leq 0}$ is closed under small sums. As $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$ is also closed under small sums, we deduce easily that the truncation functors $\tau_{>0}$ and $\tau_{\leq 0}$ preserve small sums, which implies that the homology functor H_0 has the same property. Moreover, if
$$C_0 \to \cdots \to C_n \to C_{n+1} \to \cdots$$

is a sequence of maps in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$, then $C = \mathbf{L}\varinjlim_n C_n$ fits in a distinguished triangle of shape

$$\bigoplus_n C_n \overset{1-s}{\to} \bigoplus_n C_n \to C \to \bigoplus_n C_n[1]\,,$$

where s is the map induced by the maps $C_n \to C_{n+1}$. This implies that, for any integer i, we have

$$\varinjlim_n H_i(C_n) \simeq H_i(C)$$

(where the colimit is taken in the heart of the homotopy t-structure). As the homotopy t-structure is non degenerated, this proves the two formulas

$$\mathbf{L}\varinjlim_n \tau_{>n} M \simeq M \quad \text{and} \quad \mathbf{L}\varinjlim_n \tau_{\leq n} M \simeq 0\,.$$

Let X be a smooth S-scheme of finite type, and p, q be some integer. To prove that the map

$$\mathrm{Hom}(\mathbf{Q}_S(X)(m)[i], M) \to \mathrm{Hom}(\mathbf{Q}_S(X)(m)[i], \mathbf{R}\varprojlim_n \tau_{\leq n} M)$$

is bijective, we may assume that $m = 0$ (replacing M by $M(-m)[-m]$ and i by $i-m$, and using Proposition 16.2.5). Consider the Milnor short exact sequence below, with $A = \mathbf{Q}_S(X)[i]$:

$$0 \to \varprojlim_n{}^1 \mathrm{Hom}(A[1], \tau_{\leq n} M) \to \mathrm{Hom}(A, \mathbf{R}\varprojlim_n \tau_{\leq n} M) \to \varprojlim_n \mathrm{Hom}(A, \tau_{\leq n} M) \to 0\,.$$

Using Proposition 16.2.6, as $\varprojlim^1$ of a constant functor vanishes, we get that the map

$$\mathrm{Hom}(A, M) \to \mathrm{Hom}(A, \mathbf{R}\varprojlim_n \tau_{\leq n} M)$$

is an isomorphism. This gives the isomorphism

$$M \simeq \mathbf{R}\varprojlim_n \tau_{\leq n} M\,.$$

Using the previous isomorphism, and by contemplating the homotopy limit of the homotopy cofiber sequences

$$\tau_{>n} M \to M \to \tau_{\leq n} M\,,$$

we deduce the isomorphism $\mathbf{R}\varprojlim_n \tau_{>n} M \simeq 0$. □

LEMMA 16.2.8. *We have $H_{\mathrm{B}} \in \mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$, so that we have a canonical map*

$$H_{\mathrm{B}} \to H_0(H_{\mathrm{B}})$$

in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$. In particular, for any object M in the heart of the homotopy t-structure, if M is endowed with an action of the monoid $H_0(H_{\mathrm{B}})$, then M has a natural structure of H_{B}-module in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$.

PROOF. As H_{B} is isomorphic to the motivic cohomology spectrum in the sense of Voevodsky (16.1.7), the first assertion is the first assertion of [**Mor03**, theorem 5.3.2]. Therefore, the truncation triangle for H_{B} gives a triangle

$$\tau_{>0} H_{\mathrm{B}} \to H_{\mathrm{B}} \to H_0(H_{\mathrm{B}}) \to \tau_{>0} H_{\mathrm{B}}[1]\,,$$

which gives the second assertion. For the third assertion, consider an object M in the heart of the homotopy t-structure, endowed with an action of $H_0(H_{\mathrm{B}})$. Note that $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$ is closed under tensor product, so that $H_{\mathrm{B}} \otimes^{\mathbf{L}}_S M$ is in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_{\geq 0}$. Hence we have natural maps

$$H_{\mathrm{B}} \otimes^{\mathbf{L}}_S M \to H_0(H_{\mathrm{B}} \otimes^{\mathbf{L}}_S M) \to H_0(H_0(H_{\mathrm{B}}) \otimes^{\mathbf{L}}_S M) = H_0(H_{\mathrm{B}}) \otimes^h M\,.$$

Then the structural map $H_0(H_{\mathrm{B}}) \otimes^h M \to M$ defines a map $H_{\mathrm{B}} \otimes^{\mathbf{L}}_S M \to M$ which gives the expected action (observe that, as we already know that H_{B}-modules do form a thick subcategory of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ (14.2.8), we don't even need to check all the axioms of an internal module: it is sufficient to check that the unit $\mathbf{Q} \to H_{\mathrm{B}}$ induces a section $M \to H_{\mathrm{B}} \otimes^{\mathbf{L}}_S M$ of the map constructed above). □

LEMMA 16.2.9. *We have the following exact sequence in the heart of the homotopy t-structure.*

$$H_0(\mathbf{Q}(1)[1]) \xrightarrow{\eta} H_0(\mathbf{Q}) \to H_0(H_{\text{Б}}) \to 0$$

PROOF. Using the equivalence of categories from the heart of the homotopy t-structure to the category of homotopy modules in the sense of [**Mor03**, definition 5.2.4], by virtue of Corollary 16.1.7 and [**Mor03**, theorem 5.3.2], we know that $H_0(H_{\text{Б}})$ corresponds to the homotopy module $\underline{K}_*^M \otimes \mathbf{Q}$ associated with Milnor K-theory, while $H_0(\mathbf{Q})$ corresponds to the homotopy module $\underline{K}_*^{MW} \otimes \mathbf{Q}$ associated with Milnor-Witt K-theory (which follows easily from [**Mor12**, theorems 2.11, 6.13 and 6.40]). Considering $\underline{K}_*^M$ and $\underline{K}_*^{MW}$ as unramified sheaves in the sense of Morel [**Mor12**], this lemma is then a reformulation of the isomorphism

$$K_*^{MW}(F)/\eta \simeq K_*^M(F)$$

for any field F; see [**Mor12**, remark 2.2]. □

PROPOSITION 16.2.10. *We have $H_{\text{Б}+} \simeq H_{\text{Б}}$, and the induced map $\mathbf{Q}_+ \to H_{\text{Б}}$ gives a canonical isomorphism $H_0(\mathbf{Q}_+) \simeq H_0(H_{\text{Б}})$.*

PROOF. The map $\epsilon(1)[1] : \mathbf{Q}(1)[1] \to \mathbf{Q}(1)[1]$ can be described geometrically as the morphism associated with the pointed morphism

$$\imath : \mathbf{G}_m \to \mathbf{G}_m \ , \quad t \mapsto t^{-1}$$

(see the second assertion of [**Mor03**, lemma 6.1.1]). In the decomposition

$$K_1(\mathbf{G}_m) \simeq k[t, t^{-1}]^\times \simeq k^\times \oplus \mathbf{Z}\,,$$

the map $\imath$ induces multiplication by -1 on $\mathbf{Z}$. Using the periodicity isomorphism $KGL(1)[2] \simeq KGL$, we get the identifications:

$$K_1(\mathbf{G}_m) \supset \operatorname{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty(\mathbf{G}_m)[1], KGL) \simeq \operatorname{Hom}_{KGL}(KGL, KGL) \simeq K_0(k) \simeq \mathbf{Z}\,.$$

Therefore, ϵ acts as the multiplication by -1 on the spectrum $KGL_{\mathbf{Q}}$, whence on $H_{\text{Б}}$ as well. This means precisely that $H_{\text{Б}+} \simeq H_{\text{Б}}$. By Lemma 16.2.3, the class 2η vanishes in $\mathbf{Q}_+$, so that, appyling the (t-exact) functor $M \mapsto M_+$ to the exact sequence of Lemma 16.2.9, we get an isomorphism $H_0(\mathbf{Q}_+) \simeq H_0(H_{\text{Б}+}) \simeq H_0(H_{\text{Б}})$. □

COROLLARY 16.2.11. *For any object M in the heart of the homotopy t-structure, M_+ is a Beilinson motive.*

PROOF. The object M is an $H_0(\mathbf{Q})$-module, so that M_+ is an $H_0(\mathbf{Q}_+)$-module. By virtue of Proposition 16.2.10, M_+ is then a module over $H_0(H_{\text{Б}})$, so that, by Lemma 16.2.8, M_+ is naturally endowed with an action of $H_{\text{Б}}$. □

REMARK 16.2.12. Until now, we did not really use the fact we are in a $\mathbf{Q}$-linear context (replacing $H_{\text{Б}}$ by Voevodsky's motivic spectrum, we just needed 2 to be invertible in the preceding corollary). However, the following result really uses $\mathbf{Q}$-linearity (because, in the proof, we see $\mathrm{DM}_{\text{Б}}(S)$ as a full subcategory of $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$; see Proposition 14.2.3).

THEOREM 16.2.13. *For any noetherian scheme of finite dimension S, the map $\mathbf{Q}_+ \to H_{\text{Б}}$ is an isomorphism in $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})$. As a consequence, we have a canonical equivalence of triangulated monoidal categories*

$$\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})_+ \simeq \mathrm{DM}_{\text{Б}}(S)\,.$$

This theorem has already been proved by Morel when S is the spectrum of a perfect field – where the left hand side is the rational category of Voevodsky motives. Morel announced that the category $\mathrm{D}_{\mathbf{A}^1}(S, \mathbf{Q})_+$ should be *the* category of rational motives and this theorem confirm his insight.

PROOF. Observe that, if ever $\mathbf{Q}_+ \simeq H_{\text{B}}$, we have $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_+ \simeq \mathrm{DM}_{\text{B}}(S)$: this follows from the fact that an object M of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ belongs to $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_+$ (resp. to $\mathrm{DM}_{\text{B}}(S)$) if and only if there exists an isomorphism $M \simeq M_+$ (resp. $M \simeq H_{\text{B}} \otimes^{\mathbf{L}}_S M$; see 14.2.16). It is thus sufficient to prove the first assertion.

As both $\mathbf{Q}_+$ and H_{B} are stable by pullback, it is sufficient to treat the case where $S = \mathrm{Spec}\,(\mathbf{Z})$. Using Corollary 14.3.2, we may replace S by any of its henselisations, so that, by the localization property, it is sufficient to treat the case where S is the spectrum of a (perfect) field k.

We shall prove directly that, for any object M of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$, M_+ is an H_{B}-module (or, equivalently, is H_{B}-local). Note that $\mathrm{DM}_{\text{B}}(S)$ is closed under homotopy limits and homotopy colimits in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$: indeed the inclusion functor $\mathrm{DM}_{\text{B}} \to \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$ has a left adjoint which preserves a family of compact generators, whence it also has a left adjoint (1.3.20). By virtue of Proposition 16.2.7, we may thus assume that M is bounded with respect to the homotopy t-structure. As $\mathrm{DM}_{\text{B}}(S)$ is certainly closed under extensions in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$, we may even assume that M belongs to the heart the homotopy t-structure. We conclude with Corollary 16.2.11. □

COROLLARY 16.2.14. *For any noetherian scheme of finite dimension S, if -1 is a sum of squares in all the residue fields of S, then $\mathbf{Q}_- \simeq 0$ in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$, and we have a canonical equivalence of triangulated monoidal categories*

$$\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q}) \simeq \mathrm{DM}_{\text{B}}(S)\,.$$

PROOF. It is sufficient to prove that, under this assumption, $\mathbf{Q}_- \simeq 0$. As in the preceding proof, we may replace S by any of its henselisations (4.3.9), so that, by the localization property (and by induction on the dimension), it is sufficient to treat the case where S is the spectrum of a field k. We have to check that, if -1 is a sum of squares in k, then we have $\epsilon = -1$. Using [**Mor03**, remark 6.3.5 and lemma 6.3.7], we see that, if k is of characteristic 2, we always have $\epsilon = -1$, while, if the characteristic of k is distinct from 2, we have a morphism of rings

$$GW(k) \to \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(\mathrm{Spec}(k))}(\mathbf{Q},\mathbf{Q})\,,$$

where $GW(k)$ denotes the Grothendieck-Witt ring[90] over k. This morphism sends the class of the quadratic form $-X^2$ to $-\epsilon$ and this proves the result. (For a more precise version of this, with integral coefficients, see [**Mor12**, proposition 2.13].) □

16.2.15. Recall from Example 5.3.43 that we can describe the category $\mathrm{D}_{\mathbf{A}^1,c}(S,\mathbf{Q})$ of compact objects of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ as the triangulated monoidal category obtained from

$$\left(\mathrm{K}^b\left(\mathbf{Q}(Sm/S)\right)/(BG_S \cup \mathcal{T}_{\mathbf{A}^1_S})\right)^{\natural}$$

by formally inverting the Tate twist. The operation ϵ still acts on this category and the decomposition in $+$ and $-$ part of a motive respects constructibility as this is a decomposition by direct factors. The preceding theorem gives the following description of constructible Beilinson motives:

COROLLARY 16.2.16. *For any noetherian scheme of finite dimension S, there is a canonical equivalence of triangulated monoidal categories*

$$\mathrm{DM}_{\text{B},c}(S) \simeq \mathrm{D}_{\mathbf{A}^1,c}(S,\mathbf{Q})_+$$

When -1 is a sum of square in all the residue fields of S, this equivalence can be written:

$$\mathrm{DM}_{\text{B},c}(S) \simeq \mathrm{D}_{\mathbf{A}^1,c}(S,\mathbf{Q}).$$

16.2.17. Consider the $\mathbf{Q}$-linear *étale motivic category* $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(-,\mathbf{Q})$, defined by

$$\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q}) = \mathrm{D}_{\mathbf{A}^1}(\mathrm{Sh}_{\text{ét}}(Sm/S,\mathbf{Q}))$$

(see 5.3.31). The étale sheafification functor induces a morphism of motivic categories

$$\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q}) \to \mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})\,. \tag{16.2.17.1}$$

We shall prove the following result, as an application of Theorem 16.2.13.

[90] *i.e.* the Grothendieck group of quadratic forms

THEOREM 16.2.18. *For any noetherian scheme of finite dimension S, there is a canonical equivalence of categories*

$$\mathrm{DM}_{\mathrm{B}}(S) \simeq \mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})\,.$$

As for Theorem 16.2.13, the idea of this result is from F. Morel who already proved it at least in the case of a base field.

In order prove the above Theorem, we shall study the behaviour of the decomposition (16.2.1.3) in $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})$:

LEMMA 16.2.19. *We have* $\mathbf{Q}_- \simeq 0$ *in* $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})$.

PROOF. Proceeding as in the proof of Theorem 16.2.13, we may assume that S is the spectrum of a perfect field k. By étale descent, we see that we may replace k by any of its finite extension. In particular, we may assume that -1 is a sum of squares in k. But then, by virtue of Corollary 16.2.14, $\mathbf{Q}_- \simeq 0$ in $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$, so that, by functoriality, $\mathbf{Q}_- \simeq 0$ in $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})$. □

PROOF OF THEOREM 16.2.18. Note that the functor (16.2.17.1) has a fully faithful right adjoint, whose essential image consists of objects of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ which satisfy étale descent. As any Beilinson motive satisfies étale descent (first point of 14.3.4), $\mathrm{DM}_{\mathrm{B}}(S)$ can be seen naturally as a full subcategory of $\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})$. On the other hand, by virtue of the preceding lemma, any object of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ which satisfies étale descent belongs to $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})_+$. Hence, by Theorem 16.2.13, any object of $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ which satisfies étale descent is a Beilinson motive. This achieves the proof. □

REMARK 16.2.20. If S is excellent, and if all the residue fields of S are of characteristic zero, one can prove Theorem 16.2.18 independently of Morel's theorem: this follows then directly from a descent argument, namely from Corollary 3.3.38 and from Theorem 16.1.3.

COROLLARY 16.2.21. *For any regular noetherian scheme of finite dimension S, we have canonical isomorphisms*

$$\mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q})}(\mathbf{Q}_S,\mathbf{Q}_S(p)[q]) \simeq Gr^p_\gamma K_{2p-q}(S)_{\mathbf{Q}}\,.$$

PROOF. This follows immediately from Theorem 16.2.18, by definition of DM_{B} (14.2.14). □

COROLLARY 16.2.22. *For any geometrically unibranch excellent noetherian scheme of finite dimension S, there is a canonical equivalence of symmetric monoidal triangulated categories*

$$\mathrm{D}_{\mathbf{A}^1,\text{ét}}(S,\mathbf{Q}) \simeq \mathrm{DM}(S,\mathbf{Q})\,.$$

PROOF. This follows from theorems 16.1.4 and 16.2.18. □

REMARK 16.2.23. The preceding corollary extends immediately to the case of coefficients in a $\mathbf{Q}$-algebra R (cf. Example 5.3.36 for the left hand side and Paragraph 14.2.20 for the right hand side).

COROLLARY 16.2.24. *Let S be an excellent noetherian scheme of finite dimension. An object of* $\mathrm{D}_{\mathbf{A}^1}(S,\mathbf{Q})$ *satisfies* h*-descent if and only if it satisfies étale descent.*

PROOF. This follows from theorems 16.1.3 and 16.2.18. □

17. Realizations

17.1. Tilting.

17.1.1. Let $\mathscr{M}$ be a stable perfect symmetric monoidal Sm-fibred combinatorial model category over an adequate category of $\mathcal{S}$-schemes $\mathscr{S}$, such that $\mathrm{Ho}(\mathscr{M})$ is motivic, with generating set of twists τ.

Consider a homotopy cartesian commutative monoid $\mathcal{E}$ in $\mathscr{M}$. Then $\mathcal{E}$-mod is an Sm-fibred model category, such that $\mathrm{Ho}(\mathcal{E}\text{-mod})$ is motivic, and we have a morphism of motivic categories (see 7.2.13 and 7.2.18)

$$\mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathcal{E}\text{-mod})\,,\quad M \mapsto \mathcal{E} \otimes^{\mathbf{L}} M\,.$$

In practice, all the realization functors are obtained in this way (at least over fields), which can be formulated as follows (for simplicity, we shall work here in a **Q**-linear context, but, if we are ready to consider higher categorical constructions, there is no reason to make such an assumption).

17.1.2. Consider a quasi-excellent noetherian scheme S of finite dimension, as well as two stable symmetric monoidal Sm-fibred combinatorial model categories $\mathscr{M}$ and $\mathscr{M}'$ over the category of S-schemes of finite type such that $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ are motivic (as triangulated premotivic categories). We also assume that both $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ are **Q**-linear and separated.

Consider a Quillen adjunction

$$\varphi^* : \mathscr{M} \rightleftarrows \mathscr{M}' : \varphi_* \,, \tag{17.1.2.1}$$

inducing a morphism of Sm-fibred categories

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{M}')\,. \tag{17.1.2.2}$$

We consider both $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ as endowed with their Tate twists, which defines two motivic subcategories of constructible objects $\mathrm{Ho}(\mathscr{M})_c$ and $\mathrm{Ho}(\mathscr{M}')_c$, respectively. The functor $\mathbf{L}\varphi^*$ preserves constructible objects, and thus defines a morphism of premotivic categories

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})_c \to \mathrm{Ho}(\mathscr{M}')_c\,. \tag{17.1.2.3}$$

Proposition 17.1.3. *Under the assumptions of 17.1.2, if, for any regular S-scheme of finite type X, and for any integers p and q, the map*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q]) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M}')(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q])$$

is bijective, then the morphism (17.1.2.3) *is an equivalence of premotivic categories. Moreover, if both* $\mathrm{Ho}(\mathscr{M})$ *and* $\mathrm{Ho}(\mathscr{M}')$ *are compactly generated by their Tate twists, then the morphism* (17.1.2.2) *is an equivalence of motivic categories.*

Proof. Note first that, for any separated S-scheme of finite type X, and for any integers p and q, the map

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q]) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M}')(X)}(\mathbb{1}_X, \mathbb{1}_X(p)[q])$$

is bijective. Indeed, it is equivalent to prove that the maps

$$\mathbf{R}\Gamma(X, \mathbb{1}_X(p)) \to \mathbf{R}\Gamma(X, \varphi^*(\mathbb{1}_X)(p))$$

are isomorphisms in the derived category of **Q**-vector spaces: by h-descent (3.3.37), and by virtue of Gabber's weak uniformization Theorem 4.1.2, it is sufficient to treat the case where X is regular, which is done by assumption. Let T be a S-scheme of finite type. To prove that the functor

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})_c(T) \to \mathrm{Ho}(\mathscr{M}')_c(T)$$

is fully faithful, it is sufficient to choose two small families $\mathfrak{A}$ and $\mathfrak{B}$ of objects of $\mathrm{Ho}(\mathscr{M})(T)$ such that the thick subcategory generated by $\mathfrak{A}$ (by $\mathfrak{B}$, respectively) contains $\mathrm{Ho}(\mathscr{M})(T)$, and to check that the map

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(T)}(A, B) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M}')(T)}(\varphi^*(A), \varphi^*(B))$$

are bijective, where A and B run over $\mathfrak{A}$ and $\mathfrak{B}$, respectively. By virtue of Proposition 4.2.13, it is thus sufficient to prove that, for any separated smooth morphism $f : X \to T$, for any projective morphism $g : Y \to T$, and for any integers p and q, the map

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(T)}(\mathbf{L}f_\sharp(\mathbb{1}_X), \mathbf{R}g_*(\mathbb{1}_Y)(p)[q]) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M}')(T)}(\mathbf{L}f_\sharp(\mathbb{1}_X), \mathbf{R}g_*(\mathbb{1}_Y)(p)[q])$$

is an isomorphism. Consider the pullback square

$$\begin{array}{ccc} X \times_T Y & \xrightarrow{pr_2} & Y \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & T \end{array}$$

From Proposition 2.4.53, the functor φ^* commutes with $f_!$ when f is a separated morphism of finite type. One then easily concludes using this fact and the isomorphisms (obtained by adjunction and smooth (or proper) base change)

$$\begin{aligned}\mathrm{Hom}(\mathbf{L}f_\sharp(\mathbb{1}_X), \mathbf{R}g_*(\mathbb{1}_Y)(p)[q]) &\simeq \mathrm{Hom}(\mathbb{1}_X, \mathbf{L}f^*\,\mathbf{R}g_*(\mathbb{1}_X)(p)[q])\\ &\simeq \mathrm{Hom}(\mathbb{1}_X, \mathbf{R}pr_{1,*}\,\mathbf{L}pr_2^*(\mathbb{1}_X)(p)[q])\\ &\simeq \mathrm{Hom}(\mathbb{1}_X, \mathbf{R}pr_{1,*}(\mathbb{1}_{X\times_T Y})(p)[q])\,,\\ &\simeq \mathrm{Hom}(\mathbb{1}_{X\times_T Y}, \mathbb{1}_{X\times_S Y}(p)[q])\,,\end{aligned}$$

that (17.1.2.3) is fully faithful and that $\mathrm{Ho}(\mathscr{M}')_c(T)$ is the thick subcatgory generated by the image by $\mathbf{L}\varphi^*$ of constructible objects of $\mathrm{Ho}(\mathscr{M})(T)$. In other words, the functor (17.1.2.3) is an equivalence of categories.

If, moreover, both $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ are compactly generated by their Tate twists, then the sum preserving exact functor

$$\mathbf{L}\varphi^* : \mathrm{Ho}(\mathscr{M})(T) \to \mathrm{Ho}(\mathscr{M}')(T)$$

is an equivalence at the level of compact objects, hence is an equivalence of categories (1.3.21). □

17.1.4. Under the assumptions of 17.1.2, assume that $\mathscr{M}$ and $\mathscr{M}'$ are strongly $\mathbf{Q}$-linear (7.1.4), left proper, tractable, satisfy the monoid axiom, and have cofibrant unit objects. Let $\mathcal{E}'$ be a fibrant resolution of $\mathbb{1}$ in $\mathscr{M}'(\mathrm{Spec}\,(k))$. By virtue of Theorem 7.1.8, we may assume that $\mathcal{E}'$ is a fibrant and cofibrant commutative monoid in $\mathscr{M}'$. Then $\mathbf{R}\varphi_*(\mathbb{1}) = \varphi_*(\mathcal{E}')$ is a commutative monoid in $\mathscr{M}$. Let $\mathcal{E}$ be a cofibrant resolution of $\varphi_*(\mathcal{E}')$ in $\mathscr{M}(\mathrm{Spec}\,(k))$. Using Theorem 7.1.8, we may assume that $\mathcal{E}$ is a fibrant and cofibrant commutative monoid, and that the map

$$\mathcal{E} \to \mathbf{R}\varphi_*(\mathcal{E}')$$

is a morphism of commutative monoids (and a weak equivalence by construction). We can see $\mathcal{E}$ and $\mathcal{E}'$ as cartesian commutative monoids in $\mathscr{M}$ and $\mathscr{M}'$ respectively (by considering their pullbacks along morphisms of finite type $f : X \to \mathrm{Spec}\,(k)$). We obtain the essentially commutative diagram of left Quillen functors below (in which the lower horizontal map is the functor induced by φ^* and by the change of scalars functor along the map $\varphi^*(\mathcal{E}) \to \mathcal{E}'$):

(17.1.4.1)
$$\begin{array}{ccc}\mathscr{M} & \longrightarrow & \mathscr{M}' \\ \downarrow & & \downarrow \\ \mathcal{E}\text{-}\mathrm{mod} & \longrightarrow & \mathcal{E}'\text{-}\mathrm{mod}\end{array}$$

where $\mathcal{E}$-mod and $\mathcal{E}'$-mod are respectively the model premotivic categories of $\mathcal{E}$-modules and $\mathcal{E}'$-modules (see Proposition 7.2.11).

Note furthermore that the right hand vertical left Quillen functor is a Quillen equivalence by construction (identifying $\mathscr{M}'(X)$ with $\mathbb{1}_X$-modules, and using the fact that the morphism of monoids $\mathbb{1}_X \to \mathcal{E}'_X$ is a weak equivalence in $\mathscr{M}'(X)$).

THEOREM 17.1.5. *Consider the assumptions of 17.1.4, with $S = \mathrm{Spec}\,(k)$ the spectrum of a field k. We suppose furthermore that one of the following conditions is verified.*

(i) The field k is perfect.

(ii) The motivic categories $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ are continuous and semi-separated.

Then the morphism

$$\mathrm{Ho}(\mathcal{E}\text{-}\mathrm{mod})_c \to \mathrm{Ho}(\mathcal{E}'\text{-}\mathrm{mod})_c \simeq \mathrm{Ho}(\mathscr{M}')_c$$

is an equivalence of motivic categories. Under these identifications, the morphism (17.1.2.3) *corresponds to the change of scalar functor*

$$\mathrm{Ho}(\mathscr{M})_c \to \mathrm{Ho}(\mathscr{M}')_c \simeq \mathrm{Ho}(\mathcal{E}\text{-}\mathrm{mod})_c\,, \quad M \mapsto \mathcal{E} \otimes^{\mathbf{L}} M\,.$$

If moreover both $\mathrm{Ho}(\mathscr{M})$ and $\mathrm{Ho}(\mathscr{M}')$ are compactly generated by their Tate twists, then these identifications extend to non-constructible objects, so that, in particular, the morphism (17.1.2.2) *corresponds to the change of scalar functor*

$$\mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{M}') \simeq \mathrm{Ho}(\mathcal{E}\text{-mod}) \ , \quad M \mapsto \mathcal{E} \otimes^{\mathbf{L}} M \, .$$

REMARK 17.1.6. This theorem can be thought as (a part of) a *tilting theory* for motivic (homotopy) categories. Remark that the theorem above readily implies that the morphism of motivic categories

$$\varphi^* : \mathrm{Ho}(\mathscr{M})_c \to \mathrm{Ho}(\mathscr{M}')$$

commutes with the six operations (because the, by virtue of Theorem 4.4.25, the functor $M \mapsto \mathcal{E} \otimes^{\mathbf{L}} M$ has this property, as well as the inclusion $\mathrm{Ho}(\mathscr{M}')_c \subset \mathrm{Ho}(\mathscr{M}')$).

PROOF. For any regular k-scheme of finite type X, and for any integers p and q, the map

$$\mathrm{Hom}_{\mathrm{Ho}(\mathscr{M})(X)}(\mathbb{1}_X, \mathcal{E}_X(p)[q]) \to \mathrm{Hom}_{\mathrm{Ho}(\mathscr{M}')(X)}(\mathbb{1}_X, \mathcal{E}'_X(p)[q])$$

is bijective: this is easy to check whenever X is smooth over k, which proves the assertion under condition (i), while, under condition (ii), we see immediately from Proposition 4.3.15 that we may assume condition (i). The first assertion is then a special case of the first assertion of Proposition 17.1.3. Similarly, by Proposition 7.2.7, the second assertion follows from the second assertion of Proposition 17.1.3. □

EXAMPLE 17.1.7. Let $\mathscr{M}$ be the stable *Sm*-fibred model category of Tate spectra, so that $\mathrm{Ho}(\mathscr{M}) = \mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}$, and write $\mathscr{M}_{\text{Б}}$ for the left Bousfield localization of $\mathscr{M}$ by the class of $H_{\text{Б}}$-equivalences (see 14.2.3), so that $\mathrm{Ho}(\mathscr{M}_{\text{Б}}) = \mathrm{DM}_{\text{Б}}$.

Let k be a field of characteristic zero, endowed with an embedding $\sigma : k \to \mathbf{C}$. Given a complex analytic manifold X, let $\mathscr{M}_{an}(X)$ be the category of complexes of sheaves of $\mathbf{Q}$-vector spaces on the smooth analytic site of X (i.e. on the category of smooth analytic X-manifolds, endowed with the Grothendieck topology corresponding to open coverings), endowed with its local model structure (see [**Ayo07b**, 4.4.16] and [**Ayo10**]). We shall write $\mathscr{M}^{\mathit{eff}}_{Betti}(X)$ for the stable left Bousfield localization of $\mathscr{M}_{an}(X)$ by the maps of shape $\mathbf{Q}(U \times \mathbf{D}^1) \to \mathbf{Q}(U)$ for any analytic smooth $X(\mathbf{C})$-manifold U (where $\mathbf{D}^1$ denotes the closed unit disc). We define at last $\mathscr{M}_{Betti}(X)$ as the stable model category of analytic $\mathbf{Q}(1)[1]$-spectra in $\mathscr{M}^{\mathit{eff}}_{Betti}(X)$, where $\mathbf{Q}(1)[1]$ stands for the cokernel of the map $\mathbf{Q} \to \mathbf{Q}(\mathbf{A}^{1,an} - \{0\})$ induced by $1 \in \mathbf{C}$; see [**Ayo10**, section 1].

Given a k-scheme of finite type X, we shall write

$$\mathrm{D}_{Betti}(X) := \mathrm{Ho}(\mathscr{M}_{Betti}(X)) \tag{17.1.7.1}$$

(where the topological space $X(\mathbf{C})$ is endowed with its canonical analytic structure). According to [**Ayo10**, 1.8 and 1.10], there exists canonical equivalences of categories

$$\mathrm{D}_{Betti}(X) \simeq \mathrm{Ho}(\mathscr{M}^{\mathit{eff}}_{Betti}(X)) \simeq \mathrm{D}(X(\mathbf{C}), \mathbf{Q}) \, , \tag{17.1.7.2}$$

where $\mathrm{D}(X(\mathbf{C}), \mathbf{Q})$ stands for the (unbounded) derived category of the abelian category of sheaves of $\mathbf{Q}$-vector spaces on the small site of $X(\mathbf{C})$. By virtue of [**Ayo10**, section 2], there exists a symmetric monoidal left Quillen morphism of monoidal *Sm*-fibred model categories over the category of k-schemes of finite type

$$An^* : \mathscr{M} \to \mathscr{M}_{Betti} \, , \tag{17.1.7.3}$$

which induces a morphism of motivic categories over the category of k-schemes of finite type. Hence $\mathbf{R}An_*(\mathbb{1})$ is a ring spectrum in $\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(\mathrm{Spec}\,(k))$ which represents Betti cohomology of smooth k-schemes. As D_{Betti} satisfies étale descent, it follows from Corollary 3.3.38 that it satisfies h-descent, from which, by virtue of Theorem 16.1.3, the morphism (17.1.7.3) defines a left Quillen functor

$$An^* : \mathscr{M}_{\text{Б}} \to \mathscr{M}_{Betti} \, , \tag{17.1.7.4}$$

hence gives rise to a morphism of motivic categories

$$\mathrm{DM}_{\text{Б}} \to \mathrm{D}_{Betti} \, , \tag{17.1.7.5}$$

the *Betti realization functor* of Beilinson motives.

Appyling Theorem 17.1.5 to (17.1.7.4), we obtain a commutative ring spectrum $\mathcal{E}_{Betti} = \mathbf{R}An_*(\mathbb{1})$ which represents Betti cohomology of smooth k-schemes, such that the restriction of the functor (17.1.7.5) to constructible objects corresponds to the change of scalars functors

$$\mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{Ho}(\mathcal{E}_{Betti}\text{-}\mathrm{mod})_c(X) \simeq \mathrm{D}^b_c(X(\mathbf{C}), \mathbf{Q}) \ , \quad M \mapsto \mathcal{E}_{Betti} \otimes^{\mathbf{L}} M \, . \tag{17.1.7.6}$$

It should be pointed out that, here, $\mathrm{D}^b_c(X(\mathbf{C}), \mathbf{Q})$ means the derived category of sheaves which are *constructible of geometric origin* (i.e. constructible in the algebraic sense, and not in the analytic sense).

In other words, once Betti cohomology of smooth k-schemes is known, one can reconstruct canonically the bounded derived categories of constructible sheaves of geometric origin on $X(\mathbf{C})$ for any k-scheme of finite type X, from the theory of mixed motives. We expect all the realization functors to be of this shape (which should follow from (some variant of) Theorem 17.1.5): the (absolute) cohomology of smooth k-schemes with constant coefficients determines the derived categories of constructible sheaves over any k-scheme of finite type, whatever this means. For instance, the whole theory of variations of mixed Hodge structures should be obtained from Deligne cohomology, seen as a ring spectrum in $\mathrm{DM}_{\mathrm{B}}(k)$ (or, more precisely, in $\mathscr{M}_{\mathrm{B}}(k)$).

17.2. Mixed Weil cohomologies. Let S be an excellent (regular) noetherian scheme of finite dimension, and $\mathbf{K}$ a field of characteristic zero, called the *field of coefficients.*

17.2.1. Let E be a Nisnevich sheaf of commutative differential graded $\mathbf{K}$-algebras (i.e. is a commutative monoid in the category of sheaves of complexes of $\mathbf{K}$-vector spaces). We shall write

$$H^n(X, E) = \mathrm{Hom}_{\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Q}}(X)}(\mathbf{Q}_X, E[n])$$

for any smooth S-scheme of finite type X, and for any integer n (note that, if E satisfies Nisnevich descent and is $\mathbf{A}^1$-homotopy invariant, which we can always assume, using 7.1.8, then $H^n(X, E) = H^n(E(X))$).

We introduce the following axioms :

- W1 *Dimension.*— $H^i(S, E) \simeq \begin{cases} \mathbf{K} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$
- W2 *Stability.*— $\dim_{\mathbf{K}} H^i(\mathbf{G}_m, E) = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = 1, \\ 0 & \text{otherwise.} \end{cases}$
- W3 *Künneth formula.*— For any smooth S-schemes X and Y, the exterior cup product induces an isomorphism
$$\bigoplus_{p+q=n} H^p(X, E) \otimes_{\mathbf{K}} H^q(Y, E) \xrightarrow{\sim} H^n(X \times_S Y, E) \ .$$
- W3′ *Weak Künneth formula.*— For any smooth S-scheme X, the exterior cup product induces an isomorphism
$$\bigoplus_{p+q=n} H^p(X, E) \otimes_{\mathbf{K}} H^q(\mathbf{G}_m, E) \xrightarrow{\sim} H^n(X \times_S \mathbf{G}_m, E) \ .$$

17.2.2. Under assumptions W1 and W2, we will call any non zero element $c \in H^1(\mathbf{G}_m, E)$ a *stability class.* Note that such a class corresponds to a non trivial map

$$c : \mathbf{Q}_S(1) \to E$$

in $\mathrm{D}^{\mathit{eff}}_{\mathbf{A}^1,\mathbf{Q}}(S)$ (using the decomposition $\mathbf{Q}(\mathbf{G}_m) = \mathbf{Q} \oplus \mathbf{Q}(1)[1]$). In particular, possibly after replacing E by a fibrant resolution (so that E is homotopy invariant and satisfies Nisnevich descent), such a stability class can be lifted to an actual map of complexes of presheaves. Such a lift will be called a *stability structure* on E.

Definition 17.2.3. A sheaf of commutative differential graded $\mathbf{K}$-algebras E as above is a *mixed Weil cohomology* (resp. a *stable cohomology*) if it satisfies the properties W1, W2 and W3 (resp. W1, W2 and W3′) stated above.

Proposition 17.2.4. *Let E be a stable cohomology. There exists a (commutative) ring spectrum $\mathcal{E}$ in $\mathrm{DM}_{\text{B}}(S)$ with the following properties.*

(i) For any smooth S-scheme X, and any integer i, there is a canonical isomorphism of $\mathbf{K}$-vector spaces

$$H^i(X,E) \simeq \mathrm{Hom}_{\mathrm{DM}_{\text{B}}(S)}(M_S(X), \mathcal{E}[i])\,.$$

(ii) Any choice of a stability structure on E defines a map $\mathbf{Q}(1) \to \mathcal{E}$ in $\mathrm{DM}_{\text{B}}(S)$, which induces an $\mathcal{E}$-linear isomorphism $\mathcal{E}(1) \simeq \mathcal{E}$.

Proof. One defines explicitely the commutative ring spectrum $\mathcal{E}$ as follows. First, by virtue of Theorem 7.1.8, we may assume that E is a Nisnevich sheaf of commutative differential graded algebras and is fibrant for the $\mathbf{A}^1$-local projective model structure: for any smooth S-scheme X, the two maps

$$H^n(E(X)) \to H^n_{\mathrm{Nis}}(X,E) \to H^n_{\mathrm{Nis}}(X\times \mathbf{A}^1, E)$$

are isomorphisms for any $n \in \mathbf{Z}$. Let L be the constant Nisnevich sheaf of complexes of $\mathbf{K}$-vector spaces associated to the kernel of the map induced by $S=\{1\}\subset \mathbf{G}_m$:

$$L = \ker\big(E(\mathbf{G}_m) \xrightarrow{1^*} E(S)\big)\,.$$

We remark that L is cofibrant, and one defines

$$\mathcal{E}_n = \mathit{Hom}(L^{\otimes n}, E)$$

this sheaf being endowed with an action of the symmetric group on n letters by permuting the factors on $L^{\otimes n}$. We then have canonical pairings

$$\mathcal{E}_m \otimes_{\mathbf{Q}} \mathcal{E}_n = \mathit{Hom}(L^{\otimes m}, E) \otimes_{\mathbf{Q}} \mathit{Hom}(L^{\otimes n}, E) \to \mathit{Hom}(L^{\otimes m+n}, E\otimes_{\mathbf{Q}} E) \to \mathit{Hom}(L^{\otimes m+n}, E)\,,$$

which turn the collection $\mathcal{E}=\{\mathcal{E}_n\}_{n\geq 0}$ into a commutative monoid in the category of symmetric sequences of sheaves of complexes of $\mathbf{Q}$-vector spaces; see Definition 5.3.7. On the other hand, we remark that L is the constant sheaf associated to $\Gamma(S, \mathit{Hom}(\mathbf{Q}(1)[1], E))$, from which we deduce that there is a natural map

$$L \to \mathit{Hom}(\mathbf{Q}(1)[1], E)$$

which can be transposed into a canonical map

$$\mathbf{Q}(1)[1] \to \mathit{Hom}(L, E) = \mathcal{E}_1\,.$$

This defines a canonical structure of commutative monoid in the category symmetric $\mathbf{Q}(1)[1]$-spectra on the symmetric sequence $\mathcal{E}$ (see Remark 5.3.10)[91].

By virtue of [**CD12**, Proposition 2.1.6], for any smooth S-scheme X, and any integer i, there is a canonical isomorphism of $\mathbf{K}$-vector spaces

$$H^i(X,E) \simeq \mathrm{Hom}_{\mathrm{D}_{\mathbf{A}^1,\mathbf{Q}}(S)}(M_S(X), \mathcal{E}[i])\,,$$

and any choice of a stability structure on E defines an isomorphism $\mathcal{E}(1)\simeq\mathcal{E}$. Moreover, [**CD12**, corollary 2.2.8] and Theorem 12.2.10 assert that this ring spectrum $\mathcal{E}$ is oriented, so that, by Corollary 14.2.16, $\mathcal{E}$ is an H_{B}-module, i.e. belongs to $\mathrm{DM}_{\text{B}}(S)$. □

17.2.5. Given a stable cohomology E and its associated ring spectrum $\mathcal{E}$, we can see $\mathcal{E}$ as a cartesian commutative monoid: we define, for an S-scheme X, with structural map $f: X \to S$:

$$\mathcal{E}_X = \mathbf{L}f^*(\mathcal{E})$$

(which means that we take a cofibrant replacement $\mathcal{E}'$ of $\mathcal{E}$ in the model category of commutative monoids of the category of Tate spectra, and define $\mathcal{E}_X = f^*(\mathcal{E}')$), and put

$$\mathrm{D}(X,\mathcal{E}) := \mathrm{Ho}(\mathcal{E}\text{-}\mathrm{mod})(X) = \mathrm{Ho}(\mathcal{E}_X\text{-}\mathrm{mod})\,. \tag{17.2.5.1}$$

[91]Here, we work with $\mathbf{Q}(1)[1]$-spectra. However, the paper [**CD12**] is written in the language of symmetric $\mathbf{Q}(1)$-spectra. We leave as an exercise to the reader the task of the translation, which consists to check that the functor $\{\mathcal{E}_n\}_{n\geq 0} \mapsto \{\mathcal{E}_n[n]\}_{n\geq 0}$ is a symmetric monoidal left Quillen equivalence from symmetric $\mathbf{Q}(1)[1]$-spectra to symmetric $\mathbf{Q}(1)$-spectra, which is also a right Quillen functor (and thus, in particular, preserves and detects stable $\mathbf{A}^1$-equivalences).

We thus have realization functors

$$\mathrm{DM}_{\mathrm{B}}(X) \to \mathrm{D}(X, \mathcal{E}) , \quad M \mapsto \mathcal{E}_X \otimes^{\mathbf{L}}_X M \tag{17.2.5.2}$$

which commute with the six operations of Grothendieck if ever S is the spectrum of a field (Theorem 4.4.25). Furthermore, $\mathrm{D}(-, \mathcal{E})$ is a motivic category which is $\mathbf{Q}$-linear (in fact $\mathbf{K}$-linear), separated, and continuous.

For an S-scheme X, define

$$H^q(X, E(p)) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{B}}(X)}(\mathbf{Q}_X, \mathcal{E}(p)[q]) \simeq \mathrm{Hom}_{\mathrm{D}(X,\mathcal{E})}(\mathcal{E}_X, \mathcal{E}_X(p)[q])$$

(this notation is compatible with 17.2.1 by virtue of Proposition 17.2.4).

Corollary 17.2.6. *Any stable cohomology (in particular, any mixed Weil cohomology) extends naturally to S-schemes of finite type, and this extension satisfies cohomological* h*-descent (in particular, étale descent as well as proper descent).*

Proof. This follows immediately from the construction above and from Theorem 14.3.4. □

17.2.7. We denote by $\mathrm{D}^{\vee}(S, \mathcal{E})$ the localizing subcategory of $\mathrm{D}(S, \mathcal{E})$ generated by its rigid objects (i.e. by the objects which have strong duals). For instance, for any smooth and proper S-scheme X, $\mathcal{E}(X) = \mathcal{E} \otimes^{\mathbf{L}}_S M_S(X)$ belongs to $\mathrm{D}^{\vee}(S, \mathcal{E})$; see 2.4.31.

If we denote by $\mathrm{D}(\mathbf{K})$ the (unbounded) derived category of the abelian category of $\mathbf{K}$-vector spaces, we get the following interpretation of the Künneth formula.

Theorem 17.2.8. *If E is a mixed Weil cohomology, then the functor*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, -) : \mathrm{D}^{\vee}(S, \mathcal{E}) \to \mathrm{D}(\mathbf{K})$$

is an equivalence of symmetric monoidal triangulated categories.

Proof. This is [**CD12**, theorem 2.6.2]. □

Theorem 17.2.9. *If S is the spectrum of a field, then $\mathrm{D}^{\vee}(S, \mathcal{E}) = \mathrm{D}(S, \mathcal{E})$.*

Proof. This follows then from Corollary 4.4.17. □

Remark 17.2.10. It is not reasonnable to expect the analog of Theorem 17.2.9 to hold whenever S is of dimension > 0; see (the proof of) [**CD12**, corollary 3.2.7]. Heuristically, for higher dimensional schemes X, the rigid objects of $\mathrm{D}(X, \mathcal{E})$ are extensions of some kind of locally constant sheaves (in the ℓ-adic setting, these correspond to $\mathbf{Q}_\ell$-*faisceaux lisses*).

Corollary 17.2.11. *If E is a mixed Weil cohomology, and if S is the spectrum of a field, then the functor*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, -) : \mathrm{D}(S, \mathcal{E}) \to \mathrm{D}(\mathbf{K})$$

is an equivalence of symmetric monoidal triangulated categories.

Remark 17.2.12. This result can be thought as a *tilting theory* for the spectra associated with mixed Weil cohomologies.

17.2.13. Assume that E is a mixed Weil cohomology, and that S is the spectrum of a field k. For each k-scheme of finite type X, denote by $\mathrm{D}_c(X, \mathcal{E})$ the category of constructible objects of $\mathrm{D}(X, \mathcal{E})$: by definition, this is the thick triangulated subcategory of $\mathrm{D}(X, \mathcal{E})$ generated by objects of shape $\mathcal{E}(Y) = \mathcal{E} \otimes^{\mathbf{L}}_X M_X(Y)$ for Y smooth over X (we can drop Tate twists because of 17.2.4 (ii)). The category $\mathrm{D}_c(X, \mathcal{E})$ also coincides with the category of compact objects in $\mathrm{D}(X, \mathcal{E})$; see 1.4.11. Write $\mathrm{D}^b(\mathbf{K})$ for the bounded derived category of the abelian category of finite dimensional $\mathbf{K}$-vector spaces. Note that $\mathrm{D}^b(\mathbf{K})$ is canonically equivalent to the homotopy category of perfect complexes of $\mathbf{K}$-modules, i.e. to the category of compact objects of $\mathrm{D}(\mathbf{K})$.

Corollary 17.2.14. *Under the assumptions of 17.2.13, we have a canonical equivalence of symmetric monoidal triangulated categories*

$$\mathrm{D}_c(\mathrm{Spec}\,(k)\,, \mathcal{E}) \simeq \mathrm{D}^b(\mathbf{K})\,.$$

Proof. This follows from 17.2.11 and from the fact that equivalences of categories preserve compact objects. □

Corollary 17.2.15. *Under the assumptions of 17.2.13, if E' is another* **K***-linear stable cohomology with associated ring spectrum $\mathcal{E}'$, any morphism of presheaves of commutative differential* **K***-algebras $E \to E'$ inducing an isomorphism $H^1(\mathbf{G}_m, E) \simeq H^1(\mathbf{G}_m, E')$ gives a canonical isomorphism $\mathcal{E} \simeq \mathcal{E}'$ in the homotopy category of commutative ring spectra. In particular, we get canonical equivalences of categories*

$$\mathrm{D}(X, \mathcal{E}) \simeq \mathrm{D}(X, \mathcal{E}')$$

for any k-scheme of finite type X (and these are compatible with the six operations of Grothendieck, as well as with the realization functors).

Proof. This follows from Theorem 17.2.9 and from [**CD12**, theorem 2.6.5]. □

The preceding corollary can be stated in the following way: if $\mathcal{E}$ and $\mathcal{E}'$ are two (strict) commutative ring spectra associated to **K**-linear mixed Weil cohomologies defined on smooth k-schemes E and E', respectively, then any morphism $\mathcal{E} \to \mathcal{E}'$ in the homotopy category of (commutative) monoids in the model category of **K**-linear Tate spectra is invertible. Moreover, $\mathcal{E}$ is isomorphic to $\mathcal{E}'$ if and only if E is isomorphic to E' (in the appropriate homotopy categories of commutative monoids). To be more precise (and more general), this last assertion follows immediately from Corollary 17.2.15 and from the following result.

Proposition 17.2.16. *Let* **E** *be a commutative monoid in the* $\mathbf{A}^1$*-stable model category of sheaves of complexes of symmetric* $\mathbf{Q}(1)[1]$*-spectra over the Nisnevich smooth site of k. Suppose that there exists an isomorphism $\mathbf{E}(1) \simeq \mathbf{E}$ in the homotopy category of* **E***-modules and that*

$$H^n(\mathrm{Spec}\,(k)\,, \mathbf{E}) = \begin{cases} \mathbf{K} & \textit{if } n = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $E = \mathbf{R}\Gamma(-, \mathbf{E})$ is a stable cohomology theory, and the commutative ring spectrum $\mathcal{E}$ associated to E by Proposition 17.2.4 is canonically isomorphic to **E** *in the homotopy category of (strict) commutative ring spectra.*

Proof. By virtue of Theorem 7.1.8, we may assume that **E** is (cofibrant and) fibrant. The ring spectrum **E** is defined by a symmetric sequence of complexes of Nisnevich sheaves of **K**-vector spaces $\mathbf{E}_n$, $n \geq 0$, (endowed with an action of the symmetric group on n-letters), together with maps $\sigma_n : \mathbf{E}_n(1)[1] \to \mathbf{E}_{n+1}$ inducing quasi-isomorphisms

$$\mathbf{E}_n \xrightarrow{\sim} Hom(\mathbf{K}(1)[1], \mathbf{E}_{n+1})$$

as well as pairings

$$\mathbf{E}_m \otimes_{\mathbf{K}} \mathbf{E}_n \to \mathbf{E}_{m+n}$$

satisfying a few compatibilities. In particular,

$$E = \mathbf{R}\Gamma(-, \mathbf{E}) = \mathbf{E}_0$$

is naturally endowed with a structure of Nisnevich sheaf of commutative differential graded algebras which satisfies Nisnevich descent and $\mathbf{A}^1$-homotopy invariance. Moreover, for any integer $n \geq 0$, the Nisnevich sheaf of complexes of **K**-vector spaces $\mathbf{E}_n$ also has the properties of Nisnevich descent and of $\mathbf{A}^1$-homotopy invariance, and is naturally endowed with a structure of E-module. It is clear that E is a stable cohomology theory, so that (the proof of) Proposition 17.2.4 provides a commutative ring spectrum $\mathcal{E}$ associated to it. With the notations introduced in the proof of Proposition 17.2.4, we know that $\mathcal{E}$ is made of the symmetric sequence $\{\mathcal{E}_n = Hom(L^{\otimes n}, E)\}_{n \geq 0}$, where L is the constant sheaf associated to $\Gamma(S, Hom(\mathbf{K}(1)[1], E))$. Let us define $\mathcal{L} = L(1)[1]$. We define a new symmetric sequence $\underline{\mathbf{E}}$ by the formula

$$\underline{\mathbf{E}}_n = Hom(\mathcal{L}^{\otimes n}, \mathbf{E}_n)\,, \quad n \geq 0\,,$$

where the symmetric group acts through the diagonal $\mathfrak{S}_n \to \mathfrak{S}_n \times \mathfrak{S}_n$ by permutation of the factors on $\mathcal{L}^{\otimes n}$ and by the structural action on $\mathbf{E}_n$. We see that $\underline{\mathbf{E}}$ is a commutative monoid in the category of symmetric sequences with the pairings defined by

$$Hom(\mathcal{L}^{\otimes m}, \mathbf{E}_m) \otimes_{\mathbf{K}} Hom(\mathcal{L}^{\otimes n}, \mathbf{E}_n) \to Hom(\mathcal{L}^{\otimes m+n}, \mathbf{E}_m \otimes_{\mathbf{K}} \mathbf{E}_n) \to Hom(\mathcal{L}^{\otimes m+n}, \mathbf{E}_{m+n})\,.$$

Finally, we can compose the transposition of the map $\sigma_1 : E(1)[1] \to \mathcal{E}_1$, with the structural map $\mathbf{K}(1)[1] \to Hom(L, E) = \mathcal{E}_1$, to obtain:

$$\mathbf{K}(1)[1] \to Hom(L, E) \to Hom(L, Hom(\mathbf{K}(1)[1], \mathbf{E}_1) \simeq Hom(\mathcal{L}, \mathbf{E}_1) = \underline{\mathbf{E}}_1\,.$$

This defines a structure of commutative ring spectrum on $\underline{\mathbf{E}}$. Note that L is chain homotopy equivalent to $\mathbf{K}[-1]$, so that the functors $Hom(L^{\otimes n}, -)$ preserve quasi-isomorphisms (more precisely, L is cohomologically concentrated in degree 1, and its first cohomology sheaf is the constant sheaf associated to the $\mathbf{K}$-vector space of dimension one $H^1(\mathbf{G}_m, \mathbf{E})$). Therefore, one has a quasi-isomorphism of commutative monoids of $\mathbf{K}$-linear Tate spectra $\mathcal{E} \to \underline{\mathbf{E}}$, defined by the canonical maps

$$Hom(L^{\otimes n}, E) \to Hom(L^{\otimes n}, Hom(\mathbf{K}(n)[n], \mathbf{E}_n) \simeq Hom(\mathcal{L}^{\otimes n}, \mathbf{E}_n)\,.$$

It remains to produce a quasi-isomorphism of commutative monoids of Tate spectra $\mathbf{E} \to \underline{\mathbf{E}}$. We have a structural map $\mathbf{K}(1)[1] \to Hom(L, E)$ which can be transposed into a map

$$\mathcal{L} = L(1)[1] \to E = \mathbf{E}_0\,.$$

As E is a commutative monoid and each $\mathbf{E}_n$ an E-module, we have natural maps

$$\mathcal{L}^{\otimes n} \otimes_{\mathbf{K}} \mathbf{E}_n \to E^{\otimes n} \otimes_{\mathbf{K}} \mathbf{E}_n \to E \otimes_{\mathbf{K}} \mathbf{E}_n \to \mathbf{E}_n$$

which can be transposed into $\mathfrak{S}_n$-equivariant maps

$$\mathbf{E}_n \to Hom(\mathcal{L}^{\otimes n}, \mathbf{E}_n) = \underline{\mathbf{E}}_n\,.$$

These define a morphism of commutative monoids of $\mathbf{K}$-linear Tate spectra $\mathbf{E} \to \underline{\mathbf{E}}$. It remains to check that the maps $\mathbf{E}_n \to \underline{\mathbf{E}}_n$ are quasi-isomorphisms for each $n \geq 0$. As $Hom(\mathbf{K}(n)[n], E) \simeq \mathbf{E}_n$, we can replace $\mathbf{E}_n$ by $Hom(L^{\otimes n}, E)$. The case $n = 1$ is then a reformulation of Proposition 17.2.4 (*ii*), and the general case follows by an obvious induction. □

THEOREM 17.2.17. *Under the assumptions of 17.2.13, the six operations of Grothendieck preserve constructibility in* $\mathrm{D}(-, \mathcal{E})$.

PROOF. Observe that $\mathrm{D}(-, \mathcal{E})$ is $\mathbf{Q}$-linear and separated (because DM_{B} is so, see 7.2.18), as well as pure (by Proposition 4.4.16). We conclude with 4.2.29. □

17.2.18. As a consequence, we have, for any k-scheme of finite type X, a realization functor

$$\mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_c(X, \mathcal{E})$$

and we deduce from Theorem 4.4.25 that it preserves all of Grothendieck six operations. For $X = \mathrm{Spec}\,(k)$, by virtue of Corollary 17.2.14, this corresponds to a symmetric monoidal exact realization functor

$$R : \mathrm{DM}_{\mathrm{B},c}(\mathrm{Spec}\,(k)) \to \mathrm{D}^b(\mathbf{K})\,.$$

This leads to a finiteness result:

COROLLARY 17.2.19. *Under the assumptions of 17.2.13, for any k-scheme of finite type X, and for any objects M and N in* $\mathrm{D}_c(X, \mathcal{E})$, $\mathrm{Hom}_{\mathcal{E}}(M, N[n])$ *is a finite dimensional* $\mathbf{K}$*-vector space, and it is trivial for all but a finite number of values of n.*

PROOF. Let $f : X \to \mathrm{Spec}\,(k)$ be the structural map. By virtue of 17.2.17, as M and N are constructible, the object $\mathbf{R}f_*\,\mathbf{R}Hom_X(M, N)$ is constructible as well, i.e. is a compact object of $\mathrm{D}(\mathrm{Spec}\,(k)\,, \mathcal{E})$. But $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M, N)$ is nothing else than the image of $\mathbf{R}f_*\,\mathbf{R}Hom_X(M, N)$ by the equivalence of categories given by Corollary 17.2.11. Hence $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M, N)$ is a compact object of $\mathrm{D}(\mathbf{K})$, which means that it belongs to $\mathrm{D}^b(\mathbf{K})$. □

17.2.20. For a $\mathbf{K}$-vector space V and an integer n, define

$$V(n) = \begin{cases} V \otimes_{\mathbf{K}} \mathrm{Hom}_{\mathbf{K}}(H^1(\mathbf{G}_m, E)^{\otimes n}, \mathbf{K}) & \text{if } n > 0, \\ V \otimes_{\mathbf{K}} H^1(\mathbf{G}_m, E)^{\otimes(-n)} & \text{if } n \leq 0. \end{cases}$$

Any choice of a generator in $\mathbf{K}(-1) = H^1(\mathbf{G}_m, E) \simeq H^2(\mathbf{P}^1_k, E)$ defines a natural isomorphism $V(n) \simeq V$ for any integer n. We have canonical isomorphisms

$$H^q(X, E(p)) \simeq H^q(X, E)(p)$$

(using the fact that the equivalence of Corollary 17.2.14 is monoidal). The realization functors (17.2.5.2) induce in particular cycle class maps

$$cl_X : H^q_{\mathrm{B}}(X, \mathbf{Q}(p)) \to H^q(X, E)(p)$$

(and similarly for cohomology with compact support, for homology, and for Borel-Moore homology).

Example 17.2.21. Let k be a field of characteristic zero. We then have a mixed Weil cohomology E_{dR} defined by the algebraic de Rham complex

$$E_{dR}(X) = \Omega^*_{A/k}$$

for any smooth affine k-scheme of finite type $X = \mathrm{Spec}\,(A)$ (algebraic de Rham cohomology of smooth k-schemes of finite type is obtained by Zariski descent); see [**CD12**, 3.1.5]. We obtain from 17.2.4 a commutative ring spectrum $\mathcal{E}_{dR}$, and, for a k-scheme of finite type X, we define

$$\mathrm{D}_{dR}(X) = \mathrm{D}_c(X, \mathcal{E}_{dR})\,.$$

We thus get a motivic category D_{dR}, and we have a natural definition of algebraic de Rham cohomology of k-schemes of finite type, given by

$$H^n_{dR}(X) = \mathrm{Hom}_{\mathrm{D}_{dR}(X)}(\mathcal{E}_{dR,X}, \mathcal{E}_{dR,X}[n])\,.$$

This definition coincides with the usual one: this is true by definition for separated smooth k-schemes of finite type, while the general case follows from h-descent (17.2.6) and from de Jong's Theorem 4.1.11 (or resolution of singularities à la Hironaka). We have, by construction, a *de Rham realization functor*

$$R_{dR} : \mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_{dR}(X)$$

which preserves the six operations of Grothendieck (Theorem 4.4.25). In particular, we have cycle class maps

$$H^q_{\mathrm{B}}(X, \mathbf{Q}(p)) \to H^q_{dR}(X)(p)\,.$$

Note that, for any field extension k'/k, we have natural isomorphisms

$$H^n_{dR}(X) \otimes_k k' \simeq H^n_{dR}(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}\,(k'))\,.$$

Example 17.2.22. Let k be a field of characteristic zero, which is algebraically closed and complete with respect to some valuation (archimedian or not). We can then define a stable cohomology $E_{dR,an}$ as analytic de Rham cohomology of X^{an}, for any smooth k-scheme of finite type X; see [**CD12**, 3.1.7]. As above, we get a ring spectrum $\mathcal{E}_{dR,an}$, and for any k-scheme of finite type, a category of coefficients

$$\mathrm{D}_{dR,an}(X) = \mathrm{D}_c(X, \mathcal{E}_{dR,an})\,,$$

which allows to define the analytic de Rham cohomology of any k-scheme of finite type X by

$$H^n_{dR,an}(X) = \mathrm{Hom}_{\mathrm{D}_{dR,an}(X)}(\mathcal{E}_{dR,an,X}, \mathcal{E}_{dR,an,X}[n])\,.$$

We also have a realization functor

$$R_{dR,an} : \mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_{dR,an}(X)$$

which preserves the six operations of Grothendieck.

We then have a morphism of stable cohomologies

$$E_{dR} \to E_{dR,an}$$

which happens to be a quasi-isomorphism locally for the Nisnevich topology (this is Grothendieck's theorem in the case where K is archimedian, and Kiehl's theorem in the case where K is non-archimedian; anyway, one obtains this directly from Corollary 17.2.15). This induces a canonical isomorphism

$$\mathcal{E}_{dR} \simeq \mathcal{E}_{dR,an}$$

in the homotopy category of commutative ring spectra. In particular, $E_{dR,an}$ is a mixed Weil cohomology, and, for any k-scheme of finite type, we have natural equivalences of categories

$$\mathrm{D}_{dR}(X) \to \mathrm{D}_{dR,an}(X)\ , \quad M \mapsto \mathcal{E}_{dR,an} \otimes^{\mathbf{L}}_{\mathcal{E}_{dR}} M$$

which commute with the six operations of Grothendieck and are compatible with the realization functors.

Note that, in the case $k = \mathbf{C}$, $\mathcal{E}_{dR,an}$ coincides with Betti cohomology (after tensorization by $\mathbf{C}$), so that we have canonical fully faithful functors

$$\mathrm{D}_{Betti,c}(X) \otimes_{\mathbf{Q}} \mathbf{C} \to \mathrm{D}_{dR,an}(X)$$

which are compatible with the realization functors. More precisely, it follows from Proposition 17.2.16 that the Betti spectrum $\mathcal{E}_{Betti}$, obtained by appyling Theorem 17.1.5 to Ayoub's realization functor (17.1.7.4), is the spectrum associated to $\mathbf{Q}$-linear Betti cohomology, seen as a mixed Weil cohomology, from Proposition 17.2.4. Therefore, the holomorphic Poincaré Lemma, together with Corollary 17.2.15, provide an isomorphism

$$\mathcal{E}_{Betti} \otimes_{\mathbf{Q}} \mathbf{C} \simeq \mathcal{E}_{dR,an}$$

in the homotopy category of commutative monoids of the model category of $\mathbf{C}$-linear Tate spectra. We thus have triangulated equivalences of categories

$$\mathrm{D}^b_c(X(\mathbf{C}), \mathbf{C}) \simeq \mathrm{Ho}(\mathcal{E}_{Betti} \otimes_{\mathbf{Q}} \mathbf{C}\text{-}\operatorname{mod})_c(X) \simeq \mathrm{D}_{dR,an}(X)$$

which commute with the six operations as well as with the realization functors. In particular, by the Riemann-Hilbert correspondence, $\mathrm{D}_{dR,an}(X)$ is equivalent to the bounded derived category of analytic regular holonomic $\mathcal{D}$-modules on X which are constructible of geometric origin. (A purely algebraic proof of this equivalence would furnish a new proof of the Riemann-Hilbert correspondence, using Corollary 17.2.15.)

Example 17.2.23. Let V be a complete discrete valuation ring of mixed characteristic with perfect residue field k and field of functions K. The Monsky-Washnitzer complex defines a stable cohomology E_{MW} over smooth V-schemes of finite type, defined by

$$E_{MW}(X) = \Omega^*_{A^\dagger/V} \otimes_V K$$

for any affine smooth V-scheme $X = \operatorname{Spec}(A)$ (the case of a smooth V-scheme of finite type is obtained by Zariski descent); see [**CD12**, 3.2.3]. Let $\mathcal{E}_{MW}$ be the corresponding ring spectrum in $\mathrm{DM}_{\mathrm{B}}(\operatorname{Spec}(V))$, and write $j : \operatorname{Spec}(K) \to \operatorname{Spec}(V)$ and $i : \operatorname{Spec}(k) \to \operatorname{Spec}(V)$ for the canonical immersions. As we obviously have $j^*\mathcal{E}_{MW} = 0$ (the Monsky-Washnitzer cohomology of a smooth V-scheme with empty special fiber vanishes), we have a canonical isomorphism

$$\mathcal{E}_{MW} \simeq \mathbf{R}i_*\, \mathbf{L}i^*\mathcal{E}_{MW}\,.$$

We define the rigid cohomology spectrum $\mathcal{E}_{rig}$ in $\mathrm{DM}_{\mathrm{B}}(\operatorname{Spec}(k))$ by the formula

$$\mathcal{E}_{rig} = \mathbf{L}i^*\mathcal{E}_{MW}\,.$$

This is a ring spectrum associated to a K-linear mixed Weil cohomology: cohomology with coefficients in $\mathcal{E}_{rig}$ coincides with rigid cohomology in the sense of Berthelot, and the Künneth formula for rigid cohomology holds for smooth and projective k-schemes (as rigid cohomology coincides then with cristalline cohomology), from which we deduce the Künneth formula for smooth k-schemes of finite type; see [**CD12**, 3.2.10]. As before, we define

$$\mathrm{D}_{rig}(X) = \mathrm{D}_c(X, \mathcal{E}_{rig})$$

for any k-scheme of finite type X, and put

$$H^n_{rig}(X) = \operatorname{Hom}_{\mathrm{D}_{rig}(X)}(\mathcal{E}_{rig,X}, \mathcal{E}_{rig,X}[n])\,.$$

Here again, we have, by construction, *rigid realization functors*

$$R_{rig} : \mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{D}_{rig}(X)$$

which preserve the six operations of Grothendieck (Theorem 4.4.25), as well as (higher) cycle class maps

$$H^q_{\mathrm{B}}(X, \mathbf{Q}(p)) \to H^q_{rig}(X)(p)\,.$$

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Index of terminology

Index of notations

APPENDIX A

Index of properties of $\mathscr{P}$-fibred triangulated categories

Name	Symbol	Def.	related result	Remark
additive		2.1.1		
adjoint property	(Adj)	2.2.13	2.2.14	
adjoint property for f	(Adj_f)	2.2.13		f morphism of schemes
cotransversality property		1.1.17		defined for any $\mathscr{P}$-fibred category
homotopy property	(Htp)	2.1.3		
localization property	(Loc)	2.3.2	2.4.26 6.3.15	
localization property for i	(Loc_i)	§2.3.1		i closed immersion
motivic		2.4.45	2.4.50 14.2.11	for premotivic triangulated categories, means: (Htp), (Stab), (Loc), (Adj)
oriented		2.4.38	2.4.43	for triangulated premotivic categories satisfying (wLoc)
projection formula	(PF)	2.2.13		
projection formula for f	(PF_f)	2.2.13	2.4.26	f morphism of schemes
proper base change property	(BC)	2.2.13	2.4.26	
proper base change property for f	(BC_f)	2.2.13		f morphism of schemes
purity property	(Pur)	2.4.21	2.4.26	
separated	(Sep)	2.1.7	4.2.24 4.4.21 14.3.3	
semi-separated	(sSep)	2.1.7	3.3.33	
stability property	(Stab)	2.4.4		
support property	(Supp)	2.2.5	2.2.12 2.2.14 11.4.2	
τ-compatible		4.2.20	4.2.29	τ set of twists
τ-continuous		4.3.2	6.1.13 11.1.24 14.3.1	for homotopy $\mathscr{P}$-fibred categories, τ set of twists
τ-dualizable		4.4.13	4.4.21	τ set of twists
t-descent property		3.2.5		for homotopy $\mathscr{P}$-fibred categories, t topology
transversality property		1.1.17		for any $\mathscr{P}$-fibred category
t-separated	(t-sep)	2.1.5		t topology
weak localization property	(wLoc)	2.4.7	11.4.2	
weak purity property	(wPur)	2.4.21	2.4.26 2.4.43	

ABSTRACT: We construct triangulated categories of mixed motives over a noetherian scheme of finite dimension, extending Voevodsky's definition of motives over a field. We prove that motives with rational coefficients satisfy the formalism of the six operations of Grothendieck. This is achieved by studying descent properties of motives, as well as by comparing different presentations of these categories, following insights and constructions of Beilinson, Morel and Voevodsky. Finally, we associate with any mixed Weil cohomology a system of categories of coefficients and well behaved realization functors.

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