DIMENSIONAL HOMOTOPY T-STRUCTURES IN MOTIVIC HOMOTOPY THEORY

MIKHAIL BONDARKO AND FRÉDÉRIC DÉGLISE

ABSTRACT. The aim of this work is to construct certain homotopy t-structures on various categories of motivic homotopy theory, extending works of Voevodsky, Morel, Déglise and Ayoub. We prove these t-structures possess many good properties, some analogous to that of the perverse t-structure. We also show that their heart provides a convenient extension of the theory of homotopy invariant sheaves with transfers, extending some of the main results of Voevodsky. These t-structures are closely related to Gersten weight structures as defined by Bondarko.

CONTENTS

INTRODUCTION

Background. The theory of motivic complexes over a perfect field k, invented by Voevodsky after a conjectural description of Beilinson, is based on the notion of homotopy invariant sheaves with

transfers. These sheaves have many good properties: they form an abelian category $HI(k)$ with a tensor product, their (Nisnevich) cohomology over smooth k-schemes is homotopy invariant, and they admit a Gersten resolution. An aftermath of the construction of the category of motivic complexes $DM^{eff}_-(k)$ is the existence of a canonical t-structure whose heart is exactly $HI(k)$. It was called the *homotopy t-structure* by Voevodsky in [VSF00, chap. 5]. Though this t-structure is not the motivic t-structure, its role is fundamental in the theory of Voevodsky. For example, the main conjectures of the theory have a clean and simple formulation in terms of the homotopy t-structure.[1]

Many works have emerged around the homotopy t-structure. The Gersten resolution for homotopy invariant sheaves with transfers has been promoted into an equivalence of categories between $HI(k)$ and a full subcategory of the category of Rost's cycle modules ([Ros96]) by the second-named author in [Dég11]. At this occasion, it was realized that to extend this equivalence to the whole category of Rost's cycle modules, one needed an extension of $HI(k)$ to a non effective category (with respect to the $\mathbf{G}_m$-twist), in the spirit of stable homotopy. This new abelian category was interpreted as the heart of the category now well known as $DM(k)$, the category of *stable motivic complexes*.

In the mean time, Morel has extended the construction of the homotopy t-structure to the context of the stable homotopy category $SH(k)$ of schemes over a field k (non necessarily perfect), and its non effective analogue, the S^1-stable category $SH^{S^1}(k)$ in [Mor05]. In this topological setting, the t-structure appeared as an incarnation of the *Postnikov tower* and the heart of the homotopy t-structure as the natural category in which stable (resp. S^1-stable) homotopy groups take their value. It was later understood that the difference between the heart of $SH(k)$ and $DM(k)$ was completely encoded into the action of the *(algebraic) Hopf map* (a conjecture of Morel proved in [Dég13]). The computation of the stable homotopy groups that we presently know uses this language in an essential way.

The final foundational step up to now was taken by Ayoub who extended the definition of Voevodsky and Morel to the relative setting in his Ph. D. thesis [Ayo07], in his abstract framework of *stable homotopy functors*. Under certain assumptions, he notably builds a t-structure over a scheme S with the critical property that it is "compatible with gluing": given any closed subscheme Z of X with complement U, the t-structure over S is uniquely determined by its restrictions over Z and U (cf. [Ayo07, 2.2.79]). This is the analogue property of the *perverse t-structure* on torsion étale sheaves underlined in the fundamental work [BBD82] of Beilinson, Bernstein and Deligne. Besides, when S is the spectrum of a field, Ayoub established that his t-structure agrees with that of Morel (cf. [Ayo07, 2.2.94]). After the work that was done on the foundations of motivic complexes in the relative context in [CD12, CD15b], this agreement immediately extends to the definition of Voevodsky. However, the assumptions asked by Ayoub in this pioneering work are very strong. They apply to the stable homotopy category only in characteristic 0 (integrally or with rational coefficients, cf. [Ayo07, 2.1.166, 2.1.168]), and, after the results of [CD12, CD15b], to the triangulated category of mixed motives, integrally in characteristic 0, and rationally over a field or a valuation ring (cf. [Ayo07, 2.1.171, 2.1.172]).

Our main construction in the stable case. The purpose of this text is to provide a new approach on the problem of extending the homotopy t-structure of Voevodsky and that of Morel over a base scheme S which will always be assumed to be noetherian finite dimensional. In this

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[1]Here are some classical examples:

- The *Beilinson-Lichtenbaum conjecture* (now a theorem): for any prime $l \in k^\times$, the motivic complex $\mathbf{Z}/l\mathbf{Z}(n)$ is isomorphic to $\tau_{\leq n}\mathbf{R}\pi_*(\mu_l^{\otimes,n})$ where $\tau_{\leq n}$ is the truncation functor for the homotopy t-structure and π_* the forgetful functor from étale to Nisnevich sheaves.
- The *Beilinson-Soulé conjecture*: for any integer $n > 0$, the motivic complex $\mathbf{Z}(n)$ is concentrated in degree $[1, n]$ for the homotopy t-structure.
- A weak form of *Parshin conjecture*: for any integer $n > 0$, the motivic complex $\mathbf{Q}(n)$ over a finite field is concentrated in degree n for the homotopy t-structure.

introduction, we will describe our constructions by fixing triangulated categories $\mathscr{T}(S)$ for suitable schemes S assuming they are equipped with Grothendieck's six operations.[2]

Our first technical tool, as in the work of Ayoub, is the possibility of defining a t-structure on a compactly generated[3] triangulated category by fixing a family of compact objects, called *generators* and set them as being homologically non-t-negative.[4] Then the class of t-negative objects are uniquely determined by the orthogonality property and the existence of truncation functors follow formally (see again Section 1.2 for details). Our approach differs from that of Ayoub in the choice of generators.

The guiding idea is that, as in the case of motivic complexes, motives of smooth S-schemes should be homologically non-t-negative. However, these generators are not enough to ensure the compatibility with gluing. In particular, we should be able to use non dominant S-schemes, in particular closed subschemes of S. We deal with this problem with two tools: the exceptional direct image functor $f_!$ as well as the choice of a dimension function[5] on the scheme S which will add a correcting shift in our generators, related to the dimension of the fibers of f. A dimension function on S is an application $\delta : S \to \mathbf{Z}$ which shares the following property of Krull dimension on a scheme of finite type over a field k:

$$(0.0.0.a) \qquad \delta_k(x) = \dim\left(\overline{\{x\}}\right) = \operatorname{degtr}(\kappa(x)/k)$$

where $\kappa(x)$ is the residue field of x in S. For any S-scheme X of finite type, the dimension function δ induces a canonical dimension function on X, and we let $\delta(X)$ be the maximum of this induced function on X. The last fact of which the reader must be aware is that Voevodsky's homotopy t-structure in the stable case is stable under $\mathbf{G}_m$-twists: the functor $K \mapsto K(1)[1]$ is t-exact for Voevodsky's homotopy t-structure (beware that this is false in the effective case).

Taking all these ideas together, we define the δ-*homotopy* t-*structure over* S by taking the following generators for homologically positive objects:

$$(0.0.0.b) \qquad f_!(\mathbb{1}_X)(n)[\delta(X) + n], \ f \text{ separated of finite type, } n \text{ any integer.}$$

With this definition, it is clear that the endofunctor $K \mapsto K(1)[1]$ of $\mathscr{T}(S)$ becomes t-exact. This already shows that our t-structure covers a phenomenon which is special to motives: on triangulated categories such as $D^b(S_{\text{ét}}, \mathbf{Z}/l\mathbf{Z})$, l invertible on S, it is not reasonable because if S contains l-th roots of unity, we get an isomorphism $\mathbf{Z}/l\mathbf{Z}(1) = \mathbf{Z}/l\mathbf{Z}$ so that the t-structure with the above generators is degenerated. The same argument will apply to the category of integral étale motives (cf. [Ayo14, CD15a]) and to modules over algebraic K-theory (cf. [CD12, §13.3]).

However, on categories such as SH and DM (see Example 1.3.1 for a detailed list), the resulting t-structure is very reasonable and we easily deduce, without any further assumption on S, that the δ-homotopy t-structure is compatible with gluing as well as some basic exactness: $f_* = f_!$ (resp. $f^* = f^!$) is t-exact when f is finite (étale). Note that the δ-homotopy t-structure does not really depend on the choice of δ. Rather, a change of δ shifts the t-structure. So in formulating t-exactness properties, we have to be precise about the choices of dimension functions. We refer the reader to the text for these details.

After this definition, our first theorem comes from the fact that we can find sharpen the description of the generators for the δ-homotopy t-structure in several ways. Actually in the description (0.0.0.b) we can add one of the following assumptions:

(1) f is proper (even projective).
(2) X is regular.
(3) f is proper and X is regular.

Points 1 and 2 are easily obtained (see Proposition 2.3.1) but they already yield interesting results. Most notably, they give the comparison of our definition with the classical ones. When S is the

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[2]We do not have to be more precise for describing our definitions but the reader can find a list of our main examples in Example 1.3.1.

[3]see our conventions below for recall on this property of triangulated categories;

[4]Following Morel and Ayoub we will use homological conventions for our t-structures; see Section 1.2.

[5]dimension functions appeared for the first time in [SGA4, XIV, 2.2] in the local case. They were later formalized by Gabber, cf. [PS14]. We give a recall on this notion in Section 1.1;

spectrum of a perfect field k of characteristic exponent p and δ_k the dimension function of (0.0.0.a), the δ_k-homotopy t-structure on $\mathrm{DM}(k)[1/p]$ (resp. $\mathrm{SH}(k)$) coincides with that of Voevodsky (resp. Morel): see Example 2.3.5. We also give a comparison with Ayoub's t-structure under favourable assumptions: see Corollary 2.3.10.

The strongest form of this improvement on generators, point 3 above, is obtained at the cost of using a convenient resolution of singularities (De Jong-Gabber for rational coefficients, De Jong for $\mathbf{Z}[1/p]$-coefficients and schemes of characteristic exponent p) together with good properties for the category $\mathscr{T}$ (the precise hypothesis are given in Paragraph 2.4.1). This description (Th. 2.4.3) will be a key point for our main theorem. The method is classical though it uses a trick of Riou to go beyond the cases that were known up to now. In particular, it implies an interesting result of motivic homotopy theory which was not known till now and that deserves a separate formulation:

Theorem 1 (Th. 2.4.8 and Ex. 2.4.9)**.** Let p be a prime or $p = 1$. Let $\mathrm{SH}_c(S)[1/p]$ be the constructible part (*i.e.* made of compact spectra) of the stable homotopy category $\mathrm{SH}(S)[1/p]$ over a scheme S with p inverted.

   (1) Then $\mathrm{SH}_c[1/p]$ is stable under the six operations if one restricts to schemes of characteristic exponent p and to morphisms of finite type for the operation f_*.
   (2) For any separated morphism $f : X \to S$ of finite type, the spectrum $f^!(S^0)$ is dualizing in the triangulated monoidal category $\mathrm{SH}_c(X)[1/p]$.

Up to know, this fact was only known when $p = 1$ according to [Ayo07].

The effective case. The preceding theorem is deduced from our results on the δ-homotopy t-structure because there is a strong relation between the generators of a t-structure and the generators in the whole triangulated category.[6] Indeed, the generated t-structure is left non-degenerated if and only if its generators are also generators of the triangulated category (see Lemma 1.2.8). This leads us to the following observation: in the effective case of motivic complexes, one cannot invert the Tate twist. In particular, it is natural to consider the generators (0.0.0.b) with the restriction that $n \geq \delta(X)$ (rather than $n \geq 0$, for duality reasons). Then, if we want this new t-structure to be non degenerated, it is natural to consider the triangulated localizing subcategory of $\mathscr{T}(S)$ generated by objects of the form:

$$(0.0.0.c) \qquad f_!(\mathbb{1}_X)(n)[\delta(X) + n], \; f \text{ separated of finite type}, \; n \geq \delta(X).$$

We denote by $\mathscr{T}^{\delta-\mathit{eff}}(S)$ this triangulated category and call it the *δ-effective subcategory*. The t-structure generated by the above generators is called the *effective δ-homotopy t-structure*. We show that this triangulated category, with its t-structure, has many good properties, and especially, it satisfies the gluing (or localization) property (see Cor. 2.2.10 and 2.2.17). Besides, we are able to describe its stability by many of the 6 operations (see Section 2.2 for details).

The main case of this definition is given by the category of R-linear motives $\mathrm{DM}(S, R)$ over a scheme S in the following two cases:

   • S has characteristic exponent p and $p \in R^\times$ ([CD15b]).
   • no assumptions on S and R is a $\mathbf{Q}$-algebra ([CD12]).

Then it is shown in Example 2.3.12 that when $S = \mathrm{Spec}(k)$ is the spectrum of a perfect field and δ_k the dimension function of (0.0.0.a), the triangulated category $\mathrm{DM}^{\delta-\mathit{eff}}(k, R)$ is equivalent to Voevodsky's category of motivic complexes over k. Besides, the category $\mathrm{DM}^{\delta-\mathit{eff}}(k, R)$ is invariant under purely inseparable extension. Then the category $\mathrm{DM}^{\delta-\mathit{eff}}(S, R)$ is essentially the only possible extension to a positive dimensional scheme S which satisfies the gluing property. Therefore, our construction provides a good extension of the theory of motivic complexes, with their homotopy t-structure, which was missing in motivic homotopy theory given that the natural category $\mathrm{DM}^{\mathit{eff}}(S, R)$ of [CD12] is not known to satisfies the gluing property.

The niveau filtration. As visible from the Gersten resolution of homotopy invariant sheaves with transfers and their comparison with cycle modules, Voevodsky's homotopy t-structure shares

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[6]See our conventions on triangulated categories for recall on that later notion.

an intimate relationship with the classical coniveau filtration. In fact, it was proved in [Dég12] that the coniveau spectral sequence associated with the cohomology of a smooth k-scheme with coefficients in an arbitrary motivic complex K coincide from E_2-on with the spectral sequence associated with the truncations of K for the homotopy t-structure. In [Bon10a] and [Bon15] this relationship was recast into the framework of weight structures as the coniveau filtration was interpreted as a particular weight structure called the *Gersten weight structure*.

In our generalized context, this relationship is at the heart of our main theorem. Our use of a dimension function δ and the will to work other a base scheme S rather than a field lead us to consider niveau filtration measured by the given dimension function: in other words, we filter the scheme S by looking at closed subschemes $Z \subset S$ such that $\delta(Z) \geq n$ for various integers n. Moreover, to deal with arbitrary eventually singular S-schemes, we are lead to consider homology rather than cohomology following in that point the classical work [BO74] of Bloch and Ogus. We extend their ideas by using the so-called *Borel-Moore homology* relative to the base scheme S; note that the homology considered by Bloch and Ogus is the Borel-Moore homology relative to the base field. In terms of the 6 functors formalism, Borel-Moore homology with coefficients in any coefficient object $\mathbb{E}$ in $\mathscr{T}(S)$ can be defined as follows:

$$E_{p,q}^{BM} = \mathrm{Hom}_{\mathscr{T}(X)}\left(\mathbb{1}_X(q)[p], f^!(\mathbb{E})\right).$$

When S is regular and $\mathbb{E}$ is constant this can be interpreted in good situations (for example in the situation of the above theorem) as cohomology in degree $(-p, -q)$ with coefficients in the *dualizing object* over X. Moreover, by adjunction, Borel-Moore homology corresponds to the following abelian groups:

$$\mathrm{Hom}_{\mathscr{T}(X)}\left(\mathbb{1}_X(q)[p], f^!(\mathbb{E})\right) = \mathrm{Hom}_{\mathscr{T}(S)}\left(f_!(\mathbb{1}_X(q)[p]), \mathbb{E}\right)$$

so that it is natural, following Riou, to call $f_!(\mathbb{1}_X)$ the *Borel-Moore object* (motive, spectra, etc...) associated with f (or X/S).

The six functors formalism immediately yields that Borel-Moore homology, like Chow groups, is covariant with respect to proper morphisms and contravariant with respect to étale morphisms (see Section 1.3 for recall). These ingredients altogether allows us to extend the consideration of Bloch and Ogus and to build what we call the δ-niveau spectral sequence:

$$^{\delta}E_{p,q}^1 = \bigoplus_{x \in X_{(p)}} \hat{\mathbb{E}}_{p+q,n}^{BM}(x/S) \Rightarrow \mathbb{E}_{p+q,n}^{BM}(X/S)$$

where $X_{(p)} = \{x \in X \mid \delta(x) = p\}$ and $\hat{\mathbb{E}}_{**}^{BM}(x/S)$ is computed by taking the limit of the Borel-Moore homology of $(\overline{\{x\}} \cap U)/S$ for open neighbourhoods U of x in X. A first application of the δ-niveau spectral sequence allows to compute this generalized Borel-Moore homology in case of mixed motives:

Theorem 2 (Th. 3.1.7)**.** Consider a regular scheme S and a localization R of the ring of integers $\mathbf{Z}$ such that $R = \mathbf{Q}$ or S is an $\mathbf{Q}$-scheme or (S is an $\mathbb{F}_p$-scheme and $p \in R^\times$). Let δ be the dimension function on S such that $\delta = -\mathrm{codim}_S$ (see Example 1.1.4).

Then one has a canonical isomorphism:

$$H_{2n,n}^{BM}(X/S, R) \simeq CH_{\delta=n}(X) \otimes R$$

where the left hand side is Borel-Moore motivic homology of X/S with coefficients in R and the right hand side is the Chow group of R-linear algebraic cycles $\sum_i n_i.x_i$ of X such that $\delta(x_i) = n$ (see also [Stack, Chap. 41, Def. 9.1]).

This result generalizes the case where S is the spectrum of a (perfect) field. It is a new indication of the relevance of dimension functions. As indicated in [CD15b, Rem. 7.1.12(4)], when ℓ is a prime invertible on the scheme S, this gives a generalized cycle class:

$$\sigma : CH_{\delta=n}(X) \otimes \mathbf{Q} \simeq H_{2n,n}^{BM}(X/S, \mathbf{Q}) \to H_{2n,n}^{BM,\text{ét}}(X/S, \mathbf{Q}_\ell)$$

where the left hand side is the rational ℓ-adic Borel-Moore étale cohomology of X/S and the map is induced by ℓ-adic realization functor of [CD15b, 7.2.24]. For example, when all the generic points

η of X satisfy $\delta(\eta) = n$ for a chosen integer n, we get by looking at the image of the fundamental cycle of X on the left hand side under the map σ a canonical map in the ℓ-adic derived category of Ekedahl over the small étale site of X:

$$\mathbf{Q}_\ell(n)[2n] \to f^!(\mathbf{Q}_\ell)$$

which is a generalization of the local fundamental class ([SGA4.5], when X is a closed subscheme of S) and of the fundamental class when $S = \mathrm{Spec}(k)$ (cf. [BO74]).

The δ-niveau spectral sequence will moreover be essential in the study of the δ-homotopy t-structure. In this perspective, we are lead to introduce the following definition for a given object $\mathbb{E}$ of $\mathscr{T}(S)$. Let $x : \mathrm{Spec}(E) \to S$ be an E-valued point of S, essentially of finite type. Then E is the field of functions of an integral affine S-scheme of finite type, say X. We define the *fiber δ-homology* $\hat{H}_p^\delta(\mathbb{E})$ of $\mathbb{E}$ as the application which to a point x as above and a twist $n \in \mathbf{Z}$ associates the following abelian group:[7]

(0.0.0.d) $$\hat{H}_p^\delta(\mathbb{E})(x,n) = \varinjlim_{U \subset X} \mathbb{E}^{BM}_{\delta(x)+p,\{\delta(x)-n\}}(U/S).$$

We also consider the δ-effective version of this definition, $\hat{H}_p^{\delta-eff}(\mathbb{E})$, by restriction to the twists $n \leq 0$.

In fact, the δ-niveau spectral sequence with coefficients in $\mathbb{E}$ can be rewritten in terms of the fiber δ-homology of $\mathbb{E}$. In particular, we deduce from the convergence of the δ-niveau spectral sequence that the sequence of functors $\mathbb{E} \mapsto H_p^\delta(\mathbb{E})$ indexed by integers $p \in \mathbf{Z}$ is conservative. Moreover, our main theorem can be stated as follows:

Theorem 3 (Th. 3.3.1). Under suitable assumptions on $\mathscr{T}$ satisfied in the following examples:

- k is a field of characteristic exponent p, S is a k-scheme, and $\mathscr{T}$ is $\mathrm{SH}[1/p]$ (resp. DM_R for a ring R such that $p \in R^\times$);
- R is a $\mathbf{Q}$-algebra, S is any scheme and $\mathscr{T} = \mathrm{DM}_R$;

an object $\mathbb{E}$ of $\mathscr{T}(S)$ is positive (resp. negative) for the δ-homotopy t-structure if and only if $\hat{H}_p^\delta(\mathbb{E}) = 0$ for $p \leq 0$ (resp. $p \geq 0$). Moreover the same assertion hold in the δ-effective case.

This theorem has several interesting consequences. It immediately implies the δ-homotopy t-structure, both in the stable and δ-effective case, is non degenerated, a result which was not known for Ayoub's homotopy t-structure. It also gives new exactness properties, for a morphism $f : T \to S$ essentially of finite type, a dimension function δ on S, T being equipped with the induced dimension function by δ:

- if the dimension of the fibers of f is bounded by d, $f_* : \mathscr{T}(S) \to \mathscr{T}(T)$ has homological amplitude $[0,d]$ for the δ-homotopy t-structure on each side;
- if f is separated of finite type, $f^! : \mathscr{T}(T) \to \mathscr{T}(S)$ is t-exact for the δ-homotopy t-structure on each side.

One also deduces characterization of positivity (resp. negativity) of an object $\mathbb{E}$ for the δ-homotopy t-structure in terms of vanishing of Borel-Moore homology with coefficients in $\mathbb{E}$ in a certain range. For all these facts, we refer the reader to the corollaries stated after Theorem 3.3.1.

The preceding theorem opens the way to computations. The case of the constant motive $\mathbb{1}_S$ in $\mathrm{DM}(S,R)$, under the assumptions of this theorem, is already interesting. In case S is $\mathrm{Spec}(\mathbf{Q})$ (or is regular and admits a characteristic 0 point), we obtain that $\mathbb{1}_S$ has infinitely many homology motive in non-negative degree for the δ-homotopy t-structure (Example 3.3.2). On the contrary, if S is regular, $\mathbb{1}_S$ is concentrated in degree 0 for the *effective* δ-homotopy t-structure (see Example 3.3.3). This is the advantage of the effective case of the δ-homotopy t-structure: in general, we hope interesting boundedness properties only in that case. The final computation shows that when S is singular, then $\mathbb{1}_S$ may no longer be concentrated in degree 0: thus, the homology sheaves for the effective δ-homotopy t-structure detect singularities (see Remark 3.3.4).

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[7]It is shown this definition does not depend on the chosen model X; Definition 3.2.3.

We finally deduce (Corollary 3.3.9) from the preceding theorem the following punctual characterization of the δ-homotopy t-structure. an object $\mathbb{E}$ of $\mathscr{T}(S)$ is homologically positive (resp. negative) for the δ-homotopy t-structure if and only if the following condition holds:

$$\forall x \in S, H_p^{\delta_x}(i_x^! \mathbb{E}) = 0 \text{ when } p \leq \delta(x) \text{ (resp. } p \geq \delta(x)).$$

Here δ_x is the obvious dimension function on the spectrum of the residue field $\kappa(x)$ of x in S, and $i_x^! = j^* i^!$ for the factorization $\operatorname{Spec} \kappa(x) \xrightarrow{j} \overline{\{x\}} \xrightarrow{i} S$. It is striking that this condition is exactly the same, with homological conventions, than the one characterizing the perverse t-structure in [BBD82, 4.0.1, 4.0.2].

The δ-homotopy heart. From the results already stated, one deduces several good properties of the heart $\mathscr{T}(S)^{\heartsuit}$ of the δ-homotopy t-structure. This abelian category is a Grothendieck abelian category, with a good family of generators and satisfying good functoriality properties; most notably, the gluing property. For good choices of δ, it admits a closed monoidal structure. Under the assumptions of the preceding theorem, it shares some of the fundamental properties of its model $HI(k)$. In particular, each point $\operatorname{Spec} E \to S$, with value in a field E which is finitely generated over the corresponding residue field of S, defines a fiber functor to graded abelian groups and the collection of these functors is conservative: this is the transformation which to an object of the heart $\mathbb{F}$ associates the application $\hat{H}_0^{\delta}(\mathbb{F})$ defined by formula (0.0.0.d).

Our main example, under the assumptions of the above theorem, is the R-linear triangulated categories DM_R of mixed motives (resp. $\operatorname{D}_{\mathbf{A}^1, R}$, the $\mathbf{A}^1$-derived category of Morel) for which we denote the heart by $\Pi^{\delta}(S, R)$ (resp. $\tilde{\Pi}^{\delta}(S, R)$) and $\Pi^{\delta\text{-}eff}(S, R)$ (resp. $\tilde{\Pi}^{\delta\text{-}eff}(S, R)$) in the δ-effective case. It is called the category of δ-homotopy modules (resp. generalized δ-homotopy modules) over S, and we simply add the adjective effective when considering the heart of the δ-effective category. Note that in each respective case, the effective category is a full subcategory of the general (stable) one (cf. Paragraph 4.1.3).

The category of homotopy modules is a convenient extension over an arbitrary base of the category $HI(k)$ of homotopy invariant sheaves with transfers over a perfect field k: its effective subcategory over k coincides with $HI(k)$ up to a canonical equivalence, the whole category and its effective subcategory are invariant under purely inseparable extensions and satisfy the gluing property. In fact these properties guarantee it is essentially the unique extension of the theory of Voevodsky which satisfies the gluing property. Note on the contrary that we cannot prove at the moment that generalized homotopy modules are invariant under inseparable extensions. In the other hand, over perfect field, the category $\tilde{\Pi}^{\delta}(S, R)$ is equivalent to Morel's category of homotopy modules ([Mor03, Def. 5.2.4]).

Though these two notions of homotopy modules seem very different, we prove the following comparison theorem.

Theorem 4 (see Th. 4.3.1). Let R be a ring and S be a scheme such that R is a $\mathbf{Q}$-algebra or S is a $\mathbf{Q}$-scheme. Then there is canonical fully faithful functor:

$$\Pi^{\delta}(S, R) \to \tilde{\Pi}^{\delta}(S, R)$$

whose image is made by objects on which Morel's algebraic Hopf map[8] η acts trivially.

The case where R is a $\mathbf{Q}$-algebra is not surprising given that the canonical functor $\operatorname{DM}(S, R) \xrightarrow{\gamma_*} \operatorname{D}_{\mathbf{A}^1}(S, R) = \operatorname{SH}(-S) \otimes R$ is fully faithful and its essential image has the same description through the Hopf map (cf. [CD12, Th. 16.2.13]). The case $R = \mathbf{Z}$ for $\mathbf{Q}$-schemes is much stronger given that the map γ_* is no longer full (because of the existence of the Steenrod operations). The proof of the above theorem is by reduction to the case of fields, thanks to all our preceding results, which was proven in [Dég13].

As a conclusion, the category of homotopy modules with rational coefficients seems to be the appropriate generalization of the notion of homotopy invariant sheaves with transfers over an

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[8]Recall the algebraic Hopf map is the endomorphism of the sphere spectrum induced by the following morphism: $\mathbf{G}_m \to \mathbf{G}_m, x \mapsto x^{-1}$.

arbitrary base S with a dimension function δ (the category itself does not depend on δ up to a canonical equivalence). One of the main striking aspects of these particular sheaves is their proximity with semi-abelian varieties A: the sheaf corresponding to A admits canonical transfers and is homotopy invariant. In the last part of this paper, we extend this result to the relative setting, using the works of Ancona, Pepin Lehalleur and Huber ([AHPL]) and Pepin Lehalleur ([Pep15]). Following this last work, one associates to any commutative group S-scheme G an object $\underline{G}$ of DM$(S, \mathbf{Q})$. Our last theorem is the following one:

Theorem 5 (Th. 4.4.12). Let S be a regular scheme. For any semi-abelian S-scheme G,[9] the motive $\underline{G}$ is concentrated in degree 0 for the effective δ-homotopy t-structure on DM$^{\delta-\text{eff}}(S, \mathbf{Q})$. Moreover, the functor

$$G \mapsto \underline{G}$$

from the category of semi-abelian S-schemes up to isogeny to the category of δ-effective homotopy modules with rational coefficients is fully faithful and exact.

Note also that, thanks to a theorem of Kahn and Yamazaki, we are able to completely determine the fibers of the homotopy module $\underline{G}$ at a point $x : \operatorname{Spec}(E) \to S$. In particular, we obtain: $\hat{H}_0^\delta(G)(x, 0) = G(E)$ while when $G = A$ is an abelian scheme, $\hat{H}_0^\delta(A)(x, 1)$ is the Bloch group $V(C)$ (cf. [Blo81]) attached to a curve C/E whose Jacobian is isogeneous to the abelian variety A_x over E (see Th. 4.4.11 for more details). This shows that the $\mathbf{Z}$-grading of homotopy modules has an important arithmetic meaning.

As an application of this result, we finally give the generalization of the computation of the motive of a smooth projective curve originally due to Lichtenbaum and Voevodsky. Given any smooth projective curve C/S with a section $x : S \to C$, the relative motive $M_S(C)$ in DM$(S, \mathbf{Q})$ is δ-effective and its homology for the effective δ-homotopy t-structures are computed by canonical isomorphism (depending on the section x):

$$H_i^{\delta-\text{eff}}(M_S(C)) \simeq \begin{cases} \mathbb{1}_S \oplus \underline{J(C)} & \text{if } i = 0, \\ \mathbb{1}_S\{1\} & \text{if } i = 1 \end{cases}$$

where $J(C)$ is the Jacobian scheme associated with C/S. In particular, the distinguished triangle associated with the truncation $\tau_{\geq 0}$ of the motive $M_S(C)$ gives its classical relative Künneth decomposition.

Future work. The theory developed in this paper opens the way to several interesting tracks which we plan to treat separately. The first one is the spectral sequence associated with the truncation for the δ-homotopy t-structure of motives/spectra of the form $f_*(\mathbb{E})$ for a given morphism f, which we will call the *Leray δ-homotopy spectral sequence*. We already know this spectral sequence is a generalization of Rost spectral sequence introduced in [Ros96] which was used recently by Merkurjev to prove a conjecture of Suslin. Secondly, we plan to prove a result which was conjectured by Ayoub for his perverse homotopy t-structure and which generalizes the result already obtained by the second named author in his Ph. D. thesis: the category of homotopy modules over a k-scheme S is equivalent to Rost's category of cycle modules over S (at least after inverting the characteristic exponent). This result is clear enough at the moment so that we can safely announce its proof (as well as a generalization without a base field for rational coefficients).

DETAILED PLAN

The first section is devoted to recall on the main technical tools we will be using: dimension function, the generated t-structures and various aspects of the motivic homotopy formalism.
The second section will be first devoted to the definition of our δ-homotopy t-structures, as well as the setting of δ-effective categories. These definitions are completed by examples, which we hope will help the reader, and by the study of the basic properties of our definitions with respect to the six operations, such as the gluing property. The last two subsections of Section 2 is devoted to the

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[9]for us, a semi-abelian S-scheme is a commutative group S-scheme which is an extension of an abelian S-scheme by an S-torus.

exposition of several better descriptions of the generators for the δ-homotopy t-structure (stable and effective case) as well as their application, such as the comparison to the classical definitions of Voevodsky, Morel and Ayoub.

The third section is devoted to the proof of our main theorem (number 3 in the above introduction), whose proof is given in subsection 3.3. The section starts by an extension of the ideas of Bloch and Ogus on the niveau filtration and associated spectral sequence using dimension functions. This spectral sequence is applied to get the generalized link between Borel-Moore motivic homology and Chow groups (without a base field) as stated above (theorem number 2 above). Then we define and study the *fiber δ-homology* and relate it with the E_1-term of the δ-niveau spectral sequence. After the proof of the main theorem, several examples and corollaries are stated.

The last section contains a detailed study of the heart of the δ-homotopy t-structure, both in the stable and effective case, the comparison of the hearts of motives and spectra, and finally the link with semi-abelian schemes over regular bases.

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Conventions

Schemes. In all this work, we will fix a subcategory $\mathscr{S}$ of the category of excellent schemes of finite dimension which is closed under pullbacks by morphisms of finite type, finite sums and such that if S is a scheme in $\mathscr{S}$, any quasi-projective S-scheme is in $\mathscr{S}$. Without precision, any scheme is supposed to be an object of $\mathscr{S}$.

Given a scheme S, a *point* of S will be a morphism $x : \operatorname{Spec}(E) \to S$ such that E is a finitely generated extension of the residue field of the image of x in S. When E is a given field, we also say that x is an E-valued point and denote by $S(E)$ the class of E-valued point. The field E will be called the field of definition of the point x, and sometimes denote it generically by K_x. We will also denote by κ_x the residue field of the image of x in S. Note that the extension fields K_x/κ_x is of finite transcendence degree by assumption. We will call simply *transcendence degree* of x the transcendence degree of the field extension K_x/κ_x.

When we will need to be more precise, we will say that x is a *set-theoretic point* of X if $\kappa_x = K_x$. This is also assumed by the sentence: "let $x \in X$ be a point of X".

Morphisms. Unless explicitly stated, separated (resp. smooth) morphisms of schemes are assumed to be of finite type (resp. separated of finite type). We will say *lci* for local complete intersection.

A morphism $f : X \to S$ is said to be *essentially of finite type* if X is pro-étale over an S-scheme of finite type.[10] To be consistent with this convention, we will say that an S-scheme X is *essentially separated* if it is essentially of finite type and separated. Given a scheme S in $\mathscr{S}$, we will denote as usual by $\mathscr{S}_{/S}$ the category of schemes in $\mathscr{S}$ over S.

Unless explicitly stated, quasi-finite morphisms will be assumed to be essentially of finite type.

Triangulated categories. Recall the following classical conventions. In a triangulated category $\mathscr{C}$, a class of objects $\mathcal{G}$ is called *generating* – we also say that $\mathcal{G}$ *is a generating class* of the triangulated category $\mathscr{C}$ – if for any object K of $\mathscr{C}$ one has the implication:

$$\left(\forall X \in \mathcal{G}, \forall i \in \mathbf{Z}, \operatorname{Hom}_{\mathscr{C}}(X[i], K) = 0\right) \Rightarrow K = 0.$$

---

[10]Traditionally, the condition pro-open replaces prop-étale but the arguments used here does not change in this greater generality.

When $\mathscr{C}$ admits arbitrary coproducts, we say that an object K of $\mathscr{T}$ is *compact* if for any family $(X_i)_{i \in I}$ of objects the canonical map

$$\bigoplus_{i \in I} \operatorname{Hom}(K, X_i) \to \operatorname{Hom}(K, \oplus_{i \in I} X_i)$$

is an isomorphism.

Motivic homotopy theory. We will use the language of premotivic and motivic triangulated categories developed in [CD12] as an extension of the work of Ayoub [Ayo07]. Given a premotivic triangulated category $\mathscr{T}$, objects of $\mathscr{T}(S)$ will be called *$\mathscr{T}$-spectra over S*.[11] Recall we say that $\mathscr{T}$ is *compactly generated by its Tate twist* (*op. cit.* Definition 1.3.16) if for any scheme S, the geometric $\mathscr{T}$-spectra of the form $M_S(X, \mathscr{T})(i)$ for a smooth S-scheme X and an integer $i \in \mathbf{Z}$ are compact and form a set of generators of the triangulated category $\mathscr{T}(S)$. Recall also that when $\mathscr{T}$ is a *motivic* triangulated category (*op. cit.* Definition 2.4.45), the fibred category $\mathscr{T}$ is equipped with the 6 functors formalism as stated in *op. cit.* Theorem 2.4.50.

By convention, all premotivic triangulated categories appearing in this text will be assumed to be compactly generated by their Tate twists.

We will have to use special combinations of twist and shifts, so we adopt the following conventions for any $\mathscr{T}$-spectrum K and any integer $n \in \mathbf{Z}$:

$$K\langle n \rangle := K(n)[2n],$$
$$K\{n\} := K(n)[n].$$

In the text, we will fix a triangulated motivic category $\mathscr{T}$ over $\mathscr{S}$ (compactly generated by its Tate twist). When no confusion can arise, objects of $\mathscr{T}$ will abusively be called *spectra*.

1. Preliminaries

1.1. **Reminder on dimension functions.**

1.1.1. Let X be a scheme and x, y two points of X. Recall that y is called a *specialization* of x if y belongs to the closure Z of x in X. Moreover, y is called an *immediate specialization* of x if $\operatorname{codim}_Z(y) = 1$. Recall the following definition from [PS14, 2.1.10, 2.1.6]:

Definition 1.1.2. Let X be a universally catenary scheme. A *dimension function* on X is a map $\delta : X \to \mathbf{Z}$ such that for any immediate specialization y of a point $x \in X$, $\delta(x) = \delta(y) + 1$.

In this case, we will say that (X, δ) is a *dimensional scheme*.

We will say that δ is *non-negative*, and write $\delta \geq 0$, if its image lies in the set of non-negative integers.

The following lemma is obvious according to the above definition:

Lemma 1.1.3. *If δ and δ' are dimension functions on a scheme X, the function $\delta - \delta'$ is Zariski locally constant.*

In particular, up to an element in $\mathbf{Z}^{\pi_0(X)}$, there exists at most one dimension function on X.

Example 1.1.4. If X is universally catenary and integral (resp. equicodimensional[12]) then

$$\delta(x) = - \dim(\mathcal{O}_{X,x}) = - \operatorname{codim}_X(x) \ \left(\text{resp. } \delta(x) = \dim\left(\overline{\{x\}}\right) \right)$$

is a dimension function on X – see [PS14, 2.2.2, resp. 2.4.4]. In the *resp.* case, δ is called the *Krull dimension function on X*. Note that when $\dim(X) > 0$, these two dimension functions, if defined, do not coincide.

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[11]Here the terminology differs slightly from that of [CD12], but it seems now more natural – motives should be seen as particular kind of spectra.

[12]*i.e.* the codimension of any closed point of X is equal to the dimension of its connected component. A canonical example is that of (disjoint unions) of equidimensional schemes of finite type over a field or over the ring of integers;

Remark 1.1.5. Since X is noetherian finite dimensional (according to our conventions), a dimension function on X is always bounded. So the above result implies that if X admits a dimension function, then it admits at least one non-negative dimension function.

1.1.6. Let $f : Y \to X$ be a morphism of schemes and δ be a dimension function on X.

If f is quasi-finite, then $\delta \circ f$ is a dimension function on Y (cf. [PS14, 2.1.12]).

Assume now that f is essentially of finite type. For any point $y \in Y$, $x = f(y)$, we put:

$$\delta^f(y) = \delta(x) + \mathrm{degtr}[\kappa_y/\kappa_x].$$

According to [PS14, 2.5.2] and the preceding remark, this defines a dimension function on Y – when f is pro-étale, one simply has $\delta^f = \delta \circ f$. Moreover, we will put:

$$\delta(X) = \max_{x \in X}\left(\delta^f(x)\right).$$

We easily deduce from that definition that if $Z \xrightarrow{g} Y \xrightarrow{f} X$ are morphisms essentially of finite type, then $(\delta^f)^g = \delta^{gf}$. This implies in particular that $\delta(Z) = \delta^f(Z)$.

Remark 1.1.7. According to the preceding remark, for any essentially of finite type X-scheme Y the following integers

$$\delta_-(Y) = \min_{y \in Y}(\delta(y)), \ \delta_+(Y) = \max_{y \in Y}(\delta(y)),$$

are well defined. Note that one can check that the image of δ is exactly $[\delta_-(Y), \delta_+(Y)]$. Below we will often denote $\delta_+(Y)$ just by $\delta(Y)$; yet we will never change the notation for $\delta_-(Y)$.

Recall that given a point x of a scheme X, the *dimension of X at x*, denoted by $\dim_x X$, is the integer $\lim_U \left(\dim(U)\right)$, where U runs over the open neighbourhoods of x in X. We will adopt the following definition, which is taken from several theorems of Chevalley as stated in [EGA4, section 13]:

Definition 1.1.8. Let $f : Y \to X$ be a morphism essentially of finite type. The *relative dimension of f* is the function denoted by $\dim(f)$ which to a point $y \in Y$, associates the integer $\dim_y f^{-1}f(y)$.

Beware $\dim(f)$ is not a dimension function on Y; in relevant cases (for example smooth, flat or more generally equidimensional morphisms[13]), it is locally constant. In general, we will use upper bounds on this function.

Example 1.1.9. let f be a morphism of schemes which is locally of finite type. Then $\dim(f) = 0$ if and only if it is locally quasi-finite (cf. [Stack, 28.29.5]). This can obviously be extended to the case where f is essentially of finite type.

Remark 1.1.10. It is clear that $\mathrm{degtr}(\kappa_y/\kappa_x) \leq \dim_y f^{-1}(x)$ where $x = f(y)$ and equality happens if y is a generic point of $f^{-1}(x)$ whose residue field is of maximal transcendence degree over κ_x.

Proposition 1.1.11. *Consider a cartesian square of schemes*

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ q\downarrow & & \downarrow p \\ T & \xrightarrow{f} & S \end{array}$$

made of essentially of finite type morphisms, and let δ be a dimension function on S.

(1) *If $\delta \geq 0$ then $\delta(Y) \leq \delta(X) + \delta(T)$.*

(2) *If $\dim(f) \leq d$ then $\delta(Y) \leq \delta(X) + d$.*

---

[13]Recall from [EGA4, 13.3.2] that a morphism $f : Y \to X$ is equidimensional if the following conditions are fulfilled:

- f is of finite type;
- $\dim(f)$ is Zariski locally constant on Y;
- f maps generic points of Y to generic points of X.

According to [VSF00, chap. 2, 2.1.8], a flat morphism $f : Y \to X$ is equidimensional if for any connected component Y' of Y and any generic point $x \in X$, Y'_x is equidimensional.

(3) *Assume f is surjective on generic points of S and $\dim(f) = d$.*
  Then $\delta(T) = \delta(S) + d$.

Proof. Put $h = f \circ q$. Consider a point $y \in Y$ whose image is x, t, s respectively in X, T and S. Recall that κ_y is a composite extension field of κ_x/κ_s and κ_t/κ_s ([EGA1, 3.4.9]). Thus, its transcendence degree is the sum of the transcendence degrees of this last two extension fields.

We prove (1) as follows:

$$\delta^h(y) = \delta(s) + \mathrm{degtr}(\kappa_y/\kappa_s) = \delta(s) + \mathrm{degtr}(\kappa_x/\kappa_s) + \mathrm{degtr}(\kappa_t/\kappa_s)$$
$$\leq \delta(s) + \mathrm{degtr}(\kappa_x/\kappa_s) + \delta(s) + \mathrm{degtr}(\kappa_t/\kappa_s) = \delta^p(x) + \delta^f(t) \leq \delta(X) + \delta(T);$$

and (2) as follows (using Remark 1.1.10):

$$\delta^h(y) = \delta(s) + \mathrm{degtr}(\kappa_y/\kappa_s) = \delta(s) + \mathrm{degtr}(\kappa_x/\kappa_s) + \mathrm{degtr}(\kappa_t/\kappa_s) = \delta^p(x) + \mathrm{degtr}(\kappa_t/\kappa_s)$$
$$\leq \delta(X) + d.$$

To prove point (3), it remains to show the converse inequality of (2) in the particular case $X = S$. Consider a generic point s of S such that $\delta(s) = \delta(S)$ (it exists since δ is bounded, Remark 1.1.5). By our assumption on f, $f^{-1}(s) \neq \emptyset$. Let x be a generic point of $f^{-1}(s)$ whose residue field is of maximal transcendence degree over κ_s. According to Remark 1.1.10 and the assumption on f, one has $\mathrm{degtr}(\kappa_x/\kappa_s) = d$. Then the following computation allows us to conclude:

$$\delta^f(x) = \delta(s) + \mathrm{degtr}(\kappa_x/\kappa_s) = \delta(S) + d.$$

$\square$

1.2. **Recall on t-structures.**

1.2.1. Following Morel ([Mor03]), our conventions for t-structures will be homological.[14] Apart from that, we will follow the classical definitions of [BBD82, Sec. 1.3].

This means in particular that a t-structure t on a triangulated category $\mathscr{C}$ is a pair of subcategories $(\mathscr{C}_{\geq 0}, \mathscr{C}_{<0})$ such that

(1) $\mathrm{Hom}_{\mathscr{C}}(\mathscr{C}_{\geq 0}, \mathscr{C}_{<0}) = 0$;
(2) $\mathscr{C}_{\geq 0}$ (resp. $\mathscr{C}_{<0}$) is stable by suspension $[+1]$ (resp. desuspension $[-1]$).
(3) for any object M in $\mathscr{C}$, there exists a distinguished triangle of the form

(1.2.1.a) $$M_{\geq 0} \to M \to M_{<0} \xrightarrow{+1}$$

   such that $M_{\geq 0} \in \mathscr{C}_{\geq 0}$ and $M_{<0} \in \mathscr{C}_{<0}$.

Then we denote as usual $\mathscr{C}_{\geq n} := \mathscr{C}_{\geq 0}[n]$ (resp. $\mathscr{C}_{\leq n+1} := \mathscr{C}_{<n} := \mathscr{C}_{<0}[n]$) for any integer $n \in \mathbf{Z}$. An object M of ($\mathscr{C}_{\geq 0}$ (resp. $\mathscr{C}_{<0}$)) is called *non-t-negative* (resp. *t-negative*); sometimes we indicate this by the notation $M \geq 0$ (resp. $M < 0$) and similarly for any $n \in \mathbf{Z}$.

Recall also that the triangle (1.2.1.a) is unique. Thus, we get a well-defined functor $\tau^t_{\geq 0} : \mathscr{C} \mapsto \mathscr{C}_{\geq 0}$ (resp. $\tau^t_{<0} : \mathscr{C} \mapsto \mathscr{C}_{<0}$) which is right (resp. left) adjoint to the inclusion functor $\mathscr{C}_{\geq 0} \to \mathscr{C}$ (resp. $\mathscr{C}_{<0} \to \mathscr{C}$). For any integer $n \in \mathbf{Z}$ and any object M of $\mathscr{C}$, we put more generally: $\tau^t_{\geq n}(M) = \tau^t_{\geq 0}(M[-n])[n]$ (resp. $\tau^t_{<n}(M) = \tau^t_{<0}(M[-n])[n]$).

The pair $(\mathscr{C}, t)$ is called a *t-category* and, when t is clear, it is convenient not to indicate it in the notation.

1.2.2. Let $\mathscr{C}$ be a t-category. The *heart* of $\mathscr{C}$ is the category $\mathscr{C}^{\heartsuit} = \mathscr{C}_{\leq 0} \cap \mathscr{C}_{\geq 0}$. It is an abelian category and the canonical functor:

$$H_0 = \tau_{\leq 0}\tau_{\geq 0} = \tau_{\geq 0}\tau_{\leq 0} : \mathscr{C} \to \mathscr{C}^{\heartsuit}$$

is a *homological functor* – *i.e.* converts distinguished triangles to long exact sequences. We put $H_n := H_0(.[-n])$.

We give the following definition, slightly more precise than usual.

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[14]One goes from homological to cohomological conventions by the usual rule: $\mathscr{C}^{\leq n} = \mathscr{C}_{\geq -n}$.

Definition 1.2.3. Let t be a t-structure on a triangulated category $\mathscr{C}$. We say that t is *left* (resp. *right*) *non-degenerate* if $\cap_{n\in\mathbf{Z}}\mathscr{C}_{\leq n} = \varnothing$ (resp. $\cap_{n\in\mathbf{Z}}\mathscr{C}_{\geq n} = \varnothing$). We say that t is *non-degenerate* if it is left and right non-degenerate.

Another way of stating that t is non-degenerate is by saying that the family of homology functors $(H_n)_{n\in\mathbf{Z}}$ (defined above) is conservative.

1.2.4. Consider two t-categories $\mathscr{C}$ and $\mathscr{D}$. One says that a functor $F : \mathscr{C} \to \mathscr{D}$ is *left t-exact* (resp. *right t-exact*) if it respects negative (resp. positive) objects. We will say that F is a *t-exact* if it is left and right t-exact. We say F is an *equivalence of t-categories* if it is t-exact and an equivalence of the underlying categories. Then, the functor $H_0 F$ induces an equivalence between the hearts.

More generally, for any $n \in \mathbf{Z}$, one says that F has *(homological) amplitude* less (resp. more) than n if $F(\mathscr{C}_{\leq 0}) \subset \mathscr{C}_{\leq n}$ (resp. $F(\mathscr{C}_{\geq 0}) \subset \mathscr{C}_{\geq n}$). Similarly, F has *amplitude* $[n, m]$ is F has amplitude more than n and less than m.

Recall that given a pair of adjoint functors $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$, F is right t-exact if and only if G is left t-exact. In that case, we will say that (F, G) is an *adjunction of t-categories*. According to [BBD82, 1.3.17], such an adjunction induces a pair of adjoint functors:

$$(1.2.4.a) \qquad H_0 F := (\tau_{\geq 0} \circ F) : \mathscr{C}^\heartsuit \xrightarrow{} \mathscr{D}^\heartsuit : (\tau_{\leq 0} \circ G) =: H_0 G.$$

Suppose in addition that $\mathscr{C}$ is a monoidal triangulated category with tensor product $\otimes$ and neutral object $\mathbb{1}$. One says that the tensor structure is *compatible with the t-category $\mathscr{C}$* if the bi-functor $\otimes$ sends two non negative objects to a non negative object and $\mathbb{1}$ is a non negative object. In this case, one checks that the formula:

$$K \otimes^{\mathrm{H}} L := \tau_{\leq 0}(K \otimes L) = H_0(K \otimes L)$$

defines a monoidal structure on the abelian category $\mathscr{C}^\heartsuit$ such that the functor H_0 is monoidal. If moreover the tensor structure is closed, then for any $K \geq 0$ and $L \leq 0$, one has $\underline{\mathrm{Hom}}(K, L) \leq 0$. Thus the bifunctor $\tau_{\geq 0}\underline{\mathrm{Hom}}$ defines an internal Hom in $\mathscr{C}^\heartsuit$.

1.2.5. Let $\mathscr{C}$ be a compactly generated triangulated category which admits arbitrary small co-products. We will say that a subcategory $\mathscr{C}_0$ of $\mathscr{C}$ is *stable by extensions* if given any distinguished triangle (K, L, M) of $\mathscr{C}$ such that K and M belong to $\mathscr{C}_0$, the object L also belongs to $\mathscr{C}_0$.

Given a family of objects $\mathcal{G}$ of $\mathscr{C}$, we will denote by $\langle \mathcal{G} \rangle_+$ the smallest subcategory of $\mathscr{C}$ which contains $\mathcal{G}$ and is stable by extensions, the suspension $[+1]$, and arbitrary (small) coproducts. The following Theorem was proved in [ATJLSS03, A.1].

Theorem 1.2.6. *Adopt the previous assumptions. Then $\langle \mathcal{G} \rangle_+$ is the category of non-negative objects of a t-structure $t_{\mathcal{G}}$ on $\mathcal{G}$. Moreover, for this t-structure, positive objects are stable by coproducts, and the functors $\tau_{\leq 0}$, $\tau_{\geq 0}$, H_0 commutes with coproducts.*

Definition 1.2.7. In the assumptions of the previous theorem, the t-structure $t_{\mathcal{G}}$ will be called the *t-structure generated by $\mathcal{G}$*.

A nice remark about such generated t-structure is the following elementary lemma.

Lemma 1.2.8. *Consider the notations of the above theorem. Then the following conditions are equivalent:*
  (i) *$t_{\mathcal{G}}$ is left non-degenerate;*
  (ii) *$\mathcal{G}$ is a generating family of the triangulated category $\mathscr{C}$.*

Proof. This immediately follows from the equality:

$$\cap_{n\in\mathbf{Z}}\mathscr{C}_{\geq n} = \big\{ L \in \mathscr{C} \mid \forall K \in \mathcal{G}, \forall n \in \mathbf{Z}, \mathrm{Hom}(K[n], L) = 0 \big\}$$

$\square$

Remark 1.2.9. The (right) non-degeneration of a t-structure generated as above is much more delicate. We will obtain such a result for our particular t-structures in Corollary 3.3.7. See also [Ayo07, 2.1.73] for an abstract criterion.

1.3. **Recall on motivic homotopy theory.**

Example 1.3.1. This work can be applied to the following triangulated motivic categories:

(1) The stable homotopy category SH of Morel and Voevodsky (see [Ayo07]).
(2) The $\mathbf{A}^1$-derived category $D_{\mathbf{A}^1,R}$ of Morel (see [Ayo07], [CD12]) with coefficients in a ring R.
(3) The category $\mathbf{MGL}-mod$ of $\mathbf{MGL}$-modules (cf. [CD12, 7.2.14, 7.2.18]). Its sections over a scheme S are made by the homotopy category of the model category of modules over the (strict) ring spectrum $\mathbf{MGL}_S$ in the monoidal category of symmetric spectra.
(4) The category of Beilinson motives DM_B (see [CD12]) which is equivalent (by [CD12, 16.2.22]) to the étale $\mathbf{A}^1$-derived category $DA_{\text{ét},\mathbf{Q}}$ with rational coefficients introduced in [Ayo07]. It can have coefficients in any $\mathbf{Q}$-algebra R.
(5) Let $\mathbf{MZ}$ be the ring spectrum in $SH(\text{Spec}(\mathbf{Z}))$ constructed by Spitzweck in [Spi13]. Then the category $\mathbf{MZ}-mod$ is a motivic category over the category of all schemes (see [Spi13]).
(6) Let F be a prime field of characteristic exponent p and $\mathscr{S}$ be the category of (excellent) F-schemes. The category of cdh-motivic complexes $DM_{\text{cdh}}(-, R)$ with coefficients in an arbitrary $\mathbf{Z}[1/p]$-algebra R, is a motivic category over $\mathscr{S}$ (see [CD15a]).

   Given any morphism $\text{Spec}(F) \to \text{Spec}(\mathbf{Z})$, according to [CD15a, Th. 5.1] and [Spi13, Th. 9.16], this motivic category for $R = \mathbf{Z}[1/p]$ is equivalent to $\mathbf{MZ}-mod[1/p]$ restricted to $\mathscr{S}_{/F}$.

   Recall also from [CD15a, 5.9] that for any regular F-scheme X, the canonical cdh-sheafification map

(1.3.1.a) $$DM(X, R) \to DM_{\text{cdh}}(X, R)$$

   is an equivalence of triangulated monoidal categories, where the left hand side is the non-effective version of Voevodsky's derived category of motivic complexes (see [CD09, CD12, CD15a]).

All these categories are compactly generated by their Tate twists fulfilling the requirements of our Conventions (see page 9).

   Note that DM_{cdh} is a premotivic category over the category of all noetherian schemes, but we only know it satisfies the localization property over F-schemes as explained above.

1.3.2. To gather examples (4), (5) and (6) together, we will adopt a special convention. We fix a ring R and denote by DM_R one of the following motivic categories:

(1) if R is a $\mathbf{Q}$-algebra, it is understood that $DM_R = DM_B$ and $\mathscr{S}$ is the category of all schemes;
(2) if $\mathscr{S}$ is the category of F-schemes for a prime field F with characteristic exponent p, it is understood that $p \in R^\times$ and $DM_R = DM_{\text{cdh}}(-, R)$;
(3) if $R = \mathbf{Z}$, it is implicitly understood that $DM_R = \mathbf{MZ}-mod$ and $\mathscr{S}$ is the category of all schemes.

Example 1.3.3. For any ring R satisfying one of the above assumptions, the triangulated motivic categories of the previous example are related by the following diagram:

(1.3.3.a)
$$\begin{array}{ccc} & & \mathbf{MGL}-mod \\ & \nearrow^{\phi^*} & \\ SH & & \\ & \searrow_{\delta^*} & \\ & & D_{\mathbf{A}^1,R} \xrightarrow{\gamma^*} DM_R. \end{array}$$

In the above diagram, the label used stands for the left adjoints; the right adjoints will be written with a lower-star instead of an an upper-star.

   The references are as follows: δ^*, [CD12, 5.3.35]; γ^*, [CD12, 11.2.16]; ρ^*, [CD15a, §9]; ϕ^*, [CD12, 7.2.13]. Recall also that all the right adjoint of these premotivic adjunctions are all conservative.

Remark 1.3.4. Note that there exists a premotivic adjunction:

$$\mathbf{MGL}-mod \to DM_R$$

for any ring R satisfying the assumption of 1.3.2 though the whole details have not been written down. In fact, this will follow from the following two facts:

- $\mathrm{DM}_R(S)$ is equivalent to the homotopy category of modules over the E_∞-ring spectrum $\mathbf{H}R_S$ with coefficients in R representing motivic cohomology. This is true in any of the cases of 1.3.2 – for (1) see [CD12, 14.2.11], for (2) see the construction of [CD15b, 5.1], though one does not consider E_∞-ring spectra, (3) is obvious. Note moreover that we need that the collection of E_∞-ring spectrum $\mathbf{H}R_S$ for various schemes form a cartesian section of the category of E_∞-ring spectra to be able to get the premotivic structure on $\mathbf{H}R_S - mod$ – on the model of [CD12, section 7.2].
- the ring spectrum $\mathbf{H}R_S$ is oriented, or equivalently, it is a $\mathbf{MGL}_S$-algebra which is also true in any of the cases of 1.3.2. The more difficult point is that the corresponding map

$$\mathbf{MGL}_S \to \mathbf{H}R_S$$

is a morphism of E_∞-ring spectra. The proof of this fact is indicated in [Spi13, 11.2]. The proof indicates also that this isomorphism is compatible with pullbacks in S – *i.e.* with the cartesian structure on the family of ring spectra $(\mathbf{MGL}_S)$ and $(\mathbf{H}R_S)$ indexed by schemes S in $\mathscr{S}$.

1.3.5. *Constructible $\mathscr{T}$-spectra.*

Let us consider an abstract motivic triangulated category satisfying our general assumptions (see page 9). In particular the $\mathscr{T}$-spectra of the form $M_S(X)(n)$ for a smooth S-scheme X and an integer $n \in \mathbf{Z}$ are compact.

Following the usual terminology (see [CD12, 4.2.3]), we define the category $\mathscr{T}_c(S)$ of *constructible $\mathscr{T}$-spectra over S* as the smallest thick triangulated subcategory of $\mathscr{T}(S)$ containing the $\mathscr{T}$-spectra of the form $M_S(X)(n)$ as above. Then according to [CD12, 1.4.11], a $\mathscr{T}$-spectrum is constructible if and only if it is compact.

Moreover, according to [CD12, 4.2.5, 4.2.12], constructible $\mathscr{T}$-spectra are stable by the operations f^*, $p_!$ (p separated), and tensor product.

The following terminology first appeared in [Rio06].

Definition 1.3.6. Let $p : X \to S$ be a separated morphism. We define the Borel-Moore $\mathscr{T}$-spectrum associated with X/S as the following object of $\mathscr{T}(S)$: $M^{BM}(X/S) := p_!(\mathbb{1}_X)$.

Note that according to the previous paragraph, $M^{BM}(X/S)$ is a compact object of $\mathscr{T}(S)$.

1.3.7. Let us recall that we define the Thom $\mathscr{T}$-spectrum associated with a vector space E/X as the $\mathscr{T}$-spectrum:

$$M\mathrm{Th}(E) = p_\sharp s_*(\mathbb{1}_X),$$

where p (resp. s) is the canonical projection (resp. zero section) of E/X. Recall for example that for any integer $r \geq 0$, we get under our conventions that $M\mathrm{Th}(\mathbf{A}_S^r) = \mathbb{1}_S\langle r\rangle$.

The Thom $\mathscr{T}$-spectrum $M\mathrm{Th}(E)$ is $\otimes$-invertible with inverse $s^!(\mathbb{1}_E)$. Following [Rio06, 4.1.1] (see also [Ayo07]), the functor Th can be uniquely extended to a functor from the Picard category $\underline{\mathrm{K}}(X)$ of *virtual vector bundles over X* (defined in [Del87, 4.12]) to the category of $\otimes$-invertible $\mathscr{T}$-spectra and this functor satisfies the relations:

$$M\mathrm{Th}(v + w) = M\mathrm{Th}(v) \otimes M\mathrm{Th}(w)$$
$$M\mathrm{Th}(-v) = M\mathrm{Th}(v)^{\otimes, -1}.$$

It is obviously compatible with base change. Note also that we will adopt a special notation when X is a smooth S-scheme, with structural morphism f. Then for any virtual vector bundle v over X, we put:

$$(1.3.7.a) \qquad\qquad M\mathrm{Th}_S(v) = f_\sharp\big(M\mathrm{Th}(v)\big).$$

Thom spaces are fundamental because they appear in the Thom isomorphism of a smooth morphism $f : X \to S$ with tangent bundle T_f:

$$(1.3.7.b) \qquad\qquad f^! \simeq M\mathrm{Th}(T_f) \otimes f^*.$$

Recall finally that, when $\mathscr{T}$ is oriented ([CD12, 2.4.38]), for any virtual vector bundle v over S of (virtual) rank r, there exists a canonical *Thom isomorphism*:

$$(1.3.7.c) \qquad\qquad M\mathrm{Th}_S(v) \xrightarrow{\ \sim\ } \mathbb{1}_S\langle r\rangle$$

which is coherent with respect to exact sequences of vector bundles and compatible with base change (cf. *loc. cit.*).

1.3.8. In the text below, we will use the following properties of Borel-Moore $\mathscr{T}$-spectra which follows from the six functors formalism:

(BM1) $M^{BM}(X/S)$ is contravariant in X with respect to proper morphisms of separated S-schemes.

(BM2) $M^{BM}(X/S)$ is covariant with respect to étale morphisms $f : Y \to X$ of separated S-schemes.[15]

(BM3) (*Localization*) For any closed immersion $i : Z \to X$ with complementary open immersion $j : U \to X$, one has a distinguished triangle:

$$(1.3.8.a) \qquad M^{BM}(U/S) \xrightarrow{j_*} M^{BM}(X/S) \xrightarrow{i^*} M^{BM}(Z/S) \xrightarrow{\partial_i} M^{BM}(U/S)[1].$$

(BM4) (*Purity*) For any smooth S-scheme X of relative dimension d, there exists a canonical isomorphism in $\mathscr{T}(S)$:

$$\mathfrak{p}_X : M\mathrm{Th}_S\big(-T_f\big) \to M^{BM}(X/S)$$

where T_f is the tangent bundle of f, and we have followed convention (1.3.7.a). When $\mathscr{T}$ is oriented, we get an isomorphism in $\mathscr{T}(S)$:

$$\mathfrak{p}_X : M_S(X)\langle -d\rangle \to M^{BM}(X/S)$$

where d is the relative dimension of X/S.

(BM5) (*Künneth*) For any separated S-schemes X, Y, there exists a canonical isomorphism:

$$M^{BM}(X/S) \otimes M^{BM}(Y/S) \to M^{BM}(X \times_S Y/S).$$

(BM6) (*Base change*) For any morphism $f : T \to S$, $f^* M^{BM}(X/S) = M^{BM}(X \times_S T/T)$.

Each property follows easily from the six functors formalism; using the numeration of the axioms as in [CD12, Th. 2.4.50]:

- (BM1): (2) and the unit map of the adjunction (f^*, f_*);
- (BM2): (3) and the unit map of the adjunction $(f_!, f^!)$;
- (BM3): the localization property (Loc_i);
- (BM4): point (3) *i.e.* the dual assertion of (1.3.7.b); (BM5): point (5) and (4); (BM6): point (4).

Remark 1.3.9. It easily follows from (BM4) that $\mathscr{T}$-spectra of the form $M^{BM}(X/S)$ for X/S separated are compact generators of the triangulated categoty $\mathscr{T}(S)$.

Definition 1.3.10. Let X be a separated S-scheme. For any integer $(p,q) \in \mathbf{Z}^2$, we define the *Borel-Moore homology* of X/S with coefficients in $\mathbb{E}$ and degree (p,q) as the following abelian group:

$$\mathbb{E}_{p,q}^{BM}(X/S) = \mathrm{Hom}_{\mathscr{T}(S)}\big(M^{BM}(X/S)(q)[p], \mathbb{E}\big).$$

It will also be convenient to consider two other kinds of twists and therefore use the following notations:

$$\mathbb{E}_{p,\{q\}}^{BM}(X/S) = \mathrm{Hom}_{\mathscr{T}(S)}\big(M^{BM}(X/S)\{q\}[p], \mathbb{E}\big),$$
$$\mathbb{E}_{p,\langle q\rangle}^{BM}(X/S) = \mathrm{Hom}_{\mathscr{T}(S)}\big(M^{BM}(X/S)\langle q\rangle[p], \mathbb{E}\big).$$

Besides, when dealing with a non orientable motivic category $\mathscr{T}$, it is useful to adopt the following extended notations in cohomology – note similar notations may be introduced for Borel-Moore homology but we will not used the latter in the present paper.

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[15]This covariance can be extended to the case of any smooth morphism f if one takes care about a twist by the Thom $\mathscr{T}$-spectrum of the tangent bundle of f.

Definition 1.3.11. Let X be a scheme, n an integer and v a virtual vector bundle over X. We define the $\mathscr{T}$-cohomology of X in bidegree (n, v) as the following abelian group:

$$H^{n,v}(X, \mathscr{T}) := \mathrm{Hom}_{\mathscr{T}(X)}\big(\mathbb{1}_X, \mathrm{Th}(v)[n]\big).$$

1.3.12. To simplify notations, when $f : Y \to X$ is a morphism of schemes and v is a virtual vector bundle over X, we will denote by $H^{n,v}(Y, \mathscr{T})$ the cohomology if Y with degree $(n, f^{-1}v)$. In particular, the pullback functor f^* induces a pullback morphism on cohomology:

$$f^* : H^{n,v}(X, \mathscr{T}) \to H^{n,v}(Y, \mathscr{T}).$$

We will denote by $\langle m \rangle$ the virtual bundle associated with the m-dimensional plane $\mathbf{A}^m$ over $\mathbf{Z}$. Then by definition, one gets:

$$H^{n,\langle m \rangle}(X, \mathscr{T}) = \mathrm{Hom}(\mathbb{1}_X, \mathbb{1}_X \langle m \rangle [n]) = \mathrm{Hom}(\mathbb{1}_X, \mathbb{1}_X(m)[2m + n])$$

recovering the index of twisted cohomologies that we have introduced above (for Borel-Moore homology).

Remark 1.3.13. Recall that when $\mathscr{T}$ is oriented, for any virtual X-vector bundle v of virtual rank r, there exists a canonical isomorphism (see for example [Dég14]), called the *Thom isomorphism*:

$$\mathrm{Th}(v) \xrightarrow{\sim} \langle r \rangle.$$

In particular, for any $\mathscr{T}$-spectrum $\mathbb{E}$, one has a canonical isomorphism $\mathbb{E}^{n,v}(X) \simeq \mathbb{E}^{n,\langle r \rangle}(X)$. This justifies our choice of notation, which is a useful generalization of the traditional notation to the non oriented case.

The preceding notation is natural and useful to formulate the absolute purity property that will be used in this paper.

Definition 1.3.14. A *closed pair* is a pair of schemes (X, Z) such that Z is a closed subscheme of X. One says (X, Z) is *regular* if the immersion of Z in X is regular. If X is an S-scheme, one says (X, Z) is a smooth S-pair if X and Z are smooth over S.

Given an integer $n \in \mathbf{Z}$ and a virtual vector bundle over Z, one defines the cohomology of (X, Z) – or cohomology of X with support in Z – in degree (n, v) as the abelian group:

$$H_Z^{n,v}(X, \mathscr{T}) := \mathrm{Hom}_{\mathscr{T}(X)}\big(i_*(\mathrm{Th}(-v)), \mathbb{1}_X[n]\big).$$

1.3.15. Given a morphism of schemes $f : Y \to X$, on easily checks that the pullback functor f^* induces a morphism of abelian groups:

$$f^* : H_Z^{n,v}(X, \mathscr{T}) \to H_T^{n,v}(Y, \mathscr{T})$$

where $T = f^{-1}(Z)$ and we have used the same convention as before for the group on the right hand side. Note also that we will also say f is a *cartesian morphism* of closed pairs $(Y, T) \to (X, Z)$.

1.3.16. The following observations are extracted from [Dég14]. Let (X, Z) be a regular closed pair. We will denote by $N_Z X$ (resp. $B_Z X$) the normal cone (resp. blow-up) associated with (X, Z). Recall the deformation space $D_Z X = B_{Z \times 0}(\mathbf{A}_X^1) - B_Z X$. The scheme $D_Z X$ contains $D_Z Z = \mathbf{A}_Z^1$ as a closed subscheme. It is fibered over $\mathbf{A}^1$ (by a flat morphism), and its fiber over 1 is X while its fiber over 0 is $N_Z X$. Thus we get a *deformation* diagram of closed pairs:

$$(X, Z) \xrightarrow{d_1} (D_Z X, \mathbf{A}_Z^1) \xleftarrow{d_0} (N_Z X, Z).$$

For the next definitions, we will fix a subcategory $\mathscr{S}_0$ of $\mathscr{S}$. We will call closed $\mathscr{S}_0$-pair any closed pair (X, Z) such that X and Z belongs to $\mathscr{S}_0$. We assume $\mathscr{S}_0$ satisfies the following assumptions:

(1) For any regular closed $\mathscr{S}_0$-pair (X, Z), we assume the deformation space $D_Z X$ belongs to $\mathscr{S}_0$.
(2) For any smooth morphism $X \to S$ of schemes, if S belongs to $\mathscr{S}_0$ then X belongs to $\mathscr{S}_0$.

Definition 1.3.17. Consider the above assumptions.

We say that $\mathscr{T}$ satisfies $\mathscr{S}_0$-*absolute purity* if for any regular closed $\mathscr{S}_0$-pair (X, Z), any integer n and any virtual vector bundle v over Z, the following maps are isomorphisms:

$$H_Z^{n,v}(X, \mathscr{T}) \xleftarrow{d_1^*} H_{\mathbf{A}_Z^1}^{n,v}(D_Z X, \mathscr{T}) \xrightarrow{d_0^*} H_Z^{n,v}(N_Z X).$$

When $\mathscr{S}_0$ is the category of regular schemes in $\mathscr{S}$, we simply say $\mathscr{T}$ satisfies absolute purity.

Example 1.3.18. When $\mathscr{S}_0$ is the category of smooth schemes over some scheme Σ in $\mathscr{S}$, $\mathscr{T}$ always satisfies the absolute purity property according to Morel and Voevodsky's purity theorem [MV99, Sec. 3, Th. 2.23].

1.3.19. Let S be a scheme and $\mathbb{E}$ be a $\mathscr{T}$-spectrum over S. Then for any S-scheme X in $\mathscr{S}$, we can define the $\mathbb{E}$-cohomology of X as:

$$\mathbb{E}^{n,i}(X) = \mathrm{Hom}_{\mathscr{T}(X)}(\mathbb{1}_X, f^*\mathbb{E}(i)[n])$$

– and similarly for the other indexes considered above. We will say that $\mathbb{E}$-*cohomology commutes with projective limits* if for any projective system of S-schemes $(X_i)_{i \in I}$ in $\mathscr{S}$ which admits a limit X in the category $\mathscr{S}$, the natural map

$$\left(\varinjlim_{i \in I} \mathbb{E}^{**}(X_i) \right) \to \mathbb{E}^{**}(X)$$

of bigraded abelian groups is an isomorphism. When this property is verified for $\mathbb{E}$ the constant $\mathscr{T}$-spectrum $\mathbb{1}_S$, for any scheme S in $\mathscr{S}$, we will also simply say that $\mathscr{T}$-*cohomology commutes with projective limits*.

Extending slightly the terminology of [CD12, 4.3.2],[16] we say that $\mathscr{T}$ *is continuous* if for any scheme S, and any $\mathscr{T}$-spectrum $\mathbb{E}$ over S, $\mathbb{E}$-cohomology commutes with projective limits. With this definition, all the results of [CD12, Section 4.3] work through and we will freely use them – in fact, we will mainly use Proposition 4.3.4 of *loc. cit.*

According to [CD12, 4.3.3] and [CD15a] all the motivic categories of Example 1.3.1 are continuous.

Example 1.3.20. Assume that $\mathscr{T}$-cohomology commutes with projective limits in the above sense and $\mathscr{S}$ is the category of all excellent noetherian finite dimensional schemes. Then $\mathscr{T}$ automatically satisfies the absolute purity with respect to the category of regular F-schemes for any prime field F (see [Dég14, Ex. 1.3.4(2)]). Again, this holds for all the motivic categories of Example 1.3.1.

Recall one says a cartesian morphism of regular closed pairs $(Y, T) \to (X, Z)$ is *transversal* if the induced map $N_T(Y) \to N_Z(X) \times_Z T$ is an isomorphism. The following proposition is a straightforward generalization of [CD15a, A.2.8] (see [Dég14] for more details).

Proposition 1.3.21. *Let $\mathscr{S}_0$ be a subcategory of $\mathscr{S}$ satisfying the assumptions of 1.3.16. Then the following conditions are equivalent.*

(i) *$\mathscr{T}$ is $\mathscr{S}_0$-absolutely pure.*

(ii) *For any regular closed $\mathscr{S}_0$-pair (X, Z), there exists a class $\eta_X(Z)$ in $H_Z^{0, N_Z X}(X, \mathscr{T})$. The family of such classes satisfies the following properties:*

    (a) *For a vector bundle E/Z, and (E, Z) being the closed pair corresponding to the 0-section, the class $\eta_E(Z)$ corresponds to the identity of the Thom $\mathscr{T}$-spectrum $\mathrm{Th}(-E)$ through the identification:*

$$H_Z^{0, N_Z E}(E, \mathscr{T}) \simeq H_Z^{0, E}(E, \mathscr{T}) = \mathrm{Hom}\left(\mathrm{Th}(-E), s^!(\mathbb{1}_Z)\right) \simeq \mathrm{Hom}\left(\mathrm{Th}(-E), \mathrm{Th}(-E)\right).$$

    (b) *For any transversal morphism $f : (Y, T) \to (X, Z)$ of regular $\mathscr{S}_0$-closed pairs, the following relation holds in $H_T^{0, N_Z X}(Y, \mathscr{T})$:*

$$f^*\eta_X(Z) = \eta_Y(T).$$

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[16]which applies only to triangulated motivic categories which are the homotopy category associated with a model motivic category;

(c) *For any regular closed $\mathscr{S}_0$-pair (X, Z), corresponding to the closed immersion $i : Z \to X$,*

$$\eta_X(Z) : \mathrm{Th}(-N_Z X) \to i^!(\mathbb{1}_X)$$

is an isomorphism of $\mathscr{T}(Z)$.

Moreover, when these conditions hold, the properties (a) and (b) uniquely determines the family $\eta_X(Z)$ indexed by regular closed $\mathscr{S}_0$-pairs (X, Z).

Corollary 1.3.22. *Assume $\mathscr{T}$ is $\mathscr{S}_0$-absolutely pure.*

Then for any quasi-projective local complete intersection morphism $f : Y \to X$ such that X and Y belongs to $\mathscr{S}_0$, with virtual tangent bundle τ_f, we get an isomorphism in $\mathscr{T}(Y)$:

$$\eta_f : \mathrm{Th}\left(\tau_f\right) \to f^!\left(\mathbb{1}_X\right).$$

This simply combines the preceding proposition with the relative purity isomorphism (1.3.7.b) after choosing a factorisation of f into a regular closed immersion followed by a smooth morphism.

Remark 1.3.23. It can be shown that the classes η_f are uniquely determined and in particular do not depend on the chosen factorization (see [Dég14]) but we will not use this fact here.

2. The homotopy t-structure by generators

2.1. **Definition.**

Definition 2.1.1. Let (S, δ) be a dimensional scheme (Definition 1.1.2).

The *δ-homotopy t-structure over S*, denoted by t_δ, is the t-structure on $\mathscr{T}(S)$ generated by $\mathscr{T}$-spectra of the form $M^{BM}(X/S)\{n\}[\delta(X)]$ (recall Definition 1.3.6) for any separated S-scheme X and any integer $n \in \mathbf{Z}$.

Given a morphism $f : T \to S$ essentially of finite type, we will put $t_\delta = t_{\delta f}$ as a t-structure on $\mathscr{T}(T)$.

Note also that the δ-homotopy t-structure is generated by $\mathscr{T}$-spectra of the form $M^{BM}(X/S)\langle\delta(X)\rangle\{n\}$ for any separated S-scheme X and any integer $n \in \mathbf{Z}$. In the followings, we will use this convention for the generators because it allows to treat the cases of the δ-homotopy t-structure and the effective δ-homotopy t-structure (cf. Definition 2.2.14) simultaneously.

Remark 2.1.2.   (1) Given Lemma 1.2.8 and Remark 1.3.9, t_δ is left non-degenerate.
   (2) One readily deduces from the above definition that a spectrum $\mathbb{E}$ over S is t_δ-negative if and only if for any separated S-scheme X, the $\mathbf{Z}$-graded abelian group

$$\mathbb{E}^{BM}_{p,\{*\}}(X/S)$$

is zero as soon as $p \geq \delta(X)$.

Example 2.1.3. Assume S is the spectrum of a field k of characteristic exponent p. We will show in the next section (cf. Example 2.3.5) that when $\mathscr{T} = \mathrm{DM}_{\mathrm{cdh}}[1/p]$, the δ_k-homotopy t-structure coincides with the one defined in [Dég08a] on $\mathrm{DM}(k)[1/p]$. We will also show (*ibid.*) that when $\mathscr{T} = \mathrm{SH}$ and k is a perfect field, the δ_k-homotopy t-structure on $\mathrm{SH}(k)$ coincides with that defined by Morel in [Mor12].

The premotivic category $\mathscr{T}$ is additive: if $S = \sqcup_i S_i$, $\mathscr{T}(S) = \oplus_{i \in I} \mathscr{T}(S_i)$. In particular, given any family $\underline{n} = (n_i)_{i \in I}$ of integers, we can define the *n-suspension functor* $\Sigma_{\underline{n}}$ as the sum over $i \in I$ of the n_i-suspension functor on $\mathscr{T}(S_i)$. The following lemma is obvious:

Lemma 2.1.4. *Let $(S_i)_{i \in I}$ be the family of connected components of S. Let δ and δ' be two dimension functions on S. Put $\underline{n} = \delta' - \delta$ seen as an element of $\mathbf{Z}^I$ according to Lemma 1.1.3.*

Then the functor $\Sigma_{\underline{n}} : (\mathscr{T}(S), t_\delta) \to (\mathscr{T}(S), t_{\delta'})$ is an equivalence of t-categories.

Remark 2.1.5. In other words, changing the dimension function on S only changes the δ-homotopy t-structure by some shift. However, we will keep the terminology "δ-homotopy t-structure" because it allows to make it different from the other homotopy t-structures defined earlier (Voevodsky, Morel, Déglise, Ayoub). Then the symbol δ stands for "dimensional".

Beware also that it is sometimes important to be more precise about the chosen dimension functions (in particular, about the t-exactness properties of functors).

Proposition 2.1.6. *Let S be a scheme with a dimension function δ and $f : T \to S$ be a morphism essentially of finite type.*

    (1) *The pair $f_! : (\mathscr{T}(T), t_\delta) \leftrightarrows (\mathscr{T}(S), t_\delta) : f^!$ is an adjunction of t-categories.*

    (2) *If $\delta \geq 0$, then the tensor product $\otimes_S$ is right t-exact on $(\mathscr{T}(S), t_\delta)$.*

    (3) *If $\dim(f) \leq d$, then the pair of functors $f^*[d] : (\mathscr{T}(S), t_\delta) \to (\mathscr{T}(T), t_\delta) : f_*[-d]$ is an adjunction of t-categories.*

Proof. In each cases, one has to check that the left adjoint functor sends a generator of the relevant homotopy t-structure to a non-negative object. For (1), let Y/T be a separated scheme: recall from §1.1.6 that $\delta^f(Y) = \delta(Y)$; thus $f_!(M^{BM}(Y/T)\langle\delta^f(Y)\rangle) = M^{BM}(Y/S)\langle\delta(Y)\rangle$ and this concludes. Similarly, assertion (2) (resp. (3)) follows from part (1) (resp. (2)) of Proposition 1.1.11 and property (BM5) (resp. (BM6)) of 1.3.8. $\qquad\square$

Corollary 2.1.7. *Adopt the assumptions of the previous proposition.*

    (1) *If f is étale, $f^* : (\mathscr{T}(S), t_\delta) \to (\mathscr{T}(T), t_\delta)$ is t-exact.*

    (2) *If f is finite, $f_* : (\mathscr{T}(T), t_\delta) \to (\mathscr{T}(S), t_\delta)$ is t-exact.*

    (2') *If f is proper and $\dim(f) \leq d$, then f_* has (homological) amplitude $[0, d]$ (with respect to the δ-homotopy t-structures).*

2.1.8. Consider a closed immersion $i : Z \to S$ with complementary open immersion $j : U \to S$. Recall the localization property for $\mathscr{T}$ says precisely that $\mathscr{T}(S)$ is glued from $\mathscr{T}(U)$ and $\mathscr{T}(Z)$ with respect to the six functors $i^*, i_*, i^!, j_!, j^*, j_*$. In particular, we are in the situation of [BBD82, I, 1.4.3].

From *op. cit.* 1.4.10, if t_U (resp. t_Z) is a t-structure on $\mathscr{T}(U)$ (resp. $\mathscr{T}(Z)$), there exists a unique t-structure t_{gl} whose positive objects K are characterized by the conditions:

$$i^*(K) \geq 0, j^*(K) \geq 0.$$

The t-structure t_{gl} is called the *t-structure glued from t_U and t_Z*; one also says the t-category $(\mathscr{T}(S), t_{gl})$ is *glued from $(\mathscr{T}(U), t_U)$ and $(\mathscr{T}(Z), t_Z)$*. The glued t-structure on $\mathscr{T}(S)$ is uniquely characterized by the fact that j^* and i_* are t-exact (see *op. cit.*, 1.4.12). In particular, the previous corollary immediately yields:

Corollary 2.1.9. *Consider the assumption of the previous proposition. Let $i : Z \to S$ be a closed immersion with complementary open immersion $j : U \to S$.*

Then the t-category $(\mathscr{T}(S), t_\delta)$ is obtained by glueing the t-categories $(\mathscr{T}(Z), t_\delta)$ and $(\mathscr{T}(U), t_\delta)$.

Example 2.1.10. let S be a universally catenary integral scheme with dimension function $\delta = -\operatorname{codim}_S$ (cf. Example 1.1.4).

Then we deduce from the previous corollary that for any smooth S-scheme X, the $\mathscr{T}$-spectrum $M_S(X)$ is t_δ-non-negative.

Indeed, first we can assume that X is connected. Let d be its relative dimension over S. According to the preceding corollary, this assertion is Zariski local on S. In particular, we can assume that the tangent bundle of X/S admits a trivialization. Then according to 1.3.8(BM4), $M_S(X) = M^{BM}(X/S)\langle d\rangle$. It is now sufficient to remark that $d = \delta(X)$ – because of our choice of δ along with Proposition 1.1.11(3).

Actually, this example and the previous glueing property are the main reason for our choice of generators of the δ-homotopy t-structure (see also Corollary 2.3.4).

Corollary 2.1.11. *Let (S, δ) be a dimensional scheme and E/S be a vector bundle of rank r. Then the functor $\left(M\mathrm{Th}_S(E)[-r] \otimes -\right)$ is t_δ-exact. The same result holds if one replaces E by a virtual vector bundle v over S of virtual rank r (see §1.3.7).*

It simply follows by noetherian induction on S from the previous corollary and the fact a vector bundle of rank r is generically isomorphic to $\mathbf{A}_S^r$.

Proposition 2.1.12. *Let (S, δ) be a dimensional scheme and $f : T \to S$ a smooth morphism. Then $f^!$ is t_δ-exact.*

This follows from Proposition 2.1.6, the previous corollary and the relative purity isomorphism (1.3.7.b).[17]

2.1.13. One can reinterpret the preceding proposition in terms of the classical pullback functor f^*.

Let (S, δ) be a dimensional scheme and $f : T \to S$ be a smooth morphism. Let us write δ^f for the dimension function on T induced by f (see 1.1.6). Note that since f is smooth, the function $\dim(f)$ is Zariski locally constant on T. We consider a new dimension function on T:

$$(2.1.13.\text{a}) \qquad\qquad \tilde{\delta}^f = \delta^f - \dim(f).$$

As a corollary of the previous proposition, we get:

Corollary 2.1.14. *In the notation above, the adjunction*

$$f^* : (\mathscr{T}(S), t_\delta) \leftrightarrows (\mathscr{T}(T), t_{\tilde{\delta}^f}) : f_*$$

of triangulated categories is an adjunction of t-categories such that f^* is t-exact.

Proof. We can assume that T is connected in which case $\dim(f)$ is constant equal to an integer d. By definition, one has $\tilde{\delta}^f - \delta^f = -d$. Indeed, according to the previous proposition and Lemma 2.1.4, the following composite functor is t-exact:

$$(\mathscr{T}(S), t_\delta) \xrightarrow{f^!} (\mathscr{T}(T), t_{\delta^f}) \xrightarrow{\Sigma_{-d}} (\mathscr{T}(T), t_{\tilde{\delta}^f}),$$

where $\Sigma_{-d}(K) = K[-d]$. According to 1.3.8(BM4), $f^! = f^* \otimes \mathrm{Th}(\tau_f)$, where τ_f is the tangent bundle of f, which has pure rank d. Thus, according to the beginning of the proof, we get that $f^* \otimes \mathrm{Th}(\tau_f)[-d] = f^![-d]$ is t-exact with respect to t_δ on the source and $t_{\tilde{\delta}^f}$ on the target. This concludes the proof by Corollary 2.1.11. $\qquad\square$

Example 2.1.15. Let $f : T \to S$ be a smooth morphism such that S and T are equicodimensional (for example, S integral over a field). Let $\delta = \dim_S$ (resp. $\dim_T$) be the Krull dimension function restricted to S (resp. T). Then one readily checks that $(\dim_S)^f = \dim_T + \dim(f)$, so that we have $\tilde{\delta}^f = \dim_T$ in the notation of (2.1.13.a). Thus the preceding corollary yields the following adjunction of t-categories:

$$f^* : (\mathscr{T}(S), t_{\dim_S}) \leftrightarrows (\mathscr{T}(T), t_{\dim_T}) : f_*$$

such that f^* is t-exact.

Let us remark finally the following fact.

Lemma 2.1.16. *Assume that the motivic category $\mathscr{T}$ is semi-separated.[18] Then for any finite surjective radicial morphism $f : T \to S$, the functor $f^* : \mathscr{T}(S) \to \mathscr{T}(T)$ is an equivalence of t-categories.*

This directly follows from Proposition 2.1.6 and [CD12, 2.1.9].

Example 2.1.17. The triangulated motivic category $\mathscr{T} = \mathrm{DM}_R$ in the conventions of §1.3.2 is semi-separated.

More generally, it can be shown that any triangulated motivic category $\mathscr{T}$ over a category of schemes $\mathscr{S}$ is semi-separated provided that the following two conditions hold:

- $\mathscr{T}$ is $\mathbf{Z}[N^{-1}]$-linear where N is the set of characteristic exponent of all the residue fields of schemes in $\mathscr{S}$;
- $\mathscr{T}$ is oriented.

This follows from the existence of Gysin morphisms associated with finite local complete intersection morphisms and from a local trace formula.

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[17]Under stronger assumptions, the t-exactness result of $f^!$ will be generalized in Corollary 3.3.7.

[18] This notion was first introduced by Ayoub. Recall it means that for any finite surjective radicial morphism $f : T \to S$, the functor $f^* : \mathscr{T}(S) \to \mathscr{T}(T)$ is conservative. According to [CD12, 2.1.9], it implies that f^* is in fact an equivalence of categories.

2.2. The δ-effective category.

Definition 2.2.1. Let (S, δ) be a dimensional scheme.

We define the category of δ-effective $\mathscr{T}$-spectra, denoted by $\mathscr{T}^{\delta-eff}(S)$, as the localizing triangulated subcategory[19] of $\mathscr{T}(S)$ generated by objects of the form $M^{BM}(X/S)(n)$ for any separated S-scheme X and any integer $n \geq \delta(X)$.

When $f : T \to S$ is a morphism essentially of finite type, we will put: $\mathscr{T}^{\delta-eff}(T) := \mathscr{T}^{\delta_f-eff}(T)$.

Remark 2.2.2.   (1) Arguing as in the proof of Lemma 2.1.4, we easily deduce that, up to a canonical triangulated equivalence, the category $\mathscr{T}^{\delta-eff}(S)$ does not depend on δ. This equivalence will preserve the monoidal structure if it is defined (see point (2) of the following proposition).
   (2) Our definition of δ-effectivity is closely related to the classes of morphisms B_n (and to the corresponding constructions) considered in [Pel13, §2]; yet it seems our use of dimension functions is new in this context.

Example 2.2.3.   (1) Assume $\mathscr{T}$ is oriented, the base scheme S in universally catenary and integral, and $\delta = -\operatorname{codim}_S$ (Example 1.1.4). Then for any smooth S-scheme X, the $\mathscr{T}$-spectrum $M_S(X)$ is δ-effective, as in the classical mixed motivic case — this follows from 1.3.2(BM4) and Proposition 1.1.11(3). (Actually, we do not need the orientation assumption as will be shown in in Example 2.2.11.)
   (2) Actually, as a consequence of the cancellation theorem of Voevodsky, we will see in Example 2.3.12 that when $\mathscr{T} = \mathrm{DM}[1/p]$, S is the spectrum of a field k of characteristic exponent p, δ the Krull dimension function on k, then $\mathscr{T}^{\delta-eff}(k)$ coincides with the category of Voevodsky's motivic (unbounded) complexes.

The following proposition summarizes basic facts about δ-effective spectra.

Proposition 2.2.4. *Let (S, δ) be a dimensional scheme.*
   (1) *The inclusion functors $s : \mathscr{T}^{\delta-eff}(S) \to \mathscr{T}(S)$ and $s' : \mathscr{T}^{\delta-eff}(S) \to \mathscr{T}^{\delta-eff}(S)(-1)$ (that we consider as a subcategory of $\mathscr{T}(S)$) admit right adjoints $w : \mathscr{T}(S) \to \mathscr{T}^{\delta-eff}(S)$ and $w' : \mathscr{T}^{\delta-eff}(S)(-1) \to \mathscr{T}^{\delta-eff}(S)$, respectively.*
   (2) *If $\delta \geq 0$, then $\mathscr{T}^{\delta-eff}(S)$ is stable under tensor product.*
   (3) *For any smooth S-scheme X, the $\mathscr{T}$-spectrum $M_S(X)\big(\delta(S)\big)$ is δ-effective.*
   (4) *Let $f : T \to S$ be a morphism essentially of finite and d an integer such that $\dim(f) \leq d$. Then the functor $f^*(d)$ sends $\mathscr{T}^{\delta-eff}(S)$ to $\mathscr{T}^{\delta-eff}(T)$.*
   (5) *Let $f : T \to S$ be any separated morphism. Then the exceptional direct image functor $f_!$ sends $\mathscr{T}^{\delta-eff}(T)$ to $\mathscr{T}^{\delta-eff}(S)$.*

Proof. Assertion (1) follows from Neeman's adjunction theorem, since we have assumed $\mathscr{T}(S)$ is compactly generated. Assertion (2) follows from Prop. 1.1.11(1) and §1.3.8(BM5).

Let us prove assertion (3). Since $M_S(X)$ is additive in the smooth S-scheme X, and any connected component of X dominates a connected component of S, we can assume X and S are connected. Then $f : X \to S$ has constant relative dimension, say d and we can apply Proposition 1.1.11(3) to f. Thus one gets $\delta(X) = \delta(S) + d$. Then using 1.3.8(BM4), the following spectrum is in $\mathscr{T}^{\delta-eff}(S)$:

$$M^{BM}(X/S)\langle\delta(X)\rangle = M_S(X)\langle\delta(X) - d\rangle = M_S(X)\langle\delta(S)\rangle,$$

and this concludes.

Finally, assertion (4) follows from 1.1.11(2) and 1.3.8(BM6) whereas assertion (5) is obvious. $\square$

Remark 2.2.5. The couple of functors (s, w) is analog to the couple of functors (infinite suspension, infinite loop space) of stable homotopy, though in our case, s is fully faithful. In the case where $\mathscr{T}$ is the category of motives DM_R, we refer the reader to Example 2.3.12 for more precisions.

Corollary 2.2.6. *Consider the notations of the previous definition.*

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[19]*i.e.* stable by extensions and arbitrary coproducts;

(1) *If $\delta \geq 0$, $\mathscr{T}^{\delta-eff}(S)$ has internal Hom: given any δ-effective spectra M and N over S, one has:*

$$\underline{\mathrm{Hom}}_{\mathscr{T}^{\delta-eff}(S)}(M, N) = w\,\underline{\mathrm{Hom}}_{\mathscr{T}(S)}(M, N).$$

(2) *Let $f : T \to S$ be a separated morphism. Then one has an adjunction of triangulated categories:*

$$f_! : \mathscr{T}^{\delta-eff}(T) \leftrightarrows \mathscr{T}^{\delta-eff}(S) : f^!_{\delta-eff} := w \circ f^!.$$

(3) *Let $f : T \to S$ be a morphism essentially of finite type such that $\dim(f) \leq d$. Then one has an adjunction of triangulated categories:*

$$f^*(d) : \mathscr{T}^{\delta-eff}(T) \leftrightarrows \mathscr{T}^{\delta-eff}(S) : w \circ [f_*(-d)].$$

Remark 2.2.7. When $f : T \to S$ is equidimensional (for example smooth, flat or universally open), one can improve point (3) using the same trick as that of §2.1.13. If δ is a dimension function on S, we consider the following dimension function on T: $\tilde{\delta}^f = \delta^f - \dim(f)$.

Then, according to point (3) of the above corolary, we obtain a well defined adjunction of triangulated categories:

$$(2.2.7.a) \qquad f^* : \mathscr{T}^{\delta-eff}(S) \to \mathscr{T}^{\tilde{\delta}^f-eff}(T) : w \circ f_*.$$

Note finally that if S is equicodimensional and $\delta = \dim_S$ is the Krull dimension function on S, then $\tilde{\delta}^f = \dim_T$, the Krull dimension function on T — here we use the fact f is equidimensional.

Example 2.2.8. Let (S, δ) be an arbitrary dimensional scheme. According to point (1) of the preceding proposition we obtain an adjunction of triangulated categories

$$s : \mathscr{T}^{\delta-eff}(S) \leftrightarrows \mathscr{T}(S) : w$$

such that s is fully faithful. This formally implies that $\mathscr{T}^{\delta-eff}(S)$ is the Verdier quotient of the category $\mathscr{T}(S)$ made by the full triangulated subcategory $\mathrm{Ker}(w)$ made of $\mathscr{T}$-spectra K over S such that $w(K) = 0$. Moreover, according to our conventions, the triangulated category $\mathscr{T}^{\delta-eff}(S)$ is compactly generated. This formally implies the right adjoint functor w commutes with coproducts. So the category $\mathrm{Ker}(w)$ is stable by coproducts (*i.e.* localizing).

It is difficult in general to describe concretely the category $\mathrm{Ker}(w)$. We now give examples in the particular case $\mathscr{T} = \mathrm{DM}_R$, following the conventions of point (1) or (2) in Paragraph 1.3.2. We need some facts to justify them so we postpone this justification till Paragraph 2.3.13.

Assume S is regular and X/S is a smooth projective scheme such that $\dim(X/S) = d$ for a fixed integer d. Then:

$$(2.2.8.a) \qquad w\big(M_S(X)(n)\big) = \begin{cases} M_S(X)(n) & \text{if } n \geq \delta(X) - d, \\ 0 & \text{if } n < \delta(S) - d. \end{cases}$$

This computation allows to compute the right adjoint appearing in point (3) in some particular cases. Let us give a very basic example. To fix ideas, assume S is irreducible and $\delta(S) = 0$. Of course, this implies $\mathbb{1}_S$ is δ-effective. Let P be a projective bundle over S of rank d, and $f : P \to S$ be the canonical projection. Then $\mathbb{1}_P(d)$ is δ-effective and we get:

$$w\big(f_*\big(\mathbb{1}_P(d)\big)(-d)\big) = \bigoplus_{i=0}^{d} w\big(\mathbb{1}_S(-i)[-2i]\big) = \mathbb{1}_S$$

according to the projective bundle formula and the preceding computation. Many similar computations can be obtain from the previous formula. We let them as an exercise for the reader.

2.2.9. Let $\mathscr{S}^{qf}_{/S}$ be the sub-category of $\mathscr{S}_{/S}$ made of the same objects but whose morphisms f are quasi-finite (see also Example 1.1.9).

Then the previous proposition implies that $\mathscr{T}^{\delta-eff}$ is a *Sm*-fibred triangulated subcategory of $\mathscr{T}$ over $\mathscr{S}^{qf}_{/S}$ (cf. [CD12, §1]).[20] Moreover, if $\delta \geq 0$, it is even a monoidal *Sm*-fibred triangulated sub-category of $\mathscr{T}$: for any quasi-finite S-scheme T, the tensor product $\otimes_T$ respects δ-effective spectra

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[20]In fact, point (3) of Corollary 2.2.6 shows that given a dimensional scheme S, $\mathscr{T}^{\delta-eff}$ is fibred over S-schemes essentially of finite type with respect to morphisms which are equidimensional (recall footnote 13, p. 11).

and moreover the unit $\mathbb{1}_T$ is a δ-effective spectra. In particular, the inclusion $s : \mathscr{T}^{\delta-eff}(T) \to \mathscr{T}(T)$ is monoidal, for the induced monoidal structure on the left hand side.

Finally, it is easily seen using Proposition 2.2.4 that this premotivic category satisfies the localization property.

Corollary 2.2.10. *Consider the assumptions of the preceding proposition. Then for any closed immersion $i : Z \to S$ with complementary open immersion j, and any δ-effective spectrum M, the following triangle is a distinguished triangle in $\mathscr{T}^{\delta-eff}(S)$:*

$$j_! j^*(K) \to K \to i_! i^*(K) \xrightarrow{+1}$$

Example 2.2.11. let S be a universally catenary integral scheme S with dimension function $\delta = -\operatorname{codim}_S$ (cf. Example 1.1.4).

Then for any smooth S-scheme X, the motive $M_S(X)$ is t_δ-effective. The proof is the same than that of Example 2.1.10.

As for Corollary 2.1.11, we deduce:

Corollary 2.2.12. *Let (S, δ) be a dimensional scheme and E/S be a vector bundle of rank r. Then the functor $(M\operatorname{Th}(E)[-r] \otimes -)$ preserves δ-effective spectra.*

Remark 2.2.13. Here is a complete list of functors that preserves δ-effective spectra: $f_!$ for f separated, f_* for f is proper, f^* for f quasi-finite, $f^!$ for f smooth.

All cases follow from 2.2.4 except the last one which also uses the previous corollary and the purity isomorphism $f^! \simeq M\operatorname{Th}(T_f) \otimes f^*$.

We can easily extend the definition of the δ-homotopy t-structure to the effective case:

Definition 2.2.14. Let (S, δ) be a dimensional scheme.

The (effective) δ-*homotopy t-structure over S*, denoted by t_δ is, the t-structure on $\mathscr{T}^{eff}(S)$ generated by the spectra $M^{BM}(X/S)\langle \delta(X) \rangle \{n\}$ for any separated S-scheme X and any integer $n \geq 0$.

When $f : T \to S$ is essentially of finite type, we put $t_\delta = t_{\delta f}$.

With these definitions, it is clear that the adjunction of triangulated categories (see 2.2.4(1)):

(2.2.14.a) $$s : \mathscr{T}^{\delta-eff}(S) \leftrightarrows \mathscr{T}(S) : w$$

is in fact an adjunction of t-categories.

Remark 2.2.15. As in the non-effective case, we get:
  (1) an effective spectrum $\mathbb{E}$ over S is t_δ^{eff}-negative if and only if for any separated S-scheme X the **Z**-graded abelian group $\mathbb{E}_{p,\{*\}}^{BM}(X/S)$ is zero in degree $* \geq \delta(X)$ and if $p \geq \delta(X)$;
  (2) (see Remark 2.1.2), the effective δ-homotopy t-structure is left non-degenerate;
  (3) it does not depend on the choice of the dimension function δ, up to a canonical equivalence of t-categories;
  (4) for any vector bundle E/S of rank r, the endo-functor $\left(M\operatorname{Th}_S(E)[-r] \otimes - \right)$ of $\mathscr{T}^{\delta-eff}(S)$ (see Cor. 2.2.12) is t_δ-exact.

2.2.16. We can easily transport the results of Proposition 2.1.6 to the effective case as follows:
  (1) For any separated morphism $f : T \to S$, the pair of functors

$$f_! : \left(\mathscr{T}^{\delta-eff}(T), t_\delta \right) \to \left(\mathscr{T}^{\delta-eff}(S), t_\delta \right) : w \circ f^!$$

   is an adjunction of t-categories.
  (2) If $\delta \geq 0$, the tensor product $\otimes_S$ is right t-exact on $\left(\mathscr{T}^{\delta-eff}(S), t_\delta \right)$
  (3) For any morphism $f : T \to S$ such that $\dim(f) \leq d$, the pair of functors:

$$f^*\langle d \rangle : \left(\mathscr{T}^{\delta-eff}(S), t_\delta \right) \to \left(\mathscr{T}^{\delta-eff}(S), t_\delta \right) : w \circ (f_*\langle -d \rangle)$$

   is an adjunction of t-categories.

In particular, using the adjunction of Corollary 2.2.6 together with Remark 2.2.13, we get that $f_!$ (resp. f^*) is t-exact when f is finite (resp. étale). So we get the following corollary (as for Corollary 2.1.9):

Corollary 2.2.17. *Let (S, δ) be a dimensional scheme, $i : Z \to S$ be a closed immersion with complementary open immersion $j : U \to S$. Then the t-category $(\mathscr{T}^{\delta-eff}(S), t_\delta)$ is obtained by glueing of the t-categories $(\mathscr{T}^{\delta-eff}(Z), t_\delta)$ and $(\mathscr{T}^{\delta-eff}(U), t_\delta)$.*

As in the non-effective case, we deduce that when f is smooth, the functor $w \circ f^! = f^!$ is t-exact. Moreover, the funtor f^*, with the conventions of (2.2.7.a) for the dimension functions, is t-exact. Finally, As in the non-effective case it is worth to remark the following easy lemma (use the proof of the analog Lemma 2.1.16).

Lemma 2.2.18. *Assume that the motivic category $\mathscr{T}$ is semi-separated.*
Then for any finite surjective radicial morphism $f : T \to S$, the functor $f^* : \mathscr{T}^{\delta-eff}(S) \to \mathscr{T}^{\delta-eff}(T)$ is an equivalence of t-categories.

2.3. **Improved descriptions of generators.** In this section, we give several different descriptions of the generators of the δ-homotopy t-structure and draw some corollaries for the δ-homotopy t-structure.

Proposition 2.3.1. *Let (S, δ) be a dimensional scheme.*
Then the δ-homotopy t-structure on $\mathscr{T}(S)$ (resp. $\mathscr{T}^{\delta-eff}(S)$) admits the following three families of generators for positive objects:

(1) *spectra of the form $M^{BM}(X/S)\langle \delta(X) \rangle \{n\}$ for an integer $n \in \mathbf{Z}$ (resp. $n \in \mathbf{N}$) and a proper S-scheme X;*
(2) *spectra of the form $M^{BM}(X/S)\langle \delta(X) \rangle \{n\}$ for an integer $n \in \mathbf{Z}$ (resp. $n \in \mathbf{N}$) and a regular S-scheme X;*
(3) *assuming S is regular and separated not necessarily of finite type over $\mathbf{Z}$: spectra of the form $M^{BM}(X/S)\langle \delta(X) \rangle \{n\}$ for an integer $n \in \mathbf{Z}$ (resp. $n \in \mathbf{N}$) and a regular S-scheme X such that there exists a closed S-immersion $X \to \mathbf{A}_S^r$ with trivial normal bundle.*

Proof. To treat all cases simultaneously, we denote by I the set $\mathbf{Z}$ (resp. $\mathbf{N}$) and put, using the notation of 1.2.5, $\mathscr{C} = \langle \mathcal{G} \rangle_+$ where $\mathcal{G}$ is the family describe in point (1), (2) or (3). We have to prove that for any separated morphism $f : X \to S$, $M^{BM}(X/S)\langle d \rangle$ belongs to $\mathscr{C}$ where $d = \delta(X)$.
Point (1): Because f is separated (of finite type), it admits a factorization $X \xrightarrow{j} \bar{X} \xrightarrow{p} S$ such that p is proper and j is a dense open immersion. Let $i : Z \to \bar{X}$ be the complementary closed immersion of j, with Z reduced. Then using 1.3.8(BM3), we get a distinguished triangle:

$$M^{BM}(Z/S)\langle d \rangle [-1] \to M^{BM}(X/S)\langle d \rangle \to M^{BM}(\bar{X}/S)\langle d \rangle \xrightarrow{+1}$$

Note that, since j is dense, we have: $\delta(\bar{X}) = d$ and $\delta(Z) < d$. In particular, by assumption on $\mathscr{C}$, the $\mathscr{T}$-spectra $M^{BM}(\bar{X}/S)\langle d \rangle$ and $\delta(\bar{X}) = d$ both belongs to $\mathscr{C}$. This implies $M^{BM}(X/S)\langle d \rangle$ belongs to $\mathscr{C}$ and concludes.
Point (2) and (3): We use noetherian induction on X, given that the result is obvious when X is empty. Then it is sufficient to find a dense open subscheme $U \subset X$ such that $M^{BM}(U/S)\langle d \rangle$ belongs to $\mathscr{C}$. Indeed, given that subscheme U we denote by Z its complement in X with its reduced structure of a subscheme of X. Then, applying again 1.3.8(BM3), we get the following distinguished triangle:

$$M^{BM}(U/S)\langle d \rangle \to M^{BM}(X/S)\langle d \rangle \to M^{BM}(Z/S)\langle d \rangle \xrightarrow{+1}$$

and the conclusion follows from the fact $M^{BM}(Z/S)\langle d \rangle$ belongs to $\mathscr{C}$ by noetherian induction using again the inequality $\delta(Z) < d$.
In any case, we can assume that X is reduced. Moreover, since X is excellent, it admits a dense open subscheme U which is regular. This concludes in case of point (2).
For point (3) we remark that we can even assume U is affine in addition to be a regular dense open subscheme of X. Then the structural morphism $p : U \to S$ is affine (because S is separated

over $\mathbf{Z}$, see [EGA2, 1.6.3]). In addition, reducing U again, we can assume the coherent $\mathcal{O}_S$-algebra $p_*(\mathcal{O}_U)$ is generated by global sections. Thus there exists a closed immersion $i : U \to \mathbf{A}_S^r$. This immersion is regular because U is regular by construction and $\mathbf{A}_S^r$ is regular by assumption on S. Thus the normal cone of i is a vector bundle. In particular, it is generically trivial so that there exists an open subscheme V in $\mathbf{A}_S^r$ such that $i^{-1}(V)$ is dense in U and trivialize the normal bundle of i. Thus, considering the dense open subscheme $i^{-1}(V) \subset X$, we are able to finish the proof of assertion (3). □

As a corollary of assertion (1), we get the following statement.

Corollary 2.3.2. *Let (S, δ) be a dimensional scheme.*

Then for any adjunction $\varphi^* : \mathscr{T} \leftrightarrows \mathscr{T}' : \varphi_*$ of triangulated motivic categories (cf. [CD12, 1.4.6]) and any scheme S, the functor $\varphi^* : \mathscr{T}(S) \to \mathscr{T}'(S)$ is right t_δ-exact and the functor $\varphi_* : \mathscr{T}'(S) \to \mathscr{T}(S)$ is t_δ-exact.

Proof. The first assertion follows easily from the fact φ^* commutes with functors $f_!$ (cf. [CD12, 2.4.53]). The second assertion comes from the previous proposition because φ_* commutes with $f_* = f_!$ for f proper. □

Example 2.3.3. Let R be a coefficient ring, (S, δ) be a dimensional scheme and consider the premotivic adjunctions (1.3.3.a). According to the previous corollary, all left (resp. right) adjoints of these premotivic adjunctions are right t_δ-exact (resp. t_δ-exact). Then according to (1.2.4.a), we get adjunctions of abelian categories between the homotopy hearts:

$$
\begin{array}{ccc}
 & \xrightarrow{\phi_*} & \mathbf{MGL}-mod(S)^\heartsuit \\
\mathrm{SH}(S)^\heartsuit \xleftarrow{\quad\quad} & \xleftarrow{H_0\phi^*} & \\
\xleftarrow[H_0\delta^*]{\delta_*} \mathrm{D}_{\mathbf{A}^1}(S, R)^\heartsuit & \xleftarrow[H_0\gamma^*]{\gamma_*} & \mathrm{DM}(S, R)^\heartsuit.
\end{array}
$$

Moreover, the right adjoints δ_*, γ_*, ρ_*, and ϕ_*, are all exact and conservative, thus faithful.

As a corollary of point (3) of the previous proposition, we get the following fact:

Corollary 2.3.4. *Let k be a perfect field.*

Then the δ_k-homotopy t-structure on $\mathscr{T}(k)$ is generated by spectra of the form $M(X)\{n\}$, where X/k is smooth and $n \in \mathbf{Z}$.

Indeed, given X/k as in point (3) of the previous proposition, we get:

$$(2.3.4.a) \qquad M^{BM}(X/k)\langle\delta_k(X)\rangle \overset{(BM4)}{\simeq} M\mathrm{Th}_k(-T_{X/k})\langle\delta_k(X)\rangle \simeq M(X)$$

because, under the assumptions on X, the tangent bundle of X/k is trivial of rank $\delta_k(X)$.

Example 2.3.5. Let k be a perfect field of characteristic exponent p, δ_k the canonical dimension function.

(1) Let $\mathscr{T} = \mathrm{DM}_R$ be the triangulated motivic category of point (1) or (2) of Paragraph 1.3.2. In case (2), we assume k contains F. Recall from (1.3.1.a) that $\mathrm{DM}(k, R)$ can be described using Nisnevich topology instead of cdh-topology, as was done in [CD09, 7.15].

Then according to the previous corollary and [Dég11, §5.7], the δ_k-homotopy t-structure on $\mathrm{DM}(k, R)$ coincides with the *homotopy t-structure* defined in [Dég11, Prop. 5.6].

(2) The case of the stable homotopy category $\mathscr{T} = \mathrm{SH}$.

Recall Morel has introduced in [Mor03, Th. 5.3.3] a t-structure on $\mathrm{SH}(k)$ called the *homotopy t-structure*. One notable feature of this t-structure is that its heart is equivalent to the category of *homotopy modules over k* ([Dég13, 1.2.2]), or recall below). Moreover, as remarked in [Dég13, 1.1.5], this t-structure is generated (in the sense of Definition 1.2.7) by spectra of the form: $\Sigma^\infty X_+ \wedge \mathbf{G}_m^{\wedge n}$, where X is a smooth k-scheme, Σ^∞ is the infinite suspension $\mathbf{P}^1$-spectrum functor, $\mathbf{G}_m$ stands for the sheaf of sets represented by the scheme $\mathbf{G}_m$ pointed by 1, and n is any integer.

Thus, it follows from the previous corollary that the δ_k-homotopy t-structure on $\mathrm{SH}(k)$ coincides with Morel's homotopy t-structure as defined in [Mor03].

(3) The case of the stable $\mathbf{A}^1$-derived category $\mathscr{T} = \mathrm{D}_{\mathbf{A}^1, R}$, R any ring of coefficients.

It is explain in [Mor05, Remark 8 of Introduction] how one can build the analog of the Morel's homotopy t-structure recalled in the preceding point on the effective version of the category $\mathrm{D}_{\mathbf{A}^1}(k, R)$. One deduces the stable version as in the case of the stable homotopy category of schemes and it follows from this construction that the heart is again equivalent to the category of homotopy modules over k. Again, this t-structure is generated by $\mathrm{D}_{\mathbf{A}^1, R}$-spectra of the form $M_k(X)$, where X is a smooth k-scheme.

Thus, we also obtain that the δ_k-homotopy t-structure is equivalent to Morel's homotopy t-structure on $\mathrm{D}_{\mathbf{A}^1}(k, R)$.

2.3.6. Let us underlined an interesting consequence of the previous example. Let again k be a perfect field of characteristic exponent p, invertible in a fixed coefficient ring R.

Recall from [Mor03, 5.2.4] and [Dég13, 1.1.4] that a *homotopy module* (resp. *homotopy module with transfers*) over k is a sequence $(F_n, \epsilon_n)_{n \in \mathbf{N}}$ where F_n is a Nisnevich sheaf over the category of smooth k-schemes whose cohomology is $\mathbf{A}^1$-invariant (resp. which is $\mathbf{A}^1$-invariant and admits transfers) and $\epsilon_n : F_n \to (F_{n+1})_{-1}$ is an isomorphism of sheaves where for any such a sheaf G, $G_{-1}(X) = G(\mathbf{G}_m \times X)/G(X)$, induced by the unit section of $\mathbf{G}_m$. We denote by $\Pi_*(k)$ (resp. $\Pi_*^{\mathrm{tr}}(k)$) the corresponding category, morphisms being natural transformations compatible with the grading and with the given isomorphisms ϵ_*. Recall it is a Grothendieck abelian closed monoidal category.

Then we get the following equivalences of abelian monoidal categories where the left hand sides are the heart of the relevant homotopy (or δ_k-homotopy) t-structure:

$$\mathrm{SH}(k)^\heartsuit \simeq \Pi_*(k), \quad \mathrm{D}_{\mathbf{A}^1}(k)^\heartsuit \simeq \Pi_*(k), \quad \mathrm{DM}(k)^\heartsuit \simeq \Pi_*^{\mathrm{tr}}(k).$$

The first two equivalences follow from the construction of Morel and the third one by [Dég11, 5.11].

Recall also that Morel defines the *Hopf map of a scheme S* as the morphism $\eta : \mathbf{G}_m \to S^0$ in $\mathrm{SH}(S)$, — or $\eta : \mathbb{1}_S\{1\} \to \mathbb{1}_S$ with the conventions of the present paper — induced by the morphism of schemes

$$\left(\mathbf{A}_S^2 - \{0\}\right) \to \mathbf{P}_S^1, (x, y) \mapsto [x, y].$$

We say that a spectrum $\mathbb{E}$ over S has trivial action of η if the map $\eta \wedge 1 : \mathbb{E}\{1\} \to \mathbb{E}$ is 0. This is equivalent to the fact that η has trivial action on the cohomology $\mathbb{E}^{n,m}(X)$ for any smooth S-scheme X and any couple of integers $(n, m) \in \mathbf{Z}^2$. The main theorem of [Dég13] identifies $\Pi_*^{\mathrm{tr}}(k)$ with the full subcategory of $\Pi_*(k)$ made by homotopy modules with trivial action of η.

In particular, we get the following proposition.

Proposition 2.3.7. *We use the notations of Example 2.3.3. Let k be a perfect field of characteristic exponent p and R a ring such that $p \in R^\times$. Then the following assertions hold:*

(1) *The adjunction of abelian categories between the δ_k-homotopy hearts:*

$$H_0\delta^* : \mathrm{SH}(k)^\heartsuit \leftrightarrows \mathrm{D}_{\mathbf{A}^1}(k)^\heartsuit : \delta_*$$

are mutually inverse equivalences of abelian monoidal categories.

(2) *The exact functor of abelian categories between the δ_k-homotopy hearts:*

$$\gamma_* : \mathrm{D}_{\mathbf{A}^1}(k, R)^\heartsuit \to DM(k, R)^\heartsuit$$

is fully faithful and its essential image is equivalent to the category of homotopy modules with trivial action of the Hopf map η.

2.3.8. Our t-structure is much analogous to that defined earlier by Ayoub in [Ayo07, §2.2.4, p. 365], under the name *perverse homotopy t-stucture*.

To recall the definition, one needs to fix a base scheme B and assume the following property (see [Ayo07, Hyp. 2.2.58]):

(a) For any separated morphism $f : X \to B$, the functor $f^!$ preserves constructible $\mathscr{T}$-spectra.[21]

Note this property is automatically fulfilled whenever $\mathscr{S}$ is the category of $\mathbf{Q}$-schemes (cf. [Ayo07, 2.2.33]) or $\mathscr{T}$ is $\mathbf{Q}$-linear, separated and satisfies the absolute purity property (cf. [CD12, Th. 4.2.29]).

Let S be a separated[22] B-scheme. Then the *perverse homotopy t-structure* (relative to B) on $\mathscr{T}(S)$ is the t-structure generated by spectra of the form $g_! q^!(\mathbb{1}_B)\{n\}$ for any separated morphism $g : X \to S$, q being the projection of X/B. Following Ayoub, we will denote it by ${}^p t$.

Proposition 2.3.9. *Consider the above notations along with assumption (a). Assume that B is regular and let δ be a dimension function on B.*

Then for any separated B-scheme S, the t-structure ${}^p t$ (resp. t^δ) on $\mathscr{T}(S)$ is generated by objects of the form $g_! q^!(\mathbb{1}_B)\{n\}$ (resp. $M^{BM}(X/S)\langle\delta(X)\rangle\{n\}$) for an integer $n \in \mathbf{Z}$, an affine connected regular B-scheme X which admits a closed B-embedding $i : X \to \mathbf{A}_B^r$ with trivial normal bundle, and $g : X \to S$ is a separated B-morphism.

Proof. The case of t^δ was already proved as point (3) of Proposition 2.3.1. The proof in the case of ${}^p t$ is completely similar. $\qquad\square$

Corollary 2.3.10. *Consider the assumptions of the previous proposition together with the following ones:*

(b) *S is regular with dimension function $\delta = -\operatorname{codim}_S$ (Example 1.1.4);*
(c) *$\mathscr{T}$ satisfies the absolute purity property (Definition 1.3.17).*

Then for all separated S-scheme X, the perverse homotopy t-structure ${}^p t$ and the δ-homotopy t-structure t^δ coincide on $\mathscr{T}(X)$.

Proof. We will prove that the generators of the two t-structures are the same up to isomorphism. Using the preceding proposition, the generators of ${}^p t$ (resp. t^δ) are of the form $g_! q^!(\mathbb{1}_B)\{n\}$ (resp. $M^{BM}(X/S)\langle\delta(X)\rangle\{n\}$) for an integer $n \in \mathbf{Z}$, a B-scheme X and a B-morphism g satisfying the assumptions of the preceding proposition. Then, by assumption, the virtual tangent bundle of $q : X \to B$ is isomorphic to the trivial vector B-bundle of rank d, where d is the relative dimension of q. Thus, according to the absolute purity property on $\mathscr{T}$ and Corollary 1.3.22, one gets an isomorphism:
$$q^!(\mathbb{1}_B) \simeq \mathbb{1}_S\langle d\rangle$$
and this concludes because $d = \delta(X)$ according to Proposition 1.1.11(3). $\qquad\square$

We end-up this section with corollaries of Proposition 2.3.1 about δ-effective spectra.

Corollary 2.3.11. *Let k be a perfect field and $\delta = \dim$ be the Krull dimension function on $\operatorname{Spec}(k)$.*

(1) *The category $\mathscr{T}^{\delta-\mathrm{eff}}(k)$ is the localizing triangulated subcategory of $\mathscr{T}(k)$ generated by spectra of the form $M(X)$ for a smooth k-scheme X.*
(2) *The δ-homotopy t-structure on $\mathscr{T}^{\delta-\mathrm{eff}}(k)$ is generated by spectra of the form $M(X)$ for a smooth k-scheme X.*

Proof. According to point (3) of Proposition 2.3.1, the homotopy t-structure on $\mathscr{T}^{\delta-\mathrm{eff}}(k)$ is generated, for a smooth k-scheme X by spectra of the form $M^{BM}(X/k)\langle\delta(X)\rangle\{n\}$ for a smooth k-scheme X (because k is perfect) and an integer $n \geq 0$. According to Computation (2.3.4.a), this spectrum is isomorphic to $M(X)\{n\}$. On the other hand, the later one is a direct factor of $M(X \times \mathbf{G}_m^n)$; so point (2) of the above statement follows.

By definition and according to Lemma 1.2.8, the homotopy t-structure on $\mathscr{T}^{\delta-\mathrm{eff}}(k)$ is left non-degenerate. Applying again Lemma 1.2.8, we obtain that the triangulated category $\mathscr{T}^{\delta-\mathrm{eff}}(k)$ is equal to its smallest triangulated subcategory stable by coproducts and containing objects of the form $M(X)$ for a smooth k-scheme X. This implies point (1) above. $\qquad\square$

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[21]Recall that with our assumptions, the two properties constructible and compact coincide for $\mathscr{T}$-spectra.

[22]In *loc. cit.* one assume further that S is a quasi-projective B-schemes but this restriction only appears because the functors $f_!$ and $f^!$ were only defined for quasi-projective morphisms f.

Example 2.3.12. (1) Let F be a prime field with characteristic exponent p and R be a ring such that $p \in R^\times$. Then for any perfect field k of characteristic exponent p, the equivalence (1.3.1.a), the previous corollary and the cancellation theorem of Voeovdsky (cf. [Voe10]) show that there exists a canonical equivalence of triangulated categories:

(2.3.12.a) $$\mathrm{DM}^{eff}(k, R) \to \mathrm{DM}^{\delta - eff}_{\mathrm{cdh}}(k, R)$$

where the left hand side is the triangulated category of Voevodsky's motivic complexes with coefficients in R (see [VSF00, chap. 5], [CD09]). Moreover, the corollary also shows that this functor is an equivalence of t-categories between Voevodsky's homotopy t-structure (see [VSF00, chap. 5] or [Dég11, Cor. 5.2]) and the δ-homotopy t-structure.

Note finally that through these equivalences of categories, the adjunction of triangulated categories
$$s : \mathrm{DM}^{\delta-eff}_{\mathrm{cdh}}(k, R) \leftrightarrows \mathrm{DM}_{\mathrm{cdh}}(k, R) : w$$
corresponds to the adjunction:
$$\Sigma^\infty : \mathrm{DM}^{eff}(k, R) \leftrightarrows \mathrm{DM}(k, R) : \Omega^\infty$$
of [CD09, Ex. 7.15] using again Voevodsky's cancellation theorem.

(2) In the case of the stable homotopy category $\mathscr{T} = \mathrm{SH}$, for any perfect field k, the previous corollary shows that the δ-effective category $\mathrm{SH}^{\delta-eff}(k)$ is equivalent to the essential image of the canonical functor going from S^1-spectra to $\mathbf{P}^1$-spectra.

2.3.13. We are now in position to justify computation (2.2.8.a).

Recall the assumptions of this formula: (S, δ) is an arbitrary dimensional scheme, $\mathscr{T} = \mathrm{DM}_R$ as in 1.3.2(1) or (2) and X is a smooth projective S-scheme of pure dimension d. According to 1.3.8(BM4), we get:
$$M_S(X) = M^{BM}(X/S)(d)[2d]$$
In particular, $M_S(X)(n)$ is by definition δ-effective if $n \geq \delta(X) - d$. We prove $w(M_S(X)(n))$ is zero when $n < \delta(S) - d$. This will use the following vanishing of motivic cohomology, true under our assumptions: for any regular scheme X, any couple of integers (n, m),

(2.3.13.a) $$H^{n,m}(X, R) = 0 \text{ if } m < 0.$$

According to point (3) of Proposition 2.3.1, we only need to check that for any regular quasi-projective S-scheme Y, the group
$$\mathrm{Hom}_{\mathrm{DM}(S,R)} \left(M^{BM}(Y/S)(m)[i], M_S(X)(n) \right)$$
vanishes if $m \geq \delta(Y)$, $n < \delta(S) - d$ and i is any integer. A straightforward computation, using the absolute purity property of DM_R, gives:
$$\mathrm{Hom}_{\mathrm{DM}(S,R)} \left(M^{BM}(Y/S)(m)[i], M_S(X)(n) \right) = H^{2d'-i, d'+n-m}(X \times_S Y, R)$$
where d' is the relative dimension of the quasi-projective S-scheme $X \times_S Y$, which is lci because $X \times_S Y$ and S are regular under our assumptions. Thus the required vanishing follows from (2.3.13.a), the assumptions on n and m and the fact:
$$d' = d + \delta(Y) - \delta(S).$$

Remark 2.3.14. In our knowledge of motivic homotopy theory, the vanishing (2.3.13.a) seems to be very specific to mixed motives. Indeed, it is false for cobordism, homotopy invariant K-theory, stable homotopy groups of spheres, torsion or ℓ-adic étale cohomology.

2.4. **The sharpest description of generators.**

2.4.1. The next description of generators for the δ-homotopy t-structure combines the advantages of generators in points (1) and (2) of the previous proposition. But we need the following assumption on our motivic category:

(Resol) One of the following assumptions on $\mathscr{T}$ and $\mathscr{S}$ is verified:

(i) Each integral scheme X which is essentially of finite type over a scheme in $\mathscr{S}$ admits a desingularization, i.e., there exists a proper birational morphism $X' \to X$ such that X' is regular.

(ii) $\mathscr{S}$ is made of essentially of finite type S_0-schemes, where S_0 is a noetherian excellent scheme with $\dim(S_0) \leq 3$, and $\mathscr{T}$ is the homotopy category associated with a premotivic model category which is moreover $\mathbf{Q}$-linear and separated (cf. [CD12, 2.1.7]).

(iii) $\mathscr{S}$ is made of k-schemes essentially of finite type where k is a perfect field of characteristic exponent p, $\mathscr{T}$ is $\mathbf{Z}[1/p]$-linear and there exists a premotivic adjunction:

$$\varphi^* : \mathrm{SH} \leftrightarrows \mathscr{T}.$$

Remark 2.4.2. The question whether any integral quasi-excellent scheme X admits a desingularization as in (Resol)(i) was raised in [EGA4]. It has been proved by Temkin in [Tem08] for all integral quasi-excellent $\mathbf{Q}$-schemes so that our assumption (Resol)(i) holds in our actual knowledge when $\mathscr{S}$ is a category made of (excellent) $\mathbf{Q}$-schemes.

Theorem 2.4.3. *Let (S, δ) be a dimensional scheme. Put $I = \mathbf{Z}$ (resp. $I = \mathbf{N}$). Assume that condition (Resol) holds.*

Then the homotopy t-structure on $\mathscr{T}(S)$ (resp. $\mathscr{T}^{\delta-eff}(S)$) coincides with the t-structure generated by objects of the form $M^{BM}(X/S)\langle\delta(X)\rangle\{n\}$, where X/S is proper, X is regular and $n \in I$. Moreover, when (Resol)(ii) or (Resol)(iii) holds, one can restrict to schemes X/S which are projective.

Proof. To simplify the notation of this proof, we denote by $\mathscr{T}$ the triangulated category $\mathscr{T}(S)$ (resp. $\mathscr{T}^{\delta-eff}(S)$) and we put $t = t_\delta$. We also denote by t' the t-structure generated (Def. 1.2.7) by the family $\mathcal{G}$ made of the compact objects of $\mathscr{T}$ of the form $M^{BM}(X/S)\langle\delta(X)\rangle\{n\}$ where X/S is proper in case (Resol)(i), projective in case (Resol)(ii) and (Resol)(iii), X is regular and $n \in I$. In this notation we only have to prove that

$$(2.4.3.a) \qquad\qquad\qquad \mathscr{T}_{t\geq 0} \subset \mathscr{T}_{t'\geq 0}$$

as the reverse inclusion is clear. Recall from Definition 1.2.7 that $\mathscr{T}_{t'\geq 0}$ is the smallest triangulated subcategory of $\mathscr{T}$ which contains $\mathcal{G}$ and is stable by direct summands and coproducts.

We begin this proof by a preliminary reduction that occurs only for assumption (Resol)(iii) and which uses the localization techniques of [CD15a, Appendix B]. Given any prime l, we let $\mathscr{T}_{(l)}$ be the Verdier quotient of $\mathscr{T}$ by the thick triangulated subcategory generated by the cones of maps of the form $r.1_K : K \to K$ for any integer r not divisible by l and any $\mathscr{T}$-spectrum K.

Lemma 2.4.4. *Consider the preceding notations.*

(1) *The family of projection functors $\pi_l : \mathscr{T} \to \mathscr{T}_{(l)}$, indexed by prime integers l different from p, is conservative.*

(2) *There exists a unique t-structure t_l (resp. t'_l) on $\mathscr{T}_{(l)}$ such that the functor $\pi_l : (\mathscr{T}, t) \to (\mathscr{T}_{(l)}, t_l)$ (resp. $\pi_l : (\mathscr{T}, t') \to (\mathscr{T}_{(l)}, t'_l)$) is t-exact.*

Points (1) and (2) of this lemma were proved in *loc. cit.*, B.1.7 and B.2.2 respectively. Note that (2.4.3.a) is equivalent to the assumption that for any $\mathscr{T}$-spectrum K in $\mathscr{T}_{t\geq 0}$, the $\mathscr{T}$-spectrum $\tau^{t'}_{<0}(K)$ is trivial. Thus it is sufficient to prove (2.4.3.a) after localizing at a prime $l \neq p$. This means that we can assume that $\mathscr{T}$ is $\mathbf{Z}_{(l)}$-linear for a prime $l \neq p$.

Let us go back to the main part of the proof. In any case, it is sufficient to prove that for any separated S-scheme X, the following property holds:

$$(P(X)) \qquad\qquad \forall n \in I,\ M^{BM}(X/S)\langle\delta(X)\rangle\{n\} \in \mathscr{T}_{t'\geq 0}$$

We will show this by induction on the integer $d = \dim(X)$. The case $d = 0$ is obvious. Thus we can assume that $(P(X'))$ holds for any scheme X' of dimension less than d. We will use the following lemma:

Lemma 2.4.5. *Consider the inductive assumption on d as above.*

(1) *For any dense open subscheme $U \subset X$, the property $(P(X))$ is equivalent to the property $(P(U))$.*

(2) *Let $\phi : X' \to X$ be a proper morphism such that any irreducible component of X' dominates an irreducible component of X. Given any open subscheme $U \subset X$, we denote by $\phi_U : \phi^{-1}(U) \to U$ the pullback of ϕ to U. It is proper, and induces the following morphism – 1.3.8(BM1) – of Borel-Moore $\mathscr{T}$-spectra over U:*

$$\phi_U^* : \mathbb{1}_U = M^{BM}(U/U) \to M^{BM}(V/U).$$

We assume there exists a dense open subscheme $U \subset X$ such that ϕ_U^* is a split monomorphism.

   Then property $(P(X'))$ implies property $(P(X))$.

Let us prove point (1) of this lemma. We denote by Z the reduced subscheme of X complementary to U. In $\mathscr{T}$, we get from 1.3.8(BM3) the following distinguished triangle:

$$M^{BM}(Z/S)\langle\delta(X)\rangle[-1] \to M^{BM}(X/S)\langle\delta(X)\rangle \to M^{BM}(U/S)\langle\delta(X)\rangle \to M^{BM}(Z/S)\langle\delta(X)\rangle.$$

Since U is dense in X, one gets: $\delta(X) = \delta(U)$ and $\delta(X) > \delta(Z)$, $\dim(Z) < \dim(X)$. Thus, by the inductive assumption, for any integer $n \in I$, the $\mathscr{T}$-spectrum

$$M^{BM}(Z/S)\langle\delta(X)\rangle\{n\}[-1] = M^{BM}(Z/S)\langle\delta(Z)\rangle\{\delta(X) - \delta(Z) + n\}[\delta(X) - \delta(Z) - 1]$$

belongs to $\mathscr{T}_{t'\geq 0}$. Thus point (1) of the lemma follows.

Then point (2) follows easily from point (1). Indeed, in the assumptions of point (2), we know from point (1) that $(P(X))$ is equivalent to $(P(U))$. Moreover, the open subscheme $V = \phi^{-1}(U)$ of X' is dense. So we also know that $(P(X'))$ is equivalent to $(P(V))$. If we apply $\phi_{U!}$ to the morphism ϕ_U^* in $\mathscr{T}(U)$, we get a split monomorphism $M^{BM}(U/S) \to M^{BM}(V/S)$. Thus, since $\mathscr{T}_{t'\geq 0}$ is stable by direct factors, we obtain that $(P(V))$ implies $(P(X))$ and this concludes.

Let us go back to the proof of property $(P(X))$ by induction on the dimension d of X. Since $M^{BM}(X/S) = M^{BM}(X_{red}/S)$, we can assume that X is reduced. Let $(X_\lambda)_\lambda$ be the irreducible components of X. The closed subset $Z = \cup_{\lambda\neq\mu}X_\lambda \cap X_\mu$ in X is rare. Applying point (1) of the preceding lemma, we can replace X by $X - Z$. So we can assume that X is integral. Because X/S is separated of finite type, it admits an open embedding into a proper S-scheme $\bar{X}$ that we can assume to be integral as X is integral. According to the preceding lemma, we can replace X by $\bar{X}$. In other words, we can assume X is proper over S and integral.

Let us treat the case of assumption (Resol)(i). Applying this assumption to X, we get a proper birational map $X' \to X$ such that X' is regular. Because X'/S is also proper, property $(P(X'))$ is obviously true – by definition of t'. Thus point (2) of the previous lemma shows that $(P(X))$ is true as required.

Consider the cases (Resol)(ii) and (iii). According to Chow lemma as stated in [EGA2, 5.6.1], there exists a birational map $X' \to X$ over S such that X'/S is projective. We deduce from point (2) of the preceding lemma that we can assume that X/S is projective.

We now treat the case of assumption (Resol)(ii). Under this assumption, it follows from [Tem15, 1.2.4] that there exists a projective alteration[23] $\phi : X' \to X$ such that X' is regular connected. In particular, property $(P(X'))$ is obviously true. Moreover, there exists a dense open subscheme $U \subset X$ such that U is regular and π is finite on $V = \phi^{-1}(U)$. Then, according to [CD12, 3.3.40] applied to the motivic category $\mathscr{T}$ and the finite morphism $\phi_U : V \to U$, the adjunction map $\mathbb{1}_U \to \phi_{U*}\phi_U^*(\mathbb{1}_U)$ is a split monomorphism in $\mathscr{T}(U)$. So we can apply point (2) of the preceding lemma, and we deduce property $(P(X))$.

Let us finally treat the case of assumption (Resol)(iii). Using the reduction done in the beginning of the proof, we fix a prime $l \neq p$, and prove the assertion in $\mathscr{T}_{(l)}$.[24] According to [ILO14, X, 3.5], there exists a projective generically étale alteration $\phi : X' \to X$ of generic degree d prime to l such that X' is regular. In particular, X' is projective over S and regular so property $(P(X'))$

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[23]Recall that an *alteration* is a proper surjective generically finite morphism.

[24]Note indeed that the preceding lemma can equally be applied to $\mathscr{T}_{(l)}$.

holds. Let $\phi' : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$ be the morphism induced by ϕ on generic points. Because ϕ' is finite étale, it induces a morphism in $\mathrm{SH}(K)$:

$$\phi'^* : \mathbb{1}_K \xrightarrow{ad} \phi'_* \phi'^*(\mathbb{1}_K) = \phi'_!(\mathbb{1}_L).$$

According to [LYZ13, Lemmas B.3, B.4], there exists a morphism $s : \phi'_!(\mathbb{1}_L) \to \mathbb{1}_K$ such that

$$\phi'^* \circ s = d.Id + \alpha$$

where α is a nilpotent endomorphism of $\mathbb{1}_K$ in $\mathrm{SH}(K)$. In particular, the preceding relation means that $\phi'^* \circ \frac{s}{d}$ is an isomorphism in $\mathrm{SH}(K)_{(l)}$.

By the continuity property of SH ([CD12, 4.3.2]), there exists a dense open subscheme $U \subset X$, $V = \phi^{-1}(U)$, such that s can be lifted to a map $s_U : \phi'_!(\mathbb{1}_V) \to \mathbb{1}_U$. Further, we can assume by shrinking U that $\phi_U^* \circ \frac{s_U}{d}$ is an isomorphism in $\mathrm{SH}(U)_{(l)}$.

Because $\varphi^* : \mathrm{SH} \to \mathscr{T}$ is a premotivic adjunction between motivic categories, it commutes with $f_!$ (see [CD12, 2.4.53]). Thus, we obtain that $\varphi^*\left(\frac{s_U}{d}\right)$ is a splitting of ϕ_U^* in $\mathscr{T}(U)_{(l)}$. Thus point (2) of the preceding lemma allows to conclude that $(P(X))$ holds. $\square$

Remark 2.4.6. The main application of the previous theorem will be given in the next section. Here we note that our result appears to be non-trivial already when $S = \mathrm{Spec}(k)$ (for k being a perfect field of positive characteristic p). This case of the statement (along with the easier $p = 0$ case) has found quite interesting applications in [BS15], where it was used to relate the motivic homology theory (for motives) to the so-called Chow-weight homology ones.

Now we use our theorem to deduce certain results which are of independent interest. The following corollary improves results of [Ayo07] and [CD12].

Corollary 2.4.7. *Suppose $\mathscr{T}$ satisfies assumption (Resol). Then for any scheme S, the triangulated category $\mathscr{T}_c(S)$ (resp. $\mathscr{T}(S)$) is generated[25] by spectra (resp. arbitrary coproducts of spectra) of the form $f_*(\mathbb{1}_X)(n)$ where $f : X \to S$ is a proper morphism (resp. projective under (Resol)(ii) or Resol(iii))), X a regular scheme and $n \in \mathbf{Z}$ an integer.*

This follows from the preceding proposition, Lemma 1.2.8 and Remark 2.1.2.

Here is an application of the previous corollary, which improves significantly the theory of $\mathbf{Z}[1/p]$-linear triangulated motivic categories over the category of $\mathbb{F}_p$-schemes on the model of [Ayo07] or [CD12, §4].

Theorem 2.4.8. *Suppose $\mathscr{T}$ satisfies assumption (Resol) and the absolute purity property (Definition 1.3.17).[26] Then the following properties hold:*

(1) *The category of constructible $\mathscr{T}$-spectra is stable by the six functors if one restricts to morphisms f of finite type for the functor f_*.*

(2) *For any regular scheme S and any separated morphism $f : X \to S$, the $\mathscr{T}$-spectrum $D_X = f^!(\mathbb{1}_S)$, which is constructible according to the first point, is dualizing for constructible $\mathscr{T}$-spectra over X: for any such $\mathscr{T}$-spectrum M, the natural map*

(2.4.8.a) $$M \to \underline{\mathrm{Hom}}\left(\underline{\mathrm{Hom}}(M, D_X), D_X\right)$$

is an isomorphism.

Proof. The proof of the first point follows from the following ingredients:

- Gabber's refinement of De Jong alteration theorem (cf. [ILO14, X, 3.5]).
- The argument of Riou for the existence of trace maps for finite étale morphisms in SH satisfying the degree formula (cf. [LYZ13, Lemmas B.3, B.4]).

Then the proof of Gabber (in the case of the derived category of prime to p torsion étale sheaves) work through as explained in [CD15a, proof of Th. 6.4]. The only supplementary observation is that one needs only good trace maps for finite étale morphisms as in point (2) above.

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[25]See Conventions page 9 for the definition of "generated" in the triangulated context.

[26]Recall from Example 1.3.20 that absolute purity in our case follows from the fact $\mathscr{T}$-cohomology commutes with projective limits (Paragraph 1.3.19).

The second assertion follows essentially from the proof of [CD15b, 7.3]: using the previous corollary, it is sufficient to check that the map (2.4.8.a) is an isomorphism when $M = p_!(\mathbb{1}_Y)$ where $p : Y \to X$ is a projective morphism and Y is regular. In this case, one can compute the right hand side as:

$$p_! \underline{\mathrm{Hom}} \left(\mathbb{1}_Y, \underline{\mathrm{Hom}}(\mathbb{1}_Y, (fp)^!(\mathbb{1}_S)) \right).$$

so that, replacing X by Y, we can assume that X is regular separated over S and $M = \mathbb{1}_X$. The assertion is then Zariski local in X so that we can assume in addition that X/S is quasi-projective. Because $\mathscr{T}$ satisfies absolute purity by assumption, we can apply Corollary 1.3.22 which implies that $D_X = f^!(\mathbb{1}_S) \simeq \mathrm{Th}(\tau_f)$ where τ_f is the virtual tangent bundle of f. Because the $\mathscr{T}$-spectrum $\mathrm{Th}(\tau_f)$ is $\otimes$-invertible, the result follows trivially. $\square$

Example 2.4.9. The new application of the preceding proposition is given by the case of the stable homotopy category SH$[1/p]$ restricted to the category of $\mathbb{F}_p$-schemes for a prime $p > 0$.

Thus constructible spectra with $\mathbf{Z}[1/p]$-coefficients are stable by the six operations if one restricts to excellent $\mathbb{F}_p$-schemes. Moreover, Grothendieck duality holds for these spectra. Note for example that given any field k of characteristic p, and any quasi-projective regular k-scheme X with virtual tangent bundle τ_X, the Thom space $\mathrm{Th}(\tau_X)$ is naturally a dualizing object for constructible $\mathbf{Z}[1/p]$-linear spectra over X.

Another interesting application of Corollary 2.4.7 is the following result that extends previous theorems due to Cisinski and the first named author ([CD12, 4.4.25] and [CD15a, 9.5]).

Proposition 2.4.10. *Consider a premotivic adjunction*

$$\varphi^* : \mathscr{T} \leftrightarrows \mathscr{T}'$$

of triangulated motivic categories over $\mathscr{S}$ which satisfies, the following assumptions:
  (a) *$\mathscr{S}$ is the category of F-schemes for a prime field F of characteristic exponent p.*
  (b) *$\mathscr{T}$ and $\mathscr{T}'$ are $\mathbf{Z}[1/p]$-linear, continuous (see Paragraph 1.3.19) and are the aim of a premotivic adjunction whose source is SH.*

Then for any morphism $f : X \to S$ in $\mathscr{S}$, for any $\mathscr{T}$-spectrum $\mathbb{E}$ over X, the canonical exchange transformation:

(2.4.10.a) $$\varphi^* f_*(\mathbb{E}) \to f_* \varphi^*(\mathbb{E})$$

is an isomorphism.

Proof. The proof follows that of [CD12, 4.4.25] with some improvements.

We start by treating the case where $S = \mathrm{Spec}(F)$ is the spectrum of the prime field F.

We first show that one can assume f is of finite type. Note that the functor f_* commutes with arbitrary coproducts – this follows formally from the fact its right adjoint preserves the generators which are assumed to be compact (see [CD12, 1.3.20] for more details). In particular, it is sufficient to prove that (2.4.10.a) is an isomorphism when $\mathbb{E}$ is constructible. But X can be written as a projective limit of a projective system of F-schemes of finite type

$$X_i \xrightarrow{f_i} \mathrm{Spec}(F), i \in I$$

with transition maps f_{ij}. Let us denote by $p_i : X \to X_i$ the canonical projection. Moreover, since $\mathscr{T}$ is continuous, we get an equivalence of triangulated categories ([CD12, 4.3.4]):

(2.4.10.b) $$2 - \varinjlim_{i \in I} \mathscr{T}_c(X_i) \to \mathscr{T}_c(X).$$

Thus there exists a family $(\mathbb{E}_i)_{i \in I}$ such that for all $i \in I$, $\mathbb{E}_i$ is a constructible $\mathscr{T}$-spectrum over X_i with a given identification $f_i^*(\mathbb{E}_i) \simeq \mathbb{E}$ for all index $i \in I$, and for all $i \le j$, there is a map $f_{ij}^*(\mathbb{E}_i) \to \mathbb{E}_j$ compatible with the previous identification. In particular, for all index $i \in I$, we get a canonical map:

$$f_{i*}(\mathbb{E}_i) \to f_{i*} p_{i*} p_i^*(\mathbb{E}_i) \simeq f_*(\mathbb{E}).$$

These maps are compatible with the transition map by the following morphism:

$$f_{i*}(\mathbb{E}_i) \to f_{i*} f_{ij*} f_{ij}^*(\mathbb{E}_i) \to f_{j*}(\mathbb{E}_j).$$

So we get a canonical morphism of spectra over F:

$$\psi : \operatorname{hocolim}_{i \in I} \left(f_{i*}(\mathbb{E}_i) \right) \to f_*(\mathbb{E}).$$

A direct consequence of the continuity property for $\mathscr{T}$ is the following lemma:

Lemma 2.4.11. *Under the preceding assumption, the morphism ψ is an isomorphism.*

Indeed, it is sufficient to check ψ is an isomorphism after applying the functor $\operatorname{Hom}_{\mathscr{T}(F)}(K, -)$ for any constructible $\mathscr{T}$-spectrum. Then it follows from the equivalence (2.4.10.b) and from the fact K is compact.

In particular, if we know the result for morphisms of finite type with target $\operatorname{Spec}(F)$, we can conclude for the morphism f:

$$\varphi^* f_*(\mathbb{E}) \overset{(1)}{\simeq} \varphi^* \operatorname{hocolim}_{i \in I} \left(f_{i*}(\mathbb{E}_i) \right) \overset{(2)}{\simeq} \operatorname{hocolim}_{i \in I} \left(\varphi^* f_{i*}(\mathbb{E}_i) \right)$$

$$\overset{(3)}{\simeq} \operatorname{hocolim}_{i \in I} \left(f_{i*}\left(\varphi^*(\mathbb{E}_i) \right) \right) \overset{(4)}{\simeq} f_* \varphi^*(\mathbb{E}_i)$$

where (1) (resp. (4)) follows from the preceding lemma (the obvious analog of the preceding lemma for $\mathscr{T}'$), (2) is valid because φ^* is a left adjoint thus commutes to homotopy colimits, and (3) is true by assumption.

So let us assume now that $f : X \to \operatorname{Spec}(F)$ is of finite type. According to Corollary 2.4.7, the category $\mathscr{T}(X)$ is generated by arbitrary coproducts of $\mathscr{T}$-spectra of the form $p_*(\mathbb{1}_Y)(n)$ for $p : Y \to X$ proper, Y regular and n any integer. Since f_* commutes with coproducts as we have already seen, it is sufficient to show (2.4.10.a) is an isomorphism when $\mathbb{E}$ is one of these particular $\mathscr{T}$-spectrum. Thus, because φ^* commutes with $p_* = p_!$ for p proper (see [CD12, 2.4.53]) and with twists, we only need to prove that (2.4.10.a) is an isomorphism when $\mathbb{E} = \mathbb{1}_X$ for X a regular F-scheme of finite type. In particular, $f : X \to \operatorname{Spec}(F)$ is smooth, because F is perfect. From the six functors formalism, it now formally follows that the spectrum $f_*(\mathbb{1}_X)$ is rigid, in both the monoidal categories $\mathscr{T}(k)$ and in $\mathscr{T}'(k)$, with strong dual $f_\sharp(\operatorname{Th}(-\tau_f))$ where τ_f is the tangent bundle of f (see for example [CD12, 2.4.31]). To conclude, it is sufficient now to recall that φ^* is monoidal, so it respects strong duals, and it commutes with $f_\sharp$. This concludes the proof in the case where S is the spectrum of a field.

The general case now formally follows, as in the proof of [CD12, 4.4.25]. $\qquad\square$

Remark 2.4.12. The result obtained here is stronger than its equivalent in [CD12] or [CD15a], where we had to assume f is of finite type, $\mathbb{E}$ is constructible. This is because we have used the continuity property as well as the compactly generated assumption on triangulated motivic categories. Note in particular that we do not need any semi-separatedness assumption on the triangulated motivic categories $\mathscr{T}$ and $\mathscr{T}'$.

Corollary 2.4.13. *Under the assumption of the previous propostion, the functor φ^* commutes with the 6 functors.*

This follows formally from the previous proposition (see the proof of [CD12, 4.4.25] for details).

Example 2.4.14. The corollary applies to all motivic adjunctions of the diagram (1.3.3.a) provided we restrict to equal characteristics schemes and we invert their characteristic exponent.

3. Local description of homology sheaves

3.1. **The δ-niveau spectral sequence.** Below, we extend the considerations of [BO74, §3], to the framework of motivic categories. Actually, the new point is to work in the absolute case (without a base field) and to use arbitrary dimension functions.

Definition 3.1.1. Let (X, δ) be a dimensional scheme. A δ-*flag* on X is a sequence $Z_* = (Z_p)_{p \in \mathbf{N}}$ of reduced closed subschemes of X such that $Z_p \subset Z_{p+1}$ and $\delta(Z_p) \le p$.

We will denote by $\mathbb{F}_\delta(X)$ the set of δ-flags ordered by term-wise inclusion.

Note that the ordered set $\mathbb{F}_\delta(X)$ is non-empty and cofiltered (*i.e.* two elements admit a lower bound).

3.1.2. Let us recall the framework of exact couples to fix notations (following [Dég12, §1]). An *exact couple* in an abelian (resp. triangulated, pro-triangulated[27]) category $\mathscr{C}$ *with homological conventions* is the data of bigraded objects D and E of $\mathscr{C}$ and homogeneous morphisms in $\mathscr{C}$ as pictured below:

(3.1.2.a)

$$
\begin{array}{ccc}
D & \xrightarrow[\alpha]{(1,-1)} & D \\
& \gamma \diagdown \qquad \diagup \beta & \\
(-1,0) & & (0,0) \\
& E &
\end{array}
$$

where the degrees of each morphism is indicated, and such that the above diagram is a (long) exact sequence (resp. a triangle, a pro-triangle) in each degree.

By contrast, an *exact couple with cohomological conventions* is defined by inverting the arrows in the above diagram and taking the opposite degrees.

Consider now a dimensional scheme (S, δ) and a separated S-scheme X. Denote again by δ the dimension function on X induced by that on S. For any δ-flag Z_* on X and any integer $p \in \mathbf{Z}$, we get according to Paragraph 1.3.8(BM3), a distinguished triangle:

$$
M^{BM}(Z_p - Z_{p-1}/S) \to M^{BM}(Z_p/S) \to M^{BM}(Z_{p-1}/S) \to M^{BM}(Z_p - Z_{p-1}/S)[1]
$$

in $\mathscr{T}(S)$. Moreover, these triangles are contravariantly functorial with respect to the order on δ-flags. Therefore, taking the (pseudo-)projective limit in the pro-triangulated category $\mathrm{pro} - \mathscr{T}(X)$, we get a distinguished pro-triangle:

(3.1.2.b) $\qquad G_p M^{BM}(X/S) \xrightarrow{j_{p*}} F_p M^{BM}(X/S) \xrightarrow{i_p^*} F_{p-1} M^{BM}(X/S) \xrightarrow{\partial_p} G_p M^{BM}(X/S)[1]$

where we have put:

$$
F_p M^{BM}(X/S) = \text{"}\varprojlim_{Z_*}\text{"} M^{BM}(Z_p/S), \qquad G_p M^{BM}(X/S) = \text{"}\varprojlim_{Z_*}\text{"} M^{BM}(Z_p - Z_{p-1}/S).
$$

Definition 3.1.3. Consider the above assumptions and notations.

We define the $\mathscr{T}$*-motivic* δ*-niveau exact couple associated with* X/S as the following pro-exact couple with cohomological conventions in $\mathrm{pro} - \mathscr{T}(S)$:

$$
D^{p,q} = F_p M^{BM}(X/S)[p+q], \qquad E^{p,q} = G_p M^{BM}(X/S)[p+q],
$$

and with morphisms in degree (p, q) induced by the $(p+q)$-suspension of the exact triangle (3.1.2.b).

3.1.4. Consider the notations of the above definition. Assume we are given a Grothendieck abelian category $\mathscr{A}$ and an exact functor[28]:

$$
\mathcal{H} : \mathscr{T}(S)^{op} \to \mathscr{A}.
$$

Then one can extend this functor to a pro-$\mathscr{T}$-spectrum $M_\bullet = (M_i)_{i \in I}$ over S by the formula:

$$
\hat{\mathcal{H}}(M_\bullet) = \varinjlim_{i \in I} \big(\mathcal{H}(M_i) \big)
$$

where the transition morphisms of the colimit are induced by the transition morphisms of the pro-object $P_\bullet$. Since filtered colimits are exact in $\mathscr{A}$, the functor $\hat{\mathcal{H}}$ sends distinguished pro-triangle to long exact sequences. In particular, if we apply $\hat{\mathcal{H}}$ to the pro-exact couple of the previous definition, we get an exact couple in the abelian category $\mathscr{A}$ with homological conventions, and therefore a homological spectral sequence:

(3.1.4.a) $\qquad E^1_{p,q} = \hat{\mathcal{H}}_{p+q}\big(F_p M^{BM}(X/S) \big) \Rightarrow \mathcal{H}_{p+q}\big(M^{BM}(X/S) \big).$

Note that because the dimension function δ is necessarily bounded, this spectral sequence always converges. This general definition will be especially useful in the particular case of the following definition.

---

[27]*i.e.* the category of pro-objects of a triangulated category;

[28]*i.e.* a functor sending distinguished triangles to long exact sequences

Definition 3.1.5. Consider as above a dimensional scheme (S, δ) and a separated S-scheme X. Let $\mathbb{E}$ be a $\mathscr{T}$-spectrum. We define the *δ-niveau spectral sequence of X/S with coefficients in $\mathbb{E}$* as the spectral sequence (3.1.4.a) associated with the functor $\mathcal{H} = \mathrm{Hom}(-, \mathbb{E})$.

It is useful to consider the **Z**-graded version of this spectral sequence, replacing $\mathbb{E}$ by $\mathbb{E}(-n)$ for any integer $n \in \mathbf{Z}$. Then as in [BO74, 3.7], it is easy to check it admits the following form (up to a canonical isomorphism):

$$(3.1.5.a) \qquad {}^{\delta}E^1_{p,q} = \bigoplus_{x \in X_{(p)}} \hat{\mathbb{E}}^{BM}_{p+q,n}(x/S) \Rightarrow \mathbb{E}^{BM}_{p+q,n}(X/S)$$

where $X_{(p)} = \{x \in X \mid \delta(x) = p\}$ and for any point $x \in X$, with reduced closure $Z(x)$ in X, we put:

$$(3.1.5.b) \qquad \hat{\mathbb{E}}^{BM}_{p+q,n}(x/S) := \varinjlim_{U \subset Z(x)} \mathbb{E}^{BM}_{p+q,n}(U/S)$$

where U runs over the ordered set of non-empty open subschemes of $Z(x)$. Moreover, using the functoriality properties of Borel-Moore spectra (cf. 1.3.8, (BM1) and (BM2)), one easily checks this spectral sequence is covariant with respect to proper morphisms and contravariant with respect to étale morphisms.[29]

Example 3.1.6. Let S be a regular scheme with dimension function $\delta = -\mathrm{codim}_S$ and X be a separated S-morphism.

We fix a ring of coefficients R flat over **Z** and consider the R-linear motivic category DM_R following the conventions of Paragraph 1.3.2, point (1) or (2).

Suppose $\mathbb{E} = \mathbb{1}_X$ is the constant motive over X in $\mathrm{DM}(X, R)$. Then the δ-niveau spectral sequence has the following form:

$$ {}^{\delta}E^1_{p,q} = \bigoplus_{x \in X_{(p)}} \hat{H}^{BM}_{p+q,n}(x/S, R) \Rightarrow H^{BM}_{p+q,n}(X/S, R)$$

where H^{BM}_{**} is R-linear motivic Borel Moore homology.

Suppose x is point of X such that $\delta(x) = p$. Let s be its projection to S. Note that $\mathrm{Spec}(\kappa(x)) \to S$ is a limit of lci morphisms whose relative dimension is:

$$(3.1.6.a) \qquad \mathrm{degtr}(\kappa(x)/\kappa(s)) - \mathrm{codim}_X(s) = \delta(x) = p.$$

Moreover, because S is regular, $\mathscr{T}$ satisfies absolute purity, and motivic cohomology commutes with projective limits (see §1.3.19), we deduce from Corollary 1.3.22 the following computation:

$$\hat{H}^{BM}_{p+q,n}(x/S, R) = H^{p-q,p-n}_M(\kappa(x), R)$$

where H^{**}_M denotes R-linear motivic cohomology. Note that in each cases considered above, one gets:

$$H^{p-q,p-n}_M(\kappa(x), R) = \begin{cases} 0 & \text{if } (p-q > p-n) \text{ or } (p-n < 0), \\ K^M_{p-q}(\kappa(x)) \otimes_{\mathbf{Z}} R & \text{if } q = n, \end{cases}$$

where K^M_* denotes Milnor K-theory.

In particular, the preceding spectral sequence is concentrated in the region $p \geq n$ and $q \geq n$. Moreover, as in [Dég12, 2.7], one can identify the differential

$$(3.1.6.b) \qquad {}^{\delta}E^1_{n,n} = Z_{\delta=n}(X) \otimes R \xleftarrow{\quad d^1_{n+1,n} \quad} \bigoplus_{x \in X_{(n+1)}} \left(\kappa(x)^{\times} \right) \otimes_{\mathbf{Z}} R = {}^{\delta} E^1_{n+1,n}$$

with the classical divisor class map, where $Z_{\delta=n}(X)$ is the abelian group of algebraic cycles in X of δ-dimension n (*i.e.* the free abelian group generated by $X_{(n)}$).

Thus we have proved the following proposition:

---

[29]The opposite functoriality holds for the $\mathscr{T}$-motivic δ-niveau exact couple.

Proposition 3.1.7. *Let S be a regular scheme with dimension function $\delta = -\operatorname{codim}_S$, X be a separated S-scheme, and n be an integer. Let R be a ring of coefficients.*

Assume that one of the following conditions holds:

(a) *$R = \mathbf{Q}$,*

(b) *S is a $\mathbf{Q}$-scheme and $R = \mathbf{Z}$,*

(c) *S is an $\mathbb{F}_p$-scheme and $p \in R^\times$.*

Then one has a canonical isomorphism:

$$H^{BM}_{2n,n}(X/S, R) \simeq CH_{\delta=n}(X) \otimes R$$

where the right hand side is the Chow group of R-linear algebraic cycles in X of δ-dimension n.

Remark 3.1.8.   (1) The graduation on the Chow group of cycles by a dimension function δ first appeared, to our knowledge, in [Stack, Chap. 41, Def. 9.1].

(2) Note of course that the case where S is the spectrum of a field was already well known — but with a different definition of Borel-Moore motivic homology (though equivalent, see [CD15b]).

(3) It is straightforward to show that the isomorphism of the proposition is functorial with respect to smooth pullbacks and proper push-forwards. It can also be proved it is functorial with respect to lci pullbacks. For all this, see the method of [Dég13], in particular Corollary 3.12 and Proposition 3.16.

3.1.9. Let us go back to the case of an abstract motivic triangulated category $\mathscr{T}$ satisfying our general conventions. Consider a scheme S, noetherian according to our assumptions.

Let us consider the categories $\mathscr{S}^o_{/S}$ (resp. $\bar{\mathscr{S}}^o_{/S}$) made by S-schemes which are separated (of finite type) (resp. essentially separated) and whose morphisms are étale S-morphisms. We let $\mathscr{P}^o_{/S}$ be the category of pro-objects of $\mathscr{S}^o_{/S}$ which are essentially affine.[30] Any pro-object $X_\bullet$ of $\mathscr{P}^o_{/S}$ admits a projective limit X in the category of S-schemes which is separated according to [EGA4, 8.10.5], and essentially of finite type by definition (see the conventions of this paper, p. 9). Thus we get a functor:

$$L : \mathscr{P}^o_{/S} \to \bar{\mathscr{S}}^o_{/S}, \ X_\bullet \mapsto (\varprojlim X_\bullet).$$

Lemma 3.1.10. *In the notation above the functor L is an equivalence of categories.*

Proof. The essential surjectivity of L follows from the definition of essentially of finite type and the reference [EGA4, 8.10.5(v)] (resp. [EGA4, 17.7.8]) for the fact the property of being separated (resp. étale) is compatible with projective limits. Finally, we get that L is fully faithful by applying [EGA4, 8.13.2]. $\square$

3.1.11. The preceding lemma allows to canonically extend some of the functors considered previously. First, we can define the following functor:

$$\hat{M}^{BM} : \mathscr{P}^o_{/S} \to \mathrm{pro} - \mathscr{T}(S), \ (X_i)_{i \in I} \mapsto \text{''}\varprojlim_{i \in I}\text{''} M^{BM}(X_i/S).$$

The preceding lemma tells us that it uniquely corresponds to a functor defined on $\bar{\mathscr{S}}^o_{/S}$, and it obviously extends the functor M^{BM} from separated S-schemes to essentially separated S-schemes.

Moreover, given a $\mathscr{T}$-spectrum $\mathbb{E}$, we can also define:

$$\hat{\mathbb{E}}^{BM}_{**} : \mathscr{P}^o_{/S} \to \mathscr{A}b^{\mathbf{Z}^2}, \ (X_i)_{i \in I} \mapsto \varinjlim_{i \in I} \mathbb{E}^{MB}_{**}(X_i/S).$$

Definition 3.1.12. Given any essentially separated S-scheme X, we will denote by $\hat{M}^{BM}(X/S)$ (resp. $\hat{\mathbb{E}}^{BM}_{**}(X/S)$) the unique pro-spectrum (resp. bi-graded abelian group) obtained by applying the corresponding functor defined above to any pro-scheme $X_\bullet$ in $\mathscr{P}^o_{/S}$ such that $L(X_\bullet) = X$.

The pro-spectrum $\hat{M}^{BM}(X/S)$ (resp. abelian group $\hat{\mathbb{E}}^{BM}_{**}(X/S)$) is functorially contravariant (resp. covariant) in X/S with respect to étale morphisms.

---

[30]*i.e.* the transition morphisms are affine up to a large enough index.

Remark 3.1.13. It follows from the previous lemma that the notation of this definition coincides with that of formula (3.1.5.b) when, given a point $x \in X$, one denotes abusively by x/S the essential separated S-scheme with structural morphism: $\mathrm{Spec}(\kappa(x)) \to X \to S$.

3.1.14. Consider a (noetherian) scheme S and $i : Z \to X$ a closed immersion between essentially separated S-schemes. Let us fix pro-schemes $(X_s)_{s \in I}$ and $(Z_t)_{t \in J}$ in $\mathscr{P}^o_{/S}$ such that $L(X_\bullet) = X$ and $L(Z_\bullet) = Z$.

Because X is noetherian, the ideal of Z in X is locally finitely generated and we can find indexes $s \in I$ and $t \in J$ such that the closed immersion i can be lifted to a closed immersion $i_s : Z_t \to X_s$. Thus, by reducing I and replacing J by I, one can find a morphism of pro-objects $i_\bullet : Z_\bullet \to X_\bullet$ such that for any morphism $s \to s'$ in I, the following commutative diagram is cartesian:

$$\begin{array}{ccc} Z_s & \xrightarrow{i_t} & X_s \\ \downarrow & & \downarrow \\ Z_{s'} & \xrightarrow{i_{t'}} & X_{s'}, \end{array}$$

where the vertical maps corresponds to the transition morphisms of the pro-objects $Z_\bullet$ and $X_\bullet$, and the horizontal maps are closed immersions.

Note that $U_\bullet = (X_s - Z_s)_{s \in I}$ is a pro-scheme in $\mathscr{P}^o_{/S}$ and we have $L(U_\bullet) = U$. Then, we deduce from 1.3.8(BM3) a pro-distinguished triangle:

$$\hat{M}^{BM}(U/S) \xrightarrow{j_*} \hat{M}^{BM}(X/S) \xrightarrow{i^*} \hat{M}^{BM}(Z/S) \xrightarrow{\partial_i} \hat{M}^{BM}(U/S)[1].$$

Using again [EGA4, 8.13.2], we deduce that this triangle does not depend on the lifting of i constructed above.

Given a spectrum $\mathbb{E}$, we also deduce a canonical long exact sequences of abelian groups:

$$\cdots \to \hat{\mathbb{E}}^{BM}_{n+1,m}(U/S) \xrightarrow{\partial_i} \hat{\mathbb{E}}^{BM}_{n,m}(Z/S) \xrightarrow{i_*} \hat{\mathbb{E}}^{BM}_{n,m}(X/S) \xrightarrow{j^*} \hat{\mathbb{E}}^{BM}_{n,m}(U/S) \to \cdots$$

Assuming we are given a dimension function δ on S, it is now straightforward to generalize Definitions 3.1.3 and 3.1.5 to the case of an essentially separated S-scheme X. Therefore we get:

Proposition 3.1.15. *Let (S, δ) be a dimensional scheme and X be an essentially separated S-scheme. Then there is a canonical spectral sequence (with homological conventions)*

$$^\delta E^1_{p,q} = \bigoplus_{x \in X_{(p)}} \hat{\mathbb{E}}^{BM}_{p+q,\{n\}}(x/S) \Rightarrow \hat{\mathbb{E}}^{BM}_{p+q,\{n\}}(X/S)$$

where $X_{(p)}$ stands for the set of points $x \in X$ such that $\delta(x) = p$.

3.2. **Fiber homology.** Let $x \in S(E)$ be a point. As a morphism $x : \mathrm{Spec}(E) \to S$, it is according to our convention, an essentially separated morphism. Thus, it follows from Definition 3.1.12 that the bi-graded abelian group $\hat{\mathbb{E}}^{BM}_{**}(x)$ is well defined. However, in the proof of the forthcoming theorem, we will have to be more precise. This motivates the following definition.

Definition 3.2.1. Let S be a scheme and $x \in S(E)$ be a point.

An S-*model* of x will be an affine regular S-scheme $X = \mathrm{Spec}(A)$ of finite type such that A is a sub-ring of E whose fraction field is equal to E and such that x is equal to the composite of the natural map $\mathrm{Spec}(E) \to \mathrm{Spec}(A)$ and the structural map of X/S.

We let $\mathscr{M}(x)$ be the essentially small category whose objects are S-models of x and morphisms are open immersions.

3.2.2. Consider a scheme S and a point $x \in S(E)$. Because S is assumed to be excellent according to our conventions, $\mathscr{M}(x)$ is non-empty. Moreover, it is easy to check it is a right filtering category. Thus, with the notations of Lemma 3.1.10, we have: $L\left(\text{"}\varprojlim\text{"}_{X \in \mathscr{M}(x)} X\right) = x$

Therefore, for any spectrum $\mathbb{E}$ over S, one has according to Definition 3.1.12:

$$\hat{\mathbb{E}}^{BM}_{p,\{n\}}(x) = \varinjlim_{X \in \mathscr{M}(x)^{op}} \mathbb{E}^{BM}_{p,\{n\}}(X/S).$$

We will denote by $\mathrm{Pt}(S)$ the class of points of S, seen as a discrete category.

Definition 3.2.3. Let (S, δ) be a dimensional scheme, $\mathbb{E}$ be a $\mathscr{T}$-spectrum over S and p be an integer. Consider the preceding notations. One defines the *fiber δ-homology of $\mathbb{E}$ in degree p* (or simply fiber homology) as the following functor:

$$\hat{H}_p^\delta(\mathbb{E}) : \mathrm{Pt}(S) \times \mathbf{Z} \to \mathscr{A}b, (x, n) \mapsto \hat{\mathbb{E}}_{\delta(x)+p, \{\delta(x)-n\}}^{BM}(x).$$

One also defines the *effective fiber δ-homology of $\mathbb{E}$ in degree p* as the restriction of $\hat{H}_p^\delta(\mathbb{E})$ to the discrete category $\mathrm{Pt}(S) \times \mathbf{Z}^-$, where $\mathbf{Z}^-$ is the set of non-positive integers. We will denote it by $\hat{H}_p^{\delta-\mathit{eff}}(\mathbb{E})$.

Here are a few obvious facts about this definition:

Lemma 3.2.4. *Let $\mathbb{E}$ be a spectrum over S and $p \in \mathbf{Z}$ be an integer.*

(1) *One has the relation $\hat{H}_p^\delta(\mathbb{E}[1]) = \hat{H}_{p-1}^\delta(\mathbb{E})$.*

(2) *The presheaf $\hat{H}_p^\delta(\mathbb{E})$ is covariantly functorial in $\mathbb{E}$.*

(3) *The covariant functor $\hat{H}_p^\delta$ commutes with coproducts and with twists in the following sense:*
$\hat{H}_p^\delta(\mathbb{E}\{1\})(x, n) = \hat{H}_p^\delta(\mathbb{E})(x, n + 1)$.

(4) *Given any distinguished triangle $\mathbb{E}' \xrightarrow{a} \mathbb{E} \xrightarrow{b} \mathbb{E}'' \xrightarrow{+1}$ in $\mathscr{T}(S)$, we deduce a long exact sequence of presheaf of $\mathbf{Z}$-graded abelian groups*

(3.2.4.a) $$\ldots \to H_p^\delta(\mathbb{E}') \xrightarrow{a_*} \hat{H}_p^\delta(\mathbb{E}) \xrightarrow{b_*} \hat{H}_p^\delta(\mathbb{E}'') \to \hat{H}_{p-1}^\delta(\mathbb{E}') \to \ldots$$

Moreover, the same assertions hold when $\mathscr{T}$-spectra are replaced by δ-effective $\mathscr{T}$-spectra and $\hat{H}_p^\delta$ is replaced by $\hat{H}_p^{\delta-\mathit{eff}}$.

Remark 3.2.5. let (S, δ) be a dimensional scheme, $f : T \to S$ be a separated morphism and $\mathbb{E}$ be a $\mathscr{T}$-spectrum over S.

Then for any point $x \in S(E)$, corresponding to a morphism $x : \mathrm{Spec}(E) \to S$, the following relation directly follows from the definition:

(3.2.5.a) $$\hat{H}_p^\delta(f^!\mathbb{E})(x, n) = \hat{H}^\delta(\mathbb{E})(x \circ f, n),$$

where δ abusively denotes the dimension function δ^f on T induced by δ (see Paragraph 1.1.6). Suppose that f is in addition smooth. Then it is relevant (see Paragraph 2.1.13) to introduce a new dimension function on T by the formula: $\tilde{\delta}^f = \delta^f - \dim(f)$.

(3.2.5.b) $$\hat{H}_p^{\tilde{\delta}^f}(f^*\mathbb{E})(x, n) = \hat{H}^\delta(\mathbb{E})(x \circ f, n).$$

This formula can also be extended to the following cases:

- $\mathscr{T}$ is continuous and f is essentially smooth;
- $\mathscr{T}$ is oriented and satisfies absolute purity and f is a quasi-projective morphism between regular schemes (use Corollary 1.3.22);
- $\mathscr{T}$ is continuous, oriented and satisfies absolute purity and f is an essentially quasi-projective between regular schemes.

3.2.6. Let S be a regular scheme and $x \in S(E)$ be a point. Note that the corresponding morphism $x : \mathrm{Spec}(E) \to S$ is then a localization of a quasi-projective lci morphism, given by any S-model $X \to S$ of x (Definition 3.2.1).

In particular, it admits a virtual tangent bundle τ_x in the category $\underline{K}(E)$ of virtual E-vector spaces (see §1.3.7). This virtual bundle can be computed using the cotangent complex L_x (cf. [Ill71]) of the morphism x:

$$\tau_x = \sum_i (-1)^i [H_i(L_x)] = [\Omega_x^1] - [H_1(L_x)].$$

In particular, if s denotes the image of x in S, and κ_s is the residue field of s in S, we get:

$$\tau_x = [\Omega_{E/\kappa_s}^1] - [\Gamma_{E/\kappa_s/F}] - [N_s \otimes_{\kappa_s} E]$$

where $N_s = \mathcal{M}_s/\mathcal{M}_s^2$ is the normal bundle of s in $\mathrm{Spec}(\mathcal{O}_{S,s})$, and $\Gamma_{E/\kappa_s/F}$ is the imperfection module ([Mat89, §26]) of the extension E/κ_s over their prime field F (thus it is trivial if the extension is separable).

Recall moreover from [Del87] that the category of virtual E-vector spaces is equivalent to the Picard category $\underline{Pic}(E)$ of graded line bundles over E through the determinant functor:

$$\det : \underline{K}(E) \to \underline{Pic}(E), [V] \mapsto \left(\Lambda^{\max}V, \mathrm{rk}(E)\right)$$

where $\Lambda^{\max}$ denotes the maximal exterior power and rk the virtual rank.

Proposition 3.2.7. *Assume the following conditions hold:*

(a) $\mathscr{T}$-*cohomology commutes with projective limits (see §1.3.19);*
(b) $\mathscr{T}$ *satisfies absolute purity (see Definition 1.3.17).*

Let S be a regular scheme with dimension function $\delta = -\mathrm{codim}_S$. Then for any point $x \in S(E)$ and any integers $(i,n) \in \mathbf{Z}$, there exists a canonical isomorphism

$$(3.2.7.a) \qquad \hat{H}_i^\delta(\mathbb{1}_S)(x,n) \simeq H^{-i,\langle\tau_x-r\rangle,\{n\}}(E,\mathscr{T})$$

where τ_x is the virtual tangent bundle of the essential lci morphism $x : \mathrm{Spec}\, E \to S$ and r is its virtual rank – or equivalently, $r = \delta(x)$ (see (3.1.6.a)).

Moreover we can associate to any trivialization ψ of the 1-dimensional vector space $\det(\tau_x)$ a canonical isomorphism:

$$\hat{H}_i^\delta(\mathbb{1}_S)(x,n) \simeq H^{-i,\{n\}}(E,\mathscr{T}).$$

If moreover $\mathscr{T}$ is oriented, this isomorphism is independent of the choice of ψ.

Proof. Let us choose an S-model $f : X \to S$ of the point x. As X is affine, f is quasi-projective. Because X and S are regular, f is also lci. We denote by $\tau_{X/S}$ its virtual tangent bundle over X. Thus, from the absolute purity property of $\mathscr{T}$ and Corollary 1.3.22, we get an isomorphism for any integers p, s:

$$H_{p,\{s\}}^{BM}(X/S) = \mathrm{Hom}_{\mathscr{T}(X)}(f_!(\mathbb{1}_X)\{s\}[p], \mathbb{1}_S) \simeq \mathrm{Hom}_{\mathscr{T}(X)}(\mathbb{1}_X\{s\}[p], f^!\mathbb{1}_S)$$

$$\xrightarrow{\eta_f^{-1}} \mathrm{Hom}_{\mathscr{T}(X)}(\mathbb{1}_X\{s\}[p], \mathrm{Th}(\tau_{X/S})) = H^{i,\langle\tau_{X/S}\rangle,\{s\}}(X,\mathscr{T}).$$

The class η_f being compatible with restriction along an open immersion $j : U \to X$, we get the same isomorphism after reducing X to U and the corresponding isomorphism is compatible with the pullback map j^*. Therefore we get an isomorphism:

$$\hat{H}_i^\delta(\mathbb{1}_S)(x,n) = \hat{H}_{r+i,\{r-n\}}^{BM}(x/S) = \varinjlim_{U \subset X} H_{r+i,\{r-n\}}^{BM}(U/S) \simeq \varinjlim_{U \subset X} H^{-i,\langle\tau_{U/S}-r\rangle,\{n\}}(U,\mathscr{T}).$$

Now, for any open immersion $j_{VU} : V \to U$ of open subschemes of X, we have $j_{VU}^*\tau_{U/S} = \tau_{V/S}$. Therefore assumption (a) gives the isomorphism (3.2.7.a).

The remark about the trivialization then follows from the fact det is an equivalence of categories and the last assertion follows from Remark 1.3.13. $\qquad\qquad\square$

Example 3.2.8. Let S be a regular scheme and $\delta = -\mathrm{codim}_S$.

(1) Assume $\mathscr{T} = \mathrm{DM}_R$ with the assumptions of Points (1) or (2) of Paragraph 1.3.2.
Then DM_R satisfies assumptions (a) and (b) of the previous proposition and is oriented.
Therefore one gets for any point $x \in S(E)$ the following computation:

$$\hat{H}_p^\delta(\mathbb{1}_S)(x,n) = \begin{cases} 0 & \text{if } p < 0, \\ K_n^M(E) \otimes R & \text{if } p = 0 \end{cases}$$

where K_*^M stands for the Milnor K-theory of the field E, or equivalently motivic cohomology of E in degree (n,n).

(2) Assume $\mathscr{T} = \mathrm{SH}$ and S is in addition an F-scheme for a prime field F.

Again, SH satisfies assumptions (a) and (b) of the preceding proposition. Thus one gets the following non-canonical isomorphism:

$$\hat{H}_p^\delta(\mathbf{S}_S^0)(x,n) = \begin{cases} 0 & \text{if } p < 0, \\ K_n^{MW}(E) & \text{if } p = 0. \end{cases}$$

where $K_n^{MW}(E)$ denotes the Milnor-Witt K-theory of the field E (cf. [Mor12, Introduction, Def. 21]). The case $p < 0$ follows from Morel's $\mathbf{A}^1$-connectivity theorem (cf. [Mor12, Introduction, Th. 18]) and the case $p = 0$ is the computation of the 0-th stable $\mathbf{A}^1$-homotopy group of $\mathbf{S}_S^0$ (cf. [Mor12, Introduction, Cor. 24]).

To get a canonical identification, we must introduce a twisted version of Milnor K-theory as in [Mor12], bottom of page 139. An element τ of $\underline{K}(E) \simeq \overline{\mathrm{Pic}(E)}$ can be canonically represented by a 1-dimensional vector space Λ – its determinant. We put:

$$K_n^{MW}(E,\tau) = K_n^{MW}(E) \otimes_{E^\times} \mathbf{Z}[\Lambda^\times]$$

where $\Lambda^\times$ is the set of non zero element of Λ, with its canonical action of $E^\times$, and the action of an element $u \in E^\times$ is induced by functoriality through the multiplication map by u. With this notation, one gets a canonical isomorphism:

(3.2.8.a) $$\hat{H}_0^\delta(\mathbf{S}_S^0)(x,n) \simeq K_n^{MW}(E,\tau_x)$$

where τ_x is the virtual tangent bundle of the essentially lci morphism $x : \mathrm{Spec}(E) \to S$.

3.2.9. It is useful to rewrite the δ-niveau spectral sequence using fiber homology. Let $\mathbb{E}$ be a spectrum, (S,δ) a dimensional scheme and X be an essentially separated S-scheme. Then the δ-niveau spectral sequence of X/S with coefficients in $\mathbb{E}$ defined in Proposition 3.1.15 has the following form:

(3.2.9.a) $$^\delta E_{p,q}^1 = \bigoplus_{x \in X_{(p)}} \hat{H}_q^\delta(\mathbb{E})(x,p-n) \Rightarrow \mathbb{E}_{p+q,\{n\}}^{BM}(X/S).$$

Note that the spectral sequence can actually be written as

(3.2.9.b) $$^\delta E_{p,q}^1 = \bigoplus_{x \in X_{(p)}} \hat{H}_q^\delta(\mathbb{E})(x) \Rightarrow \mathbb{E}_{p+q,\{*\}}^{BM}(X/S),$$

where it is understood to take values in graded abelian groups; then the differentials in the E^1-term are homogeneous of degree (-1).[31]

Assume moreover that $\mathbb{E}$ is δ-effective. Considering the first form of the spectral sequence $^\delta E_{p,q}^1$, we see that when $n \geq \delta(X)$, then $p - n \leq 0$ or $X_{(p)} = \varnothing$. Moreover the abutment of the spectral sequence can be computed as a group morphisms of the category $\mathscr{T}^{\delta-eff}$. Thus the effective version of the preceding spectral sequence as the form:

(3.2.9.c) $$^\delta E_{p,q}^1 = \bigoplus_{x \in X_{(p)}} \hat{H}_q^{\delta-eff}(\mathbb{E})(x) \Rightarrow \mathbb{E}_{p+q,\{*\}}^{BM}(X/S),$$

where we have restricted the gradings on the abutment to $* \geq \delta(X)$. It takes its values in $\mathbf{N}$-graded abelian groups.

Recall that, because we work with noetherian finite dimensional schemes, the preceding spectral sequences are all convergent. Thus, a corollary of their existence is the following result.

Proposition 3.2.10. *For any dimensional scheme (S,δ), the family of functors:*

$$\hat{H}_p^\delta : \mathscr{T}(S) \to \mathrm{PSh}\big(\mathrm{Pt}(S) \times \mathbf{Z}\big), p \in \mathbf{Z},$$

$$resp. \ \hat{H}_p^\delta : \mathscr{T}^{\delta-eff}(S) \to \mathrm{PSh}\big(\mathrm{Pt}(S) \times \mathbf{Z}^-\big), p \in \mathbf{Z},$$

is conservative.

The following vanishing conditions will be a key point for our main theorem 3.3.1.

---

[31]We are following the convention and homological notations of Rost in [Ros96].

Proposition 3.2.11. *The following conditions are equivalent:*

(i) *For any regular dimensional scheme (S, δ), and any $i < -\delta(S)$, $\hat{H}_i^\delta(\mathbb{1}_S) = 0$.*

(i') *For any regular scheme S, with dimension function $\delta = -\operatorname{codim}_S$, and any $i < 0$, $\hat{H}_i^\delta(\mathbb{1}_S) = 0$.*

(ii) *For any regular dimensional scheme (S, δ), any essentially separated S-scheme X, and any $p < \delta_-(X) - \delta(S)$, where $\delta_-(X) = \min_{x \in X} \delta(x)$, one has $\hat{H}_{p,\{*\}}^{BM}(X/S) = 0$.*

Proof. (i) is obviously equivalent to (i'), since any dimension function δ on a connected regular scheme satisfies the relation $\delta(x) = \delta(S) - \operatorname{codim}_S(x)$ for $x \in S$.

The fact (ii) implies (i) follows rightly by definition of $\hat{H}_i^\delta$. Let us prove the converse implication. In the situation of (ii), we look at the δ-niveau spectral sequence (3.2.9.b) in the case $\mathbb{E} = \mathbb{1}_S$. Using assumption (i), we get that ${}^\delta E_{p,q}^1 = 0$ whenever $q < -\delta(S)$ or $p < \delta_-(X)$. In particular, ${}^\delta E_{p,q}^1$ is zero if $p + q < \delta_-(X) - \delta(S)$ and this concludes. $\square$

The equivalent properties of the previous proposition will be crucial for the main theorem of this section so that we introduce the following definition.

Definition 3.2.12. We say that the motivic category $\mathscr{T}$ is *homotopically compatible* if the equivalent conditions of the preceding proposition are satisfied.

As an immediate corollary of Proposition 3.2.7, we get the following useful criterion for this property.

Proposition 3.2.13. *Assume the following conditions hold:*

(a) *$\mathscr{T}$-cohomology commutes with projective limits (see §1.3.19);*

(b) *$\mathscr{T}$ satisfies absolute purity (see Definition 1.3.17);*

(c) *For any field E such that $\operatorname{Spec}(E)$ is in our category of schemes $\mathscr{S}$, $H^{n,m}(\operatorname{Spec}(E), \mathscr{T}) = 0$ if $n > m$.*

Then $\mathscr{T}$ is homotopically compatible.

Example 3.2.14. Here are our main examples of homotopically compatible categories:

(1) Assume $\mathscr{S}$ is included in the category of schemes over a prime field F. Then all the triangulated motivic categories of Example 1.3.1, except possibly point (5) if the characteristic of F is positive, satisfies the assumptions of the previous corollary:

   - assumption (a), see [CD12, 4.3.3] (the case of SH is treated as the case of $D_{\mathbf{A}^1}$, and the case of modules then directly follows);
   - assumption (b) is then automatic (as recalled in Example 1.3.20);
   - assumption (c): in the case of SH, $\mathbf{MGL} - mod$ and $D_{\mathbf{A}^1, R}$, is a consequence of Morel's stable $\mathbf{A}^1$-connectivity theorem [Mor05, Th 3]; in the case of DM_R, under the conventions of Paragraph 1.3.2 (so that the characteristic exponent of F is invertible in R), this follows from the basic property of motivic cohomology of perfect fields and the semi-separation property of DM_R (see Lemma 2.1.16).

(2) Assume $\mathscr{S}$ is any category of schemes satisfying our conventions. Then for any $\mathbf{Q}$-algebra R, DM_R is homotopically compatible.

3.3. Main theorem.

Theorem 3.3.1. *Assume the following conditions hold:*

(a) *$\mathscr{T}$ is homotopically compatible (see Definition 3.2.12);*

(b) *Assumption (Resol) of §2.4.1 is satisfied.*

Then for any dimensional scheme (S, δ) and any $\mathscr{T}$-spectrum (resp. δ-effective $\mathscr{T}$-spectrum) $\mathbb{E}$ over S, the following conditions are equivalent:

(i) *$\mathbb{E}$ is t_δ-positive (resp. t_δ-negative).*

(ii) *For any integer $p \leq 0$ (resp. $p \geq 0$), $\hat{H}_p^\delta(\mathbb{E}) = 0$ (Definition 3.2.3).*

Moreover, when $\mathbb{E}$ is δ-effective, these conditions remain equivalent after replacing t_δ by t_δ^{eff} and $\hat{H}_p^\delta$ by $\hat{H}_p^{\delta-eff}$.

Proof. The proof is valid in the general and δ-effective case except for a change of indices. We use unified notations to treat both cases in a row:

- in the general case, we put $I = \mathbf{Z}$, $t = t_\delta$, $\mathscr{C} = \mathscr{T}(S)$, $\hat{H}_p = \hat{H}_p^\delta$;
- in the effective case, we put $I = \mathbf{N}$, $t = t_\delta$, $\mathscr{C} = \mathscr{T}^{\delta-eff}(S)$, $\hat{H}_p = \hat{H}_p^{\delta-eff}$. Below, this case will also be referred to as the *resp.* case.

Let us write $\mathscr{C}_{\geq 0}$ (resp. $\mathscr{C}_{<0}$) for the subcategory of non-t-negative (resp. t-negative) spectra in $\mathscr{C}$ and $\mathscr{C}_{H\geq 0}$ (resp. $\mathscr{C}_{H<0}$) for the subcategory of $\mathscr{C}$ made of spectra $\mathbb{E}$ such that $\hat{H}_p(\mathbb{E}) = 0$ if $p < 0$ (resp. $p \geq 0$).

Thus we have to prove $\mathscr{C}_{<0} = \mathscr{C}_{H<0}$ and $\mathscr{C}_{\geq 0} = \mathscr{C}_{H\geq 0}$.

The fact $\mathscr{C}_{<0} \subset \mathscr{C}_{H<0}$ follows from definitions – see in particular Remark 2.1.2 (resp. Remark 2.2.15). Let us prove the converse inclusion. Take a spectrum $\mathbb{E}$ in $\mathscr{C}_{H<0}$. According to the remark previously cited, we have to prove that for any separated S-scheme X, $\mathbb{E}_{\delta(X)+p,\delta(X)-n}^{BM}(X/S) = 0$ if $p > 0$ and $n \in I$. Let us rewrite the spectral sequence (3.2.9.b) (resp. (3.2.9.c)) for $\mathbb{E}$ and X/S in our notations, and for the grading $* = \delta(X) - n$, $n \in I$:

$$^\delta E_{p,q}^1 = \bigoplus_{x \in X_{(p)}} \hat{H}_q(\mathbb{E})(x, p - \delta(X) + n) \Rightarrow \mathbb{E}_{p+q,\{\delta(X)-n\}}^{BM}(X/S).$$

The $E_{p,q}^1$-term is concentrated in the region $p \leq \delta(X)$ by construction and in the region $q < 0$ by assumption on $\mathbb{E}$. In particular, $^\delta E_{p,q}^1 = 0$ if $p + q > \delta(X)$ and this concludes.

We now prove that $\mathscr{C}_{\geq 0} \subset \mathscr{C}_{H\geq 0}$. According to Lemma 3.2.4, the subcategory $\mathscr{C}_{H\geq 0}$ of $\mathscr{C}$ is stable by positive suspensions, coproducts and extensions. Thus it is sufficient to prove that the generators of the t-structure t belongs to $\mathscr{C}_{H\geq 0}$. According to assumption (b), we can apply Theorem 2.4.3 to $\mathscr{T}$. Thus, we have to show that for a proper regular S-scheme Y and an integer $m \in I$, $M^{BM}(Y/S)\langle\delta(Y)\rangle\{m\}$ belongs to $\mathscr{C}_{H\geq 0}$. Given a point x of S, we show that the abelian group $H_i^\delta\big(M^{BM}(Y/S)\langle\delta(Y)\rangle\{m\}\big)(x, n)$ is zero for $i < 0$ and $n \in I$. Note that, as Y/S is proper, we get the following computation for any model $X \in \mathscr{M}(x)$:

$$\mathrm{Hom}_{\mathscr{T}(S)}\big(M^{BM}(X/S)\{s\}[j], M^{BM}(Y/S)\big) = \mathrm{Hom}_{\mathscr{T}(Y)}\big(M^{BM}(X \times_S Y/Y)\{s\}[j], \mathbb{1}_S\big)$$
$$= \mathbb{E}_{j,\{s\}}^{BM}(X \times_S Y/Y).$$

Thus, with $j = \delta(X) - \delta(Y) + i$ and $s = \delta(X) - \delta(Y) + n - m$, we deduce:

$$H_i^\delta\big(M^{BM}(Y/S)\langle\delta(Y)\rangle\{m\}\big)(x, n) = \hat{H}_{\delta(x)-\delta(Y)+i,\{\delta(x)-\delta(Y)+n-m\}}^{BM}(Y_x/Y)$$

where Y_x is the fiber of Y at the point x of S – which is in fact essentially separated over Y. According to assumption (a), we can apply assertion (ii) of Proposition 3.2.11. This concludes the proof because $\delta(x) - \delta(Y) + i < \delta_-(Y_x) - \delta(Y)$ whenever $i < 0$.

Let us finally prove that $\mathscr{C}_{H\geq 0} \subset \mathscr{C}_{\geq 0}$. Let $\mathbb{E}$ be an object in $\mathscr{C}_{H\geq 0}$. We can consider the distinguished triangles associated with E and the t-structure t:

$$E_{\geq 0} \to \mathbb{E} \to E_{<0} \xrightarrow{+1}$$

Applying the functor H_p and Lemma 3.2.4(4), we get a sequence of presheaves of abelian groups on $\mathrm{Pt}(S) \times \mathbf{Z}$ (resp. $\mathrm{Pt}(S) \times \mathbf{Z}^-$):

$$\ldots \to \hat{H}_p(\mathbb{E}_{\geq 0}) \to \hat{H}_p(\mathbb{E}) \to \hat{H}_p(\mathbb{E}_{<0}) \to \hat{H}_{p-1}(\mathbb{E}_{\geq 0}) \to \ldots$$

According to the inclusion $\mathscr{C}_{<0} \subset \mathscr{C}_{H<0}$ already proved, we get that $\hat{H}_p(\mathbb{E}_{<0}) = 0$ if $p \geq 0$. Let $p < 0$. By assumption, $\hat{H}_p(\mathbb{E}) = 0$. According to the inclusion $\mathscr{C}_{\geq 0} \subset \mathscr{C}_{H\geq 0}$ proved just above, we also have $\hat{H}_{p-1}(\mathbb{E}_{\geq 0}) = 0$. Thus the preceding long exact sequence implies that $\hat{H}_p(\mathbb{E}_{<0}) = 0$. According to Proposition 3.2.10, we deduce that $E_{<0} = 0$ and this concludes. $\square$

This theorem has many nice consequences. Let us start with computations.

Example 3.3.2. Assume $\mathscr{T} = \mathrm{DM}_\mathbf{Q}$ following conventions of 1.3.2(1).

Let $S = \mathrm{Spec}(\mathbf{Q})$ equipped with the Krull dimension function δ. By definition, $\mathbb{1}_{\mathbf{Q}}$ is t_δ-non-negative. Moreover, we can see it is not positively bounded for the t_δ-homotopy t-structure. Actually, for any integer $n \geq 0$,

(3.3.2.a) $H_n^\delta(\mathbb{1}_{\mathbf{Q}}) \neq 0.$

Indeed, let $K/\mathbf{Q}$ be the field extension generated by the group of primitive d-th roots of unity μ_d^0. Note that according to our convention, K is a point of $\mathrm{Spec}(\mathbf{Q})$. According to Proposition 3.2.7, we get for any integer $n \geq 0$:

$$\hat{H}_n^\delta(\mathbb{1}_{\mathbf{Q}})(K, n+1) \simeq H_{\mathrm{B}}^{1,n+1}(K).$$

But this group is non-zero, since Beilinson's construction of polylogarithm yields a canonical map

$$\epsilon_{n+1} : \mu_d^0 \to H_{\mathrm{B}}^{1,n+1}(K)$$

whose composition with the regulator map is the classical polylogarithm which is non-zero. Here we refer the reader to [HW98, Cor. 9.6]. Then relation (3.3.2.a) follows from the preceding theorem.

According to [Dég08b], given any extension field $L/\mathbf{Q}$ the canonical map $H_{\mathrm{B}}^{i,n+1}(\mathbf{Q}) \to H_{\mathrm{B}}^{i,n+1}(L)$ is a (split) monomorphism. Thus relation (3.3.2.a) is true if one replaces $\mathbf{Q}$ by any characteristic 0 field, or even any regular $\mathbf{Q}$-scheme S with dimension function $\delta = -\mathrm{codim}_S$ (given again the computation of Proposition 3.2.7).

Note in particular that the δ-homology of a constructible motive will not be positively bounded in general. This makes it different from its cousin, the perverse t-structure on torsion or ℓ-adic étale sheaves.

Example 3.3.3. Assume $\mathscr{T} = \mathrm{DM}_R$ following conventions of 1.3.2(1) or (2). Let S be a regular scheme with dimension function $\delta = -\mathrm{codim}_S$.

We consider the δ-homotopy t-structure on $\mathrm{DM}^{\delta-eff}(S, R)$. This means in particular that we restrict to the $\mathbf{Z}^-$-graded part of fiber δ-homology. Applying Proposition 3.2.7, for any couple of integer $(i, n) \in \mathbf{Z} \times \mathbf{Z}^-$, we get:

$$\hat{H}_i^\delta(\mathbb{1}_S)(x, n) \simeq H_M^{-i,\{n\}}(E, R) = \begin{cases} R & \text{if } i = 0, n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\hat{H}_0^\delta(\mathbb{1}_S) = R$ as a graded abelian group concentrated in degree 0, and $\hat{H}_i^\delta(\mathbb{1}_S) = 0$ if $i \neq 0$.

Thus, the constant motive $\mathbb{1}_S$ is concentrated in degree 0 for the effective δ-homotopy t-structure – the case of a (perfect) field was already well known thanks to Example 2.3.12.

This indicates that the δ-homotopy t-structure is better behaved (with respect to bounds) on δ-effective motives over nonsingular schemes. Actually, we expect that the δ-homology of any constructible (*i.e.* compact) δ-effective motive over S is bounded (see in particular Prop. 4.4.15) but this is a deep conjecture.

Remark 3.3.4. The preceding example cannot be generalized to the singular case. Let us consider its notations and assumptions.

Let us take a field k, δ being the Krull dimension function on $\mathrm{Spec}(k)$. We look at the example where S is an algebraic k-scheme which is the union two copies of the affine line D_1 and D_2 crossing in a single rational point s. Consider the canonical closed immersions:

$$\begin{array}{ccc} s & \xrightarrow{\ k_1\ } & D_1 \\ {\scriptstyle k_2}\downarrow & {\scriptstyle k}\searrow & \downarrow{\scriptstyle i_1} \\ D_2 & \xrightarrow{\ i_2\ } & S. \end{array}$$

Then, using cdh-descent for $\mathrm{DM}(S, R)$, we get a distinguished triangle (apply [CD12, 3.3.10] to the preceding cartesian square):

$$k_*(\mathbb{1}_s)[-1] \to \mathbb{1}_S \to i_{1*}(\mathbb{1}_D) \oplus i_{2*}(\mathbb{1}_P) \to k_*(\mathbb{1}_s).$$

Applying $k^!$ to this triangle, together with the base change formula and the absolute purity (Corollary 1.3.22) of DM_R with respect to k_1 and k_2, we get the following distinguished triangle:

$$\mathbb{1}_s[-1] \to k^!(\mathbb{1}_S) \to \mathbb{1}_s\langle -1\rangle \oplus \mathbb{1}_s\langle -1\rangle \xrightarrow{(1)} \mathbb{1}_s.$$

The map labelled (1) corresponds to a cohomology class in $H_M^{2,1}(s, R) \oplus H_M^{4,2}(s, R)$ which vanishes since x is a point. Therefore, the above distinguished triangle splits and gives:

$$k^!(\mathbb{1}_S) = \mathbb{1}_s[-1] \oplus \mathbb{1}_s\{-1\}[-1] \oplus \mathbb{1}_s\{-1\}[-1].$$

Taken into account formula (3.2.5.a), and Lemma 3.2.4(1)(3), we obtain for any couple of integers $(i, n) \in \mathbf{Z} \times \mathbf{Z}^-$:

$$H_i^\delta(s, n) = \begin{cases} \mathbf{Z} & \text{if } (i, n) = (-1, 0), \\ 0 & \text{otherwise.} \end{cases}$$

The computation of the fiber δ-homology of the other points, the closed nonsingular points $S'_{(0)}$ and the two generic points $S^{[0]}$, follows from the previous example, as it can be reduced to the case of a regular base. In the end we obtain the following computation:

$$H_i^\delta(x, n) = \begin{cases} \mathbf{Z} & \text{if } \big(x = s \wedge (i, n) = (-1, 0)\big) \vee \big(x \in S^{(0)} \wedge (i, n) = (0, 0)\big) \\ & \qquad \vee \big(x \in S'_{(0)} \wedge (i, n) = (1, -1)\big), \\ \kappa(x)^\times & \text{if } x \in S'_{(0)} \wedge (i, n) = (1, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Thus the motive $\mathbb{1}_S$ has t_δ^{eff}-amplitude $[-1, 1]$ exactly.

However, it is easy using a stratification by regular locus to show that any noetherian excellent scheme has finite t_δ^{eff}-homological amplitude. In particular, it is reasonable to expect that the conjectural boundedness stated in the end of the preceding example happens also in the singular case.

Let us go back to corollaries of the preceding theorem.

Corollary 3.3.5. *Suppose that the hypothesis of the preceding theorem are fulfilled and let (S, δ) be a dimensional scheme. Then for any $\mathscr{T}$-spectrum $\mathbb{E}$ over S, we get the following equivalent conditions where the left hand side conditions refer to the δ-homotopy t-structure:*

(1) *$\mathbb{E} \geq 0$ if and only if for all separated S-scheme X, $\mathbb{E}_{p,\{*\}}^{BM}(X/S) = 0$ when $p < \delta_-(X)$.*

(2) *$\mathbb{E} \leq 0$ if and only if for all separated S-scheme X, $\mathbb{E}_{p,\{*\}}^{BM}(X/S) = 0$ when $p > \delta_+(X)$.*

(3) *$\mathbb{E} \in \mathscr{T}(S)^\heartsuit$ if and only if for all separated S-scheme X, $\mathbb{E}_{p,\{*\}}^{BM}(X/S) = 0$ when $p \notin [\delta_-(X), \delta_+(X)]$.*

Moreover, these equivalent conditions remain true when one replaces separated S-schemes by essentially separated S-schemes.

Proof. This is a straightforward consequence of the preceding theorem together with the δ-niveau spectral sequence under the form (3.2.9.b). $\qquad\square$

In the δ-effective, we need to be more precise about the $\mathbf{G}_m$-gradings.

Corollary 3.3.6. *Suppose that the hypothesis of the preceding theorem are fulfilled and let (S, δ) be a dimensional scheme. Then for any δ-effective $\mathscr{T}$-spectrum $\mathbb{E}$ over S, we get the following equivalent conditions where the left hand side conditions refer to the δ-homotopy t-structure on $\mathscr{T}^{\delta-eff}(S)$:*

(1) *$\mathbb{E} \geq 0 \Leftrightarrow$ for all separated S-scheme X, $\mathbb{E}_{p,\{n\}}^{BM}(X/S) = 0$ when $p < \delta_-(X)$ and $n > \delta_+(X)$.*

(2) *$\mathbb{E} \leq 0 \Leftrightarrow$ for all separated S-scheme X, $\mathbb{E}_{p,\{n\}}^{BM}(X/S) = 0$ when $p > \delta_+(X)$ and $n < \delta_-(X)$.*

Moreover, these equivalent conditions remain true when one replaces separated S-schemes by essentially separated S-schemes.

Again, use the preceding theorem together with the δ-niveau spectral sequence but under the form (3.2.9.a).

Corollary 3.3.7. *Suppose that the hypothesis of the preceding theorem are fulfilled and let (S, δ) be a dimensional scheme.*

(1) *For any dimensional scheme (S, δ), the t-structure t_δ (resp. t_δ^{eff}) is non-degenerate.*

(2) *The functor $w : \mathscr{T}(S) \to \mathscr{T}^{\delta-\mathrm{eff}}(S)$ of point (1) of Proposition 2.2.4 is t_δ-exact.*

(3) *The composition functor $-_{-1} : \mathscr{T}^{\delta-\mathrm{eff}}(S) \overset{\{-1\}}{\to} \mathscr{T}^{\delta-\mathrm{eff}}(S)(-1) \overset{w'}{\to} \mathscr{T}^{\delta-\mathrm{eff}}(S)$ (see the aforementioned proposition) is t_δ^{eff}-exact.*

(4) *For any separated morphism $f : T \to S$, the functor $f^!$ is t_δ-exact.*

(5) *For any morphism $f : X \to S$ such that $\dim(f) \leq d$, the functor f_* has t_δ-amplitude $[0, d]$.*

Proof. Assertion (1) follows from the previous theorem combined with Proposition 3.2.10.

Assertion (2) follows from the previous theorem given that for any spectrum $\mathbb{E}$ over S an easy computation gives:
$$\hat{H}_i^{\delta-\mathrm{eff}}\big(w(\mathbb{E})\big) = \hat{H}_i^\delta\big(\mathbb{E}\big)|_{\mathrm{Pt}(S) \times \mathbf{Z}^-}.$$

Similarly, for any effective $\mathbb{E}$ over S if we denote (essentially following [VSF00, §3.3.1]) $-_{-1}(\mathbb{E})$ by $\mathbb{E}_{-1}$, then we have
$$(\mathbb{E}_{-1})_{\delta(x)+p,\langle\delta(x)-n\rangle}^{BM}(x/S) = E_{\delta(x)+p,\langle\delta(x)-n+1\rangle}^{BM}(x/S)$$

for any $(x, n) \in \mathrm{Pt}(S) \times \mathbf{Z}^-$. This easily yields assertion (3).

Assertion (4) obviously follows from the previous theorem along with relation (3.2.5.a).

Let us prove assertion (5). In view of Proposition 2.1.6(3), one needs only to prove that f_* preserves t_δ-positive spectra. Let $\mathbb{E}$ be such a spectrum over X. A direct computation gives, for any point s of S:
$$\hat{H}_i^\delta(f_*\mathbb{E})(s, *) = \hat{\mathbb{E}}_{\delta(s)+i,\{\delta(s)-*\}}^{BM}(X_s/X)$$

where X_s denotes the fiber of f at the point s – *i.e.* the pullback of the S-scheme X along the morphism s – seen as an essentially separated scheme over X. Then the required vanishing easily follows from the preceding corollary. $\square$

3.3.8. To state the next result, we need the following definition taken from [BBD82, 2.2.12]. Let S be a scheme and $x \in S$ a set-theoretic point of S, with inclusion i_x. Then i_x can be factorized as $x \overset{j}{\to} \bar{x} \overset{i}{\to} S$, where $\bar{x}$ is the reduced closure of x in S and we put:
$$i_x^! = j^* i^!.$$

Corollary 3.3.9. *Suppose that the assumptions of the previous theorem are fulfilled and that the motivic triangulated category $\mathscr{T}$ is continuous (see Paragraph 1.3.19). Let (S, δ) be a dimensional scheme.*

(1) *For any set-theoretic point $x \in S$, the functor $i_x^!$ defined above is t_δ-exact. Moreover, for any $\mathscr{T}$-spectrum $\mathbb{E}$ over S and any finitely generated field extension K of the residue field of x, with corresponding point $x_K \in S(K)$, one has:*

(3.3.9.a) $$\hat{H}_*^\delta\big(i_x^!\mathbb{E}\big)(K, *) = \hat{H}_*^\delta(\mathbb{E})(x_K, *).$$

(2) *The family of t_δ-exact functors $i_x^! : \mathscr{T}(X) \to \mathscr{T}(x)$ indexed by set-theoretic points $x \in S$ respects and detects t_δ-positive and t_δ-negative spectra.*

Proof. Given the previous theorem, the only thing to prove is relation (3.3.9.a). Obviously, we can restrict to $\hat{H}_0^\delta(-, 0)$. By definition of $i_x^!$ and the previous remark, we can assume S is irreducible and x is the generic point of S. Let us denote by k the residue field of x. With the notations of the paragraph preceding the proof, we thus have $i_x^! = j^*$. We want to prove there exists an isomorphism of the form:
$$\psi : \mathrm{Hom}_{\mathscr{T}(x)}(\hat{M}^{BM}(K/k), j^*\mathbb{E}) \simeq \mathrm{Hom}_{\mathscr{T}(S)}(\hat{M}^{BM}(x_K), \mathbb{E}).$$

Given any S-model X of x_K, in the sense of Definition 3.2.1, it is clear that $X_k := X \times_S k$ is a k-model of E in the sense of the same definition. Moreover, it is clear that the corresponding

functor $\mathscr{M}(x_K) \to \mathscr{M}(K/k)$ is final. Therefore, one gets the following computation of the left hand side of ψ:

$$\mathrm{Hom}_{\mathscr{T}(x)}(\hat{M}^{BM}(K/k), h^*\mathbb{E}) = \varinjlim_{X \in \mathscr{M}(x_K)} \mathrm{Hom}_{\mathscr{T}(x)}(M^{BM}(X_k/k), h^*\mathbb{E}).$$

On the other hand, given any S-model X of x_K, we get by using the continuity property of $\mathscr{T}$ and [CD12, Prop. 4.3.4]:

$$\mathrm{Hom}_{\mathscr{T}(x)}(M^{BM}(X_k/k), j^*\mathbb{E}) = \varinjlim_{U \subset S} \mathrm{Hom}_{\mathscr{T}(x)}(M^{BM}(X_U/U), j_U^*\mathbb{E}) = \varinjlim_{U \subset S} \mathrm{Hom}_{\mathscr{T}(x)}(M^{BM}(X_U/S), \mathbb{E})$$

where U runs over the set of non-empty open subschemes of S, $X_U = X \times_S U$ and $j_U : U \to S$ is the obvious immersion.

The result follows from this last computation, and from the definition of $\hat{M}^{BM}(x_K)$ (in particular Lemma 3.1.10). $\qquad\square$

Let us finish with the following easy but useful corollary.

Corollary 3.3.10. *Let us assume that the hypothesis of the preceding theorem are fulfilled. Let (S, δ) be a dimensional scheme.*

Then for any $\mathscr{T}$-spectrum $\mathbb{E}$ over S, and any point x of S, the truncation functors induce a canonical isomorphism:

$$\mathrm{Hom}_{\mathscr{T}(S)}(\hat{M}^{BM}(x)\langle\delta(x)\rangle, \mathbb{E}) \simeq \mathrm{Hom}_{\mathscr{T}(S)}(\hat{M}^{BM}(x)\langle\delta(x)\rangle, H_0(\mathbb{E})).$$

This is another way of saying that $\hat{H}_0^\delta(\mathbb{E}) \simeq \hat{H}_0^\delta\big(H_0(\mathbb{E})\big)$ which follows immediately from the preceding theorem.

3.4. **On Gersten-type weight structures.** In this section we only indicate the relation of homotopy t-structures to the so-called Gersten weight structures without giving any definitions.

So, recall that Gersten weight structures were constructed in [Bon10a] for motives over a countable perfect field, and in [Bon15] for "arbitrary" motivic categories over an arbitrary perfect field (see §6 of ibid. and note that in these two papers "opposite conventions on signs on weights" were used; see Remark 2.1.2(3) of ibid.). Now, weight structures are (very roughly) "orthogonals" to t-structures, and so the main property of the aforementioned Gersten weight structures is that they are orthogonal to the corresponding homotopy t-structures (see Definition 2.4.1, Proposition 4.4.1, and Theorem 6.1.2(II.6) of ibid., noting that in ibid. the "cohomological convention" for t-structures was used).

Another (more or less, equivalent) form of the main property of Gersten weight structures (both for the ones from [Bon15] and for the "relative" ones that we will discuss here) is that their hearts should be generated by $M^{BM}(x/S)\{n\}[\delta(x)]$ for x running through the (spectra of) fields over the corresponding base scheme S. The problem is that we don't have any "reasonable" $M^{BM}(x/S)$ inside $\mathscr{C}$ whenever x is not of finite type over S; so that one has to consider a certain triangulated "homotopy completion" of $\mathscr{C}$. Thus one has to treat some model for $\mathscr{C}$, and construct a certain triangulated category $\mathscr{C}'$ of *pro-spectra* using this model. Next, there is a functor $\mathscr{C}' \to \mathrm{Pro}-\mathscr{C}$, but it is far from being a full embedding; so one has to overcome certain difficulties when computing $\mathscr{C}'$-morphism groups (whereas one requires the morphism group $\mathrm{Hom}_{\mathscr{C}}(M^{BM}(x/S)\{n\}[\delta(x) - \delta(x') + i], M^{BM}(x'/S))$ to be zero for any $x, x' \in \mathrm{Pt}(S)$ and $i < 0$; this is the $\mathscr{C}'$-substitute of the vanishing of $H_i^\delta\big(M^{BM}(Y/S)\langle\delta(Y)\rangle\{m\}\big)(x, n)$ whenever $n \in I$, $x \in \mathrm{Pt}(S)$, and Y is a proper regular S-scheme, that was used in the proof of Theorem 3.3.1).

There currently exist two distinct methods for proving this $\mathscr{C}'$-orthogonality result. The first one (applied in [Bon10a]) uses the properties of countable homotopy limits. It can be carried over to the "relative context" more or less easily; so one obtains the existence and basic properties of Gersten weight structures whenever the corresponding S possesses a Zariski cover by spectra of (at most) countable rings. Yet one can probably get rid of this restriction by using the methods of [Bon15] instead.

The existence of such a weight structure would allow to define a certain "generalized (δ-co)niveau spectral sequence" for the H-cohomology of any constructible object of $\mathscr{C}$ (where H is a cohomological functor from $\mathscr{C}$ into an AB5-abelian category) and to prove the $\mathscr{C}$-functoriality of these spectral sequences (as well as the corresponding filtrations) starting from E^2 (cf. [Bon15, §4.3]). Besides, if H is $\mathscr{C}$-representable then this spectral sequence (for the H-cohomology of any constructible c) is naturally isomorphic to the one coming from the t_δ-truncations of c (cf. Corollary 4.4.3 of ibid.).

We also note that one can easily define a "toy Gersten weight structure" w^{ATcl} on the triangulated subcategory $\mathscr{C}^{ATcl}$ of $\mathscr{C}$ generated by (the objects coming from) Artin-Tate spectra over all closed points of S (cf. §4.5 of ibid.; so, we do not need any pro-objects here). Moreover, one can easily define the natural analog t_δ^{ATcl} of t_δ for $\mathscr{C}^{ATcl}$; then w^{ATcl} is determined by the condition $\mathscr{C}^{ATcl}_{w^{ATcl}\geq 0} = \mathscr{C}^{ATcl}_{t_\delta^{ATcl}\geq 0}$ (so that w^{ATcl} is *left adjacent* to t_δ^{ATcl} in the sense of [Bon10b, Definition 4.4.1]; keep in mind that in ibid. the "cohomological conventions" for weight and t-structures are used). Thus $\mathscr{C}^{ATcl}_{w^{ATcl}\leq 0}$ is (certainly) the "left $\mathscr{C}^{ATcl}$-orthogonal" to $\mathscr{C}^{ATcl}_{t_\delta^{ATcl}\geq 0}[1]$, whereas $\mathscr{C}^{ATcl}_{w^{ATcl}\leq 0}$ is the right orthogonal to $\mathscr{C}^{ATcl}_{t_\delta^{ATcl}\geq 0}[1]$ (see Theorem 4.4.2 of loc. cit. where several other interesting properties of adjacent structures are listed; note also that the existence of w^{ATcl} and t_δ^{ATcl} easily follows from Theorem 4.5.2 of ibid.).

Note that the embedding of $\mathscr{C}^{ATcl}$ into $\mathscr{C}$ admits a right adjoint R that is a certain Verdier localization (by the corresponding results of [Nee01]); it is easily seen that R is t-exact with respect to $(t_\delta^{ATcl}, t_\delta)$.

Yet $\mathscr{C}^{ATcl}$ is not that interesting from the "relative motivic" point of view since it can be easily seen to be equivalent to the categorical product of the corresponding categories $\mathscr{T}_x^{AT}$ for x running through closed points of S. Still for certain (base) schemes some of their non-closed points are their (locally closed) subschemes (in particular, all semi-localizations of dimension 1 schemes at finite collections of points and the spectra of the rings of integers in fields with respect to any rank r discrete valuations possess this "strange" property); then one can "enlarge $\mathscr{C}^{ATcl}$" and define a "more interesting" weight structure "of Artin-Tate-Gersten type" inside $\mathscr{C}$.

Another interesting problem related to the "relative" Gersten weight structures is the construction of connecting functors between the corresponding $\mathscr{T}'(X)$ for X running through $\mathscr{S}$. The t_δ-exactness statements of Corollary 3.3.5 (parts 4 and 5) should correspond to the "dual" Gersten-weight-exactness properties of the "pro-spectral versions" of f^* and $f_!$.

Lastly we note that the description of generators in §2.4 above is closely related to the Chow weight structures w_{Chow} as constructed in [Bon10b], [Bon11], [Hí1], [Bon14], and [BI15] for various versions of DM (since these generators are certain shifts of the so-called Chow motives over S as defined in the latter two papers; one may consider motives that are constructible or not and effective or not here). In particular, note that several arguments in the latter paper are closely related to the ones used in sections 1.3 and 2.3–2.4 above. A pleasant advantage of the Chow weight structures is that their generators are (constructible) motives; so one does not need any pro-spectra (whose "motivic" versions were called *comotives* in [Bon10a]) for constructing them. Furthermore, for these weight structures admit adjacent t-structures inside the corresponding DM; so that we have $\mathrm{DM}_{w_{Chow}\geq 0} = \mathrm{DM}_{t_{Chow}\geq 0}$ for the corresponding t-structures (cf. [Bon14, §A.2]). Unfortunately, Chow weight structures do not exist for "categories of the type SH"; see [Bon15, Remark 6.3.4].

4. On δ-homotopy hearts

4.0.1. Theorem 3.3.1 is the central result in the applications of the δ-homotopy t-structure. So we will isolate its assumptions and call (*good*) the conjunction of the assumptions (a) – vanishing of certain Borel-Moore homology groups – and (b) – convenient resolution of singularities.

4.1. **Definition and functoriality properties.** We will adopt the following notations.

Definition 4.1.1. Let $\mathscr{T}$ be a motivic triangulated category satisfying our general conventions and (S, δ) be a dimensional scheme.

We will denote by $\Pi^\delta(S, \mathscr{T})$ (resp. $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$) the heart of the δ-homotopy t-structure of $\mathscr{T}(S)$ (resp. $\mathscr{T}^{\delta-eff}(S)$). Objects of $\Pi^\delta(S, T)$ (resp. $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$) will be called δ-*homotopy modules* (resp. *effective δ-homotopy modules*) with coefficients in $\mathscr{T}$.

When R is a ring of coefficients following conventions of Paragraph 1.3.2, we simply put:

$$\Pi^\delta(S, R) := \Pi^\delta(S, \mathrm{DM}_R), \quad \Pi^{\delta\text{-}eff}(S, R) := \Pi^{\delta\text{-}eff}(S, \mathrm{DM}_R),$$

$$\tilde{\Pi}^\delta(S, R) := \Pi^\delta\left(S, \mathscr{D}_{\mathbf{A}^1, R}\right), \quad \tilde{\Pi}^{\delta\text{-}eff}(S, R) := \Pi^{\delta\text{-}eff}\left(S, \mathscr{D}_{\mathbf{A}^1, R}\right).$$

Example 4.1.2. Let $S = \mathrm{Spec}(k)$ be the spectrum of a perfect field with characteristic exponent p, δ be the canonical dimension function on S and R a ring of coefficients such that $p \in R^\times$. According to Examples 2.3.5 and 2.3.12, we get the following description of the above categories:
- the category $\Pi^{\delta\text{-}eff}(k, R)$ is equivalent to the category of homotopy invariant sheaves with transfers and coefficients in R defined by Voevodsky in [VSF00, chap. 5] – and denoted by $HI(k) \otimes_{\mathbf{Z}} R$;
- the category $\Pi^\delta(k, R)$ is equivalent to the category of *homotopy modules (with transfers)* and coefficients in R of [Dég11, 1.17];
- the category $\tilde{\Pi}^\delta(k, R)$ is equivalent to the category of *(generalized) homotopy modules* with coefficients in R defined by Morel (see [Dég13, 1.2.2]).

4.1.3. *Basic properties.* Recall these categories do not depend on the choice of the dimension function δ on S up to a canonical equivalence of categories (see 2.1.4 and 2.2.15).

Moreover, using the adjunction of t-categories (2.2.14.a), we obtain an adjunction of abelian categories:

$$H_0(s) : \Pi^{\delta\text{-}eff}(S, \mathscr{T}) \leftrightarrows \Pi^\delta(S, \mathscr{T}) : H_0(w).$$

Let us assume that (good) is satisfied. Then according to Corollary 3.3.7(2), the functor $H_0(w) = w$ is exact. Moreover, because the functor $s : \mathscr{T}^{\delta-eff}(S) \to \mathscr{T}(S)$ is fully faithful and its right adjoint w is t_δ-exact, we get that $H_0(s)$ is fully faithful. Thus $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$ is a full subcategory of $\Pi^\delta(S, \mathscr{T})$; beware however it is not stable by kernel.

The categories $\Pi^\delta(S, \mathscr{T})$ and $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$ are in fact Grothendieck abelian categories because they are the hearts of a generated t-structure in a compactly generated triangulated category. Let us introduce a notation to describe the generators. According to Remark 2.2.15(4) (resp. Cor. 2.1.11), the category $\Pi^\delta(S, \mathscr{T})$ (resp. $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$) is stable by the functor $\left(M\mathrm{Th}_S(v)[-r] \otimes -\right)$ (resp. $\left(M\mathrm{Th}_S(E)[-r] \otimes -\right)$) for v a virtual vector bundle (resp. E a vector bundle) over S of rank r.

Definition 4.1.4. For any δ-homotopy module $\mathbb{F}$ of $\Pi^\delta(S, \mathscr{T})$ (resp. of $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$), we put:

$$\mathbb{F}\{v\} := \mathbb{F} \otimes M\mathrm{Th}_S(v)[-r] \quad \text{resp. } \mathbb{F}\{E\} := \mathbb{F} \otimes M\mathrm{Th}_S(E)[-r].$$

Note that if v (resp. E) is trivial of rank r, we get $\mathbb{F}\{v\} = \mathbb{F}\{r\}$ (resp. $\mathbb{F}\{E\} = \mathbb{F}\{r\}$) and this new notation is coherent with the notation for twists in the present paper. Note also that the endo-functor $(-\{v\})$ of $\Pi^\delta(S, \mathscr{T})$ (resp. $(-\{E\})$ of $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$) is an equivalence of categories (resp. right exact and fully faithful).

4.1.5. *Generators.*
With this definition, it is easy to deduce that $\Pi^\delta(S, \mathscr{T})$ (resp. $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$) is generated by δ-homotopy modules of the form

$$H_i\left(M^{BM}(X/S)\right)\{n\}, \text{ X/S separated}, (i, n) \in \mathbf{Z}^2 \text{ (resp. } (i, n) \in \mathbf{Z} \times \mathbf{N}).$$

Note in particular that, under assumption (good), the category $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$ of effective δ-homotopy modules can be described as the unique subcategory of $\Pi^\delta(S, \mathscr{T})$ containing cokernel of morphisms between arbitrary coproducts of objects of the form $H_i\left(M^{BM}(X/S)\right)\{n\}$ as above, for $n \geq 0$.

Remark 4.1.6. If one applies Proposition 2.3.1, we see that to get a family of generators as above, it is enough to consider the S-schemes X which are either proper or such that X is regular. Under assumption (Resol) of 2.4.1, we can assume both restrictions on X/S.

Note that we will get a much better result under assumption (good) in Proposition 4.2.8.

4.1.7. *Funtoriality.*

Let $f : T \to S$ be a morphism such that $\dim(f) \leq d$. Then according to point (3) of Proposition 2.1.6, we get an adjunction of abelian categories:
$$H_{-d}f^* : \Pi^\delta(S, \mathscr{T}) \leftrightarrows \Pi^\delta(T, \mathscr{T}) : H_d f_*.$$
Note also that according to Paragraph 2.2.16, the right exact functor $H_{-d}f^*\{-d\}$ preserves effective objects (image of effective δ-homotopy modules). In particular, when f is quasi-finite, we get:
$$H_0 f^* : \Pi^\delta(S, \mathscr{T}) \leftrightarrows \Pi^\delta(T, \mathscr{T}) : H_0 f_*,$$
and the functor $H_0 f^*$ preserves effective objects. Moreover, we get better properties in the following two cases, using the notation of Paragraph 4.1.5 for generators:

- when f is étale, the functor $H_0 f^* = f^*$ is exact and acts on generators as:
  $f^* H_i\big(M^{BM}(X/S)\big)\{n\} = H_i\big(M^{BM}(X \times_S T/T)\big)\{n\};$
- when f is finite, the functor $H_0 f_* = f_*$ is exact and acts on generators as:
  $f_* H_i\big(M^{BM}(Y/T)\big)\{n\} = H_i\big(M^{BM}(Y/S)\big)\{n\}.$

Assume now that f is separated. Then applying point (1) of Proposition 2.1.6, we get an adjunction of abelian categories:
$$H_0 f_! : \Pi^\delta(T, \mathscr{T}) \leftrightarrows \Pi^\delta(S, \mathscr{T}) : H_0 f^!$$
and according to Paragraph 2.2.16, the functor $H_0 f_!$ preserves effective objects. Moreover, we get the following additional informations:

- when f is smooth (resp. under assumption (good)), the functor $H_0 f^! = f^!$ is exact and acts on generators (resp. generators such that X/S is proper) as:
  $f^! H_i\big(M^{BM}(X/S)\big)\{n\} = H_i\big(M^{BM}(X \times_S T/T)\big)\{n\}.$

One gets a shadow of the 6 functors formalism. For example, if f is finite, $H_0 f_! = H_0 f_*$. One also easily gets the following result (from the 6 functors formalism satisfied by $\mathscr{T}$).

Proposition 4.1.8. *Consider a cartesian square of schemes:*
$$\begin{array}{ccc} Y & \overset{g}{\to} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ T & \overset{f}{\to} & S \end{array}$$

such that p is separated and $\dim(f) \leq d$. Then $\dim(g) \leq d$ and we get a canonical isomorphism of functors:
$$H_{-d}f^* \circ H_0 p_! \overset{\sim}{\longrightarrow} H_0 q_! \circ H_{-d}g^*$$
$$H_d p_* \circ H_0 f^! \overset{\sim}{\longrightarrow} H_0 g^! \circ H_d q_*.$$

Proof. According to the 6 functors formalism (cf. [CD12, 2.4.50]), one gets for any object $\mathbb{F}$ of $\Pi^\delta(S, ; \mathscr{T})$ the isomorphisms $f^* g_!(\mathbb{F})[d] \overset{\sim}{\to} q_! g^*(\mathbb{F})[d]$ and $p_* f^!(\mathbb{F})[d] \overset{\sim}{\to} g^! q_*(\mathbb{F})[d]$. The isomorphisms of the proposition are obtained by applying the functor $\tau_{\leq 0}$ to these isomorphisms and by using the fact the functors $f^*[d]$, $g_!$, and the like are all right t_δ-exact. $\square$

Remark 4.1.9.   (1) Considering a cartesian square as in the proposition, one can derive several relations analogous to the previous ones. For example:
- if f is étale, then for any integer $i \in \mathbf{Z}$, we get and isomorphism: $f^* \circ H_i p_! \simeq H_i q_! \circ g^*$;
- If f is smooth or under assumption (good), then for any integer $i \in \mathbf{Z}$, we get and isomorphism of functors: $H_i p_* \circ f^! \simeq g^! \circ H_i q_*.$

  (2) All these isomorphisms satisfy the usual compatibilities with respect to compositions (see [CD12, Rem. 1.1.7] for example).

4.1.10. *Monoidal structures.* Assume the dimension function δ on S is non-negative.

Then the tensor product $\otimes$ is right t_δ-exact (Proposition 2.1.6) and thus induces a closed monoidal structure $\otimes^\delta$ on $\Pi^\delta(S, \mathscr{T})$ such that for any $\mathbb{F}, \mathbb{G}$, one has:
$$\mathbb{F} \otimes^\delta_S \mathbb{G} := H_0(\mathbb{F} \otimes_S \mathbb{G}).$$

It follows from Paragraph 2.2.16 that this tensor product preserves effective objects.

We will also denote by $\underline{\mathrm{Hom}}^\delta$ the internal Hom with respect to this tensor structure. It is given by the following formula:

$$\underline{\mathrm{Hom}}^\delta_S(\mathbb{F}, \mathbb{G}) = H_0 \underline{\mathrm{Hom}}_S(\mathbb{F}, \mathbb{G}).$$

Recall that changing δ only changes $\Pi^\delta(S, \mathscr{T})$ up to a canonical equivalence of categories. But the equivalence of categories involved here will not be monoidal. In particular, the monoidal structure on $\Pi^\delta(S, \mathscr{T})$ depends on the choice of $\delta \geq 0$.

Using again the six functors formalism, one gets the following formulas.

Proposition 4.1.11. *Assume (S, δ) is a dimensional scheme such that $\delta \geq 0$. Let $f : T \to S$ be a quasi-finite and separated morphism. Then for δ-homotopy modules $\mathbb{F} \in \Pi^\delta(T, \mathscr{T})$, $\mathbb{G} \in \Pi^\delta(S, \mathscr{T})$ one has a functorial isomorphism:*

$$H_0 f_! \big(\mathbb{F} \otimes^\delta_S H_0 f^*(\mathbb{G}) \big) \xrightarrow{\sim} H_0 f_!(\mathbb{F}) \otimes^\delta_T \mathbb{G}$$

The proof is the same as the proof of the previous proposition.

Remark 4.1.12. If f is not quasi-finite, but we have an integer $d > 0$ such that $\dim(f) \leq d$, we obtain that $H_0 f_!(\mathbb{F} \otimes^\delta_S H_{-d} f^*(\mathbb{G})) = 0$ because for any δ-homotopy module $\mathbb{F}$ over S, $H_{-d} f_!(\mathbb{F}) = 0$.

The most important fact about the 6 functors formalism and δ-homotopy modules comes from the glueing property of the δ-homotopy t-structure (cf. Corollaries 2.1.9 and 2.2.17).

Proposition 4.1.13. *Let (S, δ) be a dimensional scheme and $i : Z \to S$ be a closed immersion with complementary open immersion $j : U \to S$. We denote by δ the dimension function induced on Z and U.*

Then δ-homotopy modules satisfy the glueing formalism:

$$\Pi^\delta(U, \mathscr{T}) \underset{H_0 j_*}{\overset{H_0 j_!}{\underset{\longleftarrow j^* \longrightarrow}{\rightleftarrows}}} \Pi^\delta(X, \mathscr{T}) \underset{H_0 i^!}{\overset{H_0 i^*}{\underset{\longleftarrow i_* \longrightarrow}{\rightleftarrows}}} \Pi^\delta(Z, \mathscr{T})$$

such that $\Pi^\delta(Z, T)$ is a quotient of the abelian category $\Pi^\delta(X, \mathscr{T})$ by the thick abelian subcategory $\Pi^\delta(U, \mathscr{T})$. Under (good), one has an isomorphism of functors $H_0 i^! = i^!$ and $i^!$ is exact. Moreover, all the above 6 functors preserve effective objects.

For any δ-homotopy module $\mathbb{F}$ over X, one has exact sequences:

$$0 \longrightarrow H_0 j_! j^*(\mathbb{F}) \longrightarrow \mathbb{F} \longrightarrow i_* H_0 i^*(\mathbb{F}) \longrightarrow 0,$$

$$0 \longrightarrow i_* H_0 i^!(\mathbb{F}) \longrightarrow \mathbb{F} \longrightarrow H_0 j_* j^*(\mathbb{F}) \longrightarrow 0.$$

In particular, δ-homotopy modules over a scheme are determined by their restrictions to a closed subscheme and to its open complement. From Corollary 3.3.9, we get the following stronger form.

Proposition 4.1.14. *Suppose assumpion (good) is fulfilled and $\mathscr{T}$ is continuous ([CD12, 4.3.2]). Let S be a scheme.*

(1) *For any set-theoretic point $x \in S$, the functor $i_x^!$ of Paragraph 3.3.8 induces an exact functor*

$$i_x^! : \Pi^\delta(S) \to \Pi^\delta(x)$$

which preserves δ-effective modules and acts on generators, for X/S proper, as:
$i_x^! H_i \big(M^{BM}(X/S) \big) \{n\} = H_i \big(M^{BM}(X_x/x) \big) \{n\}.$

(2) *The family of functors $i_x^! : \Pi^\delta(S) \to \Pi^\delta(x)$ for $x \in S$ is conservative.*

We end up this section with the following additional proposition, which follows directly from 2.1.16.

Proposition 4.1.15. *Assume the triangulated motivic category is semi-separated (see [CD12, 2.1.7] or Footnote 18 page 21).*

Then for any finite surjective radicial morphism $f : T \to S$ the functor $f^* : \mathscr{T}(S) \to \mathscr{T}(T)$ induces an equivalence of abelian categories:

$$f^* : \Pi^\delta(S, \mathscr{T}) \to \Pi^\delta(T, \mathscr{T}).$$

Moreover, the same is true for effective δ-homotopy modules.

Recall in particular from Example 2.1.17 that this can be applied to the categories $\Pi^\delta(-, R)$ and $\Pi^{\delta\text{-}eff}(-, R)$ when R is a ring satisfying conventions (1) or (2) of Paragraph 1.3.2.

4.2. **Fiber functors and strong generators.** The preceding proposition can also be restated using the following definition, which mimick the notion of fiber functors for sheaves:

Definition 4.2.1. Consider the notation of Definition 4.1.1. Let $\mathbb{F}$ be an object of $\Pi^\delta(S, \mathscr{T})$ (resp. $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$). For any point $x \in S(E)$ and any integer $n \in \mathbf{Z}$ (resp. $n \leq 0$), we define the fiber of $\mathbb{F}$ at the point (x, n) as the following abelian group:

$$\hat{\mathbb{F}}_n^\delta(x) := H_0^\delta(\mathbb{F})(x, n)$$

using the notation of Definition 3.2.3. When δ is clear, we will simply put: $\hat{F}_* = \hat{F}_*^\delta$.

In particular that $\hat{\mathbb{F}}_*$ is then a graded functor on the discrete category $\mathrm{Pt}(S)$ of points of S. As a consequence of the main theorem 3.3.1, we get the following important result (compare with [VSF00, chap. 3, 4.20]).

Proposition 4.2.2. *Assume that (good) is satisfied. Then the functors:*

$$\left.\begin{array}{c}\Pi^\delta(S, \mathscr{T}) \longrightarrow \widehat{\mathrm{Pt}(S) \times \mathbf{Z}} \\ \Pi^{\delta\text{-}eff}(S, \mathscr{T}) \longrightarrow \widehat{\mathrm{Pt}(S) \times \mathbf{Z}^-}\end{array}\right\}, \mathbb{F} \mapsto \hat{\mathbb{F}}_*$$

are conservative.

In other words, the objects of $\mathrm{Pt}(S) \times \mathbf{Z}$ (resp. $\mathrm{Pt}(S) \times \mathbf{Z}^-$) defines a conservative family of points for δ-homotopy modules (resp. effective δ-homotopy modules).

Remark 4.2.3. Let us assume (good) is satisfied.
  (1) According to relation (3.2.5.a), we obtain for any separated morphism $f : T \to S$, any δ-homotopy module $\mathbb{F}$ over S and any point x in $T(E)$:

$$\widehat{f^! \mathbb{F}}_*(x) = \hat{\mathbb{F}}_*(f \circ x).$$

  Moreover, when f is smooth, if one defines $\tilde{\delta}^f = \delta^f - \dim(f)$, we deduce from (3.2.5.b) the following relation:

$$\widehat{f^* \mathbb{F}}_*^{\tilde{\delta}^f}(x) = \hat{\mathbb{F}}_*^\delta(f \circ x).$$

  Note that in the case $\mathscr{T} = \mathrm{DM}_R$, this relation is also valid for f essentially quasi-projective between regular schemes (see Remark 3.2.5).
  (2) In a work in preparation, we will show when $\mathscr{T} = \mathrm{DM}_R$ or $\mathscr{T} = \mathbf{MGL}{-}mod$, that $\hat{\mathbb{F}}_*$ can be equipped with a rich functoriality, that of a cycle module over S following the definition of Rost. It will be called the *Rost transform of* $\mathbb{F}$ along the lines of [Dég11].

Let us introduce some useful definition.

Definition 4.2.4. Let $\mathscr{T}$ be a motivic triangulated category and (S, δ) be a dimensional scheme.
  Let X be a separated S-scheme, and $(X_i)_{i \in I}$ the family of its connected components. We put:
$h_0^\delta(X/S) = \oplus_{i \in I} H_0\big((M^{BM}(X_i/S)\langle \delta(X_i) \rangle)\big)$.

Example 4.2.5. Assume $\mathscr{T}$ is oriented, S is universally catenary and integral, and $\delta = -\operatorname{codim}_S$ (Ex. 1.1.4). Then for any smooth S-scheme X, $h_0^\delta(X/S) = H_0(M_S(X))$ – apply 1.3.8(BM4) as in Example 2.1.10.

4.2.6. *Functoriality properties.* Because of the functoriality properties of Borel-Moore $\mathscr{T}$-spectra (BM1) and (BM2), we obtain that the δ-homotopy module $h_0^\delta(X/S)$ is covariant with respect to proper morphisms, and contravariant with respect to smooth morphisms, of separated S-schemes. Note in particular that, because of our choice of definition and Proposition 1.1.11(3), there is no twist in the contravariant functoriality.

These functorialty property corresponds to the following formula with respect to the functoriality of the category $\Pi^\delta(S, \mathscr{T})$ (see Paragraph 4.1.7):

- For any equidimensional morphism $f : T \to S$ of relative dimension d and any proper S-scheme X, we get: $H_{-d}f^*h_0(X/S)\{n\} = h_0(X \times_S T/T)\{n+d\}$.
- For any proper morphism $f : T \to S$ and any separated T-scheme Y, we get: $H_0 f_* h_0(Y/T) = h_0(Y/S)$.

Note the first assertion follows from the base change formula and point (3) of Proposition 1.1.11. The second assertion is obvious.

Example 4.2.7. The definition above has been chosen to meet that of [Dég08a, (1.18.a)]. Consider the motivic category $\mathscr{T} = \mathrm{DM}_R$ under the conventions of point (1) or point (2) of Paragraph 1.3.2.

(1) When $S = \mathrm{Spec}(k)$ is the spectrum of a perfect field equipped with the obvious dimension function, then through the equivalence of categories of Example 2.3.5(1), the δ-homotopy module $h_0^\delta(X/k)$ for X a smooth k-scheme corresponds to the homotopy module $h_{0,*}(X)$ of [Dég08a, (1.18.a)].

(2) Let X be a regular and proper S-scheme. Then for any point x of S, one gets the following computation:

$$\widehat{h_0^\delta(X/S)}(x,0) = \mathrm{Hom}_{\mathrm{DM}(S,R)}\left(\hat{M}^{BM}(x)\langle\delta(x)\rangle, H_0 M^{BM}(X/S)\langle\delta(X)\rangle\right)$$

$$\overset{(1)}{=} \mathrm{Hom}_{\mathrm{DM}(S,R)}\left(\hat{M}^{BM}(x)\langle\delta(x)\rangle, M^{BM}(X/S)\langle\delta(X)\rangle\right)$$

$$\overset{(2)}{=} \mathrm{Hom}_{\mathrm{DM}(S,R)}\left(\hat{M}^{BM}(X_x/X)\langle\delta(x) - \delta(X)\rangle, \mathbb{1}_X\right)$$

$$\overset{(3)}{=} CH_{\dim=0}(X_x) \otimes_{\mathbf{Z}} R,$$

where dim is the Krull dimension function on X_x. Indeed, (1) follows from Corollary 3.3.10, (2) from the base change formula and because X/S is proper, (3) from Proposition 3.1.7 because X is regular.

Actually, (2) is a generalization of [Dég11, §3.8]. It also enlighten the functoriality properties of $h_0^\delta(X/S)$.

Recall that in any category $\mathscr{C}$, an object X is called compact if the functor $\mathrm{Hom}_{\mathscr{C}}(X, -)$ commutes with coproducts.

Proposition 4.2.8. *Consider the assumptions of the previous definition.*

(1) *For any separated S-scheme X and any integer $n \in \mathbf{Z}$, the δ-homotopy module $h_0^\delta(X/S)\{n\}$ is compact.*

(2) *Under assumption (good), the family of δ-homotopy modules $h_0^\delta(X/S)\{n\}$ for X/S affine, X regular and $n \in \mathbf{Z}$ (resp. $n = 0$) generates the abelian category $\Pi^\delta(S, \mathscr{T})$ (resp. $\Pi^{\delta\text{-}eff}(S, \mathscr{T})$).*

Proof. Assertion (1) follows easily from the fact $M^{BM}(X/S)\langle\delta(X)\rangle\{n\}$ is t_δ-non-negative and compact in the category $\mathscr{T}(S)$.

Consider assertion (2). We first remark that for any regular affine S-scheme X and any integer $n \geq 0$, one has:

$$h_0^\delta(X/S)\{n\} = h_0^\delta\left(X \times \mathbf{G}_m^n/S\right)/ \oplus_{i=1}^n h_0^\delta\left(X \times \mathbf{G}_m^{n-1}/S\right).$$

Thus, in the δ-effective case, we can consider all integers $n \geq 0$ in the description of the generators. Then one remarks that it is sufficient to prove that the family of functors

$$\mathrm{Hom}_{\Pi^\delta(S,\mathscr{T})}(h_0^\delta(X/S)\{n\}, -)$$

for X a proper regular S-scheme and $n \in \mathbf{Z}$ (resp. $n \geq 0$) is conservative. This now follows from Proposition 4.2.2. $\square$

Remark 4.2.9. In particular, we also obtain that a δ-homotopy module over S is compact if and only if it is a quotient of an object of the form $h_0^\delta(X/S)\{n\}$ for a proper regular S-scheme X and an integer $n \in \mathbf{Z}$.

This condition was considered (for $\mathrm{DM}(k, R)$) in [Dég11, Rem. 6.7, 2.] along with the following alternative notions of finiteness which are interesting on their own:

- a δ-homotopy module over S is said to be *constructible* (resp. *strongly constructible*) if it belongs to the smallest abelian thick subcategory of $\Pi^\delta(S, \mathscr{T})$ which contains objects of the form $H_i\big(M^{BM}(X/S)\big)\{n\}$ for a proper regular S-scheme X and integers $(i, n) \in \mathbf{Z}^2$ (resp. it is of the form $H_0(\mathbb{E})$ for a constructible $\mathscr{T}$-spectrum $\mathbb{E}$ over S).

We have the following implications: compact implies constructible and strongly constructible implies constructible but it does not seem to exist other relations.

4.3. **Comparison between δ-homotopy hearts.** As a generalization of Example 2.3.5, we get the following result:

Theorem 4.3.1. *Let R be a ring and assume that one of the two following conditions hold:*

(a) *R is a $\mathbf{Q}$-algebra.*
(b) *$\mathscr{S}$ is the category of $\mathbf{Q}$-schemes.*

Use the notation of Example 2.3.3. Then the following assertions hold for any scheme S in $\mathscr{S}$:

(1) *Under assumption (b), the adjunction of abelian categories:*

$$H_0\delta^* : \mathrm{SH}(S)^\heartsuit \leftrightarrows \tilde{\Pi}^\delta(S, \mathbf{Z}) : \delta_*$$

is an equivalence, compatible with the monoidal structure if $\delta \geq 0$.
(2) *The exact functor*

$$\gamma_* : \Pi^\delta(S, R) \to \tilde{\Pi}^\delta(S, R)$$

is fully faithful and its essential image is equivalent to the category of generalized δ-homotopy modules with trivial action of η (see Paragraph 2.3.6).

Proof. When R is a $\mathbf{Q}$-algebra, each point is a consequence of the stronger statement that it is already true for the full triangulated motivic categories involved: see [CD12, 5.3.35] for point (1) and [CD12, 16.2.13] for points (2) and (3).

So we restrict to the case of assumption (b). For each point, we will use Proposition 4.1.14.

First, let us remark that the functor $\phi = \delta_*$ (resp. $\phi = \gamma_*$) commutes with functors $i_x^!$ for any set-theoretic point $x \in S$. By definition of $i_x^!$, this boils down to the fact ϕ commutes with j^* where j is pro-open. One can check easily this follows from the continuity property of the triangulated motivic categories involved and the fact it is true when j is open.

Secondly, Corollary 2.4.13 shows that the functors $H_0\delta^*$ and $H_0\gamma^*$ commutes with $i_x^!$.

Then we can prove point (1). Indeed, we need to prove that the two adjunction maps $H_0\delta^*\delta_* \to Id$ and $Id \to \delta_* H_0\delta^*$ are isomorphisms over the base scheme S. But using Proposition 4.1.14, we need only to check that after applying the functor $i_x^!$ for any point $x \in S$. Thus we are restricted to the case where S is the spectrum of a field of characteristic 0 which follows from Proposition 2.3.7.

Finally, we remark that the fact η acts trivially on a homotopy module is preserved by any functor $i_x^!$ and detected by the family of functors $i_x^!$ for a set-theoretic point x in S.

Let us denote by $\tilde{\Pi}^\delta(S, R)/\eta$ the full subcategory of $\tilde{\Pi}^\delta(S, R)$ consisting of those objects $\mathbb{E}$ such that η acts trivially on $\mathbb{E}$ (see Paragraph 2.3.6). According to the preceding subsection and Proposition 4.1.14, the family of functors $i_x^!$ induces a conservative family of functors $\tilde{\Pi}^\delta(S, R)/\eta \to \tilde{\Pi}^\delta(x, R)/\eta$.

Since any object $\mathbb{E}$ of $\mathrm{DM}(S, R)$ defines an orientable cohomology theory, the Hopf map η acts trivially on $\mathbb{E}$. This implies that the image of $\Pi^\delta(S, R)$ by the map γ_* lies in $\tilde{\Pi}^\delta(S, R)/\eta$ and therefore we get an adjunction of abelian categories:

$$H_0\gamma^* : \tilde{\Pi}^\delta(S, R)/\eta \leftrightarrows \Pi^\delta(S, R) : \gamma_*.$$

We need to prove that these are equivalences of categories. Now, applying the conservative family of functors $i_x^!$ for any point $x \in S$, we can assume that S is the spectrum of a field of characteristic 0. This is Proposition 2.3.7. □

Remark 4.3.2. What is missing to deal with the case of fields of positive characteristic in the preceding theorem is the fact that the triangulated motivic category $\mathrm{SH}[1/p]$ is semi-separated (see footnote 18 page 21 for the definition). Indeed, this fact will immediately imply that Proposition 2.3.7 can be generalized to arbitrary fields.

4.4. **Examples and computations.**

4.4.1. Assume that $\mathscr{T}$ is absolutely pure (Definition 1.3.17).

Let S be a regular scheme and $x \in S$ a set-theoretic point. Consider the notation of paragraph 3.3.8: $\bar{x}$ is the reduced closure of x in S. According to our assumptions, there exists an open neighborhood U of x in S such that $\bar{x} \cap U$ is regular. Let us consider the following immersions: $\bar{x} \xrightarrow{j_U} \bar{x} \cap U \xrightarrow{i_U} U$. Using the absolute purity property for the closed immerison i_U, we get a fundamental class:

$$\eta_{x,U} : M\mathrm{Th}(-N_U) \to i_U^!(\mathbb{1}_U)$$

where N_U is the normal bundle of i_U. Applying the functor j_U^*, we get:

$$\eta_x : M\mathrm{Th}(-N_x) \to j_U^* i_U^!(\mathbb{1}_U) = i_x^!(\mathbb{1}_X)$$

where N_x is the normal bundle of x in $\mathrm{Spec}(\mathcal{O}_{X,x})$. Since fundamental classes are compatible with pullbacks along open immersions, this map does not depend on the choice of $U \subset S$.

Recall also that, from the six functor formalism, we get for any $\mathscr{T}$-spectra $\mathbb{E}$, $\mathbb{F}$ over S a canonical map

(4.4.1.a) $$i^*(\mathbb{E}) \otimes i^!(\mathbb{F}) \to i^!(\mathbb{E} \otimes \mathbb{F})$$

by adjunction from the canonical map:

$$i_! \big(i^*(\mathbb{E}) \otimes i^!(\mathbb{F}) \big) \xrightarrow{\sim} \mathbb{E} \otimes i_! i^!(\mathbb{F}) \xrightarrow{ad(i_!,i^!)} \mathbb{E} \otimes \mathbb{F}.$$

Definition 4.4.2. Consider the notations and assumptions above. Let $\mathbb{E}$ be a $\mathscr{T}$-spectrum over S.

For any point $x \in S$, we will say that $\mathbb{E}$ is *pure at x* if the following canonical map:

$$i_x^*(\mathbb{E}) \otimes M\mathrm{Th}(-N_x) \xrightarrow{\eta_x} i_x^*(\mathbb{E}) \otimes i_x^!(\mathbb{1}_S) \xrightarrow{(4.4.1.a)} i_x^!(\mathbb{E})$$

is an isomorphism.

On says $\mathbb{E}$ is *pure* if it is pure at all points of S. We will denote by $\mathscr{T}^{\mathrm{pur},x}(S)$ (resp. $\mathscr{T}^{\mathrm{pur}}(S)$) the full subcategory of $\mathscr{T}(S)$ made by $\mathscr{T}$-spectra which are pure at $x \in S$ (resp. pure).

Remark 4.4.3. Typical examples of non-pure $\mathscr{T}$-spectra are objects in the image of j_* for an open immersion $j : U \to S$. For example, in the case $\mathscr{T} = \mathrm{DM}_{\mathbf{Q}}$, if x is the closed point of a scheme S of codimension c, $U = S - \{x\}$, then $i_x^* j_*(\mathbb{1}_U) = \mathbb{1}_x \oplus \mathbb{1}_x(-c)[1-2c]$ while $i_x^! j_*(\mathbb{1}_U) = 0$.

The following proposition is clear:

Proposition 4.4.4. *Consider the assumptions and notations of the preceding definition.*

(1) *The subcategories $\mathscr{T}^{\mathrm{pur},x}(S)$ and $\mathscr{T}^{\mathrm{pur}}(S)$ are stable under extensions, suspensions, direct factors, and arbitrary coproducts — i.e. they are localizing triangulated subcategories of $\mathscr{T}(S)$. They are also stable under tensor product by an invertible object.*

(2) *The property of being pure is local for the Nisnevich topology on S. If $\mathscr{T}$ is separated for the étale topology[32] the same is true for the étale topology on S.*

(3) *For any smooth proper S-scheme X, the $\mathscr{T}$-spectra $M^{BM}(X/S)$ and $M_S(X)$ are pure.*

Proof. All the assertions in point (1) are clear, except possibly the assertion about coproducts; it follows from the fact $i^!$ commutes with coproducts since the functor $i_!$ respects compact (*i.e.* constructible under our assumptions) $\mathscr{T}$-spectra according to [CD12, 4.2.9].

Point (2) follows from the fact fundamental classes are compatible with pullbacks along étale morphisms, and from the fact f^* is conservative for a Nisnevich cover f (see [CD12, 2.3.8]).

For point (3), in view of 1.3.8(BM4), it is sufficient to consider the case of $M^{BM}(X/S)$. It follows from the following two facts:

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[32]*i.e.* for any étale cover $f : T \to S$, the functor f^* is conservative; see [CD12, 2.1.5];

- if one denotes by f the structural morphism of X/S, the functor $f_! = f_*$ commutes with $i_x^!$ and i^*;
- the immersion $X_x \to X$ is an essentially closed immersion of regular schemes whose fundamental class is the pullback of that of i_x along the smooth morphism f (see [Dég14]).

$\square$

4.4.5. We now recall the main constructions of [AHPL]. We consider the triangulated motivic category DM_R as in the convention of point (1) of 1.3.2. In particular, R is a $\mathbf{Q}$-algebra.[33]

Let $\mathbf{cGrp}_S$ be the category of smooth commutative group schemes over S and $\mathbf{Sh}_{\text{ét}}(S, R)$ the category of R-linear étale sheaves. Given such a group scheme G, we denote by $\underline{G}$ (see [AHPL, 2.1], except that we do not indicate the ring of coefficients in the notation for readability) the étale sheaf of R-vector spaces on Sm_S represented by G: for any smooth S-scheme X, $\Gamma(X, \underline{G}) = \mathrm{Hom}_S(X, G) \otimes_{\mathbf{Z}} R$.

Let us consider the following composite functor:

$$\mathcal{M} : \mathrm{D}(\mathbf{Sh}_{\text{ét}}(S, R)) \xrightarrow{\pi} D^{eff}_{\mathbf{A}^1, \text{ét}}(S, R) \xrightarrow{\Sigma^\infty} D_{\mathbf{A}^1, \text{ét}}(S, R) \simeq \mathrm{DM}(S, R)$$

where the first map is the canonical embedding, the second one is the $\mathbf{A}^1$-localization functor to the effective étale $\mathbf{A}^1$-derived category over S (see [Ayo07], or [CD12]), Σ^∞ is the infinite suspension functor and the last equivalence is Morel's theorem as proved in [CD12, 16.2.18]. Note that according to the theory developed in [CD12], $\mathcal{M}$ is in fact the left adjoint of a premotivic adjunction of triangulated categories.

We will denote abusively by $\underline{G}$ the image of $\underline{G/S}_R$, seen as a complex in degree 0, by this canonical functor — this is denoted by $M_1^{eff}(G/S)$ in [AHPL, 2.3]. If we assume further that S is a regular scheme with dimension function $\delta = -\mathrm{codim}_S$, then it follows from this definition that $\underline{G}$ is a δ-effective motive over S.

Let us summarize this part of the constructions of [AHPL, section 2].

Proposition 4.4.6 (Ancona, Pepin Lehalleur, Huber). *Consider the preceding notations. For any regular scheme S, there exists a canonical functor:*

$$\mathbf{cGrp}_S \to \mathrm{DM}(S, R), \ G \mapsto \underline{G}$$

which is aditive and sends exact sequences to distinguished triangles.

4.4.7. We will also use the central construction of [AHPL, D.1]. Let us consider the preceding notations and state the construction of *loc. cit.* in the particular case where we will use it. For any commutative group scheme G over S, there is a homologically non-negative complex of rational étale sheaves $A(G, R)$ together with a natural quasi-isomorphism of complexes of étale sheaves:

$$r_G : A(G, R) \to \underline{G/S}_R$$

such that for any index i, the i-th term $A(G, R)_i$ is of the form $R_S(H_i(G))$ where $H_i(G)$ is a finite coproduct of certain powers of G, seen as a smooth S-scheme Y, and $R_S(Y)$ denote as usual the R-linear étale sheaf freely represented by Y.

Moreover, for any morphism of schemes $f : T \to S$, one has the relation:

$$(4.4.7.a) \qquad\qquad H_i(G \times_S T) = H_i(G) \times_S T.$$

We can summarize this construction, a kind *cofibration resolution lemma of G*, using our slightly different notations as follows:

Proposition 4.4.8 (Ancona-Pepin Lehalleur-Huber). *Consider the above notations.*
Then one has a canonical isomorphism in $\mathrm{DM}(S, R)$:

$$\rho_G : \underline{G} = \mathcal{M}(\underline{G}) \xrightarrow{r_G^{-1}} \mathcal{M}(A(G, R)) = \mathrm{hocolim}_{i \in \mathbf{N}} M_S(H_i(G))$$

where the homotopy colimit runs over the category associated with the ordered set $\mathbf{N}$.

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[33]The constructions of *loc. cit.* are usually done only in the case where the coefficient ring $R = \mathbf{Q}$ but are valid in the case of an arbitrary $\mathbf{Q}$-algebra R.

Recall from [AHPL, 2.7] that the main corollary of this proposition is the following commutativity with base change along a morphism of schemes $f : T \to S$:[34]

(4.4.8.a) $$f^*(\underline{G}) = \underline{G \times_S T}.$$

Proof. Let us consider the étale descent model category structure on $C(\mathbf{Sh}_{\text{ét}}(S, R))$ (see [CD09]) whose homotopy category is $D(\mathbf{Sh}_{\text{ét}}(S, R))$. Since $A(G, R)$ is a bounded below complex of cofibrant objects, we get the following relation in $D(\mathbf{Sh}_{\text{ét}}(S, R))$:

$$A(G, R) = \text{hocolim}_{i \in \mathbf{N}}\big(A(G, R)_i\big) = \text{hocolim}_{i \in \mathbf{N}}\big(R_S(H_i(G))\big).$$

Then it sufficient to apply $\mathcal{M}$ to the preceding isomorphism, which commutes with homotopy colimits as a left adjoint, to this isomorphism to get the above statement. $\qquad\square$

One deduces from the preceding proposition the following result:

Proposition 4.4.9. *Consider the above notations. Then for any regular scheme S and any semi-abelian scheme G over S, the motive $\underline{G}$ of $\mathrm{DM}(S, R)$ is pure.*

Proof. According to Propositions 4.4.6 and 4.4.4(1), one needs only to consider the case where G is an abelian variety of a torus. The case of an abelian variety follows from the preceding proposition and points (1) and (3) of 4.4.4. The case of a torus follows from the étale separation property of DM_R (recall it is even separated [CD12, 14.3.3]), and points (1) and (2) of 4.4.4. $\qquad\square$

4.4.10. Recall that for a family $G_1, ..., G_n$ of semi-abelian varieties over a field k, Somekawa has introduced in [Som90] some abelian groups, now called *Somekawa K-groups*, defined by generators and relations that we will denote by:

$$K(k; G_1, ..., G_n).$$

These groups generalize both Milnor K-theory in degree n (take all $G_i = \mathbf{G}_m$) and Bloch's group attached to the Jacobian J of a smooth projective k-curve (take n=2, $G_1 = J$ and $G_2 = \mathbf{G}_m$).

Theorem 4.4.11. *Consider a regular scheme S with dimension function $\delta = -\operatorname{codim}_S$ and let G be a semi-abelian S-scheme.*

(1) *The δ-effective motive $\underline{G}$ is in the heart of the δ-homotopy t-structure on $\mathrm{DM}^{\delta-\mathit{eff}}(S, R)$.*

(2) *For any point $x \in S(E)$, and any integer $n \in \mathbf{Z}$, one has the following isomorphisms:*

$$\widehat{\underline{G}}(x, n) = \begin{cases} G_x(E) \otimes_{\mathbf{Z}} R & \text{if } n = 0, \\ L_x(E) \otimes_{\mathbf{Z}} R & \text{if } n = -1, \\ 0 & \text{if } n < -1, \\ K(E; G_x, \underbrace{\mathbf{G}_m, ..., \mathbf{G}_m}_{n \text{ times}}) \otimes_{\mathbf{Z}} R & \text{if } n > 0, \end{cases}$$

where L_x is the group of cocharacters of the toric part of the semi-abelian variety G_x over E.

Proof. According to Corollary 3.3.9, to prove point (1), we need only to prove that for any point $x \in S$, the δ-effective motive $i_x^!(\underline{G})$ is in the heart of $\mathrm{DM}^{\delta^x-\mathit{eff}}(x, R)$, where δ^x is the dimension function on x induced by the dimension function δ on S.

Because of the preceding theorem and formula (4.4.8.a), we get:

$$i_x^!(\underline{G}) = i_x^*(\underline{G})\langle -c_x \rangle = \underline{G_x}\langle -c_x \rangle$$

where c_x is the codimension of x in S. Let k be the residue field of x and δ_k the canonical dimension function on $x = \mathrm{Spec}(k)$. It is clear that we have the relation, as dimension function on x: $\delta^x = \delta_k - c_x$. In particular, the canonical functor:

$$\mathrm{DM}^{\delta^x-\mathit{eff}}(x, R) \to \mathrm{DM}^{\delta_k-\mathit{eff}}(x, R), M \mapsto M\langle c_x \rangle$$

is an equivalence of t-categories (Remarks 2.2.2 and 2.2.15(3)).

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[34]apply the proposition together with relation (4.4.7.a)

Thus, through this equivalence, the motive $i_x^!(\underline{G})$ is send to $\underline{G_x}$ in $\mathrm{DM}^{\delta_k-eff}(x, R)$. So we are restricted to the case where S is the spectrum of a field k.

The t-category $\mathrm{DM}^{\delta_k-eff}(x, R)$ is invariant under purely inseparable extensions (Lemma 2.2.18). Applying again formula (4.4.7.a), we are thus restricted to the case where k is a perfect field. Thanks to Example 2.3.12, the t-category $\mathrm{DM}^{\delta_k-eff}(x, R)$ is equivalent to Voevodsky's category $\mathrm{DM}^{eff}(k, R)$, equipped with the (standard) homotopy t-structure. Through this equivalence, according to [AHPL, 2.10], the motive $\underline{G}$ corresponds to the homotopy invariant Nisnevich sheaf represented by $G \otimes R$, with its canonical transfer structure (cf. [Org04, 3.1.2]). Therefore, it is in the heart of the homotopy t-structure and this concludes the proof of point (1). We will denote this sheaf by $\underline{G}^{tr}$.

We now consider point (2). The morphism $i_x : \mathrm{Spec}(E) \to S$ is essentially quasi-projective between regular schemes. Thus according to point (1) of Remark 4.2.3, and the fact $i_x^*(\underline{G}) = \underline{G_x}$ according to (4.4.8.a), we are reduced to the case where $S = \mathrm{Spec}\, E$, x being the tautological point.

Note that all members involved in the relation to be proved are invariant under purely inseparable extensions of the field E (for the left hand side, this follows from Lemma 2.2.18, the right hand side is obvious except for Somekawa K-groups, case which follows from the existence of norm maps and the fact we work with rational coefficients). Thus, we can assume that E is perfect.

According to what was said before, we thus are restricted to compute the homotopy module M over E associated with the homotopy invariant sheaf with transfers $\underline{G}$ (i.e. the graded sheaf $\sigma^\infty(\underline{G})$, see [Dég12, (1.18.b)]). According to [Kah14, 1.1], we get $(\underline{G})_{-1} \simeq L$ where L is the group of cocharacters of G, seen as a homotopy invariant Nisnevich sheaf with transfers. The first three relations follow because obviously $L_{-1} = 0$. Then the last relation follows from the main theorem of [KY13]. □

Theorem 4.4.12. *Under the assumptions of the preceding corollary, we get a canonical exact fully faithful functor:*

$$s\mathscr{A}_S^0 \to \Pi^{\delta\text{-}eff}(S, R),\ G \mapsto \underline{G}$$

where the source category is the category of semi-abelian S-schemes up to isogeny.

The exactness of the functor follows from 4.4.6, and the full faithfulness from the main theorem of [Pep15].

Remark 4.4.13.   (1) One can replace the abelian category $\Pi^{\delta\text{-}eff}(S, R)$ by $\Pi^\delta(S, R)$ or $\tilde{\Pi}^\delta(S, R)$ (Theorem 4.3.1).
  (2) Applying the main theorem of [AHPL, Th. 3.3], we get for any abelian S-scheme A of genus g a canonical isomorphism:

(4.4.13.a)
$$M_S(A) \xrightarrow{\sim} \bigoplus_{n=0}^{2g} \mathrm{Sym}^n(\underline{A})$$

which is a generalization of the Deninger-Murre decomposition of the Chow motive of A. Indeed, when S is a smooth k-scheme, This decomposition lives in the category of motives over S of weight 0 which is equivalent to the category of pure Chow motives according to [Jin15]. Through this equivalence $M_S(A)$ corresponds to the dual of the Chow motive of A/S and the preceding decomposition is equal to the dual of the Deninger-Murre decomposition.

When $A = E$ is an elliptic curve, the preceding theorem implies that $M_S(A)$ is concentrated in homological degree 0 and 1 for the t-structure t_δ^{eff}. In the next statement, we extend this result for S-curves of arbitrary genus.

4.4.14. Let S be a regular scheme and consider a smooth projective geometrically connected relative curve $p : C \to S$ with a given section $s : S \to C$.

In our assumptions, the motive $M_S(C)$ in $\mathrm{DM}(S, R)$ has a Chow-Künneth decomposition as in the classical case – though it is dual because Voevodsky's motives are homological: $M_S(C) =$

$p_! p^!(\mathbb{1}_S)$. First, the induced map $s_* : \mathbb{1}_S \to M_S(C)$ is split by p_*. Let us denote by $\tilde{M}_S(C, s)$ the cokernel of s_*, or equivalently, the kernel of p_*.

Recall that Voevodsky motives of smooth S-schemes are contravariant (up to a twist) with respect to projective morphisms, via the so-called Gysin morphism (cf. [Dég08a]). So, associated to the morphisms p and x, we get Gysin morphisms:

$$p^* : \mathbb{1}_S(1)[2] \to M_S(C), \ x^* : M_S(C) \to \mathbb{1}_S(1)[2]$$

satisfying the relation $x^* p^* = 0$. Thus p^* is a split monomorphism. Moreover, the composite map $p_* \circ p^*$ is a class in the motivic cohomology group $H_M^{-2,-1}(S, R)$ which is zero because S is regular. Hence p^* canonically factorizes into $\tilde{M}_S(C, s)$. Let us denote by $h_1^M(M_S(C))$ its cokernel in $\tilde{M}_S(C, s)$. We have finally obtained a canonical decomposition:

(4.4.14.a) $$M_S(C) = \mathbb{1}_S \oplus h_1^M(M_S(C)) \oplus \mathbb{1}_S(1)[2],$$

the *dual Chow-Künneth decomposition* associated with C/S.

Let us now consider the relative Jacobian scheme $J_{X/S}$ associated with C/S – the connected component of the identity of the Picard scheme associated with C/S, see [BLR90, §8.4]. The section $s \in C(S)$ gives a canonical map: $C \xrightarrow{\iota} J_{C/S}$. We deduce a canonical morphism of R-linear motives over S:

$$\varphi_s : h_1^M(M_S(C)) \to M_S(C) \xrightarrow{\iota_*} M_S(J_{C/S}) \to \underline{J_{C/S}}$$

where the first map is the canonical inclusion and second one is the canonical projection with respect to the decomposition 4.4.13.a. Another application of the preceding theorem is the following generalisation of a classical result of Voevodsky [VSF00, chap. 5, Th. 3.4.2]:

Proposition 4.4.15. *Consider the above assumption. Then the morphism φ_s is an isomorphism.*

In particular, for the dimension function $\delta = -\operatorname{codim}_S$, the δ-effective motive $M_S(C)$ is concentrated in homological degree 0 and 1 for the δ-homotopy t-structure on $\mathrm{DM}^{\delta-eff}(X, R)$ and one has canonical isomorphisms:

$$H_i^{\delta-eff}(M_S(C)) = \begin{cases} \mathbb{1}_S \oplus \underline{J(C)} & \text{if } i = 0, \\ \mathbb{1}_S\{1\} & \text{if } i = 1. \end{cases}$$

Finally, the distinguished triangle associated with the truncation function $\tau_{\geq 0}^\delta$ has the form:

$$\mathbb{1}_S \oplus \underline{J(C)} \to M_S(C) \to \mathbb{1}_S(1)[2] \xrightarrow{\partial} \mathbb{1}_S[1];$$

it splits since $\partial = 0$.

Proof. Consider the first assertion. Note first that the decompositions (4.4.14.a) and (4.4.13.a) are stable by base change: for the first one, it follows from the fact Gysin morphisms and compatible of smooth S-schemes ith base change, and for the second one by [AHPL, 2.7 and 3.3(4)]. Note the family of functors $i_{\bar{x}} : \mathrm{DM}(S, R) \to \mathrm{DM}(\bar{x}, R)$ indexed by geometric points of S is conservative: it follows directly from the following properties of DM_R: localization ([CD12, 14.2.11]), continuity ([CD12, 14.3.1]) and separation ([CD12, 14.3.3]) — see also [AHPL, Lem. A.6]. Thus to check φ_s is an isomorphism, we only need to consider its pullback over geometric points of S. In other words, we are reduced to the case of an separably closed field $S = \mathrm{Spec}(k)$. But then it follows from a classical fact about Chow-Künneth decompositions, which boils down to the identification of $CH^1(h^1(C)) = Pic^0(C)$ with $CH_1^1(J(C))$ with the notations of [Bea86] (see Prop. 3 of *loc. cit.* and the comments following it).

The remaining facts are then consequences of the preceding theorem, Example 3.3.3, and the vanishing of $H^{-1,-1}(S, R)$. $\square$

REFERENCES

[SGA4]   M. Artin, A. Grothendieck, and J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer-Verlag, 1972–1973, Séminaire de Géométrie Algébrique du Bois–Marie 1963–64 (SGA 4).

[SGA4.5]   P. Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA 41/2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.

[EGA1]   A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. I. Le langage des schémas*, Publ. Math. IHES **4** (1960).

[EGA2]   _____, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Publ. Math. IHES **8** (1961).

[EGA4]   _____, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Publ. Math. IHES **20, 24, 28, 32** (1964-1967).

[Stack]   Collective work (198 collaborators), *The Stacks Project*, version 17/7/2015, 2015.

————————

[AHPL]   G. Ancona, A. Huber and S. Pepin Lehalleur, *On the relative motive of a commutative group scheme*, Algebraic Geometry (to appear).

[ATJLSS03] L. Alonso Tarrío, A. Jeremías López, and M. J. Souto Salorio, *Construction of t-structures and equivalences of derived categories*, Trans. Amer. Math. Soc. **355** (2003), no. 6, 2523–2543 (electronic).

[Ayo07]   J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique (I, II)*, Astérisque, vol. 314, 315, Soc. Math. France, 2007.

[Ayo14]   Joseph Ayoub, *La réalisation étale et les opérations de Grothendieck*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 1, 1–145.

[BBD82]   A.A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982), 5–171.

[Bea86]   A. Beauville, *Sur l'anneau de Chow d'une variété abélienne*, Math. Ann. **273** (1986), no. 4, 647–651.

[BI15]   M.V. Bondarko and M.A. Ivanov, *On Chow weight structures for cdh-motives with integral coefficients*, Algebra i Analiz **27** (2015), no. 6, 14–40, arXiv:1506.00631.

[Blo81]   S. Bloch, *Algebraic K-theory and classfield theory for arithmetic surfaces*, Ann. of Math. (2) **114** (1981), no. 2, 229–265.

[BLR90]   S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990.

[BO74]   S. Bloch and A. Ogus, *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–201 (1975).

[Bon10a]   M.V. Bondarko, *Motivically functorial coniveau spectral sequences; direct summands of cohomology of function fields*, Doc. Math. (2010), no. Extra volume: Andrei A. Suslin sixtieth birthday, 33–117, arXiv:0812.2672v5.pdf.

[Bon10b]   _____, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory **6** (2010), 387–504, arXiv:0704.4003v8.

[Bon11]   _____, $\mathbb{Z}[\frac{1}{p}]$-*motivic resolution of singularities*, Composito Mathematica **147** (2011), 1434–1446.

[Bon14]   _____, *Weights for relative motives: relation with mixed complexes of sheaves*, International Mathematics Research Notices **17** (2014), 4715–4767, arXiv:1007.4543v6.

[Bon15]   _____, *Gersten weight structures for motivic homotopy categories; direct summands of cohomology of function fields and coniveau spectral sequences*, arXiv:1312.7493v4, 2015.

[BS15]   M.V. Bondarko and V.A. Sosnilo, *Detecting the c-effectivity of motives, their weights, and dimension via Chow-weight (co)homology: a "mixed motivic decomposition of the diagonal"*, arXiv:1411.6354v2, 2015.

[CD09]   D.-C. Cisinski and F. Déglise, *Local and stable homological algebra in Grothendieck abelian categories*, Homology, Homotopy and Applications **11** (2009), no. 1, 219–260.

[CD12]   _____, *Triangulated categories of mixed motives*, arXiv:0912.2110v3, 2012.

[CD15a]   _____, *Étale motives*, Comp. Math. **FirstView** (2015), 1–111.

[CD15b]   _____, *Integral mixed motives in equal characteristics*, Doc. Math. (2015), no. Extra volume: Alexander S. Merkurjev's sixtieth birthday, 145–194.

[Dég08a]   F. Déglise, *Around the Gysin triangle II*, Doc. Math. **13** (2008), 613–675.

[Dég08b]   _____, *Motifs génériques*, Rend. Semin. Mat. Univ. Padova **119** (2008), 173–244.

[Dég11]   _____, *Modules homotopiques*, Doc. Math. **16** (2011), 411–455.

[Dég12]   _____, *Coniveau filtration and motives*, Regulators, Contemporary Mathematics, vol. 571, 2012, pp. 51–76.

[Dég13]   _____, *Orientable homotopy modules*, Am. Journ. of Math. **135** (2013), no. 2, 519–560. MR 3038720

[Dég14]   _____, *Orientation theory in arithmetic geometry*, arXiv:1111.4203v2, 2014.

[Del87]   _____, *Le déterminant de la cohomologie*, Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), Contemp. Math., vol. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 93–177.

[Hí1]   David Hébert, *Structure de poids à la Bondarko sur les motifs de Beilinson*, Composito Mathematica **147** (2011), no. 5, 1447–1462 (French, with English summary).

[HW98]   A. Huber and J. Wildeshaus, *Classical motivic polylogarithm according to Beilinson and Deligne*, Doc. Math. **3** (1998), 27–133 (electronic).

[Ill71]   L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin-New York, 1971.

[ILO14]   L. Illusie, Y. Laszlo, and F. Orgogozo (eds.), *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*, Astérisque, vol. 361-362, S.M.F., 2014.

[Jin15]   F. Jin, *Borel-moore motivic homology and weight structure on mixed motives*, arXiv: 1502.03956, February 2015.

[Kah14]   B. Kahn, *Algebraic tori as Nisnevich sheaves with transfers*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014), no. 3, 699–715.

[KY13]   B. Kahn and T. Yamazaki, *Voevodsky's motives and Weil reciprocity*, Duke Math. J. **162** (2013), no. 14, 2751–2796.

[LYZ13]   M. Levine, Y. Yang, and G. Zhao, *Algebraic elliptic cohomology theory and flops, i*, arXiv:1311.2159, 2013.

[Mat89]   H. Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

[Mor03]   F. Morel, *An introduction to $\mathbf{A}^1$-homotopy theory*, Contemporary Developments in Algebraic K-theory, ICTP Lecture notes, vol. 15, 2003, pp. 357–441.

[Mor05]   ———, *The stable $\mathbf{A}^1$-connectivity theorems*, K-theory **35** (2005), 1–68.

[Mor12]   ———, *$\mathbb{A}^1$-algebraic topology over a field*, Lecture Notes in Mathematics, vol. 2052, Springer, Heidelberg, 2012.

[MV99]   F. Morel and V. Voevodsky, *$\mathbf{A}^1$-homotopy theory of schemes*, Publ. Math. IHES **90** (1999), 45–143.

[Nee01]   A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton Univ. Press, 2001.

[Org04]   F. Orgogozo, *Isomotifs de dimension inférieure ou égale à un*, Manuscripta Math. **115** (2004), no. 3, 339–360.

[Pel13]   P. Pelaez, *Birational motivic homotopy theories and the slice filtration*, Doc. Math. **18** (2013), 51–70.

[Pep15]   S. Pepin Lehalleur, *The motivic t-structure for relative 1-motives*, arXiv:1512.00266v1, 2015.

[PS14]   V. Pilloni and B. Stroh, *Fonctions de dimension*, Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents, Astérisque, vol. 361-362, S. M. F., 2014, pp. 279–294.

[Rio06]   J. Riou, *Opérations sur la K-théorie algébrique et régulateurs via la théorie homotopique des schémas*, Ph.D. thesis, Univ. Paris VII, 2006.

[Ros96]   M. Rost, *Chow groups with coefficients*, Doc. Math. J. (1996), 319–393.

[Som90]   M. Somekawa, *On Milnor K-groups attached to semi-abelian varieties*, K-Theory **4** (1990), no. 2, 105–119.

[Spi13]   M. Spitzweck, *A commutative $\mathbf{P}^1$-spectrum representing motivic cohomology over dedekind domains*, arXiv:1207.4078v3, 2013.

[Tem08]   Michael Temkin, *Desingularization of quasi-excellent schemes in characteristic zero*, Adv. Math. **219** (2008), no. 2, 488–522.

[Tem15]   M. Temkin, *Tame distillation and desingularization by p-alterations*, arXiv:1508.06255, 2015.

[Voe10]   Vladimir Voevodsky, *Cancellation theorem*, Doc. Math. (2010), no. Extra volume: Andrei A. Suslin sixtieth birthday, 671–685. MR 2804268 (2012d:14035)

[VSF00]   V. Voevodsky, A. Suslin, and E. M. Friedlander, *Cycles, transfers and motivic homology theories*, Annals of Mathematics Studies, vol. 143, Princeton Univ. Press, 2000.

St. Petersburg State University, Department of Mathematics and Mechanics,, Bibliotechnaya Pl. 2, 198904, St. Petersburg,, Russia
  E-mail address: m.bondarko@spbu.ru

E.N.S. Lyon - UMPA, 46, allée d'Italie, 69364 Lyon Cedex 07, France
  E-mail address: frederic.deglise@ens-lyon.fr
  URL: http://perso.ens-lyon.fr/frederic.deglise/