BOREL ISOMORPHISMS AND ABSOLUTE PURITY

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Abstract. We build an analogue of the Chern character replacing algebaric K-theory by hermitian K-theory, and motivic cohomology by the plus and minus parts of the rational sphere spectrum. We deduces that the rational sphere spectrum satisfies absolute purity and several interesting consequences.

Contents

Introduction. The *absolue purity conjecture*, stated for étale of Grothendieck torsion sheaves and by extension for ℓ-adic sheaves, has been a difficult problem since its formulation by Grothendieck in the mid-sixties – published lately in 1977, [Gro77]. For some time, only the case of one dimensional regular schemes was known thanks to Deligne, when Thomason first solved the case of ℓ-adic sheaves ([Tho84]), whose proof was later extended by Gabber to the general case (see [Fuj02]). An utlimate proof was found by Gabber, using a refinement of De Jong resolution of singularities, published in [ILO14, exp. XVI]/

The importance of this conjecture stands from its applications. First, it allows to show that constructibility (of complexes of étale sheaves) is stable under the direct image functor f_* (for f of finite type between quasi-excellent schemes). One deduces that constructibility is stable under the six operations (under very general assumptions). Then one obtains the so-called Grothendieck-Verdier duality for constructible complexes over schemes S

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with a dimension function.[1] This last point implies the existence of the (auto-dual) perverse t-structure over suitable base schemes, extending the fundamental work of [BBD82] (see [ILO14]).

For *triangulated mixed motives*, modeled on the previous étale setting by Beilinson, this conjecture was implicit in the expected property. It was first formulated and proved in the rational case by Cisinski and the first named author in [CD09]. Later the *absolute purity property* was explicitly highlighted in [CD15, Appendix], and proved for integral étale motives in [Ayo14] and [CD15]. It became apparent that this important property should hold in greater generality, and philosophically be an addition to the six functors formalism. Thus, it was conjectured in [Dég19] that this property, on the model of the algebraic K-theory spectrum, should hold for the algebraic cobordism spectrum and the sphere spectrum of Morel and Voevodsky's motivic homotopy theory.

The aim of this note is to prove the absolute purity conjecture for the rational sphere spectrum. Recall that rationally, the sphere spectrum splits into two parts, the plus and minus part. It was established in [CD09] that the plus-part agree with the rational motivic cohomology spectrum, and meanwhile satisfies the absolute purity property. We prove the general case by using a strategy similar to that of [CD09], except that one replaces Quillen's algebraic K-theory by hermitian K-theory, mainly due to Karoubi, Hornbostel and Schlichting. Indeed, thanks to work of the later, this later K-theory has all the necessary property — notably the so-called "dévissage". That being said, the crucial ingredient of our proof is to find an appropriate analogue of the Chern character for hermitian K-theory over a suitable base scheme S: noetherian, finite dimensional, with an ample family of line bundles and defined over $\mathbb{Z}[1/2]$ (see our conventions). We call it the *Borel character/isomorphism*. Its formulation allows to compute the ration hermitian K-theory spectrum $\mathrm{KQ}_{\mathbb{Q}}$ with the plus part and minus part of the rational sphere spectrum $\mathbb{Q}_{S+}$ and $\mathbb{Q}_{S-}$, which plays the role of the motivic cohomology spectrum. With these notations, the Borel character is an isomorphism of the ring spectra:

$$\mathrm{bo} : \mathrm{KQ}_{\mathbb{Q}} \to \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S+}(2m)[4m] \oplus \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S-}(4m)[8m]$$

(see Definition 2.11). Our construction uses in an essential way previous works of Ananyevskiy [Ana16] and Ananyevskiy, Levine, Panin [ALP17]. Note the Borel character will be studied further in [DFKJ]. The absolute purity conjecture for the rational sphere spectrum is then deduced from the analogous property established for hermitian K-theory: see Theorem 3.2

[1] Most notably, the existence of a dualizing complex $\mathcal{D}_S$ over S such that $D_S = \underline{\mathrm{Hom}}(-, \mathcal{D}_S)$ is an auto-equivalence of categories. The functor D_S then transforms f_* (resp. f^*) into $f_!$ (resp. $f^!$).

and its proof. An interesting application is the existence of a well-defined product for rational Chow-Witt groups of a regular base $\mathbb{Z}[|1/2]$-scheme.

A very noticeable consequence of the Borel isomorphism is that every rational spectrum, over arbitrary bases, is Sp-oriented in the sense of Panin and Walter ([PW10]). This is the $\mathbb{A}^1$-homotopy analogue of the well-known fact every rational spectrum in topology is oriented. Note that in $\mathbb{A}^1$-homotopy, there exists rational ring spectra that are non orientable in the classical sense (say: admits Chern classes): Chow-Witt groups and hermitian K-theory, rationally, already provide examples over fields with non-trivial Grothendieck-Witt groups.

Organization of the paper. . The paper is divived in three sections. In Section 1, we give some recall on ring spectra such as periodicity and representability of hermitian K-theory and Balmer's higher Witt groups (for regular schemes). In Section 2, we construct the Borel isomorphism and deduces that every rational spectrum is Sp-orientable. In Section 3, we establish the absolute purity of the rational sphere spectrum and draw some consequences.

Notations. Schemes are noetherian finite dimensional, admits an ample family of line bundles (this is to use Schlichting results in [Sch10]) and are defined over $\mathbb{Z}[1/2]$.

1. Recall on hermitian K-theory and higher Witt groups

See the reminder given by Jean in [DFKJ, Section 1].

Definition 1.1. We will denote by KQ_S the motivic ring spectrum representing hermitian K-theory over S: see [PW10].

We need the following properties:

(GW1) Given any map $f : T \to S$, $f^*\mathrm{KQ}_S = \mathrm{KQ}_T$. In other words, KQ is an absolute ring spectrum. This follows from the geometric model of hermitian K-theory using quaternionic Grassmanians (from [PW10, Th. 1.2]).

(GW2) For any regular scheme S, and any closed subscheme $Z \subset S$, such that $S - Z$ one has an isomorphism:

$$\mathrm{KQ}^{n,i}(S/S-Z) = \mathrm{GW}^{[i]}_{2i-n}(S \text{ on } Z) \tag{1.1.a}$$

where the right-hand side is Schlichting's higher Grothendieck-Witt groups: here we mean the $(2i-n)$-th homotopy group of the spectrum $\mathbb{G}W^n(\mathcal{A}_S \text{ on } Z, \mathcal{O}_S)$ of [Sch10, Def. 8 of Section 10]. In the followings, we will simply denote this spectrum by $\mathbb{G}W^{[n]}(S \text{ on } Z)$. This follows from [PW10, (1.2)].

Remark 1.2. We the twisting notation introduced for example in [DJK18], one can reformulate (1.1.a) as: $\mathrm{KQ}^n(S/S-Z, \langle i\rangle) = \mathrm{GW}^{[i]}_{-n}(S \text{ on } Z)$.

Recall the following folklore result (see [GS09]).

Proposition 1.3. *Let $\mathbb{E}$ be a motivic ring spectrum over S. Consider a pair of integers $(n,i) \in \mathbb{Z}^2$. Then the following conditions are equivalent:*

(1) *There exists an element $\rho \in \mathbb{E}_{n,i}(S)$, invertible for the cup-product on $\mathbb{E}_{**}$.*
(2) *There exists an isomorphism: $\tilde{\rho} : \mathbb{E}(i)[n] \to \mathbb{E}$.*

Definition 1.4. A pair $(\mathbb{E}, \rho)$ satisfying the equivalent conditions of the above proposition will be called an (n,i)-periodic ring spectrum over S.

An absolute (n,i)-periodic ring spectrum is an (n,i)-periodic ring spectrum over $\mathrm{Spec}(\mathbb{Z})$.

Proposition 1.5. *There exists a family of elements $\rho_S \in \mathrm{KQ}^{8,4}(S)$ indexed by schemes, stable under pullback, such that (KQ_S, ρ_S) is $(8,4)$-periodic.*

This follows from the construction of the spectrum KQ_S. The element ρ_S can be defined using [Sch10, Prop. 7], which implies that there exists a canonical isomorphism of spectra:

$$\mathbb{G}W^{[0]}(S) \simeq \mathbb{G}W^{[4]}(S).$$

Therefore using (GW1), one gets an isomorphism: $\psi_S : \mathrm{KQ}^{0,0}(S) \xrightarrow{\sim} \mathrm{KQ}^{8,4}(S)$ and we can put $\rho_S = \psi_S(1)$.

Following [Ana16], we introduce the following η-periodic spectra.

Definition 1.6 (Ananyevskiy)**.** Let $\eta : \mathbb{1}_S \to \mathbb{1}_S(-1)[-1]$ be the (desuspended) Hopf map. We define the η-periodized sphere spectrum $\mathbb{1}_S[\eta^{-1}]$ as:

$$\mathbb{1}_S[\eta^{-1}] = \mathrm{hocolim}\left(\mathbb{1}_S \xrightarrow{\eta} \mathbb{1}_S(-1)[-1] \xrightarrow{\eta(-1)[-1]} \mathbb{1}_S(-2)[-2] \xrightarrow{\eta(-2)[-2]} \dots\right).$$

Given any spectrum $\mathbb{E}$, we put: $\mathbb{E} = \mathbb{E}[\eta^{-1}]$.

We put $\mathrm{KW}_S = \mathrm{KQ}_S[\eta^{-1}]$. This defines an absolute ring spectrum.

In other words, (KW_S, η) is $(1,1)$-periodic. It is actually the $(1,1)$-periodization of KQ_S. Note also that the element $\rho_S \in \mathrm{KQ}^{8,4}(S)$ induces an element still denoted by $\rho_S \in \mathrm{KW}^{8,4}(S)$, and the above definition shows that (KW_S, ρ_S) is $(8,4)$-periodic.

From theorem 6.5 of [Ana16] (extended to regular schemes instead of smooth varieties; I believe this is formal), one gets:

Theorem 1.7 (Ananyevskiy)**.** *For any regular scheme S, there exists an isomorphism:*

$$\mathrm{KW}^{n,i}(S) \simeq W^{[n-i]}(S)$$

where the right-hand side is Balmer's higher Witt groups.

This isomorphism is contravariantly functorial in S, and induces an isomorphism of bigraded rings.

Remark 1.8. Of course, KQ and KW represents the $\mathbb{A}^1$-invariant version of higher Grothendieck-Witt and higher Witt groups respectively.

2. Rational Borel isomorphism

2.1. As KW is η-periodic, the unit map $\mathbb{1}_S \to \mathrm{KW}_S$ induces a unique morphism:

2.2. We fix an arbitrary base scheme S.

The symmetry involution permuting the factors $\mathbb{G}_m \otimes \mathbb{G}_m$ induces an involution $\epsilon : \mathbb{1}_S \to \mathbb{1}_S$. We then define two complementary projectors on $\mathbb{1}_S[1/2]$:

$$e_+ = \frac{1-\epsilon}{2}, e_- = \frac{1+\epsilon}{2},$$

and this defines Morel's decomposition: $\mathbb{1}_S[1/2] = \mathbb{1}_{S+} \oplus \mathbb{1}_{S-}$. More generally, given any spectrum $\mathbb{E}$ over S, we get a canonical decomposition:

$$\mathbb{E}[1/2] = \mathbb{E}_+ \oplus \mathbb{E}_-$$

such that ϵ acts by $+1$ (resp. -1) on E_+ (resp. $\mathbb{E}_-$).

Recall from Morel's computation that one has: $\eta = \epsilon\eta$. In particular, we get:

$$\mathbb{1}_S[1/2, \eta^{-1}] = \mathbb{1}_{S-}.$$

In view of Definition 1.6, we then deduce:

$$\mathrm{KQ}_- \simeq \mathrm{KW}[1/2].$$

Note in particular that $\mathrm{KW}_- = \mathrm{KW}[1/2]$.

Recall that (KW_S, ρ_s) is $(8,4)$-periodic. One deduces a canonical map:

$$\bigoplus_{m\in\mathbb{Z}} \mathbb{1}_S(4m)[8m] \xrightarrow{\sum_m \rho_S^m} \mathrm{KW}_S.$$

Taking the rational parts and projecting this map to the $-$-part, we finally obtain a canonical map, uniquely determined by ρ_S:

$$\psi_S : \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S-}(4m)[8m] \xrightarrow{\sum_m \rho_S^m} \mathrm{KW}_{S,\mathbb{Q}-}.$$

Note that by construction, the maps ψ_S are compatible with pullbacks in S. The following result is a corollary of a result of [ALP17, Cor. 5].

Theorem 2.3. *For any scheme S (in particular S is a $\mathbb{Z}[1/2]$-schemes), the map ψ_S is an isomorphism.*

Proof. Using the localization property and the continuity property of SH, it is sufficient to prove that for all point $x \in S$, $i_x : \operatorname{Spec} \kappa(x) \to S$ being the obvious immersion, the map $i_x^*(\psi_S)$ is an isomorphism. By compatiblity with base change, we are reduced to the map ψ_x over the field $\kappa(x)$, for which we apply [ALP17, Cor. 5]. $\square$

Definition 2.4. We denote by $\mathrm{bo}_{S,-}$ the inverse of ψ_S. It is a morphism of ring spectrum.

A remarkable corollary:

Corollary 2.5. *For any scheme S, any rational ring spectrum over S admits a canonical* Sp*-oriented. Indeed, $\mathbb{1}_{\mathbb{Q},S}$ is the universal* Sp*-orientable ring spectrum over S. In particular, the Thom space functor factorises through Deligne's Picard functor as follows:*

$$\begin{tikzcd} \underline{\mathrm{K}}(S) \arrow[r, "\mathrm{Th}_{S,\mathbb{Q}}"] \arrow[d, "(\det,\mathrm{rk})"'] & \mathrm{SH}(S)^{\otimes} \\ \underline{\mathrm{Pic}}(S) \arrow[ru, dashed, "\mathrm{Th}'_{S,\mathbb{Q}}"'] & \end{tikzcd}$$

In particular, the rational stable Thom space of a vector bundle depends only on its determinant and its rank.

This follows from the fact KQ is Sp-oriented
One deduces the following result.

Corollary 2.6. *Let $\mathbb{E}$ be an arbitrary rational ring spectrum $\mathbb{E}$ over S. For integer $(n,i) \in \mathbb{Z}^2$, a S-scheme X with structural map p (resp. p separated of finite type), and a line bundle L over X, one puts:*

$$\mathbb{E}^{n,r}(X,L) = \mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{1}_X, p^*\mathbb{E}(r)[n] \otimes \mathrm{Th}_S(L)),$$
$$\textit{resp. } \mathbb{E}_{n,r}(X/S,L) = \mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{1}_X(r)[n] \otimes \mathrm{Th}_S(L), p^!\mathbb{E}).$$

Then the following assertions hold:

- *For any smoothyfiable morphism $f : X \to S$, there exists a fundamental class $\eta_f \in \mathbb{E}_n, r(X/S, \det L_f)$ satisfying compatribility with composition and excess intersection formula.*
- *Assume $\mathbb{E}$ is absolutely pure and consider a smoothyfiable morphism $f : X \to S$ between regular schemes, with cotangent complex L_f. Then the following map:*
$$E^{n,r}(X,L) \to \mathbb{E}_{n,r}(X/S, \det(L_f) - L), x \mapsto x.\eta_f$$
is an isomorphism.

2.7. Let again S be an arbitrary scheme. Recall from [RØ16, Th. 3.4] that one has a canonical distinguished triangle:

$$\mathrm{KQ}_S(1)[1] \xrightarrow{\eta} \mathrm{KQ}_S \xrightarrow{f} \mathrm{KGL}_S \to \mathrm{KQ}_S(1)[2]$$

where KGL is the spectrum representing the homotopy invariant K-theory over S and f the *hyperbolic map.*

As $\eta_+ = 0$ and $\mathrm{KGL}_{S-} = 0$, we immediately deduces:

Proposition 2.8. *One has a split exact sequence in* $\mathrm{SH}(S)$*:*

$$0 \to \mathrm{KQ}_{S+} \xrightarrow{f} \mathrm{KGL}_S \to \mathrm{KQ}_{S+}(1)[2] \to 0.$$

In other words, $\mathrm{KGL}_S \simeq \mathrm{KQ}_{S+} \oplus \mathrm{KQ}_{S+}(1)[2]$.

Recall from Riou [Rio10, CD09] that the classical Chern character corresponds to an isomorphism of the following form in $\mathrm{SH}(S)$:

$$\mathrm{ch} : \mathrm{KGL}_{S,\mathbb{Q}} \to \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S+}(m)[2m]$$

where $\mathbb{Q}_{S+}$ is identified with either the rational motivic Eilenberg-MacLane spectrum (equivalently: the universal orientable ring spectrum).

Proposition 2.9. *The composition*

$$\mathrm{KQ}_{S,\mathbb{Q}+} \xrightarrow{f} \mathrm{KGL}_{\mathbb{Q},S} \xrightarrow{\mathrm{ch}} \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S+}(m)[2m]$$

induces an isomorphism:

$$\mathrm{KQ}_{S,\mathbb{Q}+} \xrightarrow{\mathrm{bo}_{S,+}} \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S+}(2m)[4m].$$

Proof. According to [Sch10], there is an isomorphism $\mathbb{G}W^{[0]} \simeq \mathbb{G}W^{[2]}_{\epsilon}$ where ϵ consists in taking the opposite duality. In particular, we get an isomorphism of functors $\mathbb{G}W^{[0]} \simeq \mathbb{G}W^{[4]}$ and, using the isomorphism (1.1.a), one deduces there is an element $\sigma_S \in \mathrm{KQ}^{4,2}_{+}(S)$ such that $(\mathrm{KQ}_+, \sigma_S)$ is $(4,2)$-periodic. By construction, one has $\sigma_S^2 = \rho_S$ and one can check that $f(\sigma_S) = \beta^2$. This finishes the proof. □

2.10. In particular, one gets a canonical isomorphism:
(2.10.a)

$$\mathrm{KQ}_{\mathbb{Q}} \simeq \mathrm{KQ}_{\mathbb{Q}+} \oplus \mathrm{KQ}_{\mathbb{Q}-} \xrightarrow{\mathrm{bo}_{S,+} \oplus \mathrm{bo}_{S,-}} \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S+}(2m)[4m] \oplus \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S-}(4m)[8m],$$

which, from the above constructions, is in fact an isomorphism of ring spectra.

Definition 2.11. We call the above isomorphism the Borel character and denote it by

$$\mathrm{bo} : \mathrm{KQ}_{\mathbb{Q}} \to \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S+}(2m)[4m] \oplus \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S-}(4m)[8m].$$

To my opinion, it would be better to write the right-hand term in terms of MW-motivic cohomology and ordinary motivic cohomology. Answer: the "abstract" Borel character does not involve MW-motivic cohomology as it is defined over an arbitrary base, and MW-motivic cohomology only over a perfect field. But see the applications!

Remark 2.12. It is possible to give to the above Borel character a form which is closer to the classical Chern character, if we think of its target in terms of motivic and MW-motivic cohomology. We refer the reader to our future work [DFKJ] for more details.

3. Absolute purity

3.1. Let $\mathbb{E} = (\mathbb{E}_S)$ be an absolute spectrum. Recall one says that $\mathbb{E}$ is absolutely pure ([DJK18, 4.3.11]) if for any quasi-projective lci morphism $f : Y \to X$ with cotangent complex L_f, the purity transformation:

$$\mathfrak{p}_f : \mathbb{E}_X \otimes \mathrm{Th}_X(\langle L_f \rangle) \to f^!(\mathbb{E}_X)$$

is an isomorphism.

Theorem 3.2. *The absolute spectrum* KQ *is absolutely pure.*

Proof. One uses the method of proof of [CD09, Th. 13.6.3, Rem. 13.7.5]. One needs only to consider closed immersions $i : Z \to X$ between regular schemes. Now one uses the isomorphism (1.1.a) so that we can apply Schlichting theory. Note the isomorphism (1.1.a) uses the devissage theorem [Sch10, Th. 14].

From [Sch17, 9.19], and the invariance under dg-equivalences of the Grothendieck-Witt groups associated with dg-categories with dualities, one deduces an isomorphism:

$$\mathfrak{p}'_f : \mathbb{G}W(X \text{ on } Z) \simeq \mathbb{G}W(Z, \det(-N_i)).$$

By definition of Thom spaces, and from the localization sequence in Hermitian K-theory, one deduces that the identification (1.1.a) extended to the following twisted version:

$$\mathfrak{t} : \mathrm{KQ}^n(X, v) \simeq \mathrm{GW}^{[r]}_{-n}(X, \det v)$$

for any virtual bundle v of rank r over a regular scheme X. In terms of the bivariant theory, $\mathrm{KQ}_n(X/S, v) \simeq \mathrm{GW}^{[r]}_n(X, \det(v))$. Then one identifies $\mathfrak{p}'_f$ and $\mathfrak{p}_f$ via the preceding isomorphism. This is easy: as each isomorphisms is functorial with respect to transverse base change, by deformation to the normal cone one reduces to the case of the zero section of a vector bundle. This is now just a normalization condition that one has to impose on $\mathfrak{t}$. $\square$

Corollary 3.3. *The following absolute spectra are absolutely pure:*

- *the Witt ring spectrum* KW*;*
- *the rational sphere spectrum* $\mathbb{1}_{\mathbb{Q}}$*;*
- *any strongly dualizable rational ring spectrum;*
- *any cellular rational spectrum in the sense of Dugger-Isaksen (see* [DI05, 2.10]*)*

The case of KW is clear from definition 1.6, as $i^* \otimes \mathrm{Th}(-N_i)$ and $i^!$ commutes with homotopy colimits (for $i^!$ we apply the localization property). The case of the spectra $\mathbb{Q}_{S+}$ and $\mathbb{Q}_{S-}$ follows from the Borel isomorphism (2.10.a), which shows in particular that both are direct factors of KQ_S. The other cases follow by devissage.

Example 3.4. We can then apply Corollary 2.6 to any ring spectrum of the preceding example to get duality results.

Corollary 3.5. *Let S be a regular scheme. Then for any integer $n \geq 0$, there exists an isomorphism:*

$$H^{2n,n}_{\mathbb{A}^1}(X, \mathbb{Q}) \simeq \widetilde{\mathrm{CH}}^n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where the right hand-side is the Chow-Witt group of S with coefficients in $\mathbb{Q}$. As a consequence, Chow-Witt groups when restriced to regular schemes admit products and Gysin maps with respect to projective morphisms.

This follows from the hyper-cohomology spectral sequence with respect to the δ-homotopy t-structure. Gysin morphisms follow from the construction of [DJK18].

Corollary 3.6. *Let S be a regular scheme and X be an S-scheme essentially of finite type. Then for any integer $n \geq 0$, there exists an isomorphism:*

$$H^{\mathbb{A}^1}_{2n,n}(X/S, \mathbb{Q}) \simeq \widetilde{\mathrm{CH}}_{\delta=n}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where the right hand-side is the Chow-Witt group of S of quadratic cycles sums of points x such that $\delta(x) = n$, tensored with $\mathbb{Q}$. As a consequence, these groups admits Gysin maps with respect to smoothable lci morphisms of S-schemes essentially of finite type.

This follows from the hyper-homology spectral sequence with respect to the δ-homotopy t-structure.

Let S be regular base. It is now possible to use Calmès-Fasel method to build MW-correspondences on smooth schemes over S, and to build the triangulated category of Milnor-Witt motives over S. Then one should be able to use the proof of [CD09, 16.2.13] prove that $\widetilde{\mathrm{DM}}(S, \mathbb{Q})$ is equivalent so $\mathrm{SH}(S)_{\mathbb{Q}}$.

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