Notes: The $\mathbb{A}^1$-homotopy category

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1 Introduction

The aim of the present exposé is to give a quick overview of the construction of the unstable $\mathbb{A}^1$-homotopy category of smooth schemes over a base S. Roughly, the construction goes as follow : starting from spaces Sm_S, one adds colimits by embedding in $\mathrm{Pre}(\mathrm{Sm}_S)$, the category of presheaves on Sm_S. Then, one adds homotopy colimits by embedding in $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_S)$, the category of simplicial presheaves on Sm_S. Then, one localises first with respect to the Nisnevich hypercoverings on $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_S)$, and localises again with respect to the projections $X \times \mathbb{A}^1 \to X$. This gives a model category $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_S)$. The unstable $\mathbb{A}^1$-homotopy category $\mathbb{H}$ is defined as the homotopy category of this model category.
The content of this exposé has been inspired by [AE16].

2 Model categories

Definition 1 *Let M be a category with small colimits and small limits. Given three classes W, F and C of morphism, which we call weak equivalences, fibrations and cofibrations respectively. One says the de 4-uple (M, W, F, C) is a* Model Category *if the following axioms are satisfied :*

(M1) *Given two composable morphism $X \xrightarrow{f} Y \xrightarrow{g}$, if two elements of $\{f, g, g \circ f\}$ are in W (resp. F, resp. C), then so is the third.*

(M2) *The three classes are stable by retraction, that is to say, given two morphisms f, g fitting in the following commutative diagram*

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{Id}_X"] \arrow[dr] \arrow[ddd, "f"'] & & X \arrow[dl] \arrow[ddd, "f"] \\
& X' \arrow[d, "g"] & \\
& Y' & \\
Y \arrow[ur] \arrow[rr, "\mathrm{Id}_X"'] & & Y \arrow[ul]
\end{tikzcd},$$

if g is in one of the class W, F or C, then so is f.

(M3) *Trivial fibrations (elements of $W \cap F$) have the right lifting properties with respect to cofibrations, and trivial cofibrations (elements of $W \cap C$) have the left lifting property with respect to fibration. In other words, given a diagram*[1]

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & A \arrow[d, two heads] \\ Y \arrow[r] \arrow[ur, dotted] & B, \end{tikzcd}$$

if the left arrow or the right arrow is a weak equivalence, there is a solid arrow filling the dotted arrow.

For all morphisms $f : X \to Y$, f can be factored as an acyclic cofibration $X \to Z$ followed by a fibration $h : Z \to X$, or as a cofibration $X \to Z'$ followed by a trivial fibration $Z' \to X$.

Example Let A be a ring, then $\mathrm{Ch}_{\geq 0}(A)$, the category of (homological) chain complexes concentrated on positive degrees have the structure of a model category by taking W to be the quasi-isomorphisms, F to be the maps that are degreewise surjective on strictly positive degrees, and taking $C = (W \cap F)_{\perp}$ [2].

Definition 2 *Let M be a model category. Denote by $\varnothing$ its initial element and $*$ its terminal. We say Y is* fibrant *if the unique morphism $Y \to *$ is a fibration, and we say that Y is* cofibrant *if the unique morphism $\varnothing \to Y$ is a cofibration.*
Given any object X, we can factor the morphism $X \to *$ as $X \hookrightarrow RX \twoheadrightarrow *$ where the cofibration is trivial. In such case, we say that RX is a fibrant replacement *of X. Similarly, we can factor $\varnothing \to X$ as $\varnothing \hookrightarrow QX \twoheadrightarrow X$ where the fibration is trivial. In such case, we say QX is a* cofibrant replacement *of X.*

Example Let us give an important example of model category. The category of simplicial sets $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Sets}$ has a model structure with the following data : weak equivalences are the maps of simplicial sets $f : X_\bullet \to Y_\bullet$ such that the induced map $|f| : |X_\bullet| \to |Y_\bullet|$ between geometric realization is a weak equivalence in the sense of classical homotopy theory. We define the cofibrations to be the maps that are levelwise monomorphisms. Fibrations are then defined by ${}_{\perp}(C \cap W)$, in this case, this turns out to be the Kan fibrations of simplicial sets.

Definition 3 *Given a category M and W any set of morphism, a* localization of M at W, *is a category $M[W^{-1}]$ endowed with a functor $\gamma : M \to M[W^{-1}]$ such that for all $w \in W$, $\gamma(w)$ is an isomorphism and that is universal with respect to this property. Localizations are then unique up to unique isomorphism and we often say that $M[W^{-1}]$ is the localization of M at W.*

Remark The localization of an arbitrary category at an arbitrary set of morphism always exist, but might not be small, i.e said category may not be in the same universe $\mathscr{U}$ as the category we started with, and the set of morphisms between two objects might not be a small $\mathscr{U}$-set. Often, we say that the localization "exists" to mean that it is a $\mathscr{U}$-category.

1. It is customary to denote fibrations as two-headed arrows, and cofibration as arrow with a hooked tail.
2. Given a class A of morphism, we denote $A_\perp$ to be the class of morphisms that have the left lifting property with respect to all morphisms in A and ${}_\perp A$ to be the class of morphisms that have the right lifting property with respect to all morphisms in A.

Theorem 2.1 *If (M, W, C, F) is a model category. then $M[W^{-1}]$ exists. Moreover, it is equivalent to the category* $\mathrm{Ho}(M)$*, where objects are objects of M that are both fibrant and cofibrant, and morphisms are homotopy classes of morphisms between two such objects*[3].

Let X, Y be elements of $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Sets}$, recall that their mapping space $\mathrm{map}_{\mathbf{\Delta}}(X, Y)$ is the simplicial set defined by $(\mathrm{map}_{\mathbf{\Delta}}(X, Y))_n = \mathrm{Hom}_{\mathbf{\Delta}^{\mathrm{op}}\mathrm{Sets}}(X \times \Delta^n, Y)$.

Definition 4 *Let $\mathcal{C}$ be a category. One says that the category is a* Simplicial category *if there is a simplicial mapping space bifunctor* $\mathrm{map}_{\mathcal{C}}(-,-)$ *from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to the category* $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Sets}$*, with the following properties :*

(1) *One has* $\mathrm{map}_{\mathcal{C}}(A, B)_0 = \mathrm{Hom}_{\mathcal{C}}(A, B)$ *for all $A, B \in \mathcal{C}$.*

(2) *For all A, the functor* $\mathrm{map}_{\mathcal{C}}(A, -)$ *has a left adjoint* $A \otimes - : \mathbf{\Delta}^{\mathrm{op}}\mathrm{Sets} \to \mathcal{C}$*, called the* action*.*

(3) *For all B, the functor* $\mathrm{map}_{\mathcal{C}}(-, B)$ *has a left adjoint* $B^{-} : \mathbf{\Delta}^{\mathrm{op}}\mathrm{Sets} \to \mathcal{C}^{\mathrm{op}}$ *called the* exponential*.*

Remark This definition is equivalent to the fact that $\mathcal{C}$ is enriched over $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Sets}$ and is both tensored and cotensored over it, we refer to [Lur09] A.1.3 for explanation of this terminology.

One of the most important consequence of being a simplicial category is that there is an adjunction $- \otimes X \dashv \mathrm{map}_{\mathcal{C}}(X, -)$ for all X.

Definition 5 *Let M be a model category. We say that it is a* simplicial model category *if it is a simplicial category, and that moreover, the following property, usually called "SM7", is satisfied : For all X in $\mathcal{C}$ and $i : A \hookrightarrow X$, for all $p : E \twoheadrightarrow B$, the map*

$$\mathrm{map}_M(X, E) \to \mathrm{map}_M(A, E) \times_{\mathrm{map}_M(A,B)} \mathrm{map}_M(X, B)$$

is a fibration of simplicial sets, i.e *a Kan fibration. Moreover, it is trivial if either i or p is trivial.*

Definition 6 *Let M and N be model categories. and let $F : M \leftrightarrows N : G$ be an adjoint pair. We say that it is a* Quillen adjunction *if either F respects cofibrations and trivial cofibrations or G respects fibrations and trivial fibrations*[4].

Proposition 2.2 *Given two model categories M and N, and a Quillen adjunction $F : M \leftrightarrows N : G$, there is an adjunction* $\mathbf{L}F : \mathrm{Ho}(M) \leftrightarrows \mathrm{Ho}(N) : \mathbf{R}G$*.*

Definition 7 *In the situation above, if $\mathbf{L}F \dashv \mathbf{R}G$ is an equivalence, we say that $F \dashv G$ is a* Quillen equivalence*.*

3 Bousfield Localization

Definition 8 *Let M be a simplicial model category, and I be a set of maps in M. Let $X \in M$. We say that X is* I-local *if X is fibrant and for any map $i : A \to B$ in I, the induced map* $i^* : \mathrm{map}_M(B, X) \to \mathrm{map}_M(A, M)$ *is a weak equivalence of simplicial sets.*
We say that a morphism $f : A \to B$ is an I-local weak equivalence *if for all I-local object X, the map* $f^* : \mathrm{map}_M(B, X) \to \mathrm{map}_M(A, X)$ *is a weak equivalence.*

3. We will not give the definition of the relation of homotopy between two objects in a model category, we refer the reader to [Lur09] A.2.2

4. This turns out to be equivalent.

Definition 9 *Given a model category (M, W, F, C) and a class of morphism I. We define W_I to be the set of I-local weak equivalences, $C_I = C$, and $F_I = {}_\perp(C_I \cap W_I)$. We say that (M, W_I, C_I, F_I) is the left Bousfield localization of M at I. We denote it $\mathrm{L}_I M$.*

Theorem 3.1 *If M is a simplicial model category that is left proper and combinatorial[5], then for any class I of morphism, $\mathrm{L}_I M$ is a simplicial model category, that is left proper and combinatorial.*

We refer the reader to [Lur09] A.3.7.3 for a proof of this theorem.

4 Simplicial presheaves and hypercoverings

Let $\mathcal{C}$ be an essentially small category. Let $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathcal{C})$ be the category of simplicial presheaves on $\mathcal{C}$.

Proposition and definition 4.1 *Define W to be the set of morphisms that are pointwise weak equivalences of simplicial sets, F to be the set of morphism that are are pointwise fibrations, and C to be $(W \cap F)_\perp$. The 4-uple $(\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathcal{C}), W, F, C)$ satisfies the axioms of a model category. Moreover, this model category is simplicial, combinatorial and left proper. We say that it is the* projective model structure *on $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathcal{C})$ or the* projective model category *on $\mathcal{C}$.*

This fact is proved in [Lur09] A.2.8.2 and A.2.8.4.
One has a Yoneda embedding $y : \mathcal{C} \to \mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathcal{C})$ and the projective model category is initial with respect to the model categories M with an embedding $\mathcal{C} \to M$.
Now assume that a Grothendieck topology τ is given on $\mathcal{C}$. Let $U_\bullet$ be an object of $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathcal{C})$, in other words, U_n is a presheaf of sets on $\mathcal{C}$. Then, let V be a representable object in $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathcal{C})$. We say that a map $U \to V$ is an hypercovering if every U_n is a coproduct of representable, $U_0 \to V$ is a τ-cover and, for each n. The map $U^{\Delta^n} \to U^{\partial\Delta^n}$ induced by applying the exponential functor to the inclusion $\partial\Delta^n \to \Delta^n$ induces τ-covers on degree zero. The standard example of hypercovers are the ones arising as the Čech complex associated to a τ-cover $U \to V$.

Theorem 4.2 *The left Bousfield Localization with respect to hypercovers exist. It is denoted by $\mathrm{L}_\tau\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathcal{C})$.*

The existence here comes from the fact that hypercovers form a set since we have assumed that our base category $\mathcal{C}$ is essentially small.
Remark that if τ is subcanonical, *i.e* if all representable presheaves are sheaves for τ, then, $\mathcal{C}$ embeds in $\mathrm{Ho}(\mathrm{L}_\tau\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathcal{C}))$.

5 The unstable $\mathbb{A}^1$-homotopy category of smooth schemes

We begin by applying the preceding construction with $\mathcal{C} = \mathrm{Sm}_S$ and τ is the Nisnevich topology. We recall its definition for convenience.

5. We will note define these terms. We refer to [Lur09] A.2.4.1 for the definition of a left proper model category, and to [Lur09] A.2.6.1 for the definition of a combinatorial model category. It will be enough to say that all the model categories we care about satisfy these condition

Definition 10 *A family of morphisms $\{U_i \to U\}_{i \in I}$ in Sm_S is said to be a* Nisnevich covering *if the $U_i \to U$ are étale and of finite type, and there exists a finite decreasing chain of closed subschemes $\varnothing \subset Z_n \subset \cdots \subset Z_0 = X$ of X such that the map $\coprod_i p_i^{-1}(Z_j \setminus Z_{j+1}) \to Z_j \setminus Z_{j+1}$ has a section for all $0 \leq j \leq n-1$.*

Definition 11 *Consider a commutative diagram*

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, hook] & X \end{tikzcd}$$

where p is étale, $U \hookrightarrow X$ is an open immersion. If $Z = (X \setminus U)_{\mathrm{red}}$ is such that $p^{-1}(Z) \cong Z$, we say that this square is a Nisnevich Distinguished square.

Proposition 5.1 *If S is Noetherian of finite Krull dimension, then $F \in \mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_S)$ is fibrant in $\mathrm{L}_{\mathrm{Nis}}\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_S)$ if for any Nisnevich distinguished square*

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, hook] & X, \end{tikzcd}$$

then the canonical map $F(X) \to F(U) \times_{F(U \times_X V)} F(V)$ is a weak equivalence of simplicial sets.

Definition 12 *Let $I = \{\mathbb{A}^1_S \times_S X \to X, X \in \mathrm{Sm}_S\}$. Then, the left Bousfield localization of $\mathrm{L}_{\mathrm{Nis}}\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_S)$ with respect to I exists, we denote the resulting model category by $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}\mathbb{A}^1$. It is a left proper combinatorial simplicial model category. We call it the* (unstable) $\mathbb{A}^1$-homotopy category

Example One can check that if $f, g : X \to Y$ are $\mathbb{A}^1$-homotopic, *i.e* if there is a map $H : \mathbb{A}^1 \times_S X \to Y$ such that $H(i_0 \times_S \mathrm{Id}_X) = f$ and $H(i_1 \times_S \mathrm{Id}_X) = g$, then f and g induce the same morphism in the homotopy category of $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}\mathbb{A}^1$.
One can also check that if $p : E \to X$ is a vector bundle, then p is a weak $\mathbb{A}^1$-local equivalence.

Références

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