The Gersten conjecture for Rost complexes

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Introduction

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We are going to study a corollary of Rost but in the Nisnevich site.

Contents

Cycle complexes

For a cycle module M over X, we define the complexes

$$C^p(X;M) = \coprod_{x \in X^{(p)}} M(x)$$

with differentials

$$d = d_X : C^p(X;M) \to C^{p+1}(X;M)$$

Cycle complexes

Another important remark, is that in practice, all cycle module that we are considering have a $\mathbb{Z}-$grading, so we put

$$C^p(X;M,n) = \coprod_{x\in X^{(p)}} M_{n-p}(x) \quad \left(C_p(X;M,n) = \coprod_{x\in X_{(p)}} M_{n+p}(x) \right)$$

$$C^*(X;M) = \coprod_{n\in\mathbb{Z}} C^*(X;M,n) \quad \left(C_*(X;M) = \coprod_{n\in\mathbb{Z}} C_*(X;M,n) \right)$$

Classical Chow groups

From the definition, we can see that

$$A^p(X; K_*, p) = \mathrm{CH}^p(X)$$

where K_*^M is the cycle module associated to the Milnor's K-theory.

Classical Chow groups

Classical Chow groups: If we consider the cycle module $M = K_*^M$ (Milnor's K-theory). Recalling that the classical Chow groups of $p-$codimension cycles on a variety X it is defined as the cokernel of the divisor map

$$\coprod_{x\in X^{(p-1)}} \kappa(x)^* \xrightarrow{\text{div}} \coprod_{x\in X^{(p)}} \mathbb{Z}$$

For each $n \in \mathbb{N}$ we have the complex $C^*(X;n)$ defined as follows:

$$C^p(X;n) := \coprod_{x\in X^{(p)}} K^M_{n-p}\kappa(x)$$

therefore

$$\ldots \to \coprod_{x\in X^{(p)}} K^M_{n-p}\kappa(x) \to \coprod_{x\in X^{(p+1)}} K^M_{n-(p+1)}\kappa(x) \to \ldots$$

As we said previously, there is a complex

$$\cdots \to \bigoplus_{x_{n-1}\in X^{(n-1)}} K_1^M(\kappa(x_{n-1})) \xrightarrow{d^n} \bigoplus_{x_n\in X^{(n)}} K_0^M(\kappa(x_n)) \to 0$$

Let $x_{n-1} \in X^{(n-1)}$, $Z = \overline{\{x_{n-1}\}}$ and we denote $\widetilde{Z}$ as the normalization of Z.
If $x_n \in Z^{(1)}$ then

$$\begin{array}{ccc} p^{-1}(\{x_n\}) & \hookrightarrow & \widetilde{Z} \\ \Big\downarrow \text{finite} & & \Big\downarrow p \\ \{x_n\} & \hookrightarrow & Z \end{array}$$

where $p^{-1}(\{x_n\}) = \{y_1, \ldots, y_N\}$.

$$(d^n)^{x_{n-1}}_{x_n}(\sigma) = \sum_{y_i} c_{\kappa(y_i)|\kappa(x_n)} \circ \partial_{v_i}(\sigma)$$

where v_i is the associated valuation to the discrete valuation ring $\mathcal{O}_{\widetilde{Z},y_i}$. Therefore we have that

$$K^M_*(\kappa(\widetilde{Z})) \xrightarrow{\partial_{v_i}} K^M_{*-1}(\kappa(y_i)) \xrightarrow{c_{\kappa(y_i)|\kappa(x_n)}} K^M_{*-1}(\kappa(x_n))$$

In degree 0, this map corresponds to the multiplication by the degree extension, and for degree 1 ∂_{v_i} corresponds to the valuation v_i. Hence

$$\begin{aligned}(d^n)^{x_{n-1}}_{x_n}(\sigma) &= \sum_{y_i} c_{\kappa(y_i)|\kappa(x_n)} \circ \partial_{v_i}(\sigma) \\ &= \sum_{y_i} [\kappa(y_i) : \kappa(x_n)] \cdot \mathrm{ord}_{v_i}(\sigma)\end{aligned}$$

Acyclicity for smooth local rings

We can state the main result as follow:

Theorem ([4, Theorem 6.1])

Let X be a smooth and semi-local scheme over a field k. Then

$$A^p(X; M) = 0 \quad \textit{for all } p > 0$$

Acyclicity for smooth local rings

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Theorem ([4, Theorem 6.1])

Let X be a smooth and semi-local scheme over a field k. Then

$$A^p(X; M) = 0 \quad \textit{for all } p > 0$$

This theorem is know for the following cases:

Quillen's K-theory (see [3] section 7, theorem 5.11),

étale cohomology (see [1]),

Acyclicity for smooth local rings

Let V be a vector space over k and let $\mathbb{A}(V)$ be the associated affine space. For a linear subspace W of V let

$$\begin{aligned} \pi_W : \mathbb{A}(V) &\to \mathbb{A}(V/W), \\ v &\mapsto \pi_W(v) = v + W \end{aligned}$$

Acyclicity for smooth local rings

Lemma ([4, Theorem 6.2])

Let $X \subset \mathbb{A}(V)$ be an equidimensional closed subvariety with $dim(X) = d$ and let $Y \subset X$ be a closed subvariety with $dim(Y) < d$ be a finite subset such that X is smooth in S. Then for a generic $(d-1)-$codimensional linear subspace W of V the following conditions hold.

1. *The restriction*

$$\pi_W\Big|_Y : Y \to \mathbb{A}(V/W)$$

   is finite.

2. *The restriction*

$$\pi_W\Big|_X : X \to \mathbb{A}(V/W)$$

   is locally around S smooth of relative dimension 1.

Acyclicity for smooth local rings

Proposition ([4, Proposition 6.4])

Let X be a smooth variety over a field and let $Y \subset X$ be a closed subscheme of codimension ≥ 1. Then for any finite subset $S \subset Y$ there is an open neighbourhood X' of S in X such that the map

$$i_* : A_*(Y \cap X'; M) \to A_*(X'; M)$$

is the trivial map. Here $i : Y \cap X' \to X'$ is the inclusion

Bloch's formula

In the smooth case, we can sheafify cycle modules in the following way:

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$$U \mapsto \mathcal{M}_X(U) := A^0(U; M) \subset M(\xi_X)$$

for U open subset of X.

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Corollary ([4, Corollary 6.5])

For a smooth variety X over k there are natural isomorphisms

$$A^p(X; M) = H^p_{Zar}(X; \mathcal{M}_X)$$

Nisnevich site

Let X_{Nis} be the small Nisnevich site of X. Let us define the Nisnevich sheaf $\mathcal{C}^M_{X_{\text{Nis}}}(U)$ which is the sheaf associated to $C^*(U_{\text{Nis}}, M)$.

Definition (Distinguished square)

A distinguished square is a pull-back square of schemes

$$\begin{array}{ccc} U \times_X V & \xrightarrow{j_2} & V \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j_1} & X \end{array}$$

where $j_1 : U \to X$ is an open immersion and $f : V \to X$ is an étale morphism such that the induced map from $f^{-1}((X - U)_{red})$ to $(X - U)_{red}$ is an isomorphism.

Nisnevich site

Definition

If F is a Nisnevich presheaf, then it is a sheaf if and only if for any distinguished square, we obtain a pull-back square

$$\begin{array}{ccc} F(X) & \xrightarrow{j_1^*} & F(U) \\ \downarrow{f^*} & & \downarrow{g^*} \\ F(V) & \xrightarrow{j_2^*} & F(U \times_X V) \end{array}$$

Definition

Let F be a complex of presheaf of $R-$modules. We will say that F have the Brown-Gersten property with respect to the Nisnevich topology if for any distinguished square we have that the following square

$$\begin{array}{ccc} F(X) & \xrightarrow{j_1^*} & F(U) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} \\ F(V) & \xrightarrow{j_2^*} & F(U \times_X V) \end{array}$$

is homotopy pullback.

Nisnevich site

Lemma

$\mathcal{C}^M_{X_{Nis}}$ is a Nisnevich sheaf.

Nisnevich site

Lemma

$\mathcal{C}^{M}_{X_{Nis}}$ *is a Nisnevich sheaf.*

Lemma

$\mathcal{C}^{M}_{X_{Nis}}$ *satisfies the Brown-Gersten property.*

Nisnevich site

Theorem

Let $F^M_{X_{Nis}}$ be the Nisnevich sheaf associated to $A^0(-, M)$, then the following conditions are equivalent:

1. $F^M_{X_{Nis}} \to \mathcal{C}^M_{X_{Nis}}$ *is a quasi-isomorphism.*
2. *For $n \in \mathbb{Z}$ we have that $\underline{H}^n(\mathcal{C}^M_{X_{Nis}}) = 0$ if $n \neq 0$, or $\underline{H}^n(\mathcal{C}^M_{X_{Nis}}) = F^M_{X_{Nis}}$ otherwise.*

The arguments for the proofs of the previous lemmas comes from the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{p-m}(Z,M) & \hookrightarrow & C^p(X,M) & \xrightarrow{j_1^*} & C^p(U,M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{f}^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} & & \\ 0 & \longrightarrow & C^{p-m}(T,M) & \hookrightarrow & C^p(V,M) & \xrightarrow{j_2^*} & C^p(U\times_X V,M) & \longrightarrow & 0 \end{array}$$

Nisnevich site

Proposition ([2, Proposition 1.1.10])

Let F be a complex of presheaves of R-modules. Then the following two conditions are equivalent:

1. *The complex F has the Brown-Gersten property.*
2. *For any X the canonical map*

$$H^n(F(X)) \to H^n_{Nis}(X, F_{Nis})$$

is an isomorphism of R-modules.

Conclusion

Remark

The complex $C^*(-,M)$ has the Brown-Gersten property, then $H^n_{Nis}(X, C^*(-,M)) \cong H^n(C^*(-,M))$, therefore we can conclude the following

$$\begin{array}{ccc} H^n_{Nis}(X, C^*(-,M)) & \xrightarrow{\cong} & H^n(C^*(X,M)) \\ \downarrow \cong & & \downarrow = \\ H^n(X, F^M_{Nis}) & \xrightarrow{\cong} & A^n(X;M) \end{array}$$

which give us an isomorphism between Chow groups with coefficient and the cohomology of sheaf in the Zariski and Nisnevich sites

$$A^n(X;M) \cong H^n_{Nis}(X, F^M_{Nis}) \cong H^n_{Zar}(X, F^M_{Zar})$$

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Thanks for your attention!