The axiomatic of Rost cycle modules II

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Definition

M is a cycle premodule over B if for all fields F over B, $M(F)$ is an abelian group with a $\mathbb{Z}$-grading ($M(F) = \bigoplus_{n\in\mathbb{Z}} M_n(F)$) with :

- D1 for all B-field extensions $\varphi : F \to E$ we have a restriction morphism $\varphi_* : M(F) \to M(E)$ of degree 0, also denoted $r_{E/F}$;
- D2 for all finite B-field extension $\varphi : F \to E$ we have a corestriction morphism $\varphi^* : M(E) \to M(F)$ of degree 0, also denoted $c_{E/F}$;
- D3 for all field F over B, the abelian group $M(F)$ is a left K_*F-module such that the product respects the gradings ($K_nF \bullet M_m(F) \subset M_{n+m}(F)$);
- D4 for all valuation $v : F^* \to \mathbb{Z}$ over B, we have a residue morphism $\partial_v : M(F) \to M(\kappa(v))$ of degree -1

satisfying some rules.

Definition

If M is a cycle premodule over B and π is a prime of v (i.e. $\mathfrak{m} = (\pi)$), we have a specialization morphism of degree 0
$s_v^\pi : \begin{cases} M(F) & \to & M(\kappa(v)) \\ \rho & \mapsto & \partial_v(\{-\pi\} \bullet \rho) \end{cases}$ (this uses D3 and D4).

If F is a field over $\mathrm{Spec}(k)$ then $M(F) = \bigoplus_{n\in\mathbb{N}} H^n_{dR}(F/k)$ (it is a $\mathbb{Z}$-grading with terms zero in negative degree).

D1) If $\varphi : F \to E$ is a k-field extension then we define $\varphi_* : M(F) \to M(E)$ by : $(\varphi_*)^n_n : H^n_{dR}(F/k) \to H^n_{dR}(E/k)$ is the morphism deduced from the morphism $\Omega^n_{F/k} \to \Omega^n_{E/k}$ which verifies

$$f_0 d_0(f_1) \wedge \cdots \wedge d_0(f_n) \quad \mapsto \quad \varphi(f_0) d_0(\varphi(f_1)) \wedge \cdots \wedge d_0(\varphi(f_n)).$$

D2) Let $\varphi : F \to E$ be a finite k-field extension of (finite) Galois closure $\overline{E}$ (we have $\psi : E \to \overline{E}$ and $\psi \circ \varphi : F \to \overline{E}$ is Galois). Denote $G = \mathrm{Gal}(\overline{E}/F)$.

We have a group action of G on $H^n_{dR}(\overline{E}/k)$ and for all $n \in \mathbb{N}$, $H^n_{dR}(\overline{E}/k)^G \simeq H^n_{dR}(F/k)$ (canonically).

We define $\mathrm{Tr}(\omega) = \sum_{\sigma\in G} \sigma \bullet \omega$ and $\varphi^* = \mathrm{Tr} \circ \psi_*$ (via the isomorphism).

D3) If F is a k-field and $f'_1, \dots, f'_l \in F^*$, define an additive morphism $\{f'_1, \dots, f'_l\}\bullet$ by $\{f'_1, \dots, f'_l\} \bullet (f_0 d_0(f_1) \wedge \dots \wedge d_0(f_n)) =$ $f_0 f_1'^{-1} \dots f_l'^{-1} d_0(f'_1) \wedge \dots \wedge d_0(f'_l) \wedge d_0(f_1) \wedge \dots \wedge d_0(f_n)$.

D4) Note that if $\kappa(v) \simeq k$ then $(\partial_v)^1_0 : H^1_{dR}(F/k) \to H^0_{dR}(\kappa(v)/k)$ is induced by

$$\partial : \begin{cases} \bigoplus_{f \in F} F\mathrm{d}f & \rightarrow & \kappa(v) \\ \sum_{i \in I} f_i dg_i & \mapsto & \sum_{i \in I, k \in \mathbb{Z}} a_{i,-k}\, k b_{i,k} \end{cases}$$ (with $f_i = \sum_{n \geq n_i} a_{i,n} \pi^n$ and $g_i = \sum_{n \geq m_i} b_{i,n} \pi^n$ in $\mathrm{Frac}(\widehat{O_v}) \simeq \kappa(v)((X))$, with π a prime of v, i.e. the sum of the residues of $f_i g'_i$).

For the general case, we define $H^n_{dR}(X) = H^n_{Zar}(X, \underline{\Omega^*_{X/k}})$ and $H^n_{dR}(X, Z) = H^n_{Z,Zar}(X, \underline{\Omega^*_{X/k}})$.

We have the de Rham localization sequence :

$$0 \longrightarrow H^0_{dR}(X,Z) \longrightarrow H^0_{dR}(X) \longrightarrow H^0_{Zar}(X \setminus Z, \underline{\Omega^*_{X/k}}) \xrightarrow{d_0} H^1_{dR}(X,Z)$$

$$\cdots \to H^n_{dR}(X,Z) \longrightarrow H^n_{dR}(X) \longrightarrow H^n_{Zar}(X \setminus Z, \underline{\Omega^*_{X/k}}) \xrightarrow{d_n} H^{n+1}_{dR}(X,Z)$$

We define $(\partial_v)^n_{n-1}$ to be d_n via the isomorphisms $H^n_{Zar}(X \setminus Z, \underline{\Omega^*_{X/k}}) \simeq H^n_{dR}(F)$ and $H^{n+1}_{dR}(X,Z) \simeq H^{n-1}_{dR}(Z) \simeq H^{n-1}_{dR}(\kappa(v))$ (thanks to a purity result and the facts that $\mathcal{O}_{X,Z} = \mathcal{O}_v$ and $\kappa(Z) = \kappa(v)$ (since $Z = \overline{\{x\}}$, $\mathcal{O}_{X,x} = \mathcal{O}_v$ and $F = \mathrm{Frac}(\mathcal{O}_v)$)).

The first set of rules is the following :

R1a) Whenever defined, $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$;
R1b) Whenever defined, $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$;
R1c) If $\psi : F \to L$ is a B-field extension and $\varphi : F \to E$ is a finite B-field extension, $R = L \otimes_F E$, then
$$\psi_* \circ \varphi^* = \sum_{p \in \mathrm{Spec}(R)} \mathrm{length}(R_p) \bullet (\varphi_p)^* \circ (\psi_p)_*$$ with $\varphi_p : L \to R/p$ and $\psi_p : E \to R/p$ the canonical morphisms (φ_p is finite since φ is)

Note that R1c) implies R2e) If $\varphi : E \to F$ is a finite and totally inseparable B-field extension then $\varphi_* \circ \varphi^* = \deg(\varphi) \bullet \mathrm{Id}$.

The second set of rules is the following :

R2a) Whenever defined, $\varphi_*(x \bullet \rho) = \varphi_*(x) \bullet \varphi_*(\rho)$;
R2b) Whenever defined, $\varphi^*(\varphi_*(x) \bullet \mu) = x \bullet \varphi^*(\mu)$;
R2c) Whenever defined, $\varphi^*(y \bullet \varphi_*(\rho)) = \varphi^*(y) \bullet \rho$

Note that in the expressions $\varphi_*(x)$ and $\varphi^*(y)$, the morphisms are the ones from Milnor K-theory (for instance, φ_* is the identity of $\mathbb{Z}$ or the morphism induced by φ).

Note that R2c) implies R2d) If $\varphi : E \to F$ is a finite B-field extension then $\varphi^* \circ \varphi_* = \deg(\varphi) \bullet \mathrm{Id}$.

The third set of rules is the following :

R3a) If $v : F^* \to \mathbb{Z}$ is a valuation over B and $\varphi : E \to F$ is a B-field extension such that $w = v \circ \varphi$ is a valuation over B then, denoting $\overline{\varphi} : \kappa(w) \to \kappa(v)$ the morphism induced by φ, we have $\partial_v \circ \varphi_* = |v(F)/w(E)| \bullet \overline{\varphi}_* \circ \partial_w$;
R3b) If $v : F^* \to \mathbb{Z}$ is a valuation over B and $\varphi : F \to E$ is a finite B-field extension, denoting $\varphi_w : \kappa(v) \to \kappa(w)$ the morphism induced by φ, we have $\partial_v \circ \varphi^* = \sum_w \varphi_w^* \circ \partial_w$;
R3c) If $v : F^* \to \mathbb{Z}$ is a valuation over B and $\varphi : E \to F$ is a B-field extension such that $v \circ \varphi = 0$ then $\partial_v \circ \varphi_* = 0$;
R3d) If $v : F^* \to \mathbb{Z}$ is a valuation over B and $\varphi : E \to F$ is a B-field extension such that $v \circ \varphi = 0$, and if π is a prime of v, then, denoting $\varphi : E \to \kappa(v)$ the morphism induced by φ, we have $s_v^\pi \circ \varphi_* = \overline{\varphi}_*$;
R3e) If $v : F^* \to \mathbb{Z}$ is a valuation over B, $u \in \mathcal{O}_v$ is a unit, of class $\overline{u} \in \kappa(v)$, and $\rho \in M(F)$, then $\partial_v(\{u\} \bullet \rho) = -\{\overline{u}\} \bullet \partial_v(\rho)$.

Note that R3e) implies R3f) If $v : F^* \to \mathbb{Z}$ is a valuation over B, π is a prime of v, $x \in K_n F$, and $\rho \in M(F)$, then
$\partial_v(x \bullet \rho) = \partial_v(x) \bullet s_v^\pi(\rho) + (-1)^n s_v^\pi(x) \bullet \partial_v(\rho) + \{-1\}\partial_v(x) \bullet \partial_v(\rho)$ and $s_v^\pi(x \bullet \rho) = s_v^\pi(x) \bullet s_v^\pi(\rho)$.

Definition

A morphism $\omega : M \to M'$ of cycle premodules over B of even type (respectively of odd type) is given by morphisms $\omega_F : M(F) \to M'(F)$ of degree 0 which are even : $\omega_F(-x) = \omega_F(x)$ (respectively odd : $\omega_F(-x) = -\omega_F(x)$) and satisfy :
$\varphi_* \circ \omega_F = \omega_E \circ \varphi_*$;
$\varphi^* \circ \omega_E = \omega_F \circ \varphi^*$;
$\{a\} \bullet \omega_F(\rho) = \omega_F(\{a\} \bullet \rho)$ which implies $\{a_1, \ldots, a_n\} \bullet \omega_F(\rho) = \omega_F(\{a_1, \ldots, a_n\} \bullet \rho)$ (respectively $\{a\} \bullet \omega_F(\rho) = -\omega_F(\{a\} \bullet \rho)$ which implies $\{a_1, \ldots, a_n\} \bullet \omega_F(\rho) = (-1)^n \omega_F(\{a_1, \ldots, a_n\} \bullet \rho)$);
$\partial_v \circ \omega_F = \omega_{\kappa(v)} \circ \partial_v$ (respectively $\partial_v \circ \omega_F = -\omega_{\kappa(v)} \circ \partial_v$).

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From now on, if X is an irreducible scheme, we denote by ξ_X its generic point.

If X is a normal and irreducible scheme then for each $x \in X^{(1)}$ $O_v := \mathcal{O}_{X,x}$ is a discrete valuation ring, and we denote by ∂_x the residue morphism $\partial_v : M(\kappa(\xi_X)) \to M(\kappa(x))$.

If X is a scheme and $x, y \in X$, we define $\partial^x_y : M(\kappa(x)) \to M(\kappa(y))$ by : if $y \notin \overline{\{x\}}^{(1)}$ then $\partial^x_y = 0$, else $\partial^x_y = \sum_z c_{\kappa(z)/\kappa(y)} \circ \partial_z$ with z running through the points (in finite number) of the normalization of $\overline{\{x\}}$ lying over y.

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- FD (finite support of divisors) For all normal and irreducible schemes X, $\rho \in M(\kappa(\xi_X))$ and all but finitely many $x \in X^{(1)}$, $\partial_x(\rho) = 0$;
- C (closedness) For all integral and local schemes X of dimension 2, denoting x_0 the closed point of X, $\sum_{x \in X^{(1)}} \partial_{x_0}^{x} \circ \partial_{x}^{\xi_X} = 0$.

Morphisms of cycle modules are morphisms of cycle premodules between cycle modules.

Note that in (FD), $\partial_x = \partial_x^{\xi_X}$, that if $x \notin X^{(1)}$ then $\partial_x^{\xi_X} = 0$, and that more generally (FD) implies that if $y \in X$, $\rho \in M(\kappa(y))$, then for all but finitely many $z \in X$, $\partial_z^y(\rho) = 0$.

Definition

Let M be a cycle module over X. The complex of cycles on X with coefficients in M, denoted $C_*(X; M)$, is defined by :

- for all integers $p \geq 0$, $C_p(X; M) = \bigoplus_{x \in X_{(p)}} M(\kappa(x))$;
- $d_X : C_p(X; M) \to C_{p-1}(X; M)$ is defined by $d_y^x = \partial_y^x$.

The maps d_X are well-defined and verify $d_X \circ d_X = 0$ thanks to axioms (FD) and (C).

If X is an integral scheme which verifies (FD), we define $d : M(\kappa(\xi_X)) \to \bigoplus_{x \in X^{(1)}} M(\kappa(x))$ by $d = (\partial_x^{\xi_X})_{x \in X^{(1)}}$ and

$$A^0(X; M) := \bigcap_{x \in X^{(1)}} \ker(\partial_x^{\xi_X}).$$

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H (homotopy property for $\mathbb{A}^1$) For all fields F over B, $\mathbb{A}^1_F$ verifies (FD) (hence d is well-defined) and we have the short exact sequence $0 \to M(F) \to M(F(X)) \to \bigoplus_{x \in (\mathbb{A}^1_F)_{(0)}} M(\kappa(x)) \to 0$, the second map being $r_{F(X)/F}$ and the third map being d (with $(\mathbb{A}^1_F)_{(0)}$ the points of $\mathbb{A}^1_F$ whose closure is of dimension 0);

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- **RC** (reciprocity for curves) For all fields F over B and for all proper curves X over F we have $c \circ d = 0$, with

$$c : \begin{cases} \bigoplus_{x \in X_{(0)}} M(\kappa(x)) & \to & M(F) \\ (\rho_i \in M(\kappa(x_i))) & \mapsto & \sum_i c_{\kappa(x_i)/F}(\rho_i) \end{cases}.$$

Note that if k is a perfect field and M is a cycle premodule over k then M is a cycle module over k if and only if M verifies the homotopy property for $\mathbb{A}^1$ (H).

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Indeed :

- (H) implies (FDL) (the (FD) axiom for the affine lines over fields over k) by definition;
- (H) implies (WR) (which is weak reciprocity : $\partial_\infty(A^0(\mathbb{A}^1_F; M)) = 0$ with ∂_∞ the residue morphism for the valuation at infinity of $F(X)$ over F), see Step 3 : (FD) + (H) $\Rightarrow$ (RC) in Rost's paper (p.341) and note that if you weaken (FD) into (FDL) you get (WR);

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- (FDL) and (WR) imply that M is a cycle module (see Theorem (2.3) in Rost's paper (p.338)).

Hence we only need to show that the cycle premodule of de Rham cohomology (over a field k of characterisitic zero (hence perfect)) verifies the homotopy property for $\mathbb{A}^1$ (H) in order to know that it is a cycle module.
To get (H), take the colimit over (the directed set of) the closed subschemes Z of dimension 0 of $\mathbb{A}^1_F$ of the de Rham localization sequence with $H^n_{dR}(\mathbb{A}^1_F)$ replaced by the isomorphic group $H^n_{dR}(F)$:

$$\cdots \to H^{n-2}_{dR}(Z) \xrightarrow{i_*} H^n_{dR}(F) \xrightarrow{p^*_n} H^n_{Zar}(\mathbb{A}^1_F \setminus Z, \underline{\Omega^*_{\mathbb{A}^1_F/k}}) \xrightarrow{d_n} H^{n-1}_{dR}(Z)$$

Note that p^*_n is injective since $s^*_n \circ p^*_n$ is, with $s : \mathrm{Spec}(L) \to \mathbb{A}^1_F \setminus Z$ the inclusion of a closed point; the point being closed, $f := p \circ s$ is the spectrum of a finite extension φ, hence by R2d) $\varphi^* \circ \varphi_* = \deg(\varphi) \bullet \mathrm{Id}$ and $(\varphi_*)_n = f^*_n = s^*_n \circ p^*_n$ is injective.

Note that we also have (though we won't use them) :

If M is a cycle module, X is a smooth and local scheme (we denote by x_0 its closed point), $Y \to X$ is the blow-up of X at x_0, v is the valuation corresponding to the exceptional fiber over x_0, then :

- Co (continuity) $A^0(X; M) \subset A^0(Y; M)$ i.e. $\partial_v(A^0(X; M)) = 0$;
- E (evaluation) There exists a unique morphism $\mathrm{ev} : A^0(X; M) \to M(\kappa(x_0))$ such that for all prime π of v, $r_{\kappa(v)/\kappa(x_0)} \circ \mathrm{ev} = s^{\pi}_{v|A^0(X;M)}$.

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If $\omega : M \to N$ is a morphism of cycle modules over X (of even or odd type) and U is a subset of X, we define the change of coefficients induced by ω, denoted $\omega_\sharp : \bigoplus_{x \in U} M(\kappa(x)) \to \bigoplus_{x \in U} N(\kappa(x))$, by $(\omega_\sharp)_x^x = \omega_{\kappa(x)}$ and the other components are 0.

Note that if ω is of even type (resp. of odd type) then $d_X \circ \omega_\sharp = \omega_\sharp \circ d_X$ (resp. $d_X \circ \omega_\sharp = -\omega_\sharp \circ d_X$).

Hence $\omega_\sharp$ induces a morphism between the homology groups of the complex of cycles on X with coefficients in M.

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Let $f : X \to Y$ be a morphism of schemes of finite type over a field, $p \geq 0$ be an integer, and M be a cycle module over B (hence over X and over Y). The push-forward of f, denoted $f_* : C_p(X; M) \to C_p(Y; M)$, is defined by $(f_*)^x_y = c_{\kappa(x)/\kappa(y)}$ if $y = f(x)$ and the morphism $\kappa(y) \to \kappa(x)$ induced by f is finite, 0 otherwise.

Note that $(f' \circ f)_* = f'_* \circ f_*$.

If $\omega : M \to N$ is a morphism of cycle modules over B then $\omega_\sharp \circ f_* = f_* \circ \omega_\sharp$.

If we further suppose that $f : X \to Y$ is proper, then we have $d_Y \circ f_* = f_* \circ d_X$.

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Definition

If $g : Y \to X$ is a morphism of schemes of finite type over a field of constant relative dimension d (i.e. all non-empty fibers of g are equidimensional of dimension d) then the pull-back of g, denoted $g^* : C_p(X; M) \to C_{p+d}(Y; M)$, is defined by $(g^*)_y^x = 0$ if $x \neq g(y)$, $(g^*)_y^{g(y)} = \text{length}(f^*\mathcal{O}_Y) \bullet r_{\kappa(y)/\kappa(g(y))}$ with $f : \text{Spec}(\mathcal{O}_{Y_{g(y)},y}) \to Y_{g(y)} \to Y$ ($Y_{g(y)}$ being the fiber over $g(y)$).

Interesting cases are open immersions, closed immersions and base change (the projection $X \times_{\text{Spec}(F)} \text{Spec}(E) \to X$ with $F \to E$ a morphism of fields and X of finite type over F), which all have constant relative dimension 0.

If $\omega : M \to N$ is a morphism of cycle modules over B then $\omega_\sharp \circ g^* = g^* \circ \omega_\sharp$.

If we further suppose that $g : Y \to X$ is flat then $(g' \circ g)^* = g^* \circ g'^*$ and $d_Y \circ g^* = g^* \circ d_X$.

Proposition

Note that if

$$\begin{array}{ccc} U & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & X \end{array}$$

is a cartesian square of schemes of finite type over a field and g is of constant relative dimension then $g^* \circ f_* = f'_* \circ g'^*$.

The proof uses rule R1c).

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Definition

If $a_1, \ldots, a_n \in \mathcal{O}_X(X)^*$, we define $\{a_1, \ldots, a_n\} : C_p(X; M) \to C_p(X; M)$ by $\{a_1, \ldots, a_n\}_x^x(\rho) = \{a_1(x), \ldots, a_n(x)\} \bullet \rho$ and the other components are 0.

This turns $C_p(X; M)$ into a module over the tensor algebra of $\mathcal{O}_X(X)^*$, and if X is a scheme over a field F then it turns $C_p(X; M)$ into a module over K_*F.

Note that $\{a_1, \ldots, a_n\} \circ \{b_1, \ldots, b_m\} = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$.

If $\omega : M \to N$ is a morphism of cycle modules over B of even type then $\{a_1, \ldots, a_n\} \circ \omega_\sharp = \omega_\sharp \circ \{a_1, \ldots, a_n\}$.

If $\omega : M \to N$ is a morphism of cycle modules over B of odd type then $\{a_1, \ldots, a_n\} \circ \omega_\sharp = (-1)^n \, \omega_\sharp \circ \{a_1, \ldots, a_n\}$.

Note that $\{a_1, \ldots, a_n\} \circ d_X = (-1)^n \, d_X \circ \{a_1, \ldots, a_n\}$.

Note that $f_* \circ \{f^*(a_1), \ldots, f^*(a_n)\} = \{a_1, \ldots, a_n\} \circ f_*$ and $g^* \circ \{a_1, \ldots, a_n\} = \{g^*(a_1), \ldots, g^*(a_n)\} \circ g^*$ (where if $f : Y \to X$, $f^* : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ is the associated morphism, and idem for g).

Note that if $f : Y \to X$ is a finite and flat morphism of schemes of finite type over a field and $a \in \mathcal{O}_Y(Y)^*$ then $f_* \circ \{a\} \circ f^* = \{\tilde{f}_*(a)\}$ with $\tilde{f}_* : \mathcal{O}_Y(Y)^* \to \mathcal{O}_X(X)^*$ the standard transfer map (if $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are integral then $\tilde{f}_* = (N_{\mathrm{Frac}(B)/\mathrm{Frac}(A)})_{|B^*}^{|A^*}$ (the norm)).

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Definition

If X is a scheme of finite type over a field, $i : Y \to X$ is the inclusion of a closed subscheme, $j : U := X \setminus Y \to X$ is the inclusion, we call (Y, i, X, j, U) a boundary triple and we define the boundary map associated to i, denoted $\partial_Y^U : C_p(U; M) \to C_{p-1}(Y; M)$, by $(\partial_Y^U)_y^x = \partial_y^x$ (with respect to X).

If $\omega : M \to N$ is a morphism of cycle modules over B of even type then $\partial_Y^U \circ \omega_\sharp = \omega_\sharp \circ \partial_Y^U$.

If $\omega : M \to N$ is a morphism of cycle modules over B of odd type then $\partial_Y^U \circ \omega_\sharp = -\omega_\sharp \circ \partial_Y^U$.

Note that $d_Y \circ \partial_Y^U = -\partial_Y^U \circ d_U$.

Note that if $a_1, \dots, a_n \in \mathcal{O}_X(X)^*$ then
$\partial_Y^U \circ \{j^*(a_1), \dots, j^*(a_n)\} = (-1)^n \{i^*(a_1), \dots, i^*(a_n)\} \circ \partial_Y^U$.

Let $h : X \to X'$ be a morphism of schemes of finite type over a field, Y' be a closed subscheme of X', $U' = X' \setminus Y'$ and

$$\begin{array}{ccccc} Y & \longrightarrow & X & \longleftarrow & U \\ \downarrow{\scriptstyle \overline{h}} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \overline{\overline{h}}} \\ Y' & \longrightarrow & X' & \longleftarrow & U' \end{array}$$

be two cartesian squares.

If h is proper then $\overline{h}_* \circ \partial_Y^U = \partial_{Y'}^{U'} \circ \overline{\overline{h}}_*$.

If h is flat then $\overline{h}^* \circ \partial_{Y'}^{U'} = \partial_Y^U \circ \overline{\overline{h}}^*$.

Thanks for your attention !