Real Deligne cohomology

The Goal: Exponential sequence for real Deligne cohomology,

$$\mathbb{Z}(1)_{\mathcal{D}/\mathbb{R}} \xrightarrow{\sim} \mathcal{O}_X^*[-1]$$

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· Complex case: Let $X/\mathbb{C}$ be a smooth projective variety.

Let $A \subset \mathbb{R}$ be a subring and $A(d) := (2\pi i)^d A \subseteq \mathbb{C}$

Def: The Deligne complex with coefficients in A is defined as:

$$A(d)_{\mathcal{D}} := \quad \cdots \to 0 \to \underset{\substack{\uparrow \\ \text{degree } 0}}{A(d)} \to \mathcal{O}_X \to \Omega_X^1 \to \cdots \to \underset{\substack{\uparrow \\ \text{degree } p}}{\Omega_X^{p-1}} \to 0 \to \cdots$$

$\rightsquigarrow$ Deligne cohomology

$$H^p_{\mathcal{D}}(X, A(q)) := \mathbb{H}^p(X, A(q)_{\mathcal{D}})$$

Remark: ① There is a quasi-isomorphism

$$A(d)_{\mathcal{D}} \overset{q.i.}{\simeq} \operatorname{Cone}\left(A(d) \oplus \underbrace{F^d \Omega_X^{\bullet}}_{\rightsquigarrow \text{ Hodge Filtration}} \to \Omega_X^{\bullet}\right)[-1]$$

$F^d \mathcal{E}^p = \bigoplus_{\substack{p'+q=p \\ p' \geq d}} \mathcal{E}^{p',q}$

② · For $q=0$ we have $H^q_{\mathcal{D}}(X, A) \simeq H^q_{sing}(X, A)$

· Using the short exact sequence (of sheaves)

$$x \mapsto \exp(x)$$

$$0 \to (2\pi i)\mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

we obtain a long exact sequence relating singular + $H^m(X, \mathcal{O}_X)$ and $H^*(X, \mathcal{O}_X^*)$

Also $\mathbb{Z}(1)_{\mathcal{D}} \overset{\cdot 2\pi i}{\simeq} \mathcal{O}_X^*[-1]$ using the map $x \mapsto \exp(x)$

$d=1$

$$\begin{array}{ccccccccc} \dots \to 0 & \to & \mathbb{Z}(1) & \to & \mathcal{O}_X & \to & 0 & \to & \dots \\ \downarrow & & \downarrow & & \downarrow \wr\ \exp & & \downarrow & & \\ 0 & \to & 0 & \to & \mathcal{O}_X^* & \to & 0 & \dots & \end{array}$$

Remark: Variant → Deligne-Beilinson complex, for a smooth open complex variety $\mathcal{U}$. Fix $\mathcal{U} \hookrightarrow \bar{X}$ a compactification such that $D = \bar{X} \setminus \mathcal{U}$, n.c.D.

$A(d) \rightsquigarrow Rj_* A(d)$ on $\bar{X}$

$F^d \Omega^\cdot_{\mathcal{U}} \rightsquigarrow F^d \Omega^\cdot_X(\log D)$

$\Omega^\cdot_{\mathcal{U}} \rightsquigarrow Rj_* \Omega^\cdot_X$

$A(d)_{DB} := \mathrm{cone}^\cdot(Rj_* A(d) \oplus F^d \Omega^\cdot_X(\log D) \to Rj_* \Omega^\cdot_X)[-1]$

$H^p_{BD}(\mathcal{U}, A(d)) := \mathbb{H}^p(X, A(d)_{DB})$

Real Case:

Notation:

- $An_{/\mathbb{R}}$:= denote the category of real holomorphic manifolds → objects (M, σ) $M \to$ holomorphic manifold, $\sigma \to$ anti-holomorphic involution

$(M, \sigma) \xrightarrow{F} (N, \varepsilon)$

↳ morphisms

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow \sigma & \circlearrowleft & \downarrow \varepsilon \\ M & \xrightarrow{F} & N \end{array}$$

Let $\mathcal{G} = Gal(\mathbb{C}/\mathbb{R})$ be the Galois group

- $\mathcal{G}$-Man $\to$ category of smooth manifolds with smooth G actions and equi. morphisms. $X \in G$-Man

$Cov(X)$ to be sets of open covering that are G-invariant

- $Sm/_{\mathbb{R}} \to$ smooth real algebraic varieties
- $An/_{\mathbb{C}} \to$ holomorphic manifolds
- $Sm/_{\mathbb{C}} \to$ smooth complex algebraic varieties

For $X \in Sm/_{\mathbb{R}} \rightsquigarrow X(\mathbb{C})$ complex valued points with the analytic topology

Bredon Complex: We denote A^p the sheaf of smooth complex valued differential p-forms on G-Manifolds

And $\mathcal{E}^p(X) := \{\theta \in A^p(X) \mid \overline{\sigma^*(\theta)} = \theta\}$

Invariants p-forms under the action σ on X and conjugation

- Last time we saw that there exists a morphism of complexes $\tau_p : \underbrace{A(p)_{Br}}_{\text{representation of those elements}} \to \mathcal{E}^*$

Consider $\mathcal{U} \in G$-man. and $0 \leq j \leq p$ one element in $A^j(p)_{Br}(\mathcal{U})$ is represented ~~by~~ the sum of pairs of elements $\alpha \otimes m$ where $\alpha = (a, f)$ with the following properties

1) $a: \Delta^{p-j-1} \times S \longrightarrow (\mathbb{C}^\times)^{p-1}_i \subset (\mathbb{C}^\times)^p$ is smooth and $\pi: S \to \mathcal{U}$ covering of $\mathcal{U}$ $\not{\mathbb{Z}}(1)^0_{pos}$

2) $f: \Delta^{p-j} \times S \to (\mathbb{C}^\times)^p$ is a smooth map

3) $m: S \longrightarrow A \in \underline{A}(S)$ is locally constant

If $p = 1$ then we consider representations as sums of elements of the form $f \otimes m$.

- Let $A \subset \mathbb{R}$ be a subring

i) Given $p \geq 0$ we define p-th equivariant Deligne complex $A(p)_{\mathcal{D}/\mathbb{R}}$ ~~as~~

$$A(p)_{\mathcal{D}/\mathbb{R}} := \text{cone}^\cdot\big(A(p)_{Br} \oplus F^p \mathcal{E}^* \xrightarrow{\ \varepsilon_p - \iota\ } \mathcal{E}^*\big)[-1]$$

- Given a proper manifold $X \in An_{/\mathbb{R}}$ and $p \geq 0$ we define the Deligne cohomology of X as

$$H^i_{\mathcal{D}/\mathbb{R}}(X, A(p)) := \mathbb{H}^i(X_{ep}, A(p)_{\mathcal{D}/\mathbb{R}})$$

- If $p < 0$

$$H^i_{\mathcal{D}/\mathbb{R}}(X, A(p)) := H^{i,p}_{Br}(X(\mathbb{C}), \underline{A})$$

2.- Exponential sequence: Here we want to show that $\mathbb{Z}(1)_{\mathcal{D}/\mathbb{R}} \overset{f.i.}{\sim} \mathcal{O}_X^*[-1]$

Def: Let $X \in Sm/\mathbb{R}$ and let $(\mathcal{R}_X^*, d_R)$ be the following complex

$$(\mathcal{E}^{0,0})^* \xrightarrow{\bar\partial \log} \mathcal{E}_X^{0,1} \xrightarrow{\bar\partial} \mathcal{E}_X^{0,2} \xrightarrow{\bar\partial} \cdots$$

Where $(\mathcal{E}_X^{0,0})^* \subset \mathcal{E}_X^0$ denotes the subsheaf of nowhere zero functions and $\mathcal{E}_X^{p,q} \subset \mathcal{E}_X^{p+q}$ [illegible] $(p+q)$-forms of Hodge type (p,q)

Remarks. $(\mathcal{R}_X^*, d_R)$ is a resolution of $\mathcal{O}_X^*$ and $\exp: \mathcal{O}_X \to \mathcal{O}_X^*$ induce a map

$$\exp: \mathcal{E}^{0,*} \to \mathcal{R}^*$$

Def Let $\eta: \mathbb{Z}(1)^1_{Br} \to (\mathcal{E}_X^{0,0})^*$ be the map that for $\mathbb{Z}(1)^1_{Br}(\mathcal{U})$

$$f \otimes m \longmapsto \hat{\pi}_!(f^m)$$

$$\eta(f\otimes m)(u) = \prod_{s \in \pi^{-1}(u)} f(s)^{m(s)}$$

$$\begin{tikzcd}
\Delta^{p-j}\times S \arrow[r, "1\times\pi"] \arrow[dr, "\hat{\pi}"'] & \Delta^{p-j}\times \mathcal{U} \arrow[d, "pr"] \\
 & \mathcal{U}
\end{tikzcd}$$

Prop (Proposition 4.5 [dS-LF]) There exists a quasi-isomorphism $\xi : \mathbb{Z}(1)_{\mathcal{D}/\mathbb{R}} \longrightarrow \mathcal{R}^*[-1]$

Proof Let X be a projective real variety

$\xi : \mathbb{Z}(1)_{\mathcal{D}/\mathbb{R}} \longrightarrow \mathcal{R}^*[-1]$

$$\mathbb{Z}(1)_{\mathcal{D}/\mathbb{R}}.$$

$$\begin{tikzcd}
\to 0 \arrow[r] & \mathbb{Z}(1)^0_{Br} \arrow[r, "d^{-1}"] \arrow[d] & \mathbb{Z}(1)^1_{Br} \oplus \mathcal{E}^{1,0}_X \oplus \mathcal{E}^0_X \arrow[r, "d^0"] \arrow[d, "\eta \cdot \exp"] & \mathcal{E}^{1,1} \oplus \mathcal{E}^{2,0} \oplus \mathcal{E}^1 \arrow[r] \arrow[d, "p^{0,1}"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \mathcal{R}^0 \arrow[r] & \mathcal{R}^1 \arrow[r] & \dots
\end{tikzcd}$$

where $\eta \cdot \exp$ is the map $(f, \omega, h) \longmapsto \eta(f)\exp(h)$

and $p^{0,i}$ is the projection from $\mathcal{E}^i \to \mathcal{E}^{0,i}$

- The proof $\to$ Just notice that both complexes are concentrated in degree 1
- $H^1(\xi)$ is an iso on the stalks
- Surjectivity $H^1(\mathcal{R}^*[-1]) = H^0(\mathcal{R}^*) \simeq \mathcal{O}^*_X$

Let $x \in X$ and let $g \in \mathcal{O}^*_{X,x}$ there exists h such that $\exp(h) = g$

- Set $\alpha = 0$ in $\mathbb{Z}(1)_{Br,x}$ and set $\omega = \frac{\partial g}{g} \in \mathcal{E}^{1,0}_{X,x}$
- $d^0(\alpha, \omega, h) = 0$ and $\xi(\alpha, \omega, h) = g$.

$$\tau_1 : \mathbb{Z}(1)_{Br} \longrightarrow \mathcal{E}_X^1.$$

<u>Injectivity</u> Let $(\alpha, \omega, h) \in \mathbb{Z}^1_{\mathcal{D}/\mathbb{R}, U}$ such that

$\xi(\alpha, \omega, h) = 0$ and $d^0(\alpha, \omega, h) = 0$

$\eta(\alpha)\exp(h)$ $\Rightarrow$ $-h$ is a logarithm of $\eta(\alpha)$ $(*)$

degree: $\omega + dh + \tau_1^1 \alpha = 0$

$\omega = -dh - \tau_1^1\alpha$

$\tau_1^1(\alpha) = \hat{\pi}_! [p_2^*(m) \cdot f^*(\frac{dt}{t})]$ $\underset{(**)}{=} d\log\eta(\alpha)$

$$\begin{tikzcd} U & \Delta^{p-j} \times U \arrow[l, "p_1"'] & \Delta^{p-j} \times S \arrow[l, "1\times\pi"'] \arrow[r, "f"] \arrow[d, "p_2"] & (\mathbb{C}^\times)^p \\ & & S \arrow[r, "m"] & \mathbb{Z} \end{tikzcd}$$

$\omega = -dh - \tau_1^1 \alpha$

$(*) + (**) \Rightarrow \omega = 0.$

So now take $\sum f_i \otimes m_i$ as a representation of α and we can choose $\log f_i$ such that

$$\sum m_i \log f_i = -h$$

$\beta := \exp(-t \sum m_i \log f_i) : \Delta^1 \times S \longrightarrow \mathbb{C}^\times$

$\beta \in \mathbb{Z}(1)^0_{Br, U}.$

$d^{-1}_{\mathcal{D}/\mathbb{R}}(\beta) = (\alpha, \omega, h)$ $\blacksquare$

<u>Corollary 1</u>: $\mathbb{Z}(1)_{\mathcal{D}/\mathbb{R}} \overset{q.i.}{\simeq} \mathcal{O}_X^\times[-1]$

<u>Corollary 2</u>: Let X be smooth proper real algebraic variety. Then there is a long exact sequence

$$\cdots \to H^{n,1}_{Br}(X(\mathbb{C}), \mathbb{Z}) \to H^n(X, \mathcal{O}_X) \to H^n(X, \mathcal{O}_X^\times) \to H^{n+1,1}_{Br}(X(\mathbb{C}), \mathbb{Z}) \to \cdots$$

Idea: $\mathbb{Z}(1)_{D,R} = \mathrm{cone}\{\mathbb{Z}(p)_{Br} \oplus F^1\mathcal{E}^\bullet \longrightarrow \mathcal{E}_X^\bullet\}[-1]$

So we have a long exact sequence $H^n(X, \mathcal{O}_X^*)$

$$\cong$$

$$\to H^{n+1}_{Br}(X(\mathbb{C}), \mathbb{Z}) \oplus \mathbb{H}^n(X, F^1\mathcal{E}^\bullet) \to \mathbb{H}^n(X, \mathcal{E}_X^\bullet) \to \mathbb{H}^{n+1}_{D,R}(X, \mathbb{Z}(1)) \to \cdots$$

$$\cdot\ H^{n+1}_{Br}(X(\mathbb{C}), \mathbb{Z}) \to \underbrace{\mathrm{coker}\left(\mathbb{H}^n(X, F^1\mathcal{E}^\bullet) \to \mathbb{H}^n(X, \mathcal{E}_X^\bullet)\right)}_{\text{is injective}} \to H^n(X, \mathcal{O}_X^*)$$

$$\mathrm{coker}(\cdots) \cong \mathbb{H}^n(X, \mathcal{E}_X^{0,*}) \cong H^n(X, \mathcal{O}_X)$$

and $\mathrm{cone}(F^1\mathcal{E}_X^\bullet \longrightarrow \mathcal{E}_X^\bullet) \simeq \mathcal{E}_X^{0,*}$