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The Generalized Lefschetz Trace Formula

With an Application to the Riemann hypothesis

Groupe de travail
Paris 13

Contents

Conventions

For a field k, a *variety over* k will be a finite type separated k-scheme. A *subvariety* of a variety is a closed subscheme. A *curve over* k is a variety over k pure of dimension 1, i.e., all irreducible components are 1-dimensional. A *surface over* k is a variety over k pure of dimension 2.

In this talk all varieties will be over a finite field or an algebraically closed field.

Groupe de travail

This talk was given at Paris 13 for the groupe de travail: *L'hypothése de Riemann d'après Deligne.* The first part of this groupe de travail is concerned with the article (commonly referred to as) WeilI. It contains a proof of the Riemann hypothesis over finite fields.

One of the main tools in Deligne's proof of the Riemann hypothesis is the generalized Lefschetz trace formula for ℓ-adic sheaves on varieties over finite fields. This talk aims at explaining the statement and proof of this generalized Lefschetz trace formula.

Please let me know of any typo's, mistakes, incomplete arguments, etc. My email is ariyanjavan at gmail.com

CHAPTER 1

The generalized trace formula

1. Introduction

Fix a finite field k and an algebraic closure $\overline{k}$. Fix a prime number ℓ which is invertible in k, i.e., $\ell \neq \operatorname{char} k$. Let X_0 be a variety over k. We let X denote the base change $X_0 \times_k \overline{k}$. The Frobenius correspondance on X is denoted by Frob.

THEOREM 1.1. Suppose that X_0 is smooth and projective over k. For a finite field extension $k \subset K$ of degree n, it holds that

$$\# X_0(K) = \sum_i (-1)^i \operatorname{Tr}((\mathrm{Frob}^n)^*, H^i_c(X_{et}, \mathbf{Q}_\ell).$$

PROOF. This follows from the properties of a Weil cohomology (such as Poincaré duality, Künneth formula, etc.) and the fact that the fixed point subscheme $X^{\mathrm{Frob}^n} = X_0(K)$. □

By the theorem below, the hypothesis of smooth and projective is not necessary. In fact, this Theorem is a generalization of Theorem 1.1 for ℓ-adic sheaves due to Grothendieck.

Let $\mathcal{F}_0$ be a $\mathbf{Q}_\ell$-sheaf on X_0 and let $\mathcal{F}$ be the pullback to X.

THEOREM 1.2. **(Generalized Trace Formula)** For every positive integer n, we have that

$$\sum_{x \in X^{\mathrm{Frob}^n}} \operatorname{Tr}((\mathrm{Frob}^n)^*, \mathcal{F}_x) = \sum_i (-1)^i \operatorname{Tr}((\mathrm{Frob}^n)^*, H^i_c(X, \mathcal{F})).$$

EXAMPLE 1.3. If $\dim X_0 = 0$, the statement is trivial. Write it out to see the link between traces and fixed points explicitly.

As was explained in the second talk of this groupe de travail, the rationality of Grothendieck's zeta function follows from this theorem.

COROLLARY 1.4. Grothendieck's zeta function $L(X_0, \mathcal{F}_0, t)$ is rational.

This talk will give the main ideas of the proof of this theorem. Due to lack of time, we will not be able to present all the details of the proof.

Nevertheless, for the sake of completion and personal interest, we have added a chapter with details of some Lemma's on derived categories that we will use in the proof. We have also added a chapter giving some applications and examples of the trace formula.

2. Perfect complexes

Let Λ be a left noetherian (possibly noncommutative!) ring. Let $K(\Lambda)$ be the homotopy category of left Λ-modules and $D(\Lambda)$ the derived category.

DEFINITION 1.5. The full subcategory of $K(\Lambda)$ whose objects are bounded complexes of finite type projective left Λ-modules is denoted by $K_{\mathrm{parf}}(\Lambda)$. The functor $K_{\mathrm{parf}}(\Lambda) \longrightarrow D(\Lambda)$ is fully faithful. Its essential image is denoted by $D_{\mathrm{parf}}(\Lambda)$. An object of $D_{\mathrm{parf}}(\Lambda)$ is called a *perfect complex*.

We give two examples.

EXAMPLE 1.6. Let A be a commutative noetherian ring. For elements $x_1, \dots, x_n$ in A and E the free A-module of rank n with basis $(e_1, \dots, e_n)$, we define the *Koszul complex* $K^A(x_1, \dots, x_n)$ associated to the sequence $(x_1, \dots, x_n)$ to be

$$0 \longrightarrow \Lambda^n E \xrightarrow{d} \Lambda^{n-1} E \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^1 E = E \xrightarrow{d} \Lambda^0 E = A \longrightarrow 0\,.$$

Here the boundary map $d : \Lambda^p E \longrightarrow \Lambda^{p-1} E$ is given by

$$d(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} e_{i_1} \wedge \ldots \wedge \hat{e_{i_j}} \wedge \ldots \wedge e_{i_p}.$$

The reader may verify that $d^2 = 0$. Note that for any permutation σ of the set $\{1, \dots, n\}$, the Koszul complex $K^A(x_1, \dots, x_n)$ is isomorphic to the Koszul complex $K^A(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. An element $x \in A$ is called *regular* if the multiplication by x is injective. A sequence $(x_1, \dots, x_n)$ of elements $x_1, \dots, x_n \in A$ is said to be a *regular sequence* if x_1 is regular and the image of x_i in $A/(x_1 A + \ldots + x_{i-1} A)$ is regular for all $i = 2, \dots, n$. Let $(x_1, \dots, x_n)$ be a sequence in A and let I be the ideal generated by it. Assume $I \neq A$. If $(x_1, \dots, x_n)$ is regular, the augmented Koszul complex

$$0 \longrightarrow \Lambda^n E \xrightarrow{d} \Lambda^{n-1} E \xrightarrow{d} \cdots \xrightarrow{d} E \xrightarrow{d} A \longrightarrow A/I \longrightarrow 0$$

is exact. Therefore, the A-module A/I (viewed as a complex in degree 0) is a perfect complex.

EXAMPLE 1.7. Let A be a noetherian local regular ring with unique maximal ideal $\mathfrak{m}$. Then $A/\mathfrak{m}$ is a perfect complex of A-modules by the above example. (Any system of parameters of A forms a regular sequence.)

EXAMPLE 1.8. Let k be a field. Suppose that $A = k[x, y]/(xy)$ and $\mathfrak{m} = (x, y)A$. Let $k = A/\mathfrak{m}$ be the corresponding residue field. Consider the infinite resolution of free A-modules

$$\cdots \xrightarrow{g} A^2 \xrightarrow{h} A^2 \xrightarrow{g} A^2 \xrightarrow{h} A^2 \xrightarrow{g} A^2 \xrightarrow{f} A \longrightarrow k \longrightarrow 0\,.$$

Here

$$f : (s, t) \mapsto sx + ty,\ \ g : (s, t) \mapsto (sy, tx) \text{ and } h : (s, t) \mapsto (sx, ty).$$

Note that

$$\mathrm{Tor}_i^A(k, k) = \begin{cases} k & \text{if} \quad i = 0 \\ k^2 & \text{if} \quad i > 0 \end{cases}.$$

To prove this, note that after tensoring the above resolution with k the maps become zero. This shows that k does not have a finite projective resolution of A-modules. Else the $\mathrm{Tor}_i^A(k, -)$ functors would be identically zero for $i \gg 0$. Therefore, the A-module k is not a perfect complex of A-modules.

REMARK 1.9. Why perfect complexes? The answer lies in the following facts. Firstly, as we will show in the next section, we can define the trace of an endomorphism of a perfect complex. Furthermore, the compactly supported Euler characteristic of a constructible $\mathbf{Z}/\ell^n$-sheaf on a variety (over a finite field or algebraically closed field) is a perfect complex of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules. We will prove these facts in the following sections.

3. Traces and perfect complexes

Let Λ be a left noetherian ring. (We will only be concerned with $\mathbf{Z}/\ell^n\mathbf{Z}[G]$, where G is a finite group.)

DEFINITION 1.10. Let H be the subgroup of Λ^+ generated by the elements $ab - ba$. We define the additive group $\Lambda^\# := \Lambda^+/H$.

EXAMPLE 1.11. For $\Lambda = \mathbf{Z}/\ell^n\mathbf{Z}[G]$, we have that

$$\Lambda^\# = \bigoplus_{\text{conjugacy classes of } G} \mathbf{Z}/\ell^n\mathbf{Z}.$$

REMARK 1.12. We all know how to define the trace of an endomorphism $f : \Lambda^r \longrightarrow \Lambda^r$. To force the "usual" equality $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$, we will consider the image of the usual trace in $\Lambda^\#$. We will now be more precise.

Free modules: Let $M \cong \Lambda^r$ and $f : M \longrightarrow M$ an endomorphism. We define $\mathrm{Tr}(f) = \mathrm{Tr}(f, M)$ as the image of $\sum f_{ii}$ in $\Lambda^\#$. Here $(f_{ij})_{i,j=1}^r$ is the matrix associated to f. For

$$\Lambda^r \xrightarrow{f} \Lambda^s \xrightarrow{g} \Lambda^r ,$$

one easily checks that $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$.

Projective modules: Let P be a projective (finite type) left Λ-module. One can choose an isomorphism $\alpha : P \oplus Q \longrightarrow \Lambda^r$. Let $f' = \alpha \circ (f \oplus 0) \circ \alpha^{-1}$, i.e., the composition

$$\Lambda^r \xrightarrow{\alpha^{-1}} P \oplus Q \xrightarrow{f\oplus 0} P \oplus Q \xrightarrow{\alpha} \Lambda^r .$$

We define $\mathrm{Tr}(f, P) = \mathrm{Tr}(f)$ to be $\mathrm{Tr}(f')$. This is well-defined, i.e., independent of the choices made.

$\mathbf{Z}/2$-graded projective modules: A $\mathbf{Z}/2$-*grading* on a left Λ-module E is a decomposition $E = E_- \oplus E_+$ into left Λ-modules. A decomposition $A = A_+ \oplus A_-$ of a Λ-algebra A into left Λ-modules is said to be a $\mathbf{Z}/2$-*grading* if $A_+A_+ \subset A_+$, $A_-A_- \subset A_+$, $A_+A_- \subset A_-$ and $A_-A_+ \subset A_-$.

EXAMPLE 1.13. Let E be a free Λ-module. Let Λ^*E denote the exterior algebra. We have an obvious $\mathbf{Z}/2$-grading of Λ^*E which we denote by

$$\Lambda^*E = \Lambda^{even}E \oplus \Lambda^{odd}E.$$

For a projective finite type $\mathbf{Z}/2$-graded modulo $P = P_+ \oplus P_-$ and $f : P \longrightarrow P$, we define

$$\mathrm{Tr}(f) = \mathrm{Tr}(f, P) := \mathrm{Tr}(f_{00}) - \mathrm{Tr}(f_{11}).$$

Equivalently, let τ in $\mathrm{End}(P)$ be ± 1 on $P_\pm$. Then $\mathrm{Tr}(f)$ is just the trace of τf (forgetting the $\mathbf{Z}/2$-grading).

Let us explain why we do this. A *pure element* of a $\mathbf{Z}/2$-graded algebra $A = A_+ \oplus A_-$ is an element of either A_+ or A_-. The *degree* of a pure element $a \in A$, denoted by $\deg a$, is defined as

$$\deg a = \begin{cases} 0 & a \in A_+ \quad \text{(we call these even elements)} \\ 1 & a \in A_- \quad \text{(we call these odd elements)} \end{cases}$$

For pure elements a and b of A, we define the *supercommutator* of a and b, (also) denoted by $[a, b]$, as

$$[a, b] = ab - (-1)^{\deg a \deg b} ba.$$

Given a $\mathbf{Z}/2$-graded Λ-module E, we can define a natural $\mathbf{Z}/2$-grading on the Λ-algebra $\mathrm{End}(E)$ as follows. Let τ in $\mathrm{End}(E)$ be ± 1 on $E_\pm$. Let $\mathrm{End}_+(E) = \{A \in \mathrm{End}(E) \mid A\tau = \tau A\}$ and $\mathrm{End}_-(E) = \{A \in \mathrm{End}(E) \mid A\tau = -\tau A\}$. For A in $\mathrm{End}(E)$, we define the *supertrace*[1] of A by

$$\mathrm{Tr}_S(A) = \mathrm{Tr}(\tau A).$$

Now, just as the trace vanishes on commutators, the supertrace vanishes on supercommutators:

PROPOSITION 1.14. The supertrace Tr_S vanishes on the supercommutator.

PROOF. By the identity $\mathrm{Tr}_S(f) = \mathrm{Tr}(f_{00}) - \mathrm{Tr}(f_{11})$, it suffices to treat the following three cases.

(1) Let f and g be even. Then the supercommutator of f and g is just the commutator of f and g. Furthermore, the action of τ is trivial in this case and therefore the equality follows from the properties of the usual trace.

[1] This terminology comes from so-called supersymmetric quantum field theories.

(2) Let f be even and g be odd (or vice versa). The supercommutator of A and B is an odd element of $\mathrm{End}(E)$. Since odd elements have only off-diagonal entries it follows that the supercommutator $\mathrm{Tr}_S([f,g]) = 0$.

(3) Let f and g be odd. Note that this implies that f and g anticommute with τ. The usual commutator

$$[\tau f, g]_{usual} = \tau f g - g \tau f = \tau f g + \tau g f = \tau [g, f]_{super}.$$

Now,

$$\mathrm{Tr}_S([f,g]_{super}) = \mathrm{Tr}(\tau [f,g]_{super}) = \mathrm{Tr}([\tau f, g]_{usual}) = 0. \qquad \square$$

We will only use this Proposition for morphisms of degree ± 1. The supertrace is just the anticommutator in this case.

Bounded complexes of projective modules: To a bounded complex

$$P_\bullet = \cdots \longrightarrow P_n \xrightarrow{f_n} P_{n+1} \xrightarrow{f_{n+1}} \cdots$$

of projective (finite type) left Λ-modules, we associate the $\mathbf{Z}/2$-graded Λ-module $P := P^{\text{even}} \oplus P^{\text{odd}}$. We define $\mathrm{Tr}(f, P_\bullet)$ to be the (super)trace of the morphism $f : P \longrightarrow P$.

Perfect complexes: We have that for any object of K in $K_{\text{parf}}(\Lambda)$ and endomorphism $f : K \longrightarrow K$ in $K(\Lambda)$, the trace $\mathrm{Tr}(f, K) = \mathrm{Tr}(f)$ is well-defined. To prove this, we note that $\mathrm{Tr}(dH + Hd) = 0$, i.e., the supertrace vanishes on the supercommutator. Consequently, for any perfect complex K and endomorphism $f : K \longrightarrow K$ in $D(\Lambda)$, we have that $\mathrm{Tr}(f, K) = \mathrm{Tr}(f)$ is well-defined.

4. Constructible finite Tor dimension complexes

In this section we define the generalized notion of a perfect complex on a noetherian scheme.

Let X be a noetherian scheme and let Λ be a (left and right) noetherian ring. (A flat sheaf of Λ-modules is a sheaf whose stalks are flat Λ-modules.)

Definition 1.15. We define $D^b_{ctf}(X, \Lambda)$ to be the full subcategory of $D^-(X, \Lambda)$ whose objects are isomorphic to bounded complexes of flat constructible sheaves of Λ-modules. Here $D(X, \Lambda)$ is

$$D(\text{category of sheaves of left } \Lambda\text{-modules on } X).$$

Example 1.16. For k a separably closed field, we have that $D_{\text{parf}}(\Lambda)$ is isomorphic to $D^b_{ctf}(\operatorname{Spec} k, \Lambda)$. (This is trivial.) More generally, for a field k, we have that $D_{\text{parf}}(\Lambda[G])$ is isomorphic $D^b_{ctf}(\operatorname{Spec} k, \Lambda)$, where $G = \mathrm{Gal}\ (k^s/k)$ is the absolute Galois group of k. (This might be wrong.)

Lemma 1.17. Let K be an object of $D^-(X, \Lambda)$. Then K is an object of $D^b_{ctf}(X, \Lambda)$ if and only if $H^i(K)$ is constructible for all i and K is of finite Tor dimension. $\square$

Corollary 1.18. An object K of $D^-(\Lambda)$ is a perfect complex if and only if $H^i(K)$ is of finite type and K is of finite Tor dimension. $\square$

Theorem 1.19. Let $f : X \longrightarrow Y$ be a finite type separated morphism of noetherian schemes. Let K be an object of $D^b_{ctf}(X, \Lambda)$. Then $Rf_! K$ is an object of $D^b_{ctf}(Y, \Lambda)$.

Proof. We sketch the proof. We will use that $Rf_!$ commutes with the tensor product. (Insert reference to Appendix.) That is, for any right Λ-module, the natural morphism

$$N \otimes^L Rf_! K \longrightarrow Rf_!(N \otimes^L K)$$

is an isomorphism. By the spectral sequence

$$E_2^{pq} = R^p f_! H^q(K) \Longrightarrow H^{p+q} Rf_! K$$

and the finiteness theorem for $Rf_!$, we have that $Rf_!K$ is an object of $D^b(X, \Lambda)$ and has constructible cohomology. By Lemma 1.17, it suffices to show that $Rf_!K$ is of finite Tor dimension. By the "commutativity" mentioned above, we have that

$$H^i(N \otimes^L Rf_!K) = H^i(Rf_!(N \otimes^L K)).$$

Now, consider the spectral sequence

$$E_2^{pq} = R^p f_! H^q(N \otimes^L K) \Longrightarrow H^{p+q} Rf_!(N \otimes^L K).$$

This is the above spectral sequence for $N \otimes^L K$. By Lemma 1.17, we have that $H^1(N \otimes^L K)$ vanishes universally for q small enough. Th e result now follows. $\square$

REMARK 1.20. We will use this theorem only when $Y = \operatorname{Spec} k$. In this case, we use the notation as given in the definition below.

DEFINITION 1.21. Let k be an algebraically closed field and let X be a variety k. Write $f : X \longrightarrow \operatorname{Spec} k$ for the structure morphism. For any object K of $D^b_{ctf}(X, \Lambda)$, where Λ is a left noetherian ring, we definef[2] $R\Gamma_c(K) := Rf_!(K)$. By Theorem 1.19 and Example 1.16, it is a perfect complex.

5. Local Lefschetz number equals global Lefschetz number

The goal of this talk is to prove the generalized trace formula (Theorem 1.2). We will show that Theorem 1.2 follows from the so-called Strong "Local is Global" Theorem.

THEOREM 1.22. **(Strong "Local is Global")** Let X_0 be a variety over $\mathbf{F}_q$ and let Λ be a noetherian torsion ring which gets killed by some integer coprime with p. Let K_0 be an object of $D^b_{ctf}(X_0, \Lambda)$. Then, for any n, it holds that

$$\sum_{x \in X^{\mathrm{Frob}^n}} \mathrm{Tr}((\mathrm{Frob}^n)^*, K_x) = \mathrm{Tr}((\mathrm{Frob}^n)^*, R\Gamma_c(X, K)).$$

Let us explain how the generalized Lefschetz formula (Theorem 1.2) follows from this Theorem.

PROOF. **(Theorem 1.22 implies Theorem 1.2)** We only treat the case $n = 1$. The general case is similar. Fix a prime number ℓ invertible in $\mathbf{F}_q$.

Let X_0 be a variety over $\mathbf{F}_q$ and let $\mathcal{F}_0$ be a $\mathbf{Q}_\ell$-sheaf on X_0. We have that $\mathcal{F}_0 = \mathcal{G}_0 \otimes \mathbf{Q}_\ell$, where $\mathcal{G}_0$ is a torsion free $\mathbf{Z}_\ell$-sheaf. We write $\mathcal{G}$ for the pullback of $\mathcal{G}_0$ to $X = X_0 \otimes_{\mathbf{F}_q} \overline{\mathbf{F}_q}$. Similarly, we write $\mathcal{F}$ for the pull-back of $\mathcal{F}_0$ to X. Recall that $\mathcal{G}$ is a projective system of sheaves $(\mathcal{G}_n)_{n \geq 0}$ with $\mathcal{G}_n$ a constructible sheaf of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules such that the transition morphism $\mathcal{G}_n \longrightarrow \mathcal{G}_{n-1}$ factorises through an isomorphism

$$\mathcal{G}_n \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \longrightarrow \mathcal{G}_{n-1}.$$

Firstly, let us apply Theorem 1.22 to $\mathcal{G}_n$ viewed as a constructible finite Tor dimension[3] complex concentrated in degree 0. This gives us that

$$\begin{aligned}
\mathrm{Tr}(\mathrm{Frob}^*, R\Gamma_c(X, \mathcal{G}_n)) &= \sum_{x \in X^{\mathrm{Frob}}} \mathrm{Tr}(\mathrm{Frob}^*, (\mathcal{G}_n)_x) = \sum_{x \in X^{\mathrm{Frob}}} \mathrm{Tr}(\mathrm{Frob}^*, \mathcal{G}_x) \mod \ell^{n+1} \\
&= \sum_{x \in X^{\mathrm{Frob}}} \mathrm{Tr}(\mathrm{Frob}^*, \mathcal{F}_x) \mod \ell^{n+1}.
\end{aligned}$$

The second equality follows from the isomorphism

$$\mathcal{G}_n \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \longrightarrow \mathcal{G}_{n-1}.$$

[2] In my opinion, one could call this the compactly supported Euler characteristic of K.

[3] I didn't explain why this is true during the talk. Drew gave an argument. One takes a stratification for the constructible sheaf $\mathcal{G}$ and considers the short exact sequence associated to the biggest stratum. Using this short exact sequence, we may assume $\mathcal{G}$ is locally constant. A simple arguement shows that we may assume $\mathcal{G}$ to be constant and then the statement follows from some commutative algebra (for $\mathbf{Z}/\ell^n$-modules)

The last equality follows from the fact that the trace doesn't change if we extend scalars to $\mathbf{Q}_\ell$. (Right?)

Now, we take limits at both sides to see that

$$\begin{aligned}\lim_{\leftarrow} \operatorname{Tr}(\mathrm{Frob}^*, R\Gamma_c(X, \mathcal{G}_n)) &= \lim_{\leftarrow} \sum_{x \in X^{\mathrm{Frob}}} \operatorname{Tr}(\mathrm{Frob}^*, \mathcal{F}_x) \mod \ell^{n+1} \\ &= \sum_{x \in X^{\mathrm{Frob}}} \operatorname{Tr}(\mathrm{Frob}^*, \mathcal{F}_x).\end{aligned}$$

Thus, to prove Theorem 1.2, it suffices to show that

$$\lim_{\leftarrow} \operatorname{Tr}(\mathrm{Frob}^*, R\Gamma_c(X, \mathcal{G}_n)) = \sum (-1)^i \operatorname{Tr}(\mathrm{Frob}^*, H^i_c(X, \mathcal{F})).$$

To this end, we take a small detour. We define

$$K_n = R\Gamma_c(X, \mathcal{G}_n).$$

This is an object of $D_{\mathrm{parf}}(\mathbf{Z}/\ell^{n+1}\mathbf{Z})$. We have to show that

$$\lim_{\leftarrow} \operatorname{Tr}(\mathrm{Frob}^*, K_n) = \sum (-1)^i \operatorname{Tr}(\mathrm{Frob}^*, H^i_c(X, \mathcal{F})).$$

Here we used that the Frobenius induces endomorphisms

$$\mathrm{Frob}^* : K_n \longrightarrow K_n$$

in $D_{\mathrm{parf}}(\mathbf{Z}/\ell^{n+1})$. We may and do assume that K_n is realized as a bounded complex of free finite type $\mathbf{Z}/\ell^{n+1}$-modules and that the above endomorphism is realized in the category of complexes. We still let Frob^* denote this endomorphism. Define

$$K = \lim_{\leftarrow} K_n.$$

This is a complex of free $\mathbf{Z}_\ell$-modules and $K_n \cong K \otimes_{\mathbf{Z}_\ell} \mathbf{Z}/\ell^{n+1}$. We denote the Frobenius on K by Frob^* again. Furthermore, since every projective system of finite groups satisfies the Mittag-Leffler condition, it holds that

$$H^i(K) = \lim_{\leftarrow} H^i(K_n).$$

In particular, we have that $H^i(K)$ is a finite type $\mathbf{Z}_\ell$-module. Note[4] that

$$H^i_c(X, \mathcal{F}) = (\lim_{\leftarrow} H^i(K_n)) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell = H^i(K) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

Therefore, we have to show that

$$\lim_{\leftarrow} \operatorname{Tr}(\mathrm{Frob}^*, K_n) = \sum (-1)^i \operatorname{Tr}(\mathrm{Frob}^*, H^i(K) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell) = \sum (-1)^i \operatorname{Tr}(\mathrm{Frob}^*, H^i(K)).$$

This follows from an argument using the trace of a filtered perfect complex. (See appendix. It's not that complicated.) □

Remark 1.23. You might have noticed that in the above proof, we only used Theorem 1.22 with $\Lambda = \mathbf{Z}/\ell^n\mathbf{Z}$. It will be in the course of the proof of Theorem 1.22 that we will use representation rings $\mathbf{Z}/\ell^n\mathbf{Z}[G]$ of finite groups (i.e., noncommutative rings).

[4] Recall that this is how one shows the finite dimensionality of the $\mathbf{Q}_\ell$-vector space $H^i_c(X, \mathcal{G})$.

6. Formal properties of the Local and Global trace

Our goal is to prove Theorem 1.2. In the previous section, we explained that it suffices to prove Theorem 1.22. We will prove this Theorem for $n = 1$. (The general case is similar.) Thus, we will prove the following

THEOREM 1.24. **(Strong "Local is Global")** Let X_0 be a variety over $\mathbf{F}_q$ and let Λ be a noetherian torsion ring which gets killed by some integer coprime with p. Let K_0 be an object of $D^b_{ctf}(X_0, \Lambda)$. Then we have that

$$\sum_{x \in X^{\text{Frob}}} \operatorname{Tr}(\text{Frob}^*, K_x) = \operatorname{Tr}(\text{Frob}^*, R\Gamma_c(X, K)).$$

In this section we show that Theorem 1.24 follows from

THEOREM 1.25. **(Weak "Local is Global")** Let X_0 be a variety over $\mathbf{F}_q$ and let Λ be a noetherian torsion ring which gets killed by a power of a prime ℓ which is invertible in $\mathbf{F}_q$. Suppose that X_0 is a smooth geometrically integral affine curve over k and that $X_0(k) = \emptyset$. For $K = \mathcal{F}$ a finite locally constant sheaf of Λ-modules on X whose stalks are finite projective Λ-modules, we have that

$$0 = \operatorname{Tr}(\text{Frob}^*, R\Gamma_c(X, K)).$$

DEFINITION 1.26. For X_0 a variety over $\mathbf{F}_q$ Λ a noetherian torsion ring which gets killed by some integer coprime with p and K_0 an object of $D^b_{ctf}(X_0, \Lambda)$, we define the *local Lefschetz number* by

$$T'(X_0, K_0) = \sum_{x \in X^{\text{Frob}}} \operatorname{Tr}(\text{Frob}^*, K_x).$$

Moreover, we define the *global Lefschetz number* by

$$T''(X_0, K_0) = \operatorname{Tr}(\text{Frob}^*, R\Gamma_c(X, K)).$$

REMARK 1.27. The goal is to show that $T'(X_0, K_0) = T''(X_0, K_0)$. Hence the name "Local is Global". The following three Lemma's will show that T' and T'' satisfy the same formal properties.

LEMMA 1.28. **(Excision)** Let K_0 be an object of $D^b_{ctf}(X_0, \Lambda)$. Let $U_0 \subset X_0$ be an open subset with complement Y_0. Then

$$T'(X_0, K_0) = T'(U_0, K_0|_{U_0}) + T'(Y_0, K_0|_{Y_0})$$

and

$$T''(X_0, K_0) = T''(U_0, K_0|_{U_0}) + T''(Y_0, K_0|_{Y_0}).$$

PROOF. This is straightforward to verify for T'. For T'' one uses filtered derived complexes. □

LEMMA 1.29. Let K_0 be an object of $D^b_{ctf}(X_0, \Lambda)$. Suppose that K_0 is a bounded complex of constructible flat modules. Then

$$T'(X_0, K_0) = \sum (-1)^i T'(X_0, K_0^i)$$

and

$$T''(X_0, K_0) = \sum (-1)^i T''(X_0, K_0^i).$$

PROOF. Again, this is easy for T'. For T'' we use a filtration again. □

LEMMA 1.30. Let K_0 be an object of $D^b_{ctf}(X_0, \Lambda)$. If $\dim X_0 = 0$, we have that $T'(X_0, K_0) = T''(X_0, K_0)$.

PROOF. Omitted. □

We are now ready to show that Theorem 1.25 implies Theorem 1.24.

THEOREM 1.31. Theorem 1.24 follows from Theorem 1.25, i.e., the Weak "Local is Global" Theorem implies the Strong "Local is Global" Theorem.

PROOF. Let X_0 be a variety over $\mathbf{F}_q$ and let Λ be a noetherian torsion ring which gets killed by some integer coprime with p. Let K_0 be an object of $D^b_{ctf}(X_0, \Lambda)$. Assuming Theorem 1.25, we want to show that

$$\sum_{x \in X^{\mathrm{Frob}}} \mathrm{Tr}(\mathrm{Frob}^*, K_x) = \mathrm{Tr}(\mathrm{Frob}^*, R\Gamma_c(X, K)).$$

We begin with the reductions for X_0.

(1) By noetherian induction and Lemma 1.28, we may and do assume that X_0 is affine.
(2) Since X_0 is affine, there exists[5] a morphism $f : X_0 \longrightarrow Y_0$ with $\dim Y \leq \dim X - 1$ such that each scheme-theoretic fibre $f^{-1}(y)$ is of dimension ≤ 1. By induction on $\dim X$ and the proper[6] base change theorem ([**?**, Chapter I, Theorem 8.7], we may and do assume that $\dim X_0 \leq 1$.
(3) By Lemma 1.28 and Lemma 1.30, we may and do assume that $\dim X_0 = 1$.
(4) We may and do assume that X_0 is an integral affine curve over $\mathbf{F}_q$. In fact, write $X_0 = X_1 \cup \ldots \cup X_m$ for the decomposition into irreducible components. Then the mutual intersections are finite sets of closed points which we may throw away by Lemma 1.30. Applying Lemma 1.28 with $U = \emptyset$ and $Y = X_{red}$ allows us to assume X_0 is reduced.
(5) We may and do assume that X_0 is smooth over $\mathbf{F}_q$. In fact, combining Lemma 1.28 and Lemma 1.30, we may throw away any finite set of points. Since there are only a finite number of singular points, we may assume that X_0 is nonsingular. Since $\mathbf{F}_q$ is perfect, we may assume that X_0 is smooth over $\mathbf{F}_q$.
(6) We may and do assume that X_0 is a smooth geometrically integral affine curve. We just have to show that we may assume X_0 is geometrically connected. But how? (I think one can consider the decomposition as before. The intersections are closed points. But why can we throw these away?
(7) We may and do assume that X_0 is a smooth geometrically integral affine curve without rational points. In fact, we only have to show the last statement. In fact, combining Lemma 1.28 and Lemma 1.30 again, we may throw away any finite set of points. But there are only a finite number of rational points.

Now, we have almost reduced to Theorem 1.25. Firstly, the assumption on K follows from the definition of $D^b_{ctf}(X, \Lambda)$ and Lemma 1.29. Finally, the assumption on Λ follows from considering its primary decomposition. Since there are no rational points, the fixed point subscheme X^{Frob} is empty. Therefore, the formula we want to prove reduces to

$$0 = \mathrm{Tr}(\mathrm{Frob}^*, R\Gamma_c(X, K)).$$

This is precisely the statement of Theorem 1.25. □

7. Informal intermezzo on remaining steps

We want to prove the Weak "Local is Global" Theorem. This will conclude our proof of Grothendieck's generalized Lefschetz theorem (Theorem 1.2). The proof of the Weak "Local is Global" Theorem proceeds in two steps. The first step uses Weil's classical trace formula. The second step uses general facts concerning traces associated to finite étale covers.

[5] You don't need Noether's normalization lemma for this, I believe. You just put X_0 in an affine space of minimal dimension. Then you take the projection. Where is Noether?

[6] This has never been written down for this Theorem. In fact, Grothendieck gives the details of such a computation but for a different theorem. Namely, Grothendieck does it in his proof of the cohomological interpretation of the L-function.

8. Weil's classical trace formula

We will formulate Weil's theorem using the language of intersection theory on smooth projective surfaces (over algebraically closed fields). See [**Har**, Chapter V.1] and [**Liu**, Chapter 9.1].

Let X be a smooth projective integral curve over an algebraically closed field k.

DEFINITION 1.32. The diagonal of X is the morphism $\Delta_X : X \longrightarrow X \times_k X$. The corresponding divisor in $A^1(X \times_k X) = CH^1(X \times_k X)$ is denoted by $[\Delta_X]$. For a non-constant morphism $f : X \longrightarrow X$, we let $[\Gamma_X] \in A^1(X \times X)$ be the class of the graph of f. We define

$$v(f) := \deg_{X\times X}\left([\Gamma_f] \cdot [\Delta_X]\right).$$

Note that $v(f)$ is the number of fixed points of f counted with their multiplicity if f has isolated fixed points.

REMARK 1.33. A non-constant morphism of smooth projective connected curves $f : X \longrightarrow Y$ over an algebraically closed field is finite and flat. It is finite because it is surjective. Furthermore, it is flat because the local rings of a nonsingular curve are discrete valuation rings.

EXAMPLE 1.34. The base field is $\overline{\mathbf{F}_p}$. A *line* in $\mathbf{P}^2$ is a subvariety given by an equation of the form $ax + by + cz = 0$. Two distinct lines intersect in a unique point. Let X be the union of two distinct lines. There is an obvious automorphism of X. Namely the automorphism f which switches the lines. Although X is singular, note that $v(f) = 1$.

As we mentioned in the previous section, we will use the following

THEOREM 1.35. **(Weil's classical formula)** Let $f : X \longrightarrow X$ be a morphism with isolated fixed points. Let ℓ be a prime number which is invertible in k, i.e., ℓ and the characteristic of k are coprime. Then

$$v(f) = \sum_{i=0}^{2}(-1)^i \operatorname{Tr}(f^*, H^i(X, \mathbf{Q}_\ell)).$$

PROOF. We already mentioned that this follows from the properties of a Weil cohomology in the proof of Theorem 1.1. $\square$

EXAMPLE 1.36. Let X be the union of two distinct lines and let f be the automorphism of X given by switching the lines. Here are the cohomology groups:

$$H^0(X, \mathbf{Q}_\ell) = \mathbf{Q}_\ell, \quad H^1(X, \mathbf{Q}_\ell) = 0 \qquad H^2(X, \mathbf{Q}_\ell) = \mathbf{Q}_\ell \oplus \mathbf{Q}_\ell.$$

The action of f on $H^0(X, \mathbf{Q}_\ell)$ is given by the identity $\mathrm{id} : \mathbf{Q}_\ell \longrightarrow \mathbf{Q}_\ell$. The action of f on $H^2(X, \mathbf{Q}_\ell)$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One easily checks that $\operatorname{Tr}(f^*, H^*(X, \mathbf{Q}_\ell)) = 1$. The unique point of intersection!

REMARK 1.37. Of course, Weil didn't use étale cohomology explicitly in his formulation. Let us explain his formulation for $k = \overline{\mathbf{F}_q}$. We use the notation of the above Theorem. Note that

$$\sum_{i=0}^{2}(-1)^i \operatorname{Tr}(f^*, H^i(X, \mathbf{Q}_\ell)) = 1 - \operatorname{Tr}(f^*, H^1(X, \mathbf{Q}_\ell)) + \deg f.$$

Weil's classical formula states that

$$v(f) = 1 - \operatorname{Tr}(f^*, \operatorname{Jac}(X)) + \deg f,$$

where $\operatorname{Jac}(X)$ is the Jacobian of X over k. Let us show that this equality is equivalent to the above. In fact, it suffices to show that we have a functorial isomorphism $\operatorname{Jac}(X)[\ell] \cong H^1(X, \mu_\ell)$. (The trace of f^* on $\operatorname{Jac}(X)$ modulo ℓ^n is given by the trace of f^* on $\operatorname{Jac}(X)[\ell^n]$ over $\mathbf{Z}/\ell^n\mathbf{Z}$.) Since

$H^1(X_{et}, \mathbf{G}_m) = \mathrm{Pic}(X)$ (for any scheme X), the long exact sequence of cohomology associated to the Kummer sequence

$$\{1\} \longrightarrow \mu_{\ell^n} \longrightarrow \mathbf{G}_m \longrightarrow \mathbf{G}_m \longrightarrow \{1\}$$

shows that

$$H^1(X_{et}, \mu_{\ell^n}) = \mathrm{Pic}(X)[\ell^n].$$

One then concludes via the short exact sequence

$$0 \longrightarrow \mathrm{Jac}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathbf{Z} \longrightarrow 0$$

and the choice of an isomorphism of $\mu_{\ell^n}(k)$ with $\mathbf{F}_{\ell^n}$.

REMARK 1.38. The goal of this groupe de travail is Deligne's proof of the Riemann hypothesis for varieties over finite fields. In the case of smooth projective integral curves, we can combine Weil's classical formula with the Hodge index theorem for surfaces to prove that the Riemann hypothesis holds for smooth projective geometrically integral curves over some finite field. (Actually, to prove the rationality of the zeta function, one can just as easily use the Riemann-Roch theorem[7].)

9. The first key formula

We now prove the first formula we need to prove the Weak "Local is Global" Theorem.

LEMMA 1.39. **(Formula 1)** Let X_0 be a smooth irreducible affine curve over $\mathbf{F}_q$ with no rational points and let Λ be a noetherian torsion ring which gets killed by a power of a prime ℓ. We assume ℓ is invertible in $\mathbf{F}_q$. Let $f : Y_0 \longrightarrow X_0$ be a finite étale Galois morphism with Galois group H and Y_0 connected. Then, for any h in H, we have that

$$\sum_{i=0}^{2} (-1)^i \operatorname{Tr}((\mathrm{Frob} \circ h^{-1})^*, H^i_c(Y, \mathbf{Q}_\ell)) = 0.$$

PROOF. Consider a nonsingular projective compactification $\overline{Y}$ (resp. $\overline{X}$) of Y (resp. X). We have the following diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"] & \overline{Y} \arrow[d, "\overline{f}"] \\ X \arrow[r] & \overline{X} \end{tikzcd} ,$$

where the horizontal maps are open immersions. Let h be in H. Consider the endomorphism $\mathrm{Frob} \circ h^{-1}$ of Y. Note that $\overline{Y}$ is a smooth projective integral curve and that the fixed points of $\mathrm{Frob} \circ h^{-1}$ in $\overline{Y} - Y$ are of multiplicity 1. Therefore, the fixed points of $\mathrm{Frob} \circ h^{-1}$ on Y are the fixed points of $\mathrm{Frob} \circ h^{-1}$ on $\overline{Y}$ not lying in $\overline{Y} - Y$. Let us show that $\mathrm{Frob} \circ h^{-1}$ has no fixed points on Y. In fact, suppose that $\overline{y}$ is a fixed point of $\mathrm{Frob}_Y \circ h^{-1}$. Since

$$\overline{f}(\overline{y}) = \overline{f}(\mathrm{Frob}_Y\, h^{-1}\overline{y}) = \mathrm{Frob}_X\, \overline{f}(h^{-1}\overline{y}) = \mathrm{Frob}_X\, \overline{f}(\overline{y}),$$

we have that $\overline{f}(\overline{y})$ is fixed under Frob_X. Therefore, we have that $\overline{f}(\overline{y})$ is a rational point of $\overline{X}$. Since X has no rational points, we see that $\overline{y}$ doesn't lie above X. Therefore, it lies above $\overline{X} - X$. Therefore, it lies in $\overline{Y} - Y$. In particular, we have that

$$v_{\overline{Y}}(\mathrm{Frob} \circ h^{-1}) - v_{\overline{Y}-Y}(\mathrm{Frob} \circ h^{-1}) = 0.$$

Write $j : Y \longrightarrow \overline{Y}$ for the open immersion and consider the associated short exact sequence

$$0 \longrightarrow j_! \mathbf{Q}_\ell \longrightarrow \mathbf{Q}_\ell \longrightarrow \mathbf{Q}_\ell|_{\overline{Y}-Y} \longrightarrow 0.$$

Recalling the sign convention, the long exact sequence associated to this sequence shows that

$$\sum_{i=0}^{2} (-1)^i \operatorname{Tr}((\mathrm{Frob} \circ h^{-1})^*, H^i_c(Y, \mathbf{Q}_\ell))$$

[7] I'm planning on writing this out one day.

equals

$$\sum_{i=0}^{2}(-1)^i \operatorname{Tr}((\mathrm{Frob}\circ h^{-1})^*, H^i_c(\overline{Y}, \mathbf{Q}_\ell)) - \sum_{i=0}^{2}(-1)^i \operatorname{Tr}((\mathrm{Frob}\circ h^{-1})^*, H^i_c(\overline{Y}-Y, \mathbf{Q}_\ell)).$$

(Compare this to Lemma 1.28.) By Weil's classical formula and Example 1.3, we conclude that

$$\sum_{i=0}^{2}(-1)^i \operatorname{Tr}((\mathrm{Frob}\circ h^{-1})^*, H^i_c(Y, \mathbf{Q}_\ell)) = v_{\overline{Y}}(\mathrm{Frob}\circ h^{-1}) - v_{\overline{Y}-Y}(\mathrm{Frob}\circ h^{-1}) = 0. \qquad \square$$

10. Traces, finite étale covers and the last step

We now give the last step of the proof of the Weak "Local is Global" Theorem.

Let X_0 be a smooth irreducible affine curve over $\mathbf{F}_q$ with no rational points and let Λ be a noetherian torsion ring which gets killed by a power of a prime ℓ. We assume ℓ is invertible in $\mathbf{F}_q$. Just as in the statement of Theorem 1.25, let $K = \mathcal{F}$ be a finite locally constant sheaf of Λ-modules on X whose stalks are finite projective Λ-modules. To finish our proof, we need to show that

$$\operatorname{Tr}(\mathrm{Frob}^*, R\Gamma_c(X,\mathcal{F})) = 0.$$

Definition 1.40. Let H be a finite group and $H^+ = H \times \mathbf{Z}_{\geq 0}$. For h in H^+, we define

$$Z_h = \{h' \in H \mid h'h = hh'\}.$$

Lemma 1.41. **(Formula 2)** Let $f : Y_0 \longrightarrow X_0$ be a finite étale Galois morphism with Y_0 connected. Let H be the corresponding Galois group. Suppose that $f^*\mathcal{F}_0$ is constant. Then, we have that

$$\operatorname{Tr}(\mathrm{Frob}^*, R\Gamma_c(X,K)) = \sum_h{}' \frac{1}{\#Z_h}\sum(-1)^i \operatorname{Tr}((\mathrm{Frob}\circ h^{-1})^*, H^i_c(Y, \mathbf{Q}_\ell)) \cdot \operatorname{Tr}_\Lambda(h, M).$$

Here M is the *valeur constante* $H^0(Y_0, f^*\mathcal{F}_0)$ of $\mathcal{F}_0$. Note that M is a $\Lambda[H]$-module which is projective as a Λ-module. Also, we let $\operatorname{Tr}_\Lambda(h,M)$ be the image of $\operatorname{Tr}(h,M) \in \Lambda[H]^\#$ in $\Lambda^\#$ via the natural morphism $\Lambda[H]^\# \longrightarrow \Lambda^\#$. The sum $\sum_h'$ is over the conjugacy classes of H.

Assuming this Lemma, we can now prove Theorem 1.25.

Proof. **(Weak "Local is Global")** We can choose a finite étale Galois morphism $f : Y_0 \longrightarrow X_0$ with Y connected such that $f^*\mathcal{F}_0$ is constant. (Here's a short argument for why this exists. Firstly, being locally constant is equivalent to being representable as a functor. Then being constructible is equivalent to being represented by a finite étale scheme over X_0. Galois theory then accounts for the existence of f as above.) Now, substitute Formula 1 into Formula 2 to conclude that

$$\begin{aligned}\operatorname{Tr}(\mathrm{Frob}^*, R\Gamma_c(X,\mathcal{F})) &= \sum_h{}' \frac{1}{\#Z_h}\Big(\sum(-1)^i \operatorname{Tr}((\mathrm{Frob}\circ h^{-1})^*, H^i_c(Y, \mathbf{Q}_\ell))\Big) \cdot \operatorname{Tr}_\Lambda(h, M) \\ &= \sum_h{}' \frac{1}{\#Z_h}\sum(-1)^i 0 \cdot \operatorname{Tr}_\Lambda(h,M) = 0. \quad \square\end{aligned}$$

Due to lack of time, I didn't manage to write down nor present the proof of Lemma 1.41. Needless to say, it is not very hard and consists simply of some "canonical" computations with groups. It is also the **only** place of the proof which uses noncommutative rings.

CHAPTER 2

Application of the Lefschetz formula to equivariant varieties

Deligne and Lusztig used Grothendieck's generalized trace formula to study equivariant varieties. Since this illustrates the use of our Theorem a bit more (outside the realm of the Riemann hypothesis) I decided to write some notes on it.

Let k be an algebraic closure of $\mathbf{F}_p$ (p prime number) and let X be a finite type separated k-scheme.

As a consequence of the Lefschetz trace formula we will give a proof of the following Theorem (see [**DeLu**, Paragraph 3]). (Recall that the action of a group G with identity e on a set X is said to be *free* if for all g in G and x in X, we have that $gx = x$ if and only if $g = e$.)

THEOREM 2.1. Let $\sigma : X \longrightarrow X$ be an automorphism of X. We assume σ to be of finite order. Then

(1) For any prime number $l \neq p$, we have that
$$\mathrm{Tr}(\sigma^*, H_c^*(X, \mathbf{Q}_\ell))$$
is an integer independent of ℓ.

(2) Assume the action on X of the cyclic group H generated by σ to be free. Also, assume that the order of H is divisible by a prime number $l \neq p$. Then
$$\mathrm{Tr}(\sigma^*, H_c^*(X, \mathbf{Q}_\ell) = 0.$$

(3) Write $\sigma = su$, where s and u are powers of σ such that $\mathrm{ord}(s)$ is coprime with p and $\mathrm{ord}(u)$ is a power of p. Then
$$\mathrm{Tr}(\sigma^*, H_c^*(X, \mathbf{Q}_\ell) = \mathrm{Tr}(u^*, H_c^*(X^s, \mathbf{Q}_\ell).$$

Before we give the proof, let us give some corollaries and do some examples.

COROLLARY 2.2. The compactly supported Euler characteristic $\sum(-1)^i \dim_{\mathbf{Q}_\ell} H_c^i(X, \mathbf{Q}_\ell)$ of X is independent of ℓ.

PROOF. Apply (1) with $\sigma = \mathrm{id}_X$. □

COROLLARY 2.3. Let $\pi : Y \longrightarrow X$ be a finite étale morphism such that $\deg \pi$ and p are coprime. Then
$$e_c(Y) = \deg \pi e_c(X).$$

PROOF. Firstly, we may assume that Y and X are connected. Also, we may assume that π is Galois. That is, there is a finite group G acting freely on Y such that X is the quotient Y/G. In fact, assume the Corollary to be true for Galois covers. Let $Z \longrightarrow Y \longrightarrow X$ be the Galois closure of $\pi : Y \longrightarrow X$. Let G be its group and let H be the subgroup corresponding to the cover $Z \longrightarrow Y$, i.e., we identify Y with Z/H and X with Z/G. Then
$$e_c(X) = \frac{1}{|G|} e_c(Z) = \frac{|H|}{|G|} e_c(Y) = \frac{1}{\deg \pi} e_c(Y).$$
Therefore, the result follows in the general case.

Thus, we have a finite group G acting freely on Y such that $X = Y/G$. Note that $\deg \pi = |G|$. Apply (2) to see that $\mathrm{Tr}(g, H_c^*(Y)) = 0$ for any $g \neq e$ in G. By character theory for $\mathbf{Q}_\ell[G]$, we may conclude that the element

$$[H_c^*(Y, \mathbf{Q}_\ell)] := \sum (-1)^i [H_c^i(Y, \mathbf{Q}_\ell)]$$

in the Grothendieck group $K_0(\mathbf{Q}_\ell[G])$ of finitely generated $\mathbf{Q}_\ell[G]$-modules is given by an integer multiple of $[\mathbf{Q}_\ell[G]]$; the class of the regular representation. So we may write

$$[H_c^*(Y, \mathbf{Q}_\ell)] = m[\mathbf{Q}_\ell[G]],$$

where $m \in \mathbf{Z}$. Now, note that $H_c^i(X, \mathbf{Q}_\ell) = \left(H_c^i(Y, \mathbf{Q}_\ell)\right)^G$ for any $i \in \mathbf{Z}$. Therefore, we have that

$$[H_c^*(X, \mathbf{Q}_\ell)] = m$$

in $K_0(\mathbf{Q}_\ell[G])$. In particular, we see that $e_c(X) = \dim_{\mathbf{Q}_\ell} [H_c^*(X, \mathbf{Q}_\ell)] = m$. We conclude that

$$e_c(Y) = \dim_{\mathbf{Q}_\ell} [H_c^*(Y, \mathbf{Q}_\ell)] = m|G| = e_c(X)|G| = \deg \pi e_c(X). \qquad \square$$

Example 2.4. Consider the finite étale morphism $\mathbf{A}_k^1 \longrightarrow \mathbf{A}_k^1$ given by $z \mapsto z^p - z$. This is a finite étale morphism. We see that the hypothesis on the degree of π is necessary in Corollary 2.3.

Question 1. Does an analogous theorem hold in the case of ℓ-adic sheaves? The proof seems to use only the Lefschetz trace formula for Frobenius and $\mathbf{Q}_\ell$ and some general facts on automorphisms of $\mathbf{Q}_\ell$.

Proof. **(Theorem 2.1)** Here's the idea. Consider $\mathrm{Frob} \circ h$, where $h \in H$. This is the Frobenius corresponding to some way of lowering the base field from $\mathbf{F}_q$ to $\mathbf{F}_{q^n}$. So we can apply the generalized Lefschetz trace formula. Since Frob^* and h^* commute as morphisms on the $\mathbf{Q}_\ell$-vector spaces $H^i(X, \mathbf{Q}_\ell)$, some easy linear algebra finishes the proof of 1. To prove 2 one uses that we may realize the trace $\mathrm{Tr}(h^*, H_c^*(X, \mathbf{Q}_\ell)$ as the character of a virtual projective $\mathbf{Z}_\ell[H]$-module. (This uses Theorem 1.19 in an essential way!) Finally, 3 follows from 2. $\square$

CHAPTER 3

Derived categories

Here we should explain what we mean by a filtered derived category and the trace of an endomorphism of a perfect complex in the filtered derived category.

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