# Online Scheduling

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# Motivation

Decision making with incomplete information.

- Jobs arrive as a sequence, maybe even over time. Future jobs unknown.
- Processing times are unknown.
- Processor breakdowns or maintenance intervals are unknown.

Compute schedules that represent good approximate solutions.

# Competitive analysis

Online setting: jobs sequence

| $A$: Online algorithm | OPT: Offline algorithm |
| --- | --- |
| $A(\sigma)$ | $OPT(\sigma)$ |

$A$ is $c$-competitive if for all job sequences $\sigma$

$$A(\sigma) \leq c \cdot OPT(\sigma).$$

## Problems settings

$m$ processors/machines

$n$ jobs $J_1, \ldots, J_n$ $\quad p_{ij} =$ processing time $J_i$ on machine $j$

- Machine environments: identical: $p_{ij} = p_i$
  related: $p_{ij} = p_i / s_j$ for some processing time $p_i$ and machine speed $s_j$
  unrelated: $p_{ij}$ may take arbitrary values

- Online models: jobs arrive as a sequence/list
  jobs arrive over time

- Job duration: permanent or temporary.

- Objective functions:
  Makespan $C_{\max}$ sum of completion times $\sum_i c_i$ $\quad \sum_i w_i c_i$
  Flow time $f_i = c_i - a_i$ $\quad \sum_i f_i$ $\quad \sum_i w_i f_i$
  Stretch $s_i = f_i / p_i$ $\quad \sum_i s_i$ $\quad \sum_i w_i s_i$

# Graham's problem

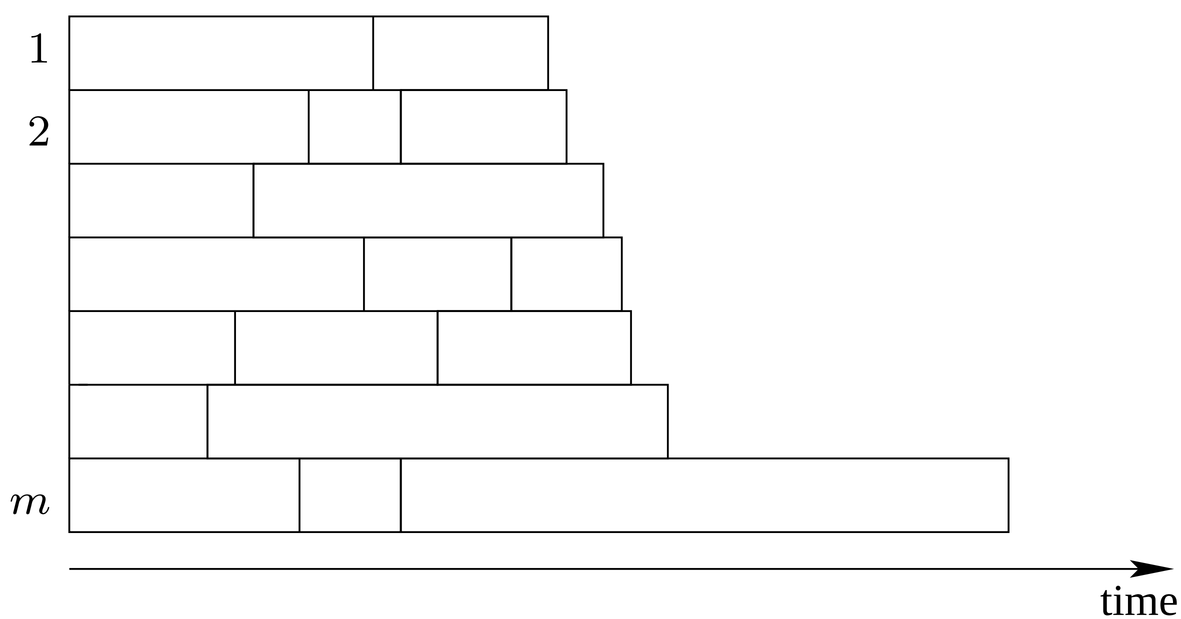
Graham 1966

$m$ identical parallel machines.

Jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$.

Jobs arrive one by one.   Preemption not allowed.

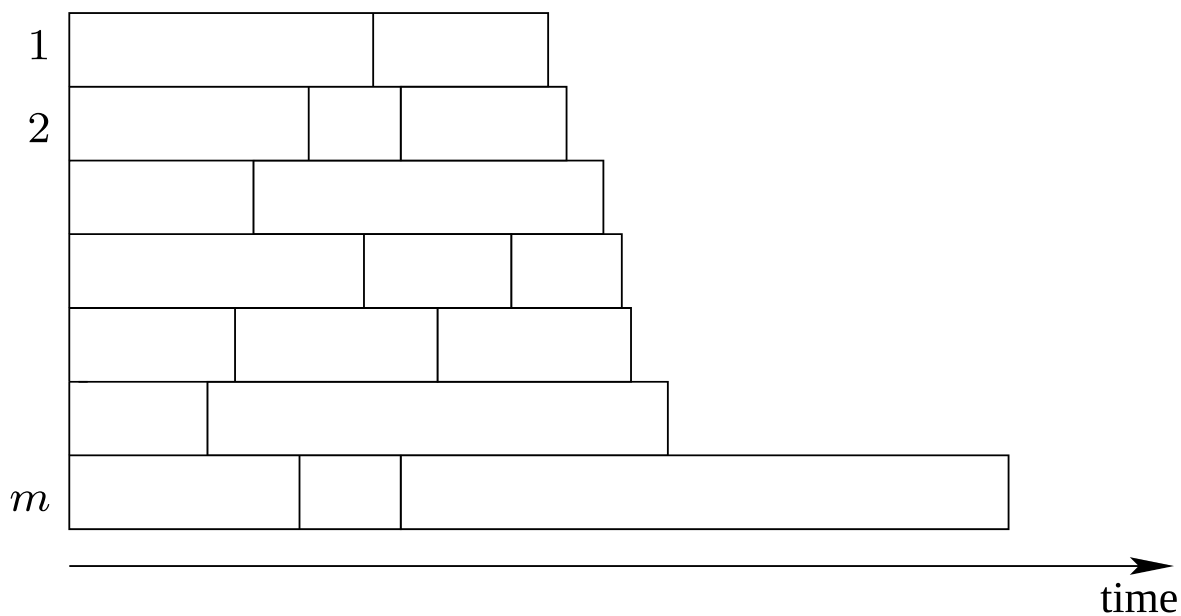
Minimize makespan.



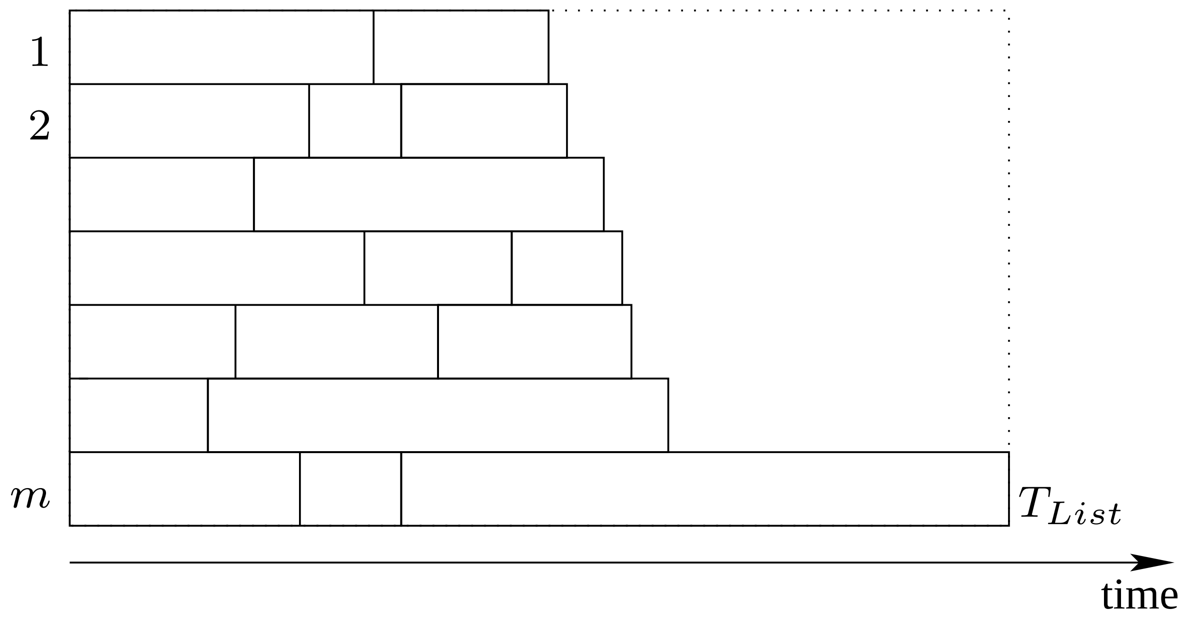
# Graham's problem

List: Schedule any new job on least loaded machine.

Thm: List is $(2 - \frac{1}{m})$-competitive.

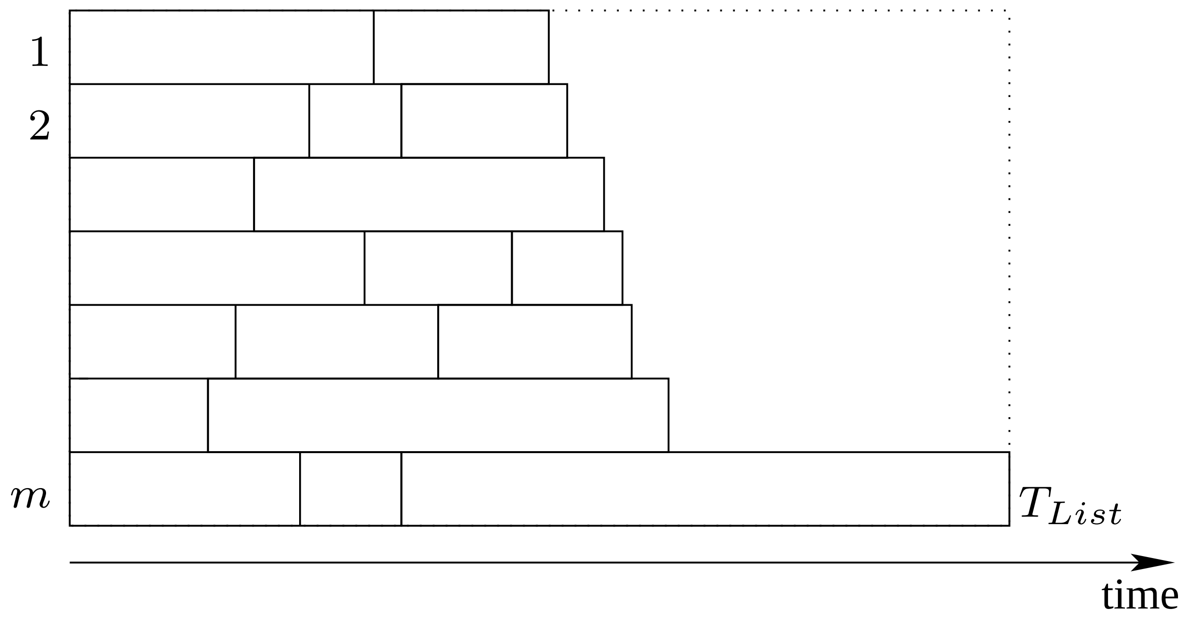


# Analysis




$$mT_{List} \leq \sum_{i=1}^{n} p_i + (m-1) \max_{1 \leq i \leq n} p_i$$

# Analysis



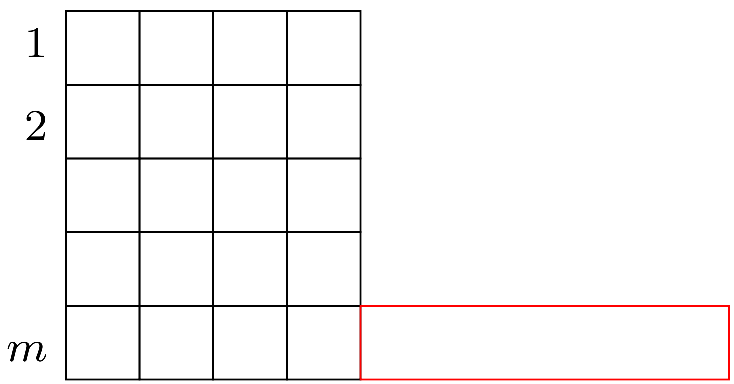

$$mT_{List} \leq \sum_{i=1}^{n} p_i + (m-1) \max_{1 \leq i \leq n} p_i$$

$$T_{List} \leq \frac{1}{m}\sum_{i=1}^{n} p_i + (1 - \frac{1}{m}) \max_{1 \leq i \leq n} p_i \leq T_{OPT} + (1 - \frac{1}{m})T_{OPT}$$

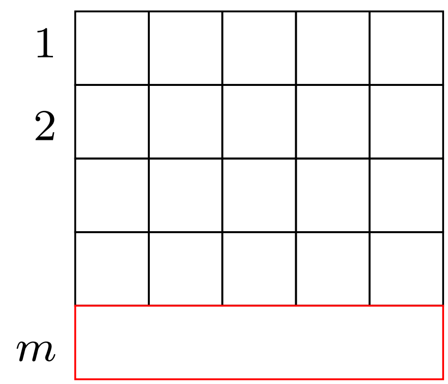
# Lower bound

Thm: List is not better than $(2 - \frac{1}{m})$-competitive.

$m(m-1)$ jobs of size 1        1 job of size $m$




List




OPT

$$T_{List} = (m-1) + m = 2m - 1$$

$$T_{List} = m$$

# Improvements

List best possible for $m = 2, 3$

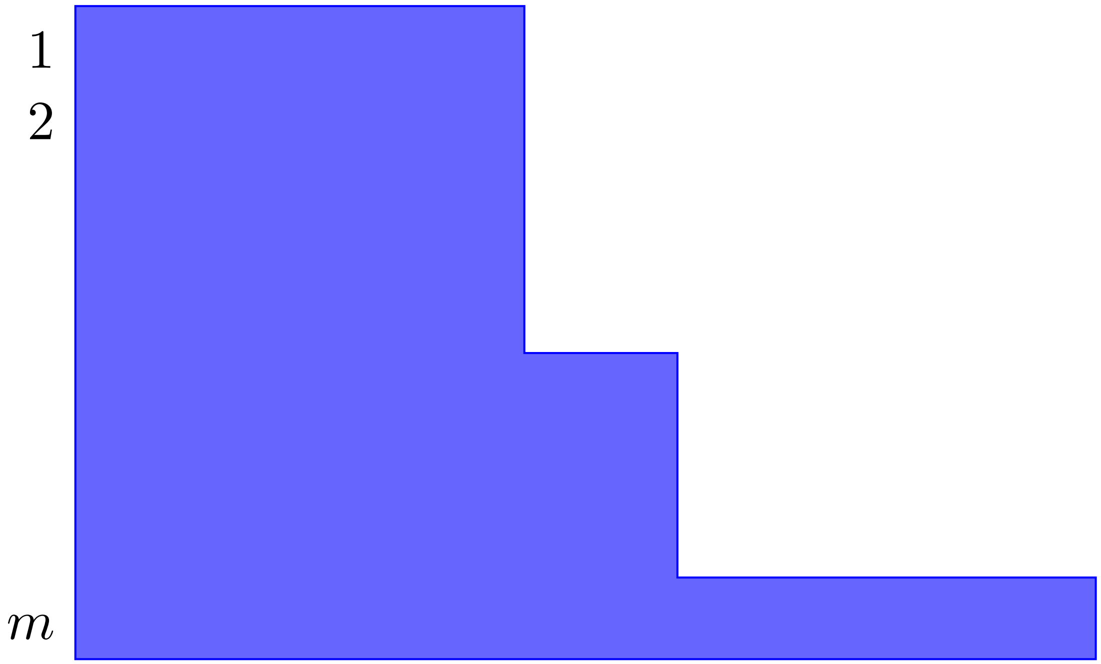
Upper bounds

- 1.986 Bartal, Fiat, Karloff, Vohra STOC 92
- 1.945 Karger, Phillips, Torng SODA 94
- 1.923 Albers STOC 97
- 1.9201 Fleischer, Wahl ESA 00

Lower bounds

- 1.701 Faige, Kern, Turan 89
- 1.837 Bartal, Karloff, Rabani 94
- 1.852 Albers 97
- 1.853 Gormey, Reingold, Torng, Westbrook 00
- 1.88 Rudin 01

# Imbalanced schedules



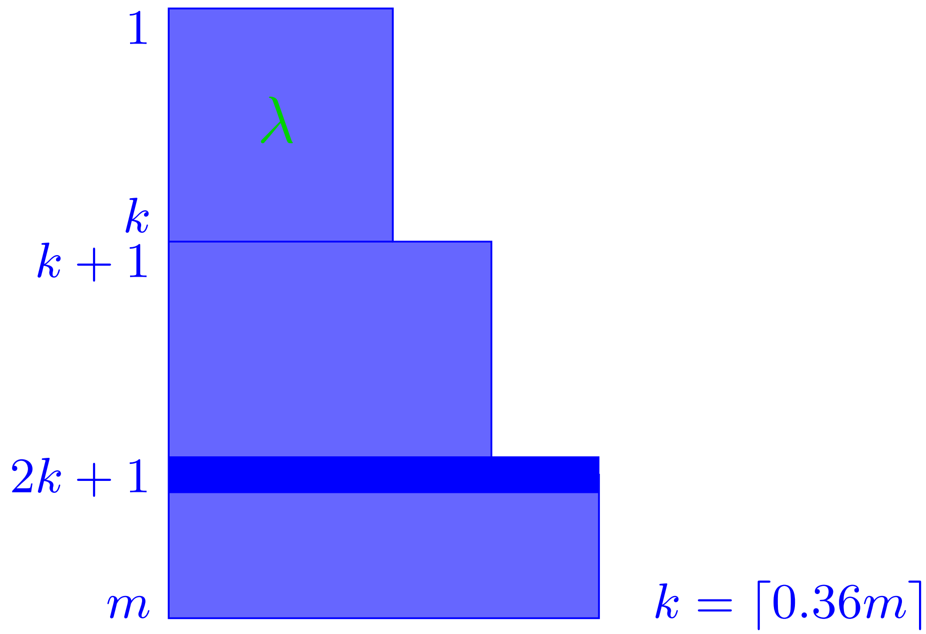

Makespan higher than necessary at this point in time

# Algorithm Imbal

Sort machines in order of non-decreasing load

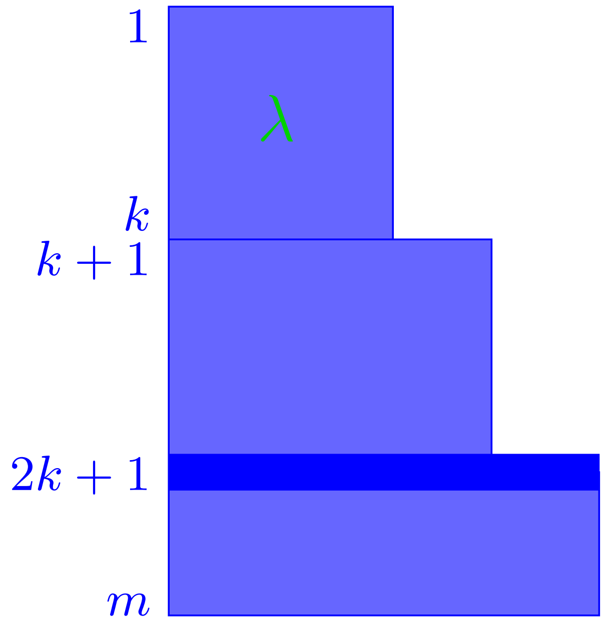
Load = processing times of jobs assigned to it

$l_i$ = load of $i$-th smallest machine




$\lambda = \frac{1}{k}\sum_{i=1}^{k} l_i$ average load on the $k$ smallest machines

# Algorithm Imbal

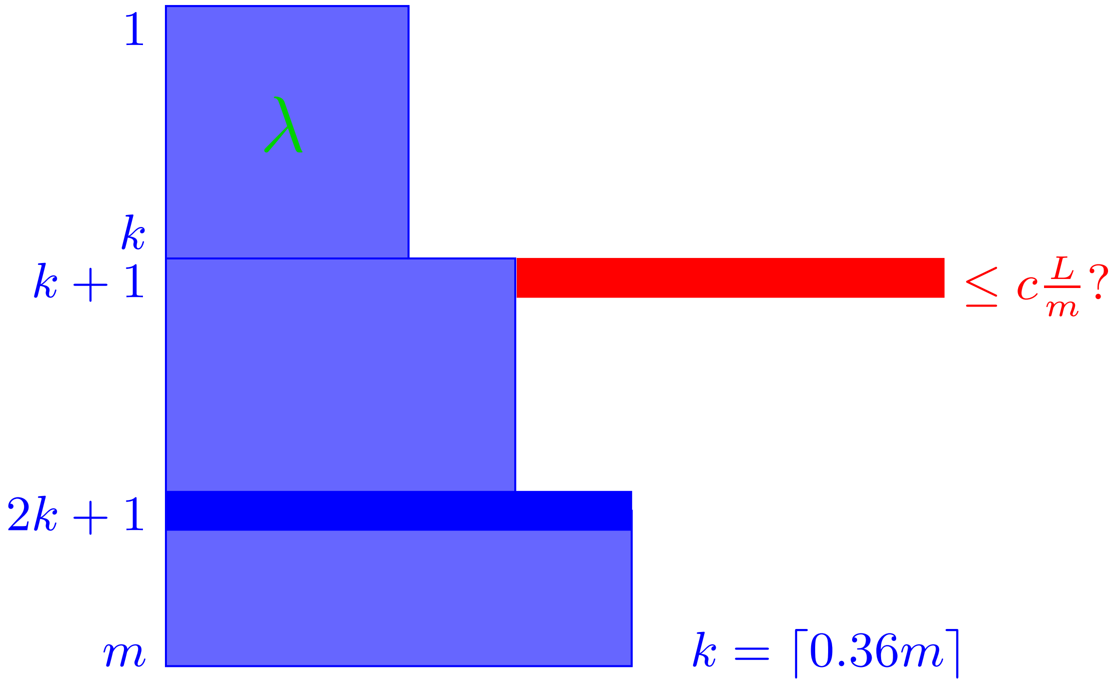



$k = \lceil 0.36m \rceil$

Try to maintain

$$\lambda \leq \alpha \cdot l_{2k+1} \qquad \alpha = 0.456$$

# Algorithm Imbal




**Algorithm:** $c = 1.9201$ $\quad L = \sum_i l_i$

Schedule new job on machine $k+1$ if

- $\lambda > \alpha \cdot l_{2k+1}$ and resulting load $\leq c \cdot \frac{L}{m}$

Otherwise schedule job on machine 1.

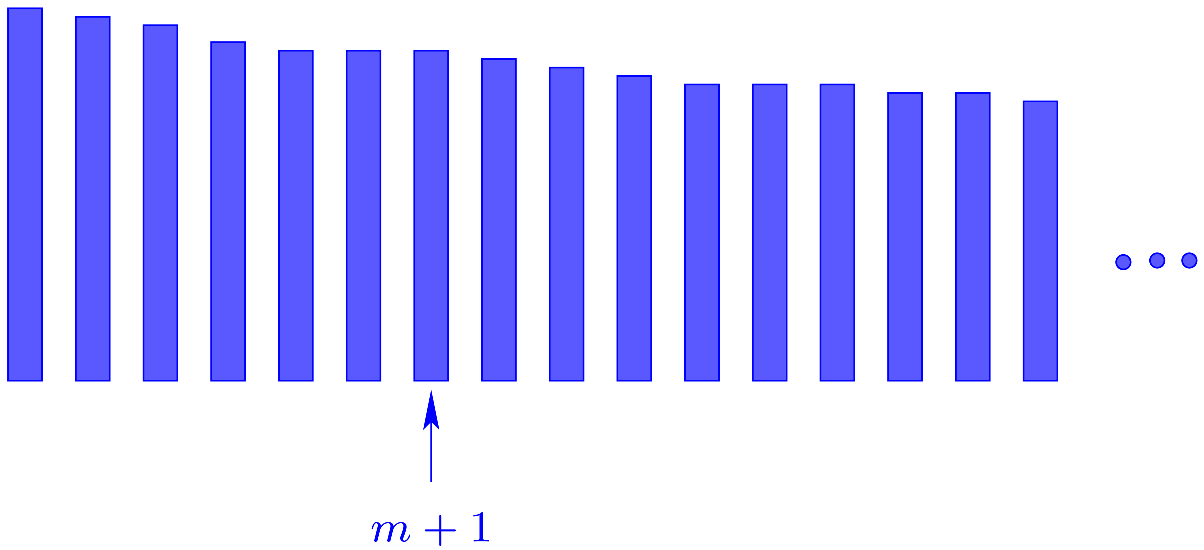
# Room for improvement?

OPT is at least

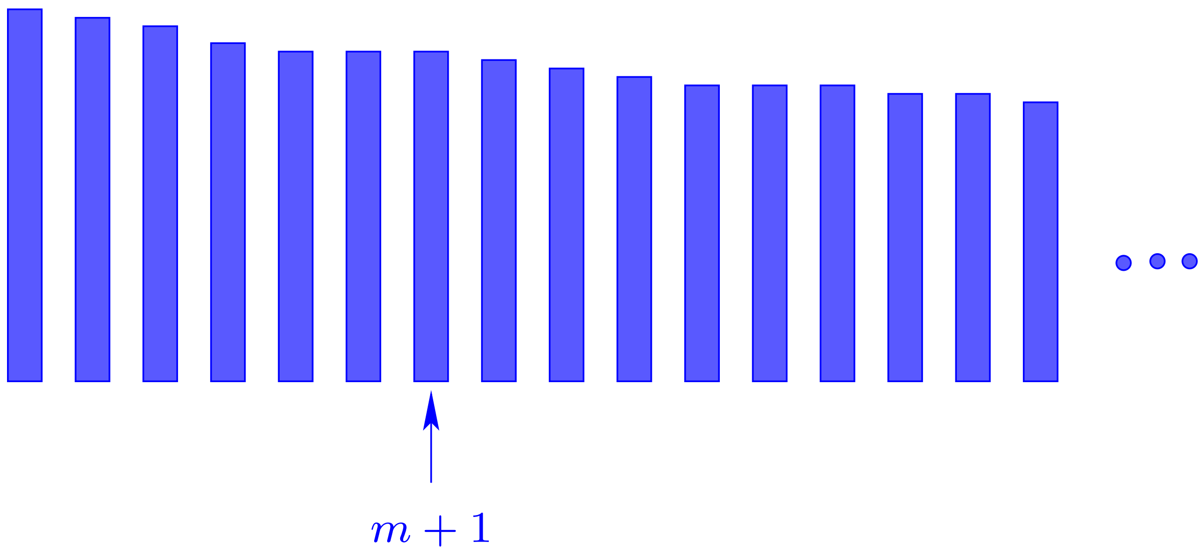
(a) $\dfrac{1}{m}\displaystyle\sum_{i=1}^{n} p_i$

(b) $\displaystyle\max_{1 \leq i \leq n} p_i$

(c) $2\times$ proc. time of $(m+1)$-st largest job



# Room for improvement?

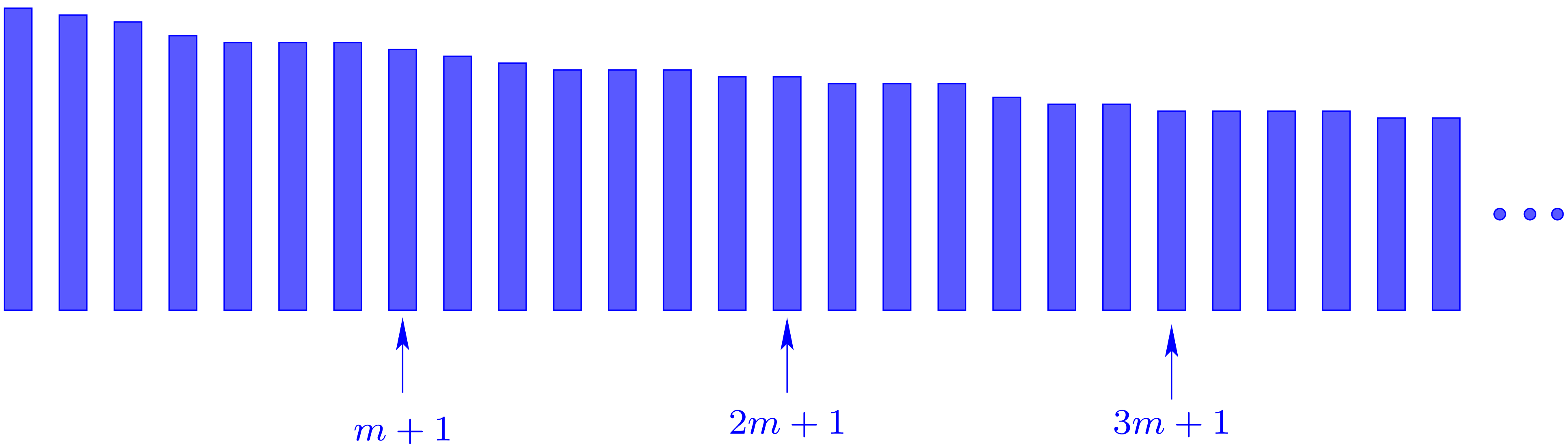



**Thm**: Impossible to prove competitive ratio smaller than 1.919 using

(1) $\frac{1}{m}\sum_{i=1}^{n} p_i$

(2) Set of $m+1$ largest proc. times

# Room for improvement?




(3) $(im+1)$-st to $(im-i+1)$-st largest jobs, $i = 1\ldots, \lfloor\frac{(n-1)}{m}\rfloor$

**Thm**: Impossible to prove competitive ratio smaller than 1.917 using (1–3).

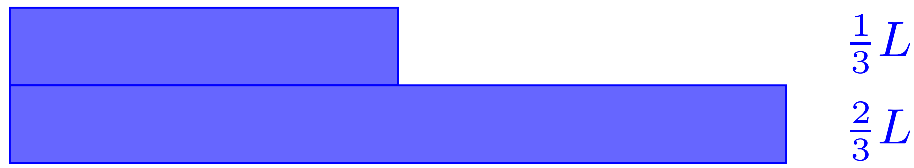
# Randomization

- $m = 2$ : 4/3-competitive algorithm. This is optimal.
  Bartal, Fiat, Karloff, Vohra STOC 92

- Lower bound of $e/(e-1) \approx 1.58$, for any $m$.
  Chen, van Vliet, Woeginger IPL 94; Sgall IPL 97

- BIT algorithm that is 1.916-competitive, for any $m$.
  Albers STOC 02

# 2 machines



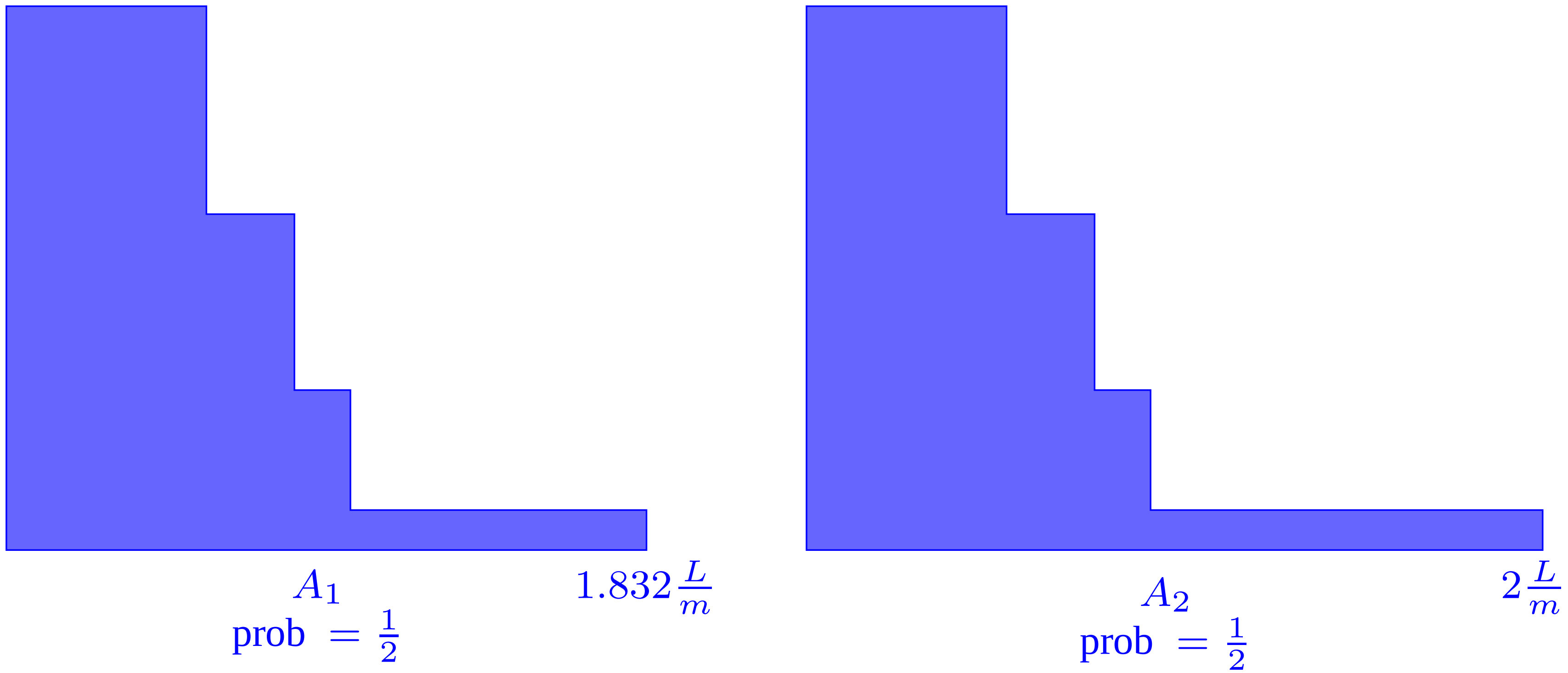

Discrepancy = load difference on the two machines

$L$ = total proc. times of all jobs arrived so far.

Algorithm: Maintain set of all schedules generated so far, together with their probabilities. When a new job arrives:

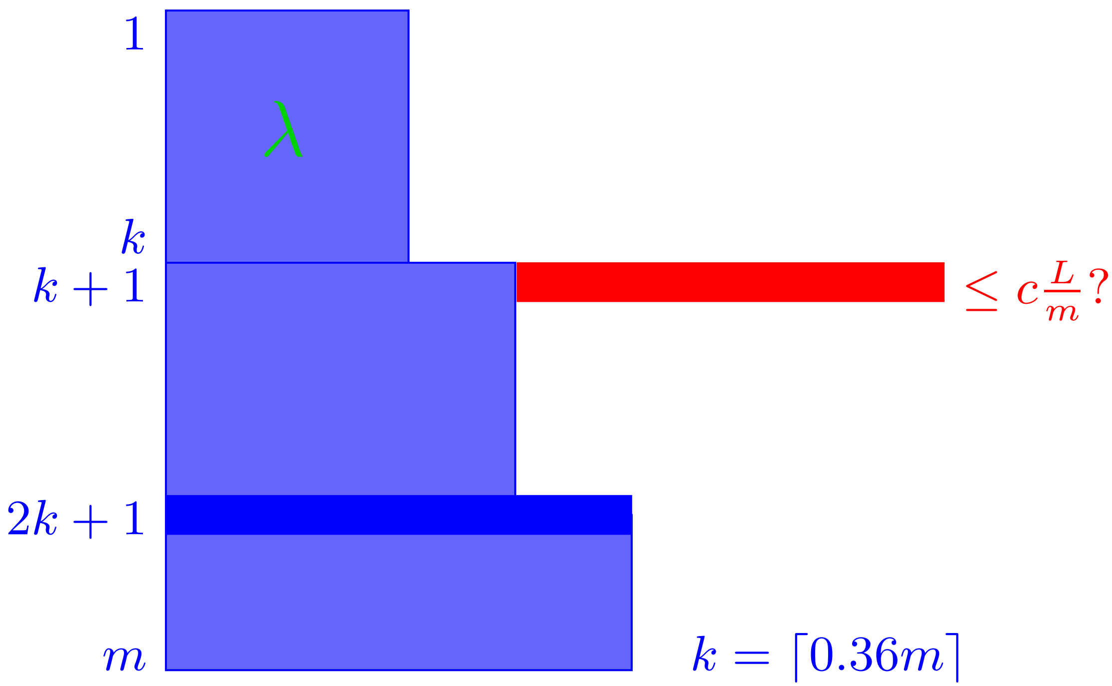
- $E_1$ = discrepancy if job is put on short machine in each schedule.
  $E_2$ = discrepancy if job is put on tall machine in each schedule.
- Determine $p$, $0 \leq p \leq 1$, such that $pE_1 + (1-p)E_2 = \frac{1}{3}L$.
- If $p$ exists, with prob. $p$ schedule on short machine and with prob. $1-p$ on the tall machine in each schedule.
- If such $p$ does not exist, schedule on short machine.

# BIT algorithm




$$\mathsf{E}[BIT(\sigma)] \leq \frac{1}{2}(1.832 + 2)\frac{L}{m} = 1.916\frac{L}{m}$$

# Algorithm $A_1$



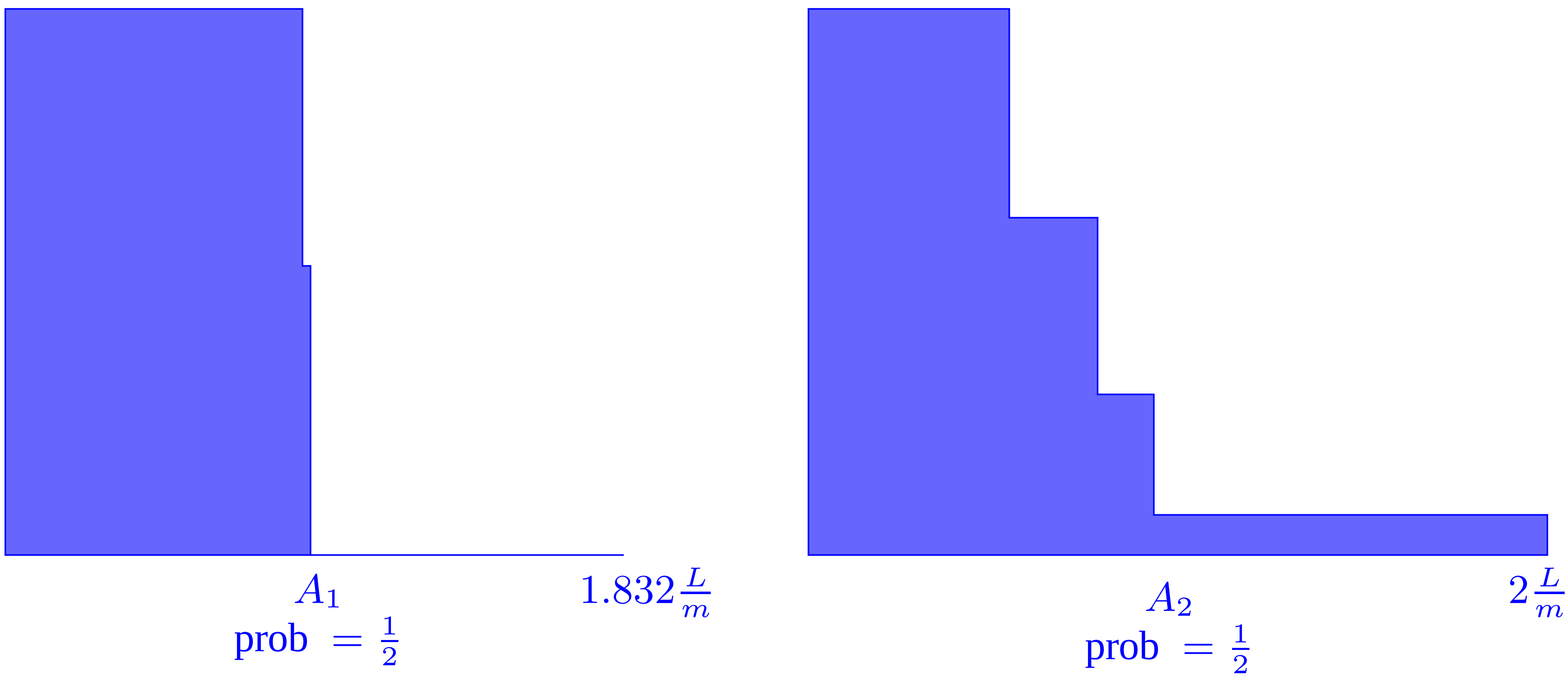

**Algorithm:** $c = 1.832$ $\quad k = \lceil 0.36m \rceil$ $\quad \alpha = 0.56$

Schedule new job on machine $k+1$ if

- $\lambda > \alpha \cdot l_{2k+1}$ and resulting load $\leq c \cdot \frac{L}{m}$

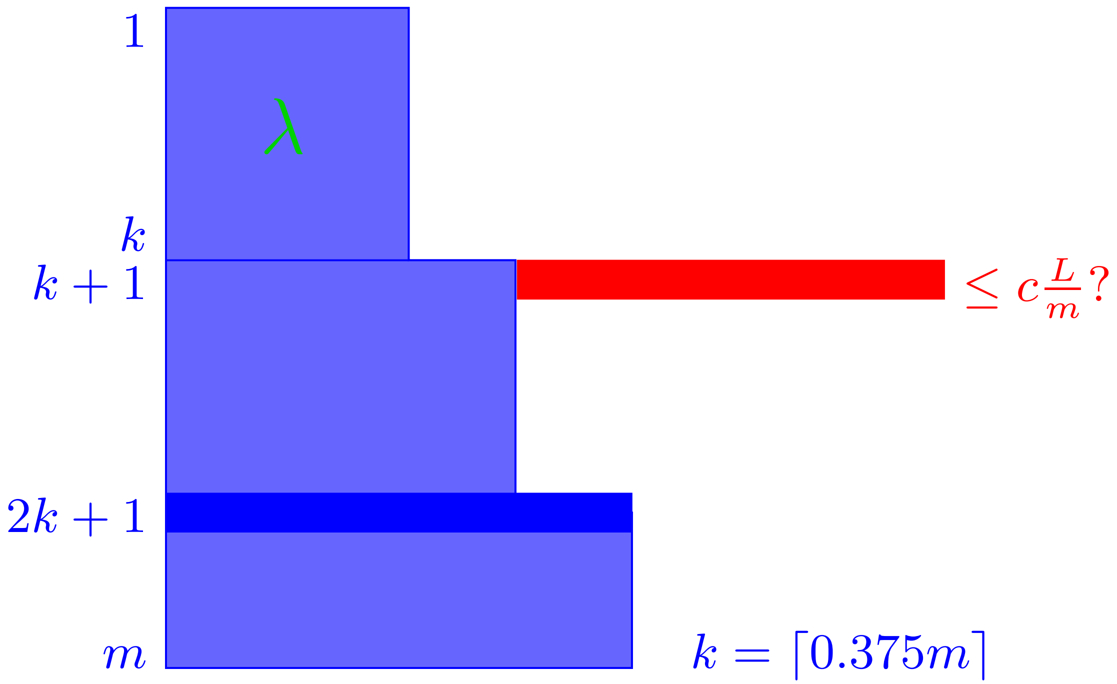
Otherwise schedule job on machine 1.

# Problem




$A_1$'s schedule is nearly balanced/flat.

# Algorithm $A_2$




**Algorithm:** $k = \lceil 0.375m \rceil \quad \alpha = 0.45$

Keep track of $A_1$'s schedule. If $A_1$ not flat then $c = 2$ else $c = 1.885$.

Schedule new job on machine $k+1$ if

- $\lambda > \alpha \cdot l_{2k+1}$ and resulting load $\leq c \cdot \frac{L}{m}$

Otherwise schedule job on machine 1.

## Extensions

Preemption allowed: Tight upper and lower bounds of $e/(e-1)$.

$l_p$ norm: Minimize $(\sum_{i=1}^{m} l_i^p)^{1/p}$, where $l_i$ is the load of $i$-th machine.
Motivation: Balance load among the machines.
Line breaks in LaTeX are computed based on $l_2$ and $l_3$ norms.

$p = \infty$: Makespan

$p = 2$: List is $\sqrt{4/3}$-competitive.

# Open problems

Develop improved deterministic algorithms.

Develop randomized algorithms that beat the deterministic lower bound, for all $m$.

## Jobs arrive over time

Job sequence $J_1, J_2, \ldots, J_n$

$J_i$: arrival time $a_i$

processing time $p_i$

possible weight $w_i$

preemption may be allowed

- Clearvoyant scheduling: When $J_i$ arrives, $p_i$ is known.
  Classical manufacturing;   Web server delivering static Web pages.

- Non-clearvoyant scheduling: When $J_i$ arrives, $p_i$ is unknown and becomes known only when job finishes.
  Scheduling in operating systems.

# Algorithms

Clearvoyant

- SRPT: Shortest Remaining Processing Time   Run job with least remaining work.
- SJF: Shortest Job First   Run job with least initial work.
- HDF: Highest Density First   Run job with highest ratio $w_i/p_i$

# Algorithms

Non-clearvoyant

- FIFO: First In First Out or First Come First Served. Run job with earliest arrival time.
- RR: Round Robin or Processor Sharing or Equi-Partition
  Devote an equal amount of processing resources to all jobs.
  Multi-processor environment: Assign $J_i$ to processor $i \bmod m$.
- SETF: Shortest Elapsed Time First Run the job that has been processed the least. Amounts to RR when there are ties.
- MLF: Multi-Level Feedback Mimick SETF while keeping the number of preeemptions per job logarithmic.
  Queues $Q_0, Q_1, \ldots$ with target processing times $T_j = 2^j$.
  $J_i$ is processed for $T_j - T_{j-1}$ time units while in $Q_j$ and before being promoted to $Q_{j+1}$. Always run job in the front of lowest non-empty queue.

# Clearvoyant scheduling

Job sequence $J_1, J_2, \ldots, J_n$

$J_i$: arrival time $a_i$

processing time $p_i$

preemption allowed

Total flow time: $\sum_{i=1}^{n} f_i$

- 1 machine: SRPT is 1-competitive.
- $m$ machines: SRPT is $O(\min\{\log P, \log n/m\})$-competitive, where $P = p_{\max}/p_{\min}$. This ratio is best possible.

  Resource augmentation: SPRT is $1/s$-competitive using speed $s \geq 2 - \frac{1}{m}$.

  No migration: $O(\min\{\log P, \log n/m\})$-competitive algorithm.

# Clearvoyant scheduling

Total stretch: $\sum_{i=1}^{n} f_i/p_i$

- 1 machine: SRPT is 2-competitive.
  Any algorithm has competitiveness $c \geq 1.036$.

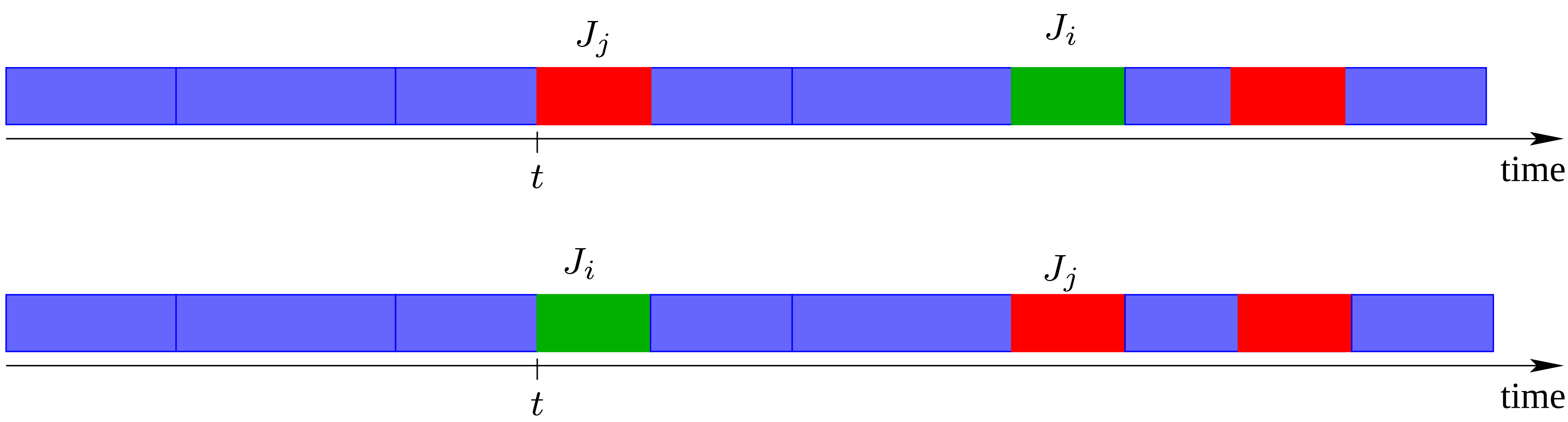
- $m$ machines: SRPT is 14-competitive.
  Any algorithm has competitivess $1.093 \leq c \leq 9.82$.

  No migration: $17.32$-competitive algorithm.

# 1 machine

SRPT is optimal.

Consider optimal schedule. Suppose that it violates SRPT first at time $t$.




Use slots where $J_i$ or $J_j$ are scheduled to first process $J_i$ and then $J_j$.

Swap can only decrease the total flow time.

## $m$ machines

**Thm:** SRPT is $O(\log P)$-competitive where $P = p_{\max}/p_{\min}$

For any scheduler $S$ let $F_S$ be the total flow time on given job instance.
$N_S(t)$ = #active jobs in $S$ at time $t$
$M_S(t)$ = #active machines in $S$ at time $t$

$$F_S = \int_t N_S(t)dt$$

**Observation 1:** $F_S \geq \sum_i p_i$

**Observation 2:** $\int_t M_S(t)dt = \sum_i p_i$

# Analysis SRPT

$T$ = times when all $m$ machines are active.

**Main Lemma:** $t \in T \quad N_{SRPT}(t) \leq m(2 + \ln P) + N_{OPT}(t)$

$$F_{SRPT} \quad = \quad \int_t N_{SRPT}(t)dt \;=\; \int_{t \notin T} N_{SRPT}(t)dt + \int_{t \in T} N_{SRPT}(t)dt$$

# Analysis SRPT

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**Main Lemma:** $t \in T \quad N_{SRPT}(t) \leq m(2 + \ln P) + N_{OPT}(t)$

$$\begin{aligned} F_{SRPT} &= \int_t N_{SRPT}(t)dt = \int_{t \notin T} N_{SRPT}(t)dt + \int_{t \in T} N_{SRPT}(t)dt \\ &\leq \int_{t \notin T} M_{SRPT}(t)dt + \int_{t \in T} ((2 + \ln P)M_{SRPT}(t) + N_{OPT}(t))dt \end{aligned}$$

# Analysis SRPT

$T$ = times when all $m$ machines are active.

**Main Lemma:** $t \in T \quad N_{SRPT}(t) \leq m(2 + \ln P) + N_{OPT}(t)$

$$\begin{aligned}
F_{SRPT} &= \int_t N_{SRPT}(t)dt = \int_{t \notin T} N_{SRPT}(t)dt + \int_{t \in T} N_{SRPT}(t)dt \\
&\leq \int_{t \notin T} M_{SRPT}(t)dt + \int_{t \in T} ((2 + \ln P)M_{SRPT}(t) + N_{OPT}(t))dt \\
&\leq (2 + \ln P) \int_t M_{SRPT}(t)dt + F_{OPT} \leq (2 + \ln P) \sum_i p_i + F_{OPT}
\end{aligned}$$

# Analysis SRPT

**Lemma:** For any $t$ and $i$ with $N_{OPT}(t) + 2m + i \leq N_{SRPT}(t)$, the ($N_{OPT}(t) + 2m + i$)-th largest remaining processing time of SRPT is at least $p_{\min}(\frac{m}{m-1})^i$.

**Main Lemma:** $t \in T \quad N_{SRPT}(t) \leq m(2 + \ln P) + N_{OPT}(t)$

**Proof Main Lemma:** $p_{\min}(\frac{m}{m-1})^i \leq p_{\max}$ and hence

$$i \leq \log_{m/(m-1)} P$$

$$N_{SRPT}(t) \leq N_{OPT}(t) + 2m + \log_{m/(m-1)} P \leq N_{OPT}(t) + 2m + m \ln P$$

since $\log_{m/(m-1)} P = \log_{(m/(m-1))^{m\frac{1}{m}}} P$

# Analysis SRPT

**Lemma:** For any $t$ and $i$ with $N_{OPT}(t) + 2m + i \leq N_{SRPT}(t)$, the ($N_{OPT}(t) + 2m + i$)-th largest remaining processing time is at least $p_{\min}(\frac{m}{m-1})^i$.

**Proof:**

- $VOL_A(t, x)$ = total volume of remaining processing time $\leq x$.
  $VOL_{SPRT}(t, x) - VOL_{SPRT}(t, x) \leq mx$
- Prove lemma for $N_{OPT}(t) = 0$.
- Any additional job in $N_{OPT}(t)$ can increase $N_{SRPT}(t)$ by only 1.

# Eliminating migration

$O(\min\{\log P, \log n\})$-competitive algorithm

- Job classes: class $k$ contains jobs with initial processing time in $[2^k, 2^{k+1})$.
- When new job of class $k$ arrives, assign it to machine that was assigned the smallest load of class-$k$ jobs (List scheduling).
- On each processor execute SRPT.

# Non-clearvoyant scheduling

Job sequence $J_1, J_2, \ldots, J_n$

$J_i$: arrival time $a_i$

processing time $p_i$ unknown

preemption allowed

1 machine, total flow time: $\sum_{i=1}^{n} f_i$

- Any deterministic algorithm is $\Omega(n^{1/3})$-competitive.
- SETF is $(1 + 1/\epsilon)$-competitive using speed $1 + \epsilon$.
- Randomized algorithms: $\Theta(\log n)$-competitive.

# Load balancing

$m$ machines, restricted machine model

$J_1, \ldots, J_n$

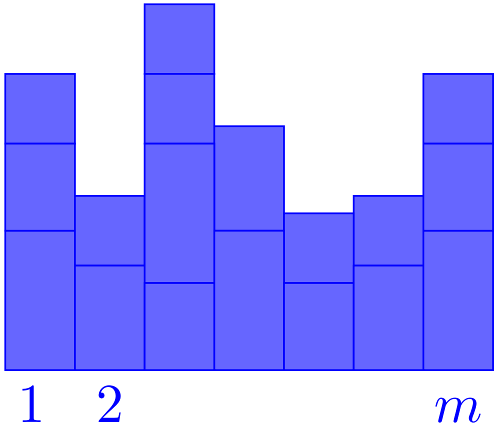
$J_i$: arrival time $a_i$

load $l_i$

unknown duration

executable on any machine in $M_i \subseteq \{1, \ldots, m\}$

Goal: min. maximum load on the machines



# Load balancing

**Thm:** Any det. algorithm is $\Omega(\sqrt{m})$-competitive even if all loads are 1.
Azar, Broder, Karlin   FOCS 92

**Thm:** Robin-Hood is $(2\sqrt{m}+2)$-competitive.
Azar, Kalyanasundaram, Plotkin, Pruhs, Waarts   J. Algorithms 97

## Lower bound

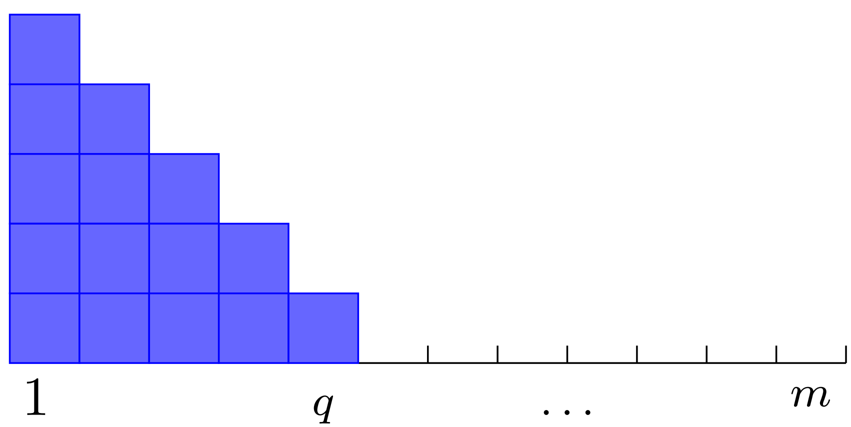
**Thm:** $A$ online algorithm $\Longrightarrow c_A \geq \Omega(\sqrt{m})$

**Proof:** $L_j(t)$: $A$'s load of $j$-th machine just prior to arrivals at $t$

Order machines s.t. $L_1(t) \geq \ldots \geq L_{q(t)}$ $\quad L_j(t) = 0 \quad j > q(t)$

Construct job sequence s.t. $m$ active jobs

OPT's load = 1 $\quad L_j(t) \geq L_{j+1}(t) + 1 \quad 1 \leq j \leq q - 1$



# Lower bound

Construct job sequence s.t. $m$ active jobs

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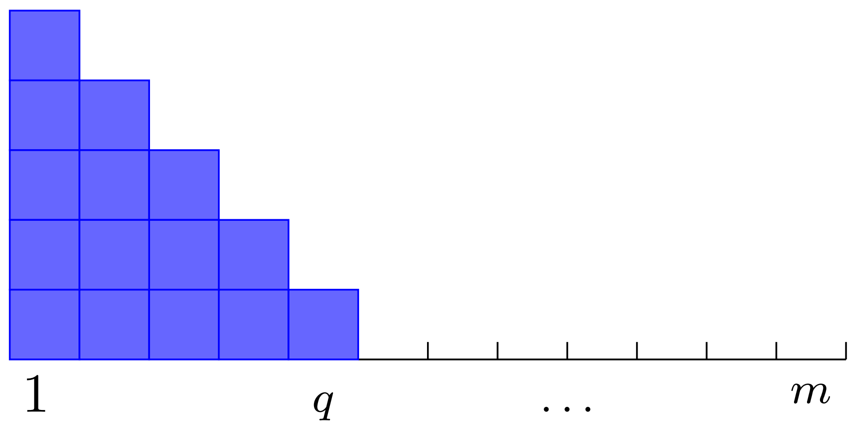



$$m = \sum_{j=1}^{q} L_i(t) \leq \sum_{j=0}^{q-1} (l-j) \leq \frac{l(l+1)}{2} \leq l^2$$

because $q \leq l$ and $l \geq 1$.

$$l \geq \sqrt{m}$$

# Construction

Consider

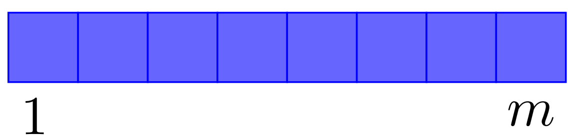
$$L = (L_1(t), \ldots, L_{q(t)}(t))$$

$$L' = (L_1(t'), \ldots, L_{q(t')}(t'))$$

$L < L'$ if $L_k(t) = L_k(t')$ for $k < j$ and $L_j(t) < L_j(t')$

Stages

**Base case:**



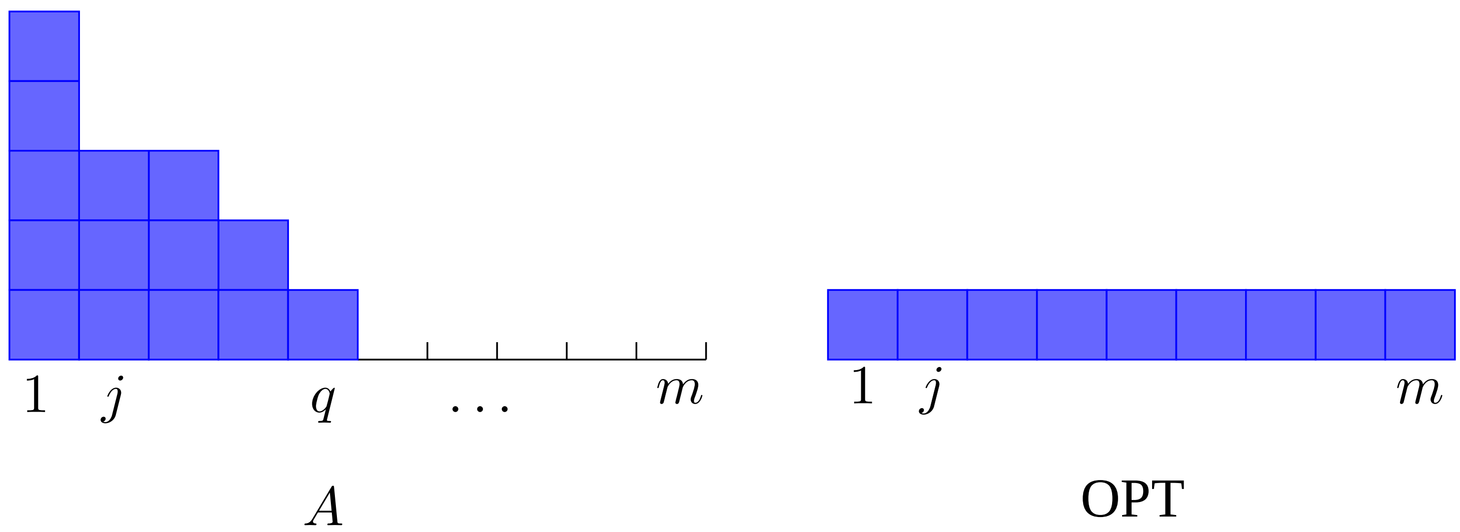
# Construction

**Inductive step:**

At the end of $k$-th stage $L^k = (L_1^k, \ldots, L_{q_k}^k)$

If $L_j^k \geq L_{j+1}^k + 1$, we are done. Otherwise construct $L^{k+1}$ with $L^{k+1} > L^k$

Invariant: OPT has job $i$ on machine $j$ and $L_j^k > 0$, then $A$ has $i$ on $j$.




Determine smallest $j$ with $L_j^k = L_{j+1}^k$

# Construction

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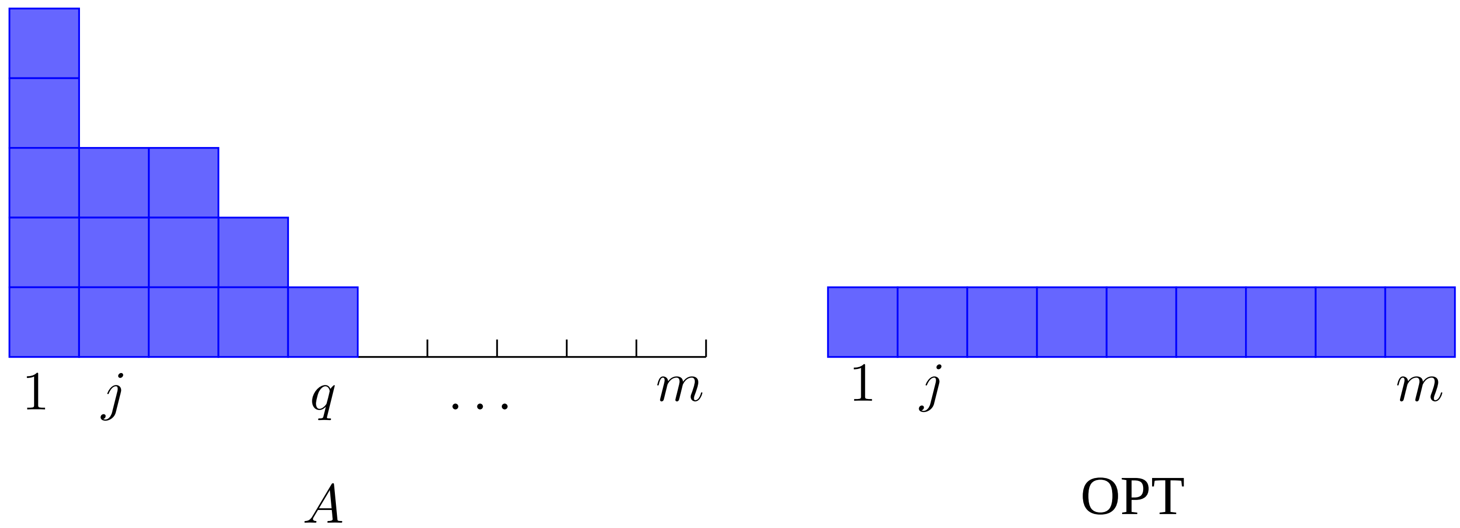


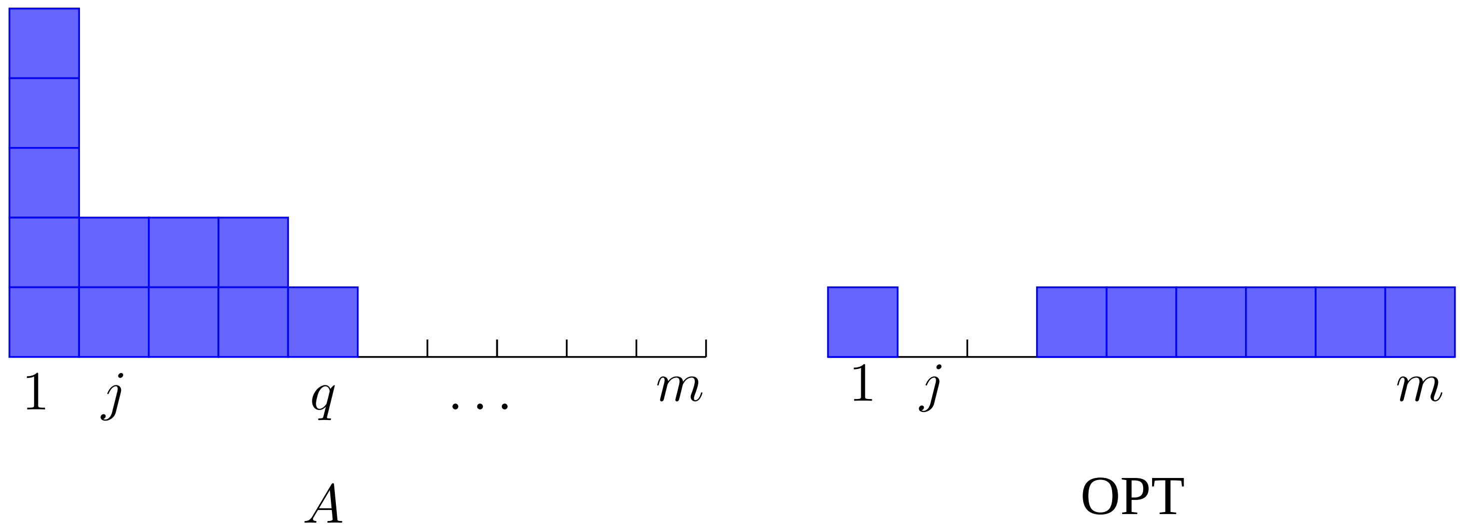

Determine smallest $J$ with $L_j^k = L_{j+1}^k$

- Terminate OPT's jobs on $j, j+1$
- Create $J$ with allowable machines $\{j, j+1\}$.
  Assume $A$ assigns to $j$. Then OPT assigns to $j+1$.

# Construction

**Inductive step:**

Invariant: OPT has job $i$ on machine $j$ and $L_j^k > 0$, then $A$ has $i$ on $j$.

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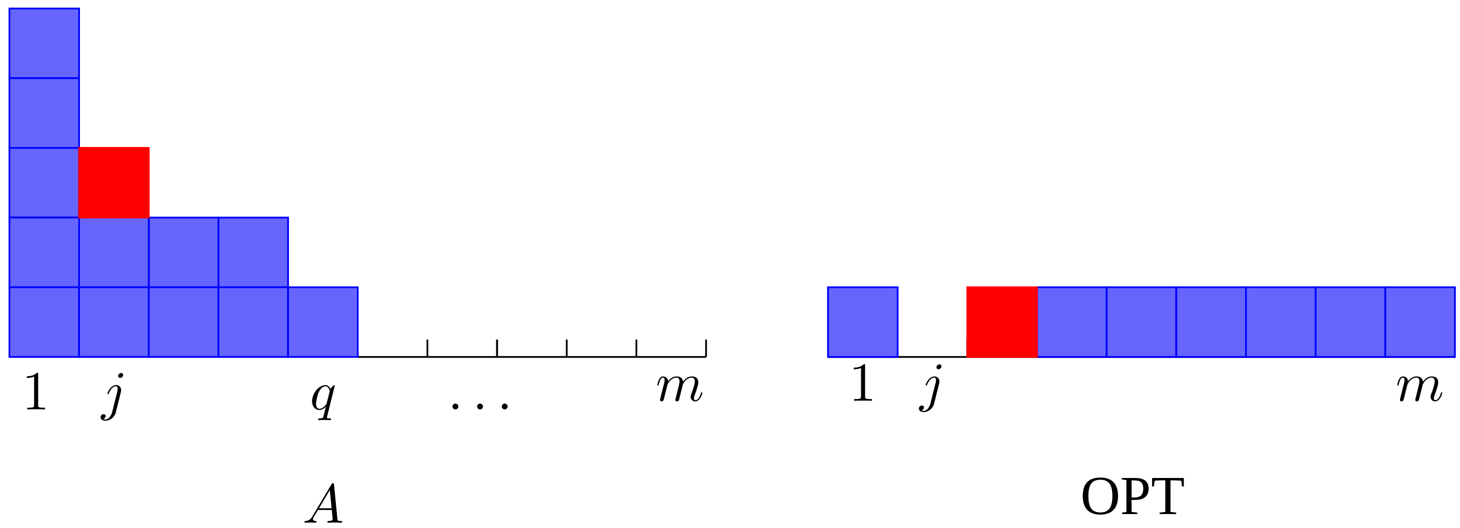
# Construction

**Inductive step:**

Invariant: OPT has job $i$ on machine $j$ and $L_j^k > 0$, then $A$ has $i$ on $j$.




Determine smallest $j$ with $L_j^k = L_{j+1}^k$

- Terminate OPT's jobs on $j, j+1$
- Create $J$ with allowable machines $\{j, j+1\}$.
  Assume $A$ assigns to $j$. Then OPT assigns to $j+1$.

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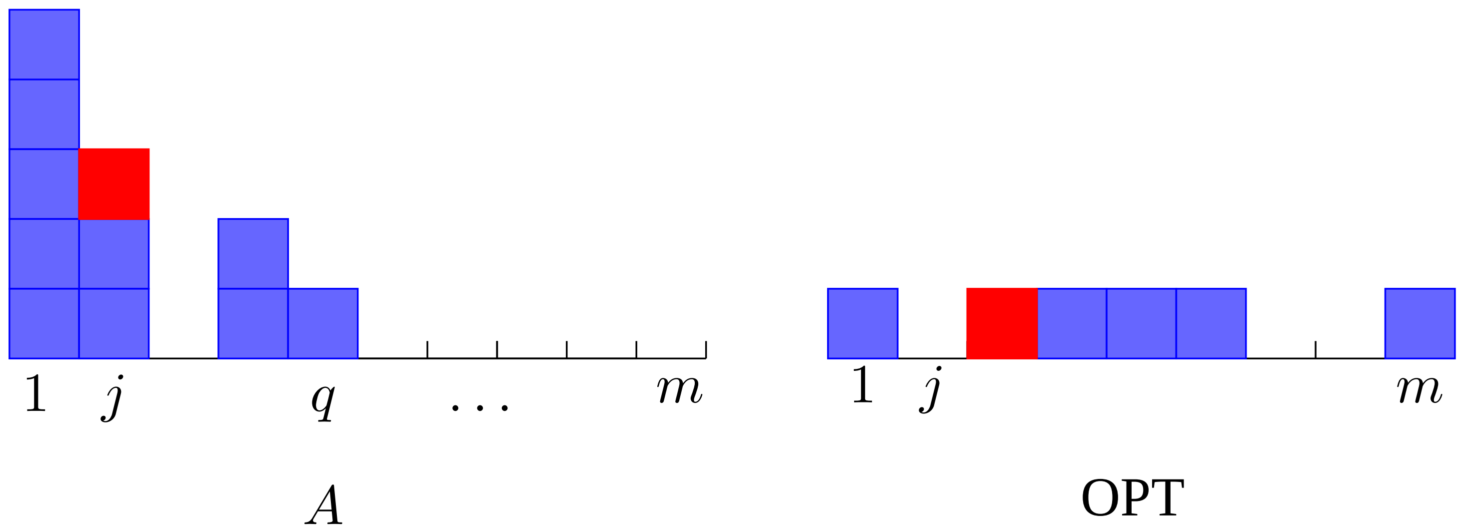


- Terminate $A$'s jobs on $j + 1$.
- For each of OPT's idle machines, create job that can only be executed on that machine.

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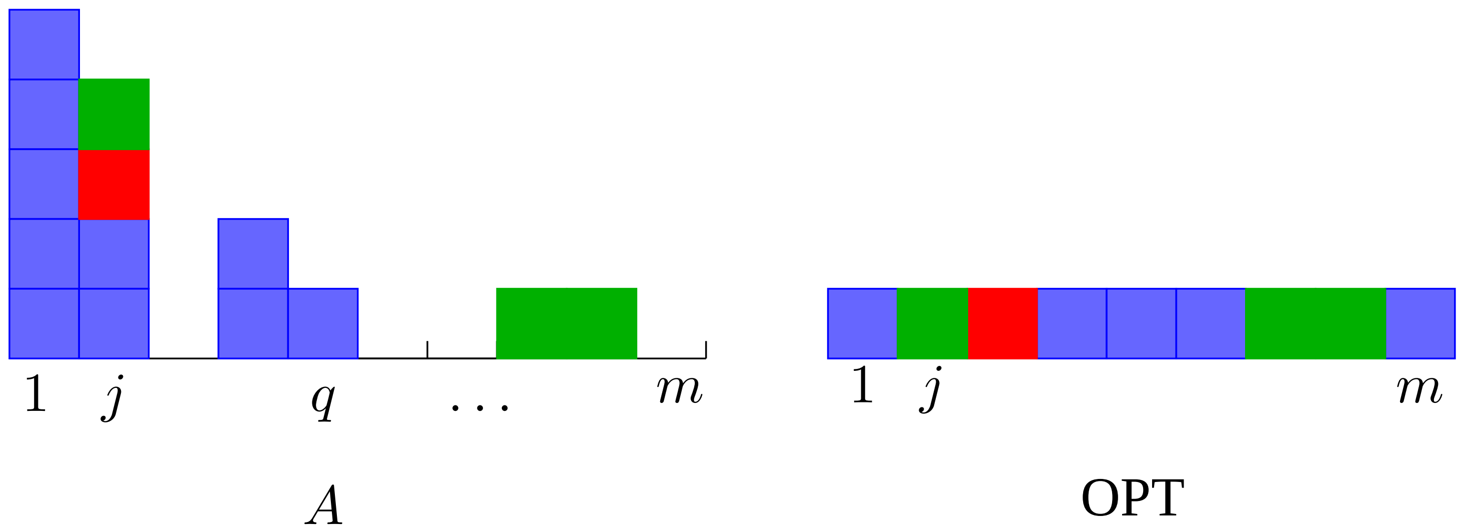


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# Algorithm Robin Hood

$\sigma_i = J_1, \ldots, J_i$

$L_j(a_i)$: load on machine $j$ just prior to $a_i$

Lower bound on OPT$(\sigma_i)$

$$B(a_0) = 0 \qquad B(a_i) = \max\{B(a_{i-1}),\ \ l_i,\ \ \frac{1}{m}(l_i + \sum_j L_j(a_i))\}$$

Machine is rich if $L_j(a_i) \geq \sqrt{m}B(a_i)$ and poor otherwise.

ROBIN HOOD: Assign $J_1$ to any machine.

$J_i$: Assign to poor machine if possible. Otherwise to rich machine that most recently became rich.

**Thm:** ROBIN HOOD is $(2\sqrt{m} + 2)$-competitive.

## Proof

At any time most $\sqrt{m}$ rich machines.

$$mB(a_i) \geq \sum_j L_j(a_i)$$

If there were more than $\sqrt{m}$ rich machines, then $\sum_j L_j(a_i) > \sqrt{m}\sqrt{m}B(a_i)$.

$J_i$ arrives and assigned to machine $j_0$

1. $j_0$ poor

$$L_{j_0}(a_i) + l_i < \sqrt{m}B(a_i) + \mathsf{OPT}(\sigma_i) \leq (\sqrt{m} + 1)\mathsf{OPT}(\sigma_i)$$

# Proof

## 2. $j_0$ rich

$a_{t(i)}$: last time when $j_0$ became rich

$S = \{$jobs assigned to $j_0$ during $(a_{t(i)}, a_j)\}$

$k \in S$ could only be assigned to machines that were rich at time $a_{t(i)}$

$$h = |\bigcup_{k \in S} M_k| \leq \sqrt{m}$$

$$\sum_{k \in S} l_k \leq h\mathsf{OPT}(\sigma_i) \leq \sqrt{m}\mathsf{OPT}(\sigma_i)$$

$$\begin{aligned} L_{j_0}(a_i) + l_i &\leq L_{j_0}(a_{t(i)}) + l_{t(i)} + \sum_{k \in S} l_k + l_i \; < \; \sqrt{m}B(a_{t(i)}) + l_{t(i)} + \sum_{k \in S} l_k + l_i \\ &\leq (2\sqrt{m} + 2)\mathsf{OPT}(\sigma_i) \end{aligned}$$

# Summary

- Online settings: jobs form a list; jobs arrive over time
- Classical and new objectives: makespan, total flow, total stretch, $l_p$ norms, load balancing
- Old and new algorithms: List, SRPT, more sophisticated strategies
- Power of randomization and resource augmentation