Cyclic scheduling: an introduction

Claire Hanen

Claire.Hanen@lip6.fr

Université Paris X Nanterre / LIP6 UPMC

Summary

- Definition
- Uniform precedence constraints
  - Feasibility
  - Periodic schedules
  - Earliest schedule
- Resource constraints
  - Periodicity and Patterns
  - Complexity
  - approximation
  - ILP formulations
- Perspectives

Definition

- A set of tasks $\mathcal{T}$ to be repeated many times assumed infinite
- For $i \in \mathcal{T}$, $< i, k >$ denotes k^{th} occurrence of i
- An infinite schedule σ defines:
- $\forall k \geq 0 \; t^{\sigma}(< i, k >)$ starting time of $< i, k >$
- Resources for each task execution

Optimizing

- Minimizing the average cycle time :

$$A(\sigma) = \max_{i \in \mathcal{T}} A(\sigma, i) = limsup_{k \rightarrow +\infty} \frac{t^{\sigma}(< i, k >)}{k}$$

- Maximizing the throughput:

$$D(\sigma) = \frac{1}{A(\sigma)}$$

- For a given average cycle time, minimizing the amount of resources per time units (ex: nb of processors, nb of registers,...)

Applications and models

- Implementing loops on parallel architectures
  - $\longrightarrow$ compiling, code generation
  - $\longrightarrow$ embedded applications

  litterature on software pipelining and dataflow computations
- Mass production
  - $\longrightarrow$ Cyclic shop problems
  - $\longrightarrow$ Hoist scheduling problem.

  litterature on cyclic scheduling in production systems
- Models of parallelism
  - $\longrightarrow$ timed Petri Nets
  - $\longrightarrow$ Graphs
  - $\longrightarrow$ Max + algebra

  litterature on parametric paths, timed event graphs, Max +

Infinite schedule?

- Algorithms-> finite description
- Static schedule

  Must be regular /time and resources
- Dynamic policy

  examples: earliest schedule, regular priorities on resources assignment

Loop example

Assume arrays A, B, C, D, and that the processor can compute several instructions in parallel :

for I = 2 to N do

$B(I) = A(I-1) + 1$	task 1	$< 4, k-1 >$ precedes $< 1, k >$
$C(I) = B(I) + 5$	task 2	$< 1, k >$ precedes $< 2, k >$
$D(I) = B(I-2) * D(I)$	task 3	$< 1, k-2 >$ precedes $< 3, k >$
$A(I) = C(I-2) + D(I)$	task 4	$< 2, k-2 >, < 1, k >$ precede $< 2, k >$

Precedence constraints are uniforms: if $< i, k >$ precedes $< j, l >$ then for all integers δ, $< i, k + \delta >$ precedes $< j, l + \delta >$

Cyclic scheduling with uniform precedence

- A set $\mathcal{T}$ of n generic tasks with processing times $p_1, \ldots, p_n$
- non reentrance: $< i, k >$ precedes $< i, k+1 > \forall k \geq 1$
- A multi-graph $G = (\mathcal{T}, A)$ of uniform constraints
- For each arc $a \in A$,

  A value called length $L(a) \in Z$ also mentioned as latency or delay

  A value called height $H(a) \in \mathbb{Z}$ also called dependence distance

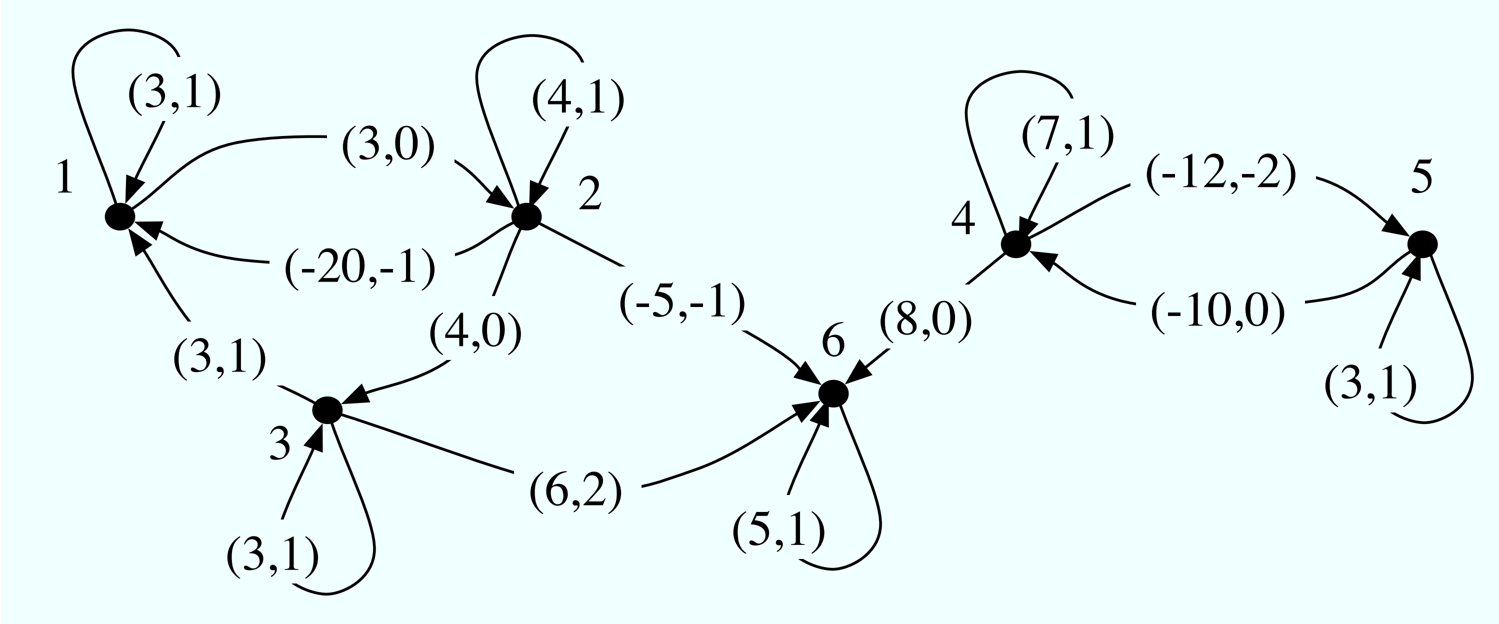
$$b(a) \xrightarrow{(L(a), H(a))} e(a)$$

- If $i = b(a)$ et $j = e(a)$, $\forall k \geq 1$, $t^\sigma(< i, k >) + L(a) \leq t^\sigma(< j, k + H(a) >$
- Find an infinite feasible schedule minimizing the average cycle time.

Definition 1. *The Length of path μ, denoted $L(\mu) =$ sum of arcs length. Height of μ, denoted by $H(\mu) =$ sum of arcs height.*

Remark 1. *Non-reentrance can be expressed with uniform constraints : arcs (i, i) with length p_i and height 1.*

Example



Negative length can be used to model deadlines :

$t^\sigma(< 5, k >) \leq t^\sigma(< 4, k >) + 10$

Questions

- Feasibility
- Construction and properties of schedules.
  - Periodic schedules
  - Earliest schedule
- Performance

Feasibility

For any path μ from i to j,

$$\forall k \geq \max(1, 1 - H(C)), \quad t^\sigma(<i,k>) + L(\mu) \leq t^\sigma(<j, k + H(\mu)>)$$

For any circuit C of G, and task i in C:

$$\forall k \geq \max(1, 1 - H(C)), \quad t^\sigma(<i,k>) + L(C) \leq t^\sigma(<i, k + H(C)>)$$

Non reentrance $\Rightarrow$

Lemma 1. *[Lee 05][Munier 06] If G is feasible then for any circuit C such that $H(C) \leq 0$, $L(C) \leq 0$. If $H \in \mathbb{N}$ this condition is sufficient [many authors] [Chretienne 85]*

- If $H(C) > 0$, $k = qH(C) + r$ then $t^\sigma(<i,k>) \geq qL(C) + t^\sigma(<i,r>)$.
- If $H(C) < 0$ and $k = -qH(C) + r$ then $t^\sigma(<i,k>) \leq -qL(C) + t^\sigma(<i,r>)$

Lemma 2. *If $H(C) \geq 0$,* $A(\sigma, i) = \limsup_{k \to +\infty} \dfrac{t^\sigma(<i,k>)}{k} \geq \dfrac{L(C)}{H(C)}$.

If $H(C) < 0$, $A(\sigma, i) \leq \dfrac{L(C)}{H(C)}$

Feasibility and performance bounds

For a circuit C the Index of C: $\alpha(C) = \dfrac{L(C)}{H(C)}$

- $\mathcal{C}^+(i)$ set of circuits C s.t. $i \in C$ and $H(C) \geq 0$. $\quad \mathcal{C}^+ = \cup_{i \in \mathcal{T}} \mathcal{C}^+(i)$
- $\mathcal{C}^-(i)$ set of circuits C s.t. $i \in C$ and $H(C) < 0$. $\quad \mathcal{C}^- = \cup_{i \in \mathcal{T}} \mathcal{C}^-(i)$

Corollary 1. *If G is feasible then for any task i, and for any schedule σ*

$$\max_{C \in \mathcal{C}^+(i)} \alpha(C) \leq A(\sigma, i) \leq \min_{C \in \mathcal{C}^-(i)} \alpha(C)$$

A critical circuit is a circuit $C^* \in \mathcal{C}^+$ s.t $\alpha(C)$ is maximum : lower bound on $A(\sigma)$.

Periodic schedules

Definition 2. *A schedule σ is periodic if each task i has a period w_i such that:*

$$t^\sigma(< i, k >) = t^\sigma(< i, 1 >) + (k-1)w_i$$

Let a be an arc from $b(a)$ to $e(a)$

$$\forall k, \quad t^\sigma(< b(a), 1 >) + (k-1)w_{b(a)} + L(a) \leq t^\sigma(< e(a), 1 >) + (k-1)w_{e(a)} + H(a)w_{e(a)}$$

$$\Leftrightarrow t^\sigma(< e(a), 1 >) - t^\sigma(< b(a), 1 >) \geq L(a) - w_{e(a)}H(a) + (k-1)(w_{b(a)} - w_{e(a)})$$

Existence of a periodic schedule

Lemma 3. *for any arc a, $w_{b(a)} \leq w_{e(a)}$.*

We denote by $C_1, \ldots, C_q$ the strong components of G.

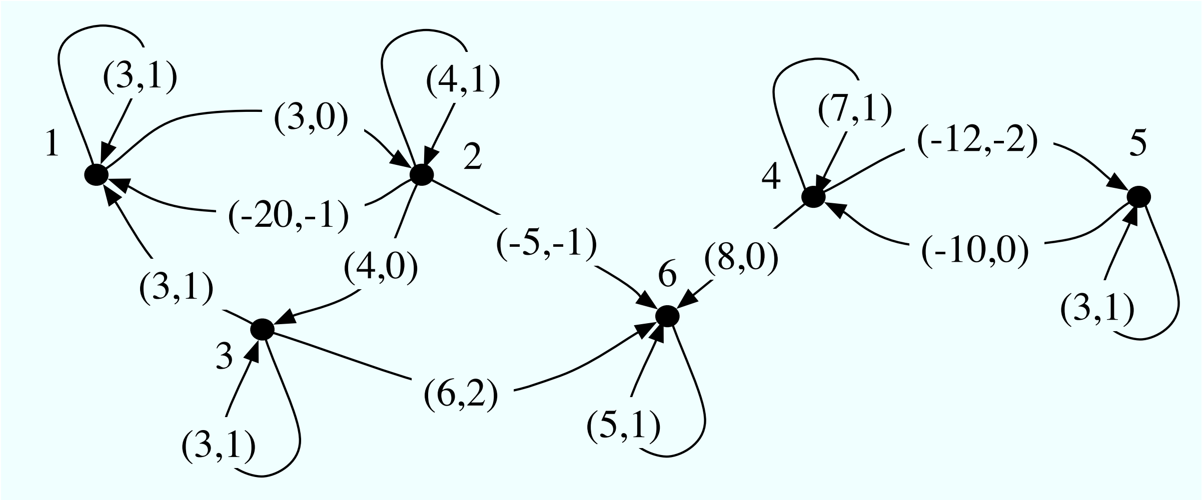
Let $\alpha^+(C_s) = \max_{C \in C_s, H(C) > 0} \alpha(C)$ and $\alpha^-(C_s) = \min_{C \in C_s, H(C) < 0} \alpha(C)$

Corollary 2.

$$\forall s, \forall i, j \in C_s, w_i = w_j = W_s \quad and \quad \alpha^+(C_s) \leq W_s \leq \alpha^-(C_s)$$

Moreover, if there is an arc a s.t. $b(a) \in C_s, e(a) \in C_{s'}$ then $W_s \leq W_{s'}$

Example



- $C_1 = \{1, 2, 3\}, C_2 = \{4, 5\}, C_3 = \{6\}$
- $\alpha^+(C_1) = 10, \alpha^-(C_1) = 17, \quad \alpha^+(C_2) = 7, \alpha^-(C_2) = 11, \quad \alpha^+(C_3) = 5$
- $10 \leq W_1 \leq 17, \quad 7 \leq W_2 \leq 11, \quad 5 \leq W_3,$
- $W_3 \geq W_2, W_3 \geq W_1$
- $W_1 = 10, W_2 = 7, W_3 = 10$. Notice that $W_1 = W_2 = W_3 = 10$ is also a solution.

Feasibility

Theorem 1. *[Munier 06] G is feasible if and only if there exists a periodic schedule.*

Idea : If no periodic schedule exists, build paths μ_x between i and j in G such that $H(\mu_x)$= h and $\lim_{x \to +\infty} L(\mu_x) \to +\infty$, which contradicts

$$t^\sigma(< j, k >) - t^\sigma(< i, k + h >) \geq L(\mu_x)$$

for some k.

Computation of a periodic schedule

For a strong connected graph G and a given W:

$$\forall a, \quad t^\sigma(< e(a), 1 >) - t^\sigma(< b(a), 1 >) \geq L(a) - W.H(a)$$

- Let $V_W(a) = L(a) - W.H(a)$
- If (G, V_W) has positive circuits then infeasibility.
- otherwise $t^\sigma(< i, 1 >) =$ longest path to i in (G, V_W) is a solution.
- Bellman-Ford algorithm

Computation of critical circuits

[Dasdan et al 99]

- Polynomial algorithms :

  Binary search on W: at each step check if (G, V_W) has positive circuits $0(nm(\log n + \log \max_a(L(a), H(a))))$ [Lawler 79][Gondran-Minoux 85]

  Linear programming with primal dual approach $O(n^2 m)$ [Burns 91]

- An efficient pseudo polynomial algorithm: Howard's algorithm[Cochet-Terrasson et al 98]

  $W =$ lower bound

  At each step check if $G, V_W)$ has positive circuit C with breadth first search. If $H(C) <= 0$ stop. Otherwise set $W = \alpha(C)$.

  complexity $O(m.X)$, X product of degrees of nodes.

omputation of an optimal periodic schedu

- Compute the strong components of G $O(n+m)$
- Check feasibility for each component C_s, and the critical circuit value α_s.$O(nm\log(n)+\log(V_{max}))$
- Compute the reduced graph of components.$O(n+m)$
- Sort the components by topological order.$O(n+m)$
- for each component C_s.

  if C_s has no predecessor, set $W_s = \alpha_s$.

  if $C_{x_1}, \ldots, C_{x_r}$ are predecessors of C_s. Let $\beta_s = \max_i W_{x_i}$

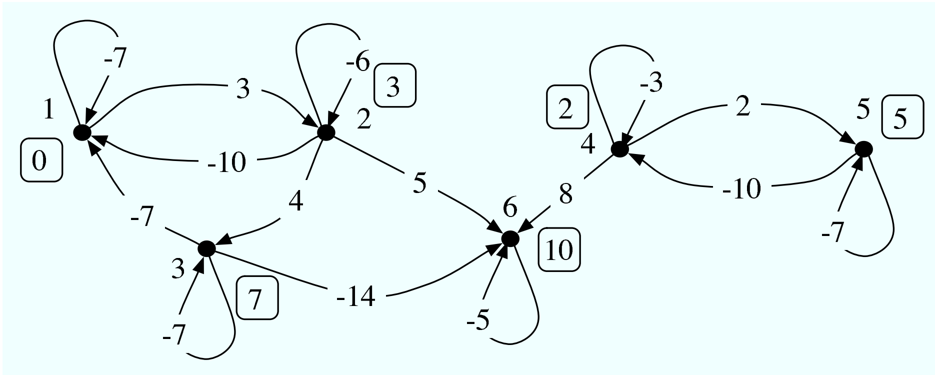
  If $\alpha_s \geq \beta_s$ set $W_s = \alpha_s$.

  otherwise check if (C_s, V_{β_s}) has some positive circuit.$O(nm)$

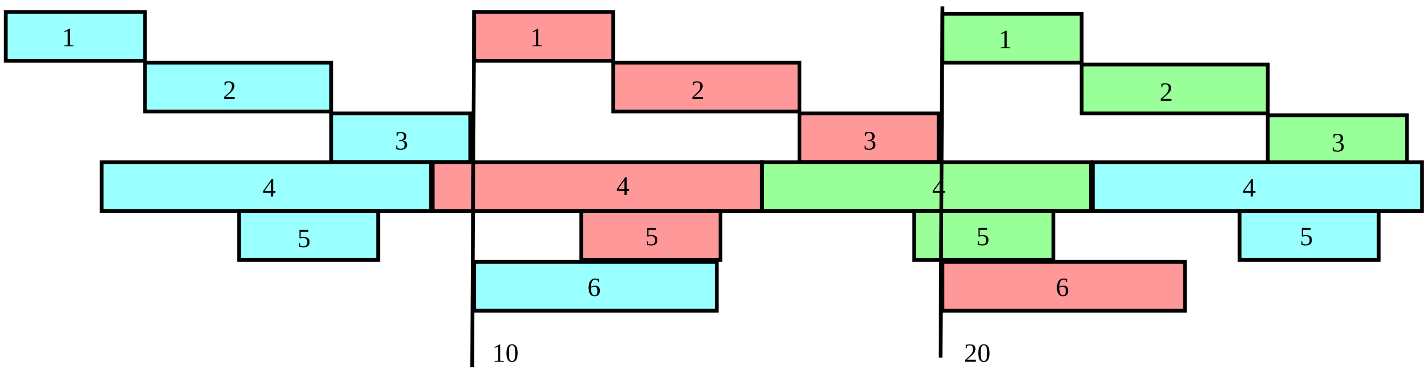
  if so,infeasibility otherwise set $W_s = \beta_s$.
- Compute the longest paths from a dummy source node to any node i on G with on each component value V_{W_s} and on each intermediate arc between $C_s, C_{s'}$ value $V_{W_{s'}}$. $O(nm)$
- $t^\sigma(<i,1>) =$ longest path to i.

Example

Graph with $L - W_s H$ values:



A periodic schedule: period of 1,2,3,6 is 10, period of 4,5 is 7



Positive heights

The problem has been studied earlier [Chretienne 85][many authors]

Theorem 2. *For any $w \geq \max_{C\ circuit} \alpha(C)$ there is a periodic schedule, all tasks have period w.*

No need to compute strong components.

K-periodic schedules

Definition 3. *In a K-periodic schedule σ, a schedule of K_i occurrences of task i is repeated every W_i time units:*

$$\forall l \geq l_0, \quad t^{\sigma}(< i, l + K_i >) = t^{\sigma}(< i, l >) + W_i$$

Lemma 4. *The average cycle time of task i in a $K-$periodic schedule is* $A(\sigma, i) = \frac{W_i}{K_i}$.

Un ordonnancement 2-périodique (pour chaque tâche).

1	3	4	1	7	3	4	1	3	4	1	7	3	4
2	5	6	2	5	6	7	2	5	6	2	5	6	7

Properties of the earliest schedule

[Romanovskii 67,Chretienne 85, Munier 06]

Theorem 3. *If G is feasible, the earliest schedule σ^* is K-periodic and*

$$A(\sigma^*) = \max_{C \in \mathcal{C}^+} \alpha(C)$$

After a transitory time, the earliest schedule becomes regular and its behavior is ruled by the critical circuit index.

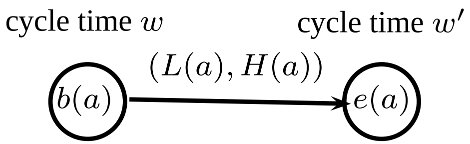
Remark 2. *K might be large $\leq$ product of heights of critical circuits.*

- Open questions

  Length of the transitory time?

  Exact measure of K.

Stability/boundedness



If $w < w'$ then $b(a)$ produces pieces or results faster than $e(a)$.

Unstability

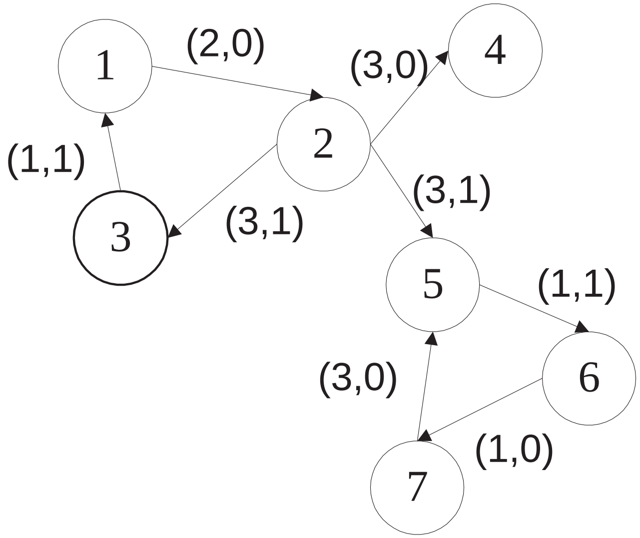
Moreover the iteration delay:

$$D^{\sigma}(k) = \max_{i \in \mathcal{T}} t^{\sigma}(< i, k >) - \min_{i \in \mathcal{T}} t^{\sigma}(< i, k >) \rightarrow_{k \rightarrow +\infty} +\infty$$

This might occur in earliest schedule and in periodic schedules for general uniform graphs.

Example

if task 4 has processing time 1:



- There are 3 strong components $C_1 = \{1,2,3\}, C_2 = \{4\}, C_3 = \{5,6,7\}$
- $\alpha(C_1) = 3, \alpha(C_2) = 1, \alpha(C_3) = 5$
- The earliest schedule of $i \leq 4$ is $2-$periodic with period 6.
- The earliest schedule of $\{5,6,7\}$ is $1-$periodic with period 5.
- Unstable schedule
- Here a static 1-periodic schedule with period 5 can be built.

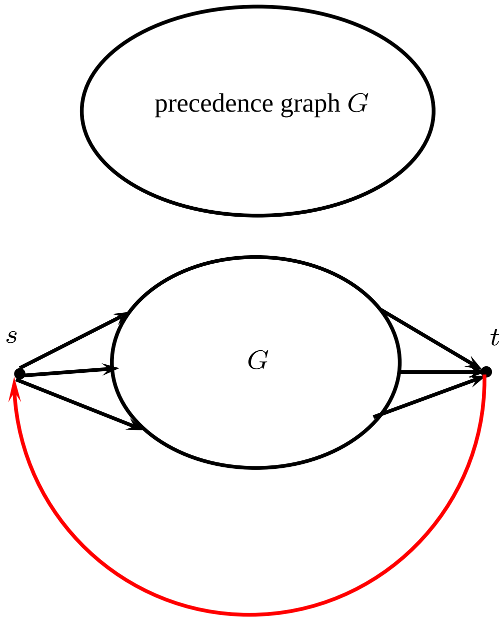
Cyclic problems with resources

- Complexity of a cyclic problem/ its non cyclic version.
- Tools for constructing periodic schedules: circuits and patterns
  Polynomial problems
- Are periodic schedules optimal?
- Decomposed scheduling: a general approach
- Approximation of decomposed scheduling
- ILP formulations

Complexity

Consider a problem $resources|prec|C_{max}$ which is NP-hard.

A simple Reduction:



- Add two dummy nodes to G (source s, sink t with unit proce
- set $H(a) = 0$ for all arcs in G and arcs from s and arcs to t.
- Add an arc from t to s with height 1.
- Any schedule σ' of G' is a sequence of schedules of G.

- Its average cycle time $A(\sigma')$ is the mean of makespan of G schedules
- $A(\sigma') \leq B \Leftrightarrow C_{max}(G) \leq B - 2$

Corollary 3. $P|uniform\,prec, p_i = 1|A, pre-assigned\,processors|uniform\,prec|A$, *cyclic job-shop with uniform constraints are NP-hard*

Circuits

Consider a uniform graph G =circuit, such that for any arc a $L(a) = p_{b(a)}$: usual precedence constraints.

Lemma 5. *In any schedule of G no more than $H(G)$ tasks are performed in parallel.*

Corollary 4. *[Munier 91] The problem $P|circuit, L(a) = p_{b(a)}|A$ is solvable in polynomial time.*

Idea : if $H(G) \leq m$ then any schedule meets the resource constraint. If $H(G) > m$, then we can reduce the height of some arcs so that $H(G) = m$ without modifiying the lower bound on

$$A(\sigma) \geq \max(\max_{i \in \mathcal{T}} p_i, \frac{\sum_{i \in \mathcal{T}} p_i}{m})$$

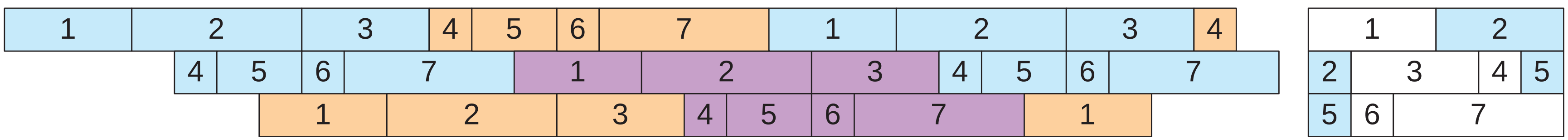
Patterns

Let σ be a periodic schedule with unique period w.

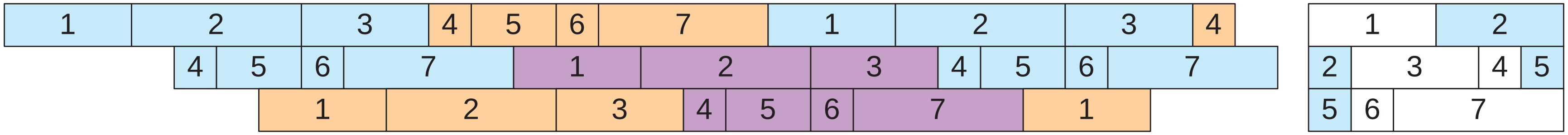
$$t^\sigma(< i, k >) = t^\sigma(< i, 1 >) + (k-1)w$$

Definition 4. *The* Pattern *of σ is defined by: $\pi^\sigma(i) = t^\sigma(< i, 1 >) mod w$. The* iteration setting *of σ is $\eta^\sigma(i) \in \mathbb{Z}$, s.t.*

$$t^\sigma(< i, 1 >) = \pi^\sigma(i) + \eta^\sigma(i)w$$



Patterns



- The pattern defines the schedule of tasks in an interval $[kw, (k+1)w]$ for enough large k
- The iteration setting indicates which occurrences of tasks are involved in this interval
- In the interval $[kw, (k+1)w]$, i starts at $kw + \pi^\sigma(i)$ its occurrence $< i, k+1-\eta^\sigma(i) >$

Feasibility of a pattern

Question: Given a pattern π,and a period w is there an iteration setting η such that the uniform constraints are met by the periodic schedule?

For an arc a, $t(< e(a), 1 >) - t(< b(a), 1 >) \geq L(a) - wH(a)$.

Lemma 6. *An iteration setting satisfies for any arc* a

$$\eta(e(a)) - \eta(b(a)) \geq \left\lceil \frac{L(a) + \pi(b(a)) - \pi(e(a))}{w} \right\rceil - H(a) = E_{w,\pi}(a)$$

Lemma 7. *An iteration setting exists iff* $(G, E_{w,\pi})$ *has no positive circuit. can be checked in polynomial time*

Polynomial problems

Corollary 5. *if G has no other circuit than the non-reentrance loops, any pattern has an iteration setting.*

Theorem 4. *[Munier 91] $P|acyclic\, uniform\, prec|A$ is sovlable in polynomial time.*

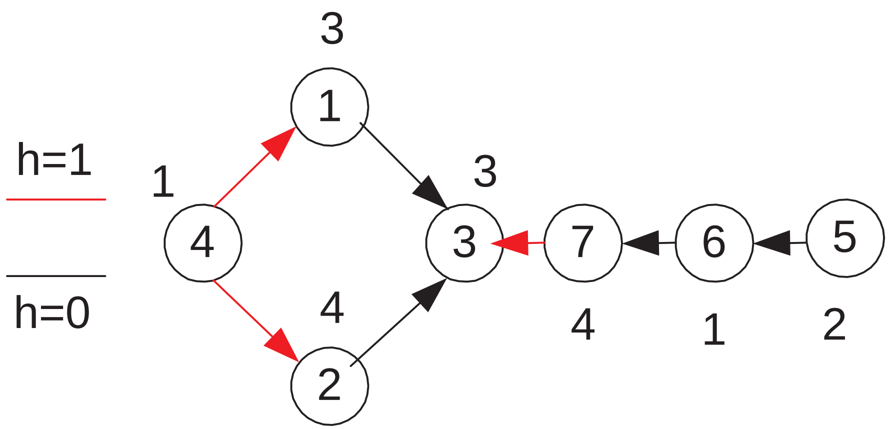
Idea: Consider tasks as independent, and schedule them on m processors using Mc-Naughton algorithm (preemptive scheduling). Use the schedule as a Pattern of a periodic schedule and compute the iteration setting.

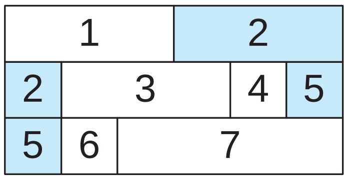
Theorem 5. *$dedicated\, processors|acyclic\, uniform\, prec|A$ is solvable in polynomial time. In particular non-reentrant job-shop or flow-shop.*

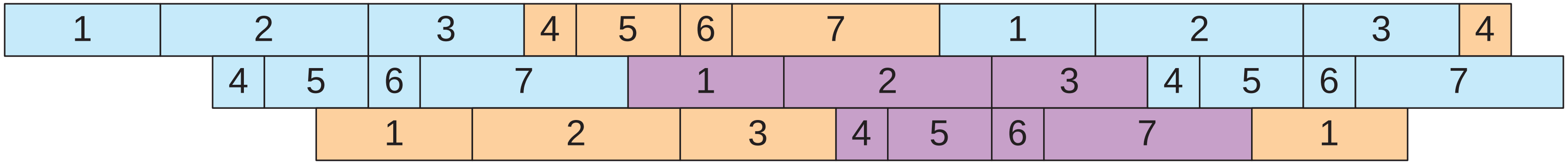
Idea: schedule operations on each machine as independent. Set $W = C_{max}$ (which is a lower bound on $A(\sigma)$ in this case). Use the schedule as a Pattern of a periodic schedule and compute the iteration setting.

[Robert, Legrand] also used similar ideas to build schedules for a broadcasting problem on a heterogeneous platform with net contentions.

Example

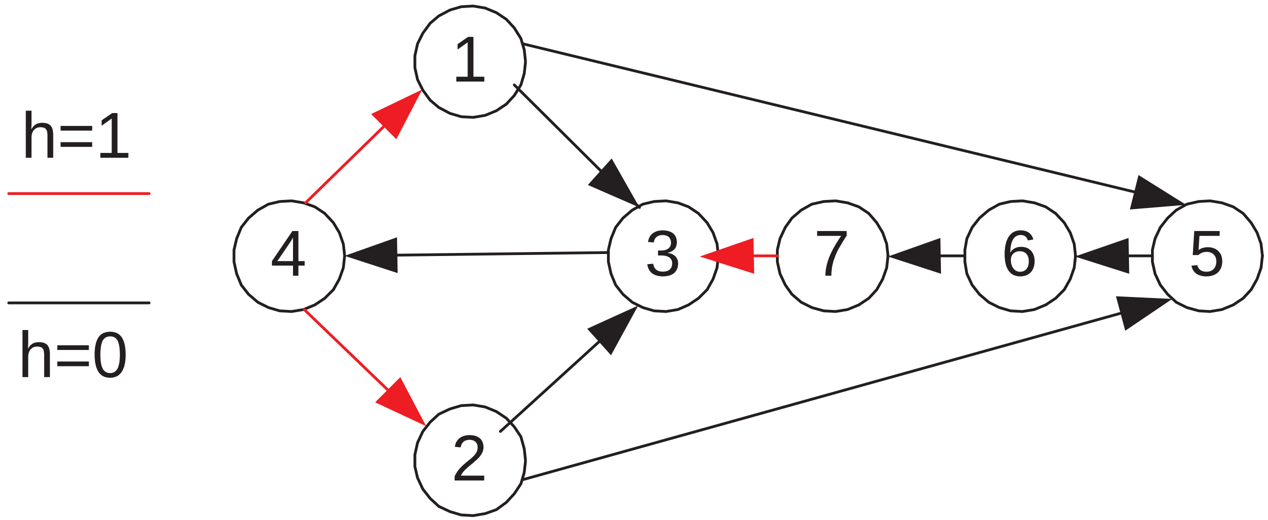

h=1
h=0
1
3
3
4
4
1
2
1
4
3
2
7
6
5


1 2
2 3 4 5
5 6 7


1 2 3 4 5 6 7 1 2 3 4
4 5 6 7 1 2 3 4 5 6 7
1 2 3 4 5 6 7 1

Optimality of periodic schedules?

In general, periodic schedules are not optimal schedules. Example: cyclic problem with two processors



1	3	4	1	7	3	4	1	3	4	1	7	3	4
2	5	6	2	5	6	7	2	5	6	2	5	6	7

However, periodic schedules are simple to implement and easier to compute.

ecomposed scheduling: a general approac

Also known as Decomposed software pipelining[Eisenbeis, Darte, Gasperoni, Schwiegelsohn, de dinechin, Munier,...]

- Build a non cyclic schedule S of $\mathcal{T}$ that meets the resource constraints and eventually some precedence constraints.
- Consider this schedule as a pattern, and set $W = C_{max}(S)$
- Build a feasible iteration setting for the pattern. (if possible)

OR

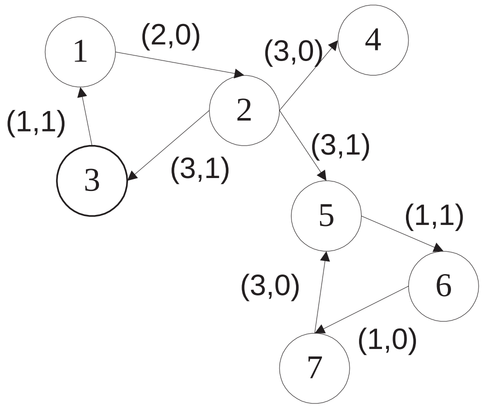
- Build an iteration setting.
- Deduce from the iteration setting precedence relations between tasks in a pattern
- Build a non cyclic schedule S of $\mathcal{T}$ that meets the resource constraints and these precedence constraints.
- Consider this schedule as a pattern, and set $W = C_{max}(S)$

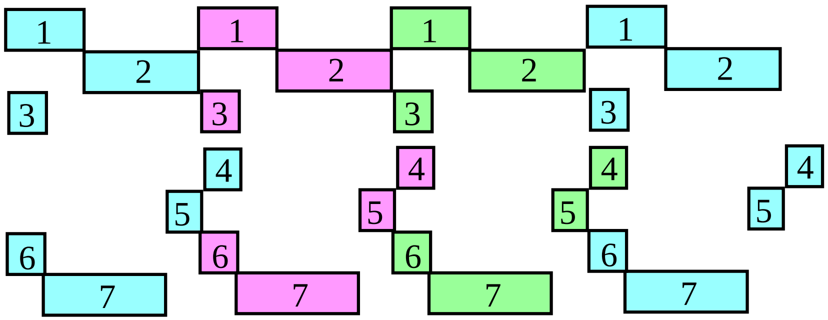
Gasperoni-Schwiegelsohn algorithm

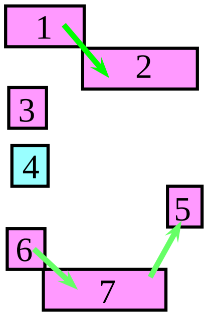
An algorithm for $P|uniform\ with\ L(a) = p_{b(a)}|A$.

- Compute an optimal schedule σ^∞ on infinitely many processors.
- Consider the pattern π^{σ^∞} and remove arcs a from G such that $\pi^{\sigma^\infty}(b(a)) + p_{b(a)} > \pi^{\sigma^\infty}(e(a)) \rightarrow G'$.
- Schedule G' on the m processors according to a list scheduling algorithm $\rightarrow$ schedule S.
- Set S as a pattern with $w = C_{max}(S)$ and combine with the iteration setting of $\sigma^\infty \rightarrow$ periodic feasible schedule σ.

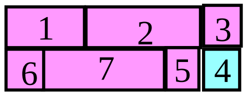
Example



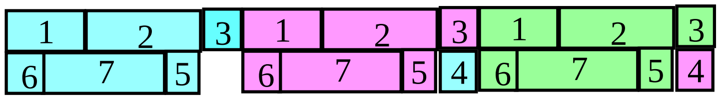




G' has 3 arcs



Schedule on 2 processors



periodic schedule with period 6

Bounds

Theorem 6. *[Gasperoni-Schwiegelsohn]*

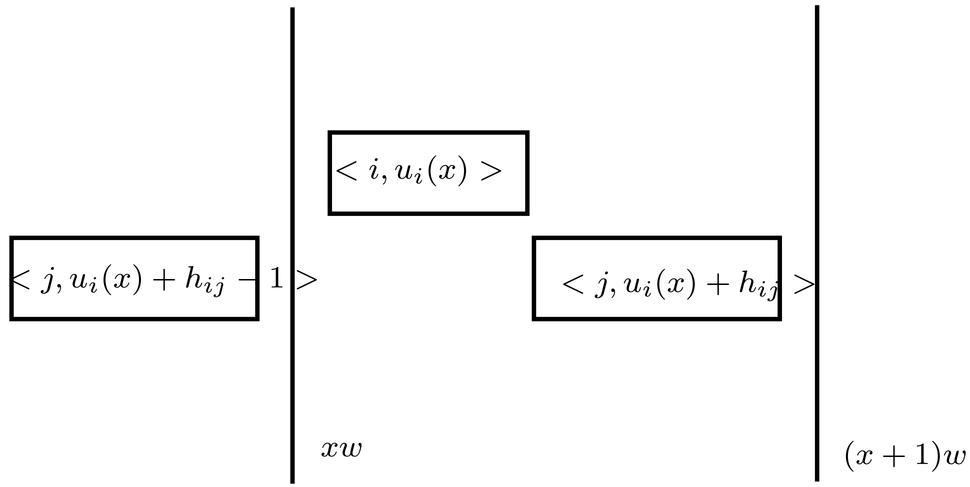
$$A(\sigma) \leq (2 - \frac{1}{m})A(\sigma^{opt}) + \max_{i \in \mathcal{T}} p_i$$

[Darte et al] generalized the algorithm by choosing a convenient G' using retiming.

Disjunctive constraints

Assume that two tasks i and j use the same resource: $\forall k, l,$ either
$t^\sigma(< i, k >) + p_i \leq t^\sigma(< j, l >)$ or $t^\sigma(< j, l >) + p_j \leq t^\sigma(< i, k >)$.

- If σ is periodic with period w,
- in time interval $[xw, (x+1)w)]$ there is only one occurrence $< i, u_i(x) >$.
- $< j, u_i(x) + h_{ij} >$ next occurrence of j.
- $< j, u_i(x) + h_{ij} - 1 >$ precedes $< i, u_i(x) >$.



Hence setting $h_{ij} \in \mathbb{Z}$ as a variable, the disjunctive constraints can be expressed as:

$$\begin{cases} t^\sigma(< j, 1 >) - t^\sigma(< i, 1 >) & \geq p_i - wh_{ij} \\ t^\sigma(< i, 1 >) - t^\sigma(< j, 1 >) & \geq p_j - wh_{ji} \\ h_{ij} + h_{ji} & = 1 \end{cases}$$

ILP model

General form of constraints:

$$t^{\sigma}(< e(a), 1 >) - t^{\sigma}(< b(a), 1 >) \geq L(a) - wH(a)$$

Heights might be either fixed or variables. Additional linear constraints on variable heights might be added.

Remark 3. *For a fixed w, the constraint is linear.*

Remark 4. *Once disjunctive variables of this ILP are fixed, it remains a uniform graph scheduling problem.*

Many problems with disjunctive resources (shop-problems, but also Hoist scheduling problems)[Roundy],[Brucker], [Levner et al] [Chu et al][Lei et al]can be formulated, for a fixed w as an mixed integer linear program.

Algorithms for disjunctive problems

- Branch and bound algorithms

  For cyclic job-shop problems [Roundy,Hanen, Kampmeyer and Brücker]

  For hoist scheduling[Lei and Wang, Chu and Proth]

- Metaheuristics [Kampmeyer and Brücker]

Some Open problems

- Transitory state of the earliest schedule.
- Complexity of the cyclic problem with unit lengths an 2 processors.
- How to use results on scheduling problems in their cyclic version?
  - Efficiency of Jackson algorithm for the 1-machine problem
  - Specific list schedules for parallel machines problems.
- Defining regular dynamic policy and study their behaviour. Regularity of the schedule, performance bounds.