Divisible load theory

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Overview

1. The context
2. Bus-like network: classical resolution
3. Bus-like network: resolution under the divisible load model
4. Star-like network
5. Multi-round algorithms
6. Conclusion

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Context of the study

- Scientific computing: large needs in computation or storage resources.
- Need to use systems with “several processors”:
  - Parallel computers with shared memory.
  - Parallel computers with distributed memory.
  - Clusters.
  - Heterogeneous clusters.
  - Clusters of clusters.
  - Network of workstations.
  - The Grid.
- Problematic: to take into account the heterogeneity at the algorithmic level.

New platforms, new problems

Execution platforms: Distributed heterogeneous platforms (network of workstations, clusters, clusters of clusters, grids, etc.)

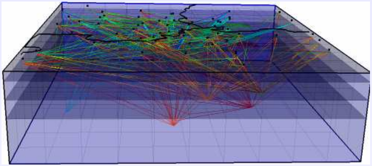
New sources of problems

- Heterogeneity of processors (computational power, memory, etc.)
- Heterogeneity of communications links.
- Irregularity of interconnection network.
- Non dedicated platforms.

We need to adapt our algorithmic approaches and our scheduling strategies: new objectives, new models, etc.

An example of application: seismic tomography of the Earth

- Model of the inner structure of the Earth



- The model is validated by comparing the propagation time of a seismic wave in the model to the actual propagation time.
- Set of all seismic events of the year 1999: $817,101$
- Original program written for a parallel computer:

```
if (rank = ROOT)
    raydata ← read $n$ lines from data file;
MPI_Scatter(raydata, $n/P$, ..., rbuff, ...,
                ROOT, MPI_COMM_WORLD);
compute_work(rbuff);
```

Applications covered by the divisible load model

Applications made of a very (very) large number of fine grain computations.

Computation time proportional to the size of the data to be processed.

Independent computations: neither synchronizations nor communications.

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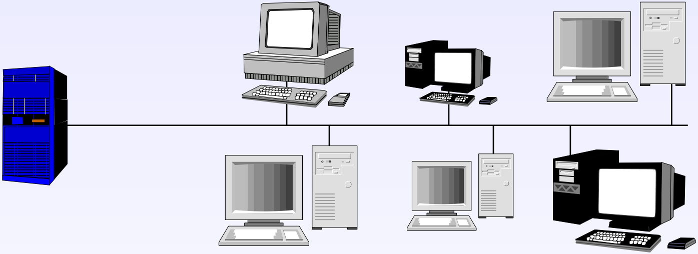
Bus-like network



- The links between the master and the slaves all have the same characteristics.
- The slave have different computation power.

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  Length of a unit-size work on processor P_i: w_i.
  Computation time on P_i: $n_i w_i$.

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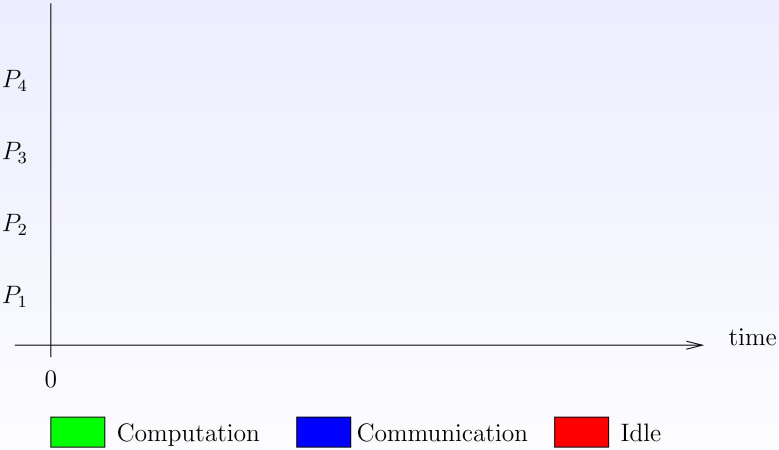
- Time needed to send a unit-message from P_1 to P_i: c.
  One-port bus: P_1 sends a *single* message at a time over the bus, all processors communicate at the same speed with the master.

Behavior of the master and of the slaves (illustration)



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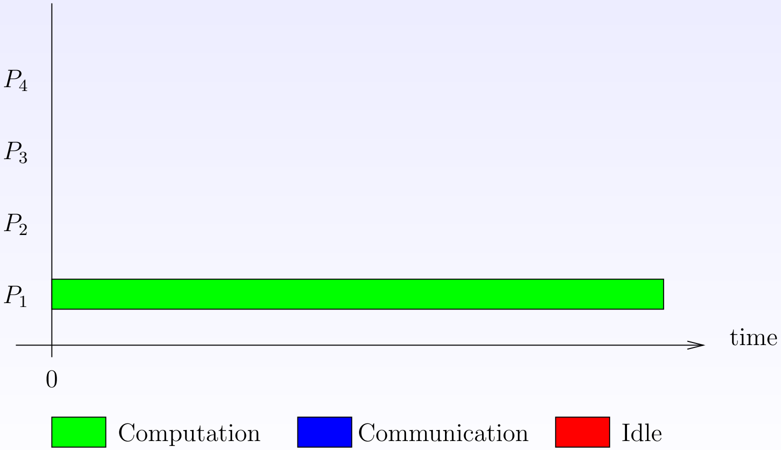

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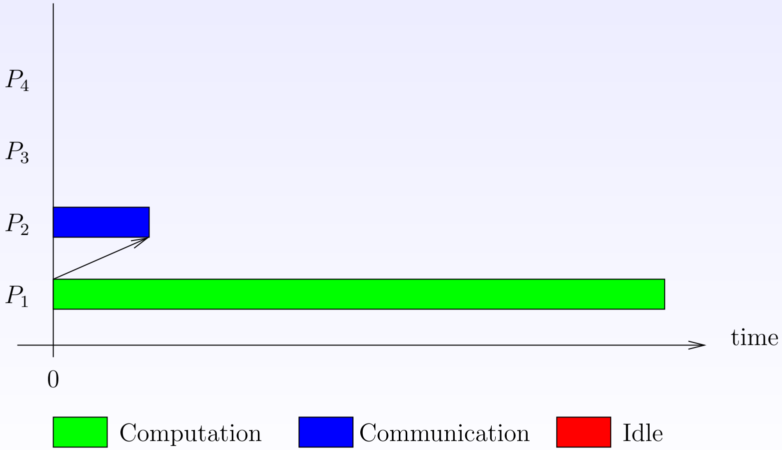

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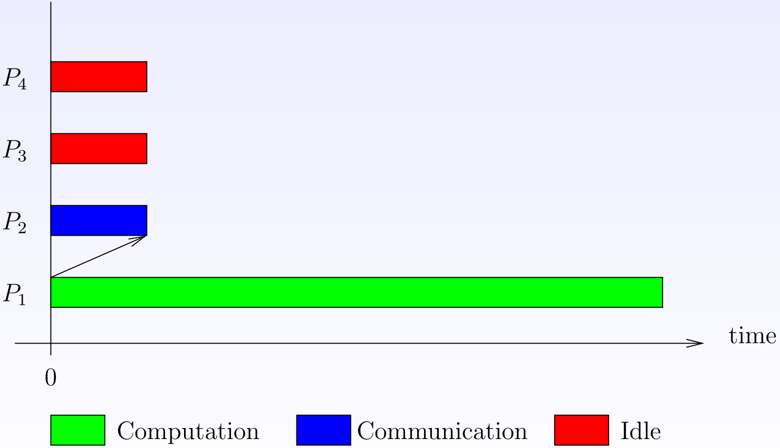

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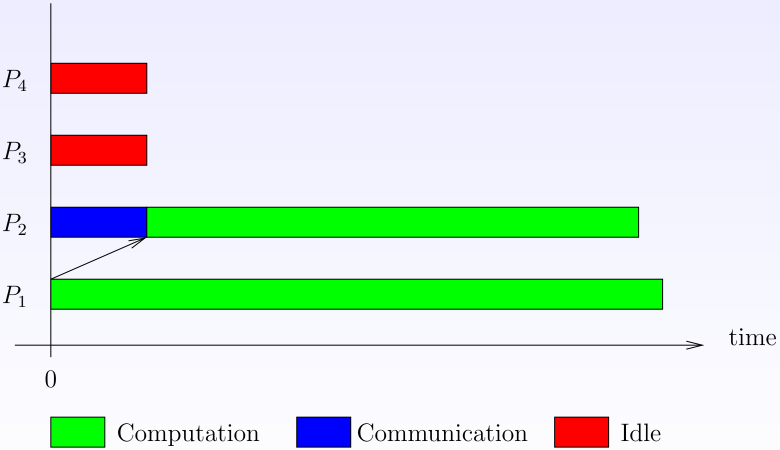

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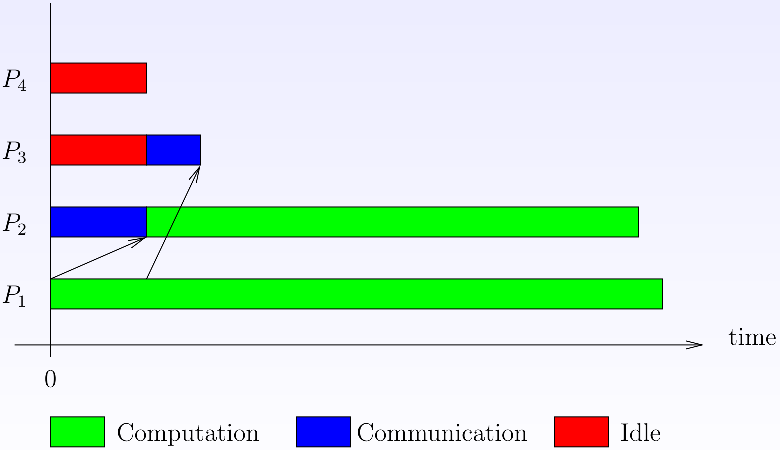

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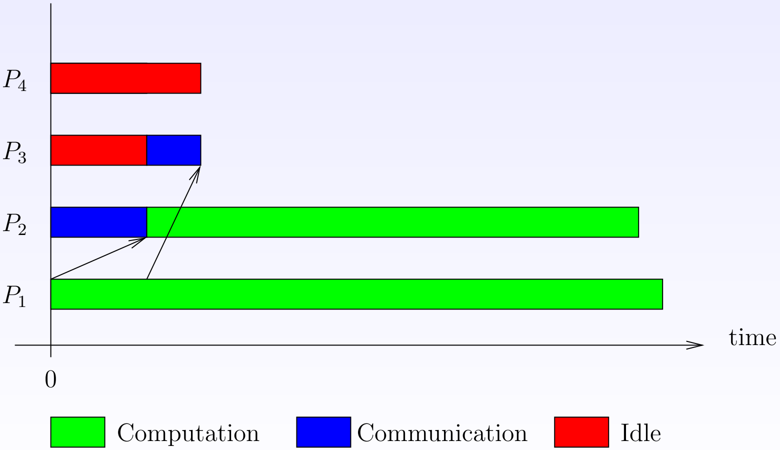

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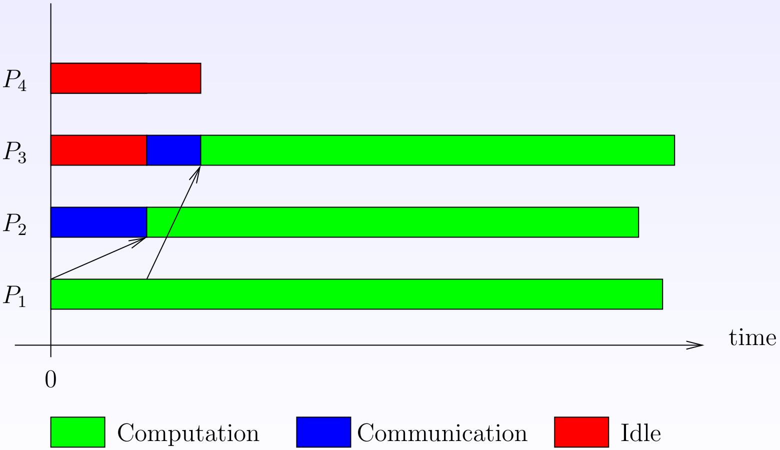

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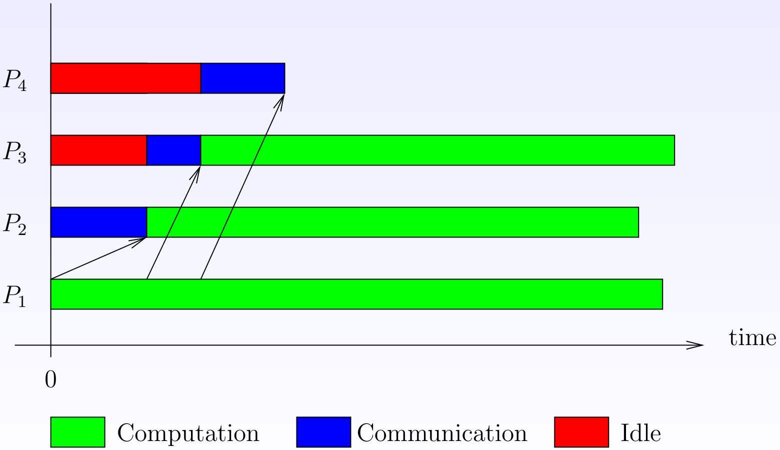

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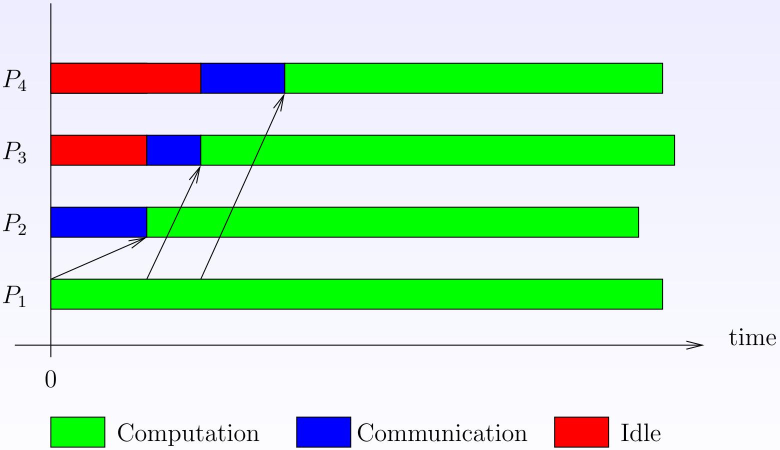

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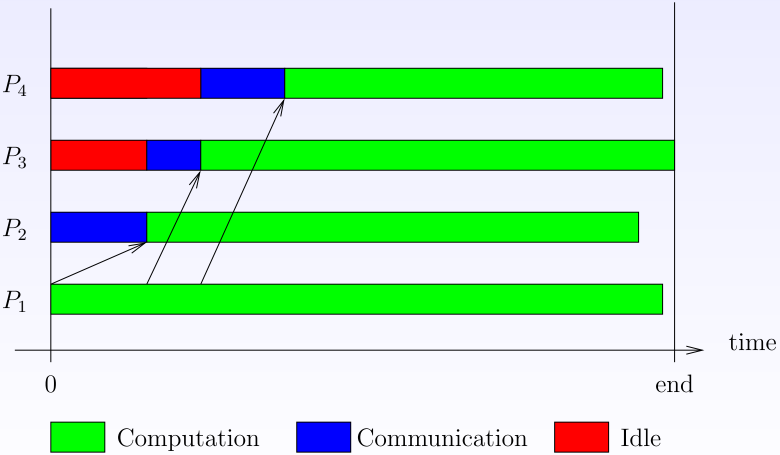

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- The master sends its chunk of n_i data to processor P_i in a single sending.
- The master sends their data to the processors, serving one processor at a time, in the order P_2, ..., P_p.
- During this time the master processes its n_1 data.
- A slave does not start the processing of its data before it has received all of them.

Equations

- P_1: $T_1 = n_1.w_1$

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- P_i: $T_i = \sum_{j=2}^{i} n_j.c + n_i.w_i$ for $i \geq 2$
- P_i: $T_i = \sum_{j=1}^{i} n_j.c_j + n_i.w_i$ for $i \geq 1$ with $c_1 = 0$ and $c_j = c$ otherwise.

Execution time

$$T = \max_{1 \leq i \leq p} \left(\sum_{j=1}^{i} n_j.c_j + n_i.w_i \right)$$

We look for a data distribution n_1, ..., n_p which minimizes T.

Execution time: rewriting

$$T = \max\left(n_1.c_1 + n_1.w_1, \max_{2 \leq i \leq p}\left(\sum_{j=1}^{i} n_j.c_j + n_i.w_i\right)\right)$$

$$T = n_1.c_1 + \max\left(n_1.w_1, \max_{2 \leq i \leq p}\left(\sum_{j=2}^{i} n_j.c_j + n_i.w_i\right)\right)$$

An optimal solution for the distribution of W_{total} data over p processors is obtained by distributing n_1 data to processor P_1 and then optimally distributing $W_{\text{total}} - n_1$ data over processors P_2 to P_p.

Algorithm

```
$\textit{solution}[0,p] \leftarrow \mathrm{cons}(0, NIL)$; $\textit{cost}[0,p] \leftarrow 0$
for $d \leftarrow 1$ to $W_{\text{total}}$ do
    $\textit{solution}[d,p] \leftarrow \mathrm{cons}(d, NIL)$
    $\textit{cost}[d,p] \leftarrow d \cdot c_p + d \cdot w_p$
for $i \leftarrow p-1$ downto $1$ do
    $\textit{solution}[0,i] \leftarrow \mathrm{cons}(0, \textit{solution}[0,i+1])$
    $\textit{cost}[0,i] \leftarrow 0$
    for $d \leftarrow 1$ to $W_{\text{total}}$ do
        $(sol, min) \leftarrow (0, \textit{cost}[d,i+1])$
        for $e \leftarrow 1$ to $d$ do
            $m \leftarrow e \cdot c_i + \max(e \cdot w_i, \textit{cost}[d-e,i+1])$
            if $m < min$ then
                $(sol, min) \leftarrow (e, m)$
        $\textit{solution}[d,i] \leftarrow \mathrm{cons}(sol, \textit{solution}[d-sol,i+1])$
        $\textit{cost}[d,i] \leftarrow min$
return $(\textit{solution}[W_{\text{total}},1], \textit{cost}[W_{\text{total}},1])$
```

Complexity

- **Theorical complexity**

$$O(W_{\text{total}}^2 \cdot p)$$

- **Complexity in practice**
  If $W_{\text{total}} = 817,101$ and $p = 16$, on a Pentium III running at 933 MHz: more than two days... (in 2002)
  (Optimized version ran in 6 minutes.)

Disadvantages

- Cost
- Solution is not reusable
- Solution is only partial

We do not need the solution to be so precise

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Notation

- A set P_1, ..., P_p of processors
- P_1 is the master processor: initially, it holds all the data.
- The overall amount of work: W_{total}.
- **Processor P_i receives an amount of work $\alpha_i W_{\textbf{total}}$ with $\alpha_i W_{\textbf{total}} \in \mathbb{Q}$ and $\sum_i \alpha_i = 1$.**
  Length of a unit-size work on processor P_i: w_i.
  Computation time on P_i: $\alpha_i W_{\text{total}} w_i$.
- Time needed to send a unit-message from P_1 to P_i: c.
  One-port model: P_1 sends a *single* message at a time, all processors communicate at the same speed with the master.

Equations

For processor P_i (with $c_1 = 0$ and $c_j = c$ otherwise):

$$T_i = \sum_{j=1}^{i} \alpha_j W_{\mathsf{total}}.c_j + \alpha_i W_{\mathsf{total}}.w_i$$

$$T = \max_{1 \le i \le p} \left(\sum_{j=1}^{i} \alpha_j W_{\mathsf{total}}.c_j + \alpha_i W_{\mathsf{total}}.w_i \right)$$

We look for a data distribution α_1, ..., α_p which minimizes T.

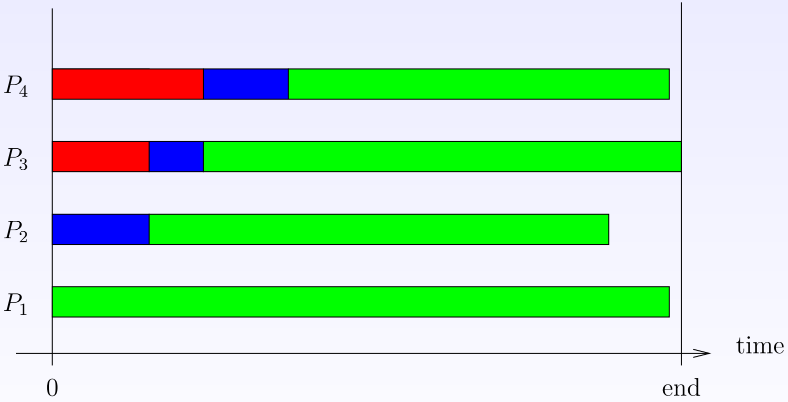
Properties of load-balancing

Lemma

In an optimal solution, all processors end their processing at the same time.

Demonstration of lemma 1

Two slaves i and $i+1$ with $T_i < T_{i+1}$.



We decrease α_{i+1} by ϵ.

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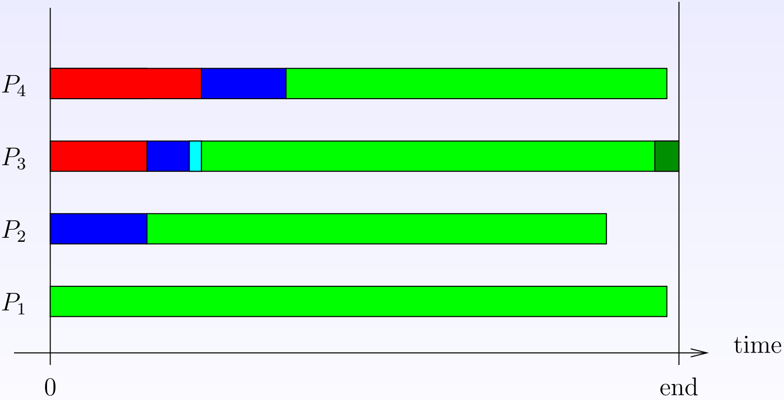

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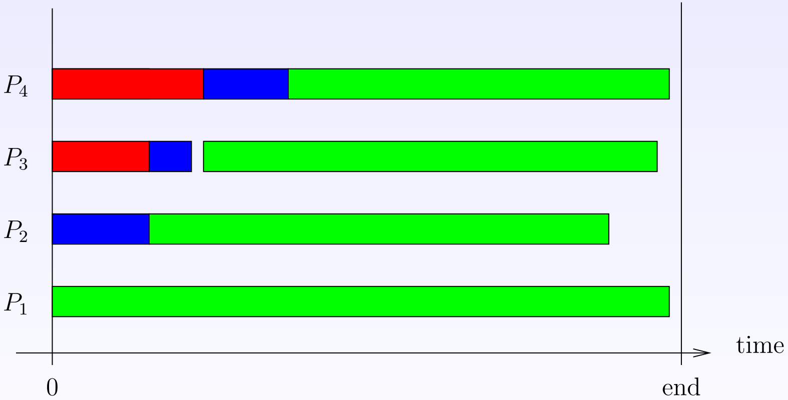

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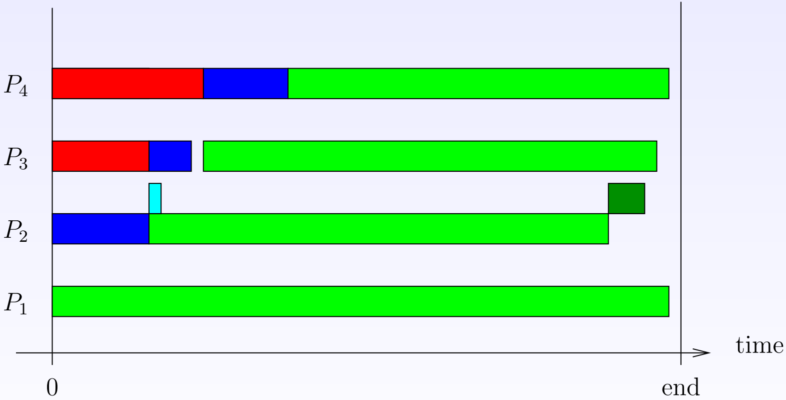

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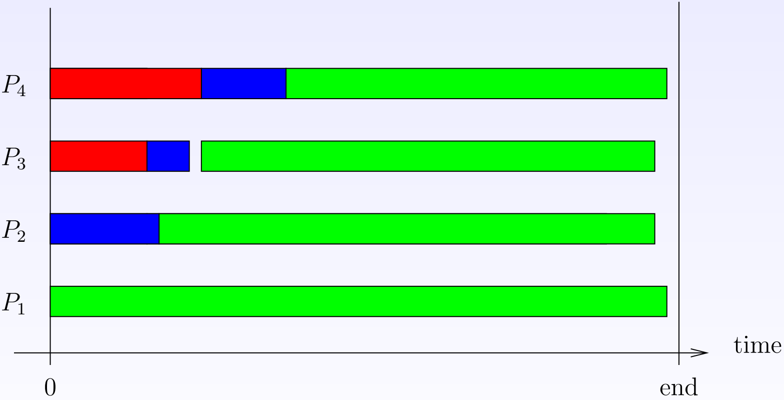

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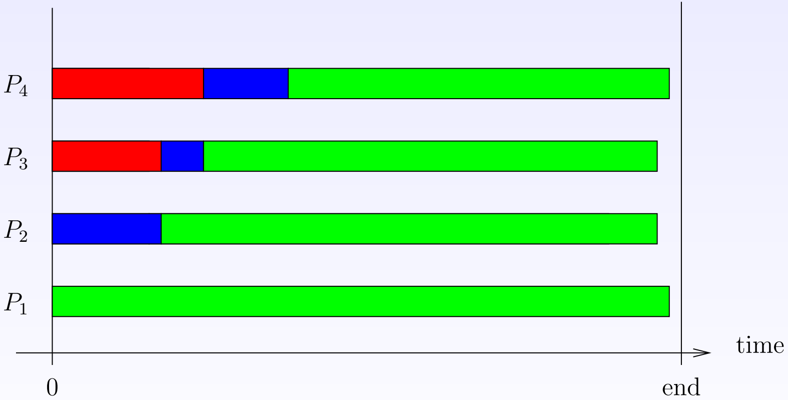

The communication time for the following processors is unchanged.

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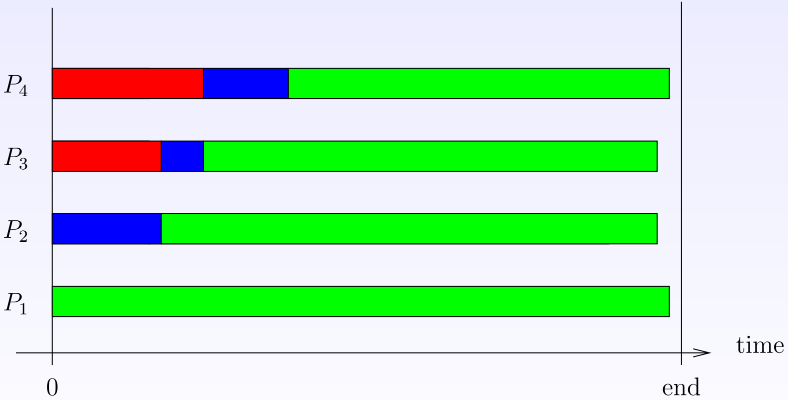

We end up with a better solution!

Demonstration of lemma 1 (continuation and conclusion)

- Ideal: $T'_i = T'_{i+1}$.
  We choose ϵ such that:

$$
\begin{aligned}
&(\alpha_i + \epsilon)W_{\mathsf{total}}(c + w_i) = \\
&\qquad (\alpha_i + \epsilon)W_{\mathsf{total}}c + (\alpha_{i+1} - \epsilon)W_{\mathsf{total}}(c + w_{i+1})
\end{aligned}
$$

- The master stops before the slaves: absurde.
- The master stops after the slaves: we decrease P_1 by ϵ.

Property for the selection of ressources

Lemma

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Demonstration: this is just a corollary of lemma 1...

Resolution

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$\sum_{i=1}^{n} \alpha_i = 1$.

$$\alpha_1 \left(1 + \frac{w_1}{c + w_2} + \ldots + \prod_{k=2}^{j} \frac{w_{k-1}}{c + w_k} + \ldots \right) = 1$$

Impact of the order of communications

How important is the influence of the ordering of the processor on the solution ?

?

No impact of the order of the communications

Volume processed by processors P_i and P_{i+1} during a time T.

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Processor P_i**:** $\alpha_i(c + w_i)W_{\text{total}} = T$. Therefore $\alpha_i = \frac{1}{c+w_i}\frac{T}{W_{\text{total}}}$.

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Thus $\alpha_{i+1} = \frac{1}{c+w_{i+1}}(\frac{T}{W_{\text{total}}} - \alpha_i c) = \frac{w_i}{(c+w_i)(c+w_{i+1})}\frac{T}{W_{\text{total}}}$.

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Processors P_i **and** P_{i+1}**:**

$$\alpha_i + \alpha_{i+1} = \frac{c + w_i + w_{i+1}}{(c+w_i)(c+w_{i+1})}$$

Choice of the master processor

We compare processors P_1 and P_2.

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Total volume processed:

$$\alpha_1 + \alpha_2 = \frac{c + w_1 + w_2}{w_1(c + w_2)} = \frac{c + w_1 + w_2}{cw_1 + w_1 w_2}$$

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Minimal when $w_1 < w_2$.
Master = the most powerfull processor (for computations).

Conclusion

- Closed-form expressions for the execution time and the distribution of data.
- Choice of the master.
- The ordering of the processors has no impact.
- All processors take part in the work.

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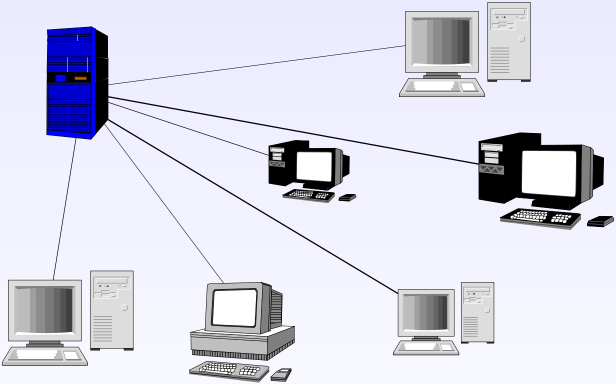
6. Conclusion

Star-like network



- The links between the master and the slaves have *different* characteristics.
- The slaves have different computational power.

Notation

- A set P_1, ..., P_p of processors
- P_1 is the master processor: initially, it holds all the data.
- The overall amount of work: W_{total}.
- Processor P_i receives an amount of work $\alpha_i W_{\text{total}}$ with $\sum_i n_i = W_{\text{total}}$ with $\alpha_i W_{\text{total}} \in \mathbb{Q}$ and $\sum_i \alpha_i = 1$.
  Length of a unit-size work on processor P_i: w_i.
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- **Time needed to send a unit-message from P_1 to P_i: c_i.**
  One-port model: P_1 sends a *single* message at a time.

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Processors must be served by decreasing bandwidths.

Ressource selection

Lemma

In an optimal solution, all processors work.

Demonstration of lemma 3

We take an optimal solution. Let P_k be a processor which does not receive any work: we put it last in the processor ordering and we give it a fraction α_k such that $\alpha_k(c_k + w_k)W_{\text{total}}$ is equal to the processing time of the last processor which received some work.

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Why should we put this processor last ?

Load-balancing property

Lemma

In an optimal solution, all processors end at the same time.

Demonstration of lemma 4

- Most existing proofs are false.

$$\textsc{Minimize } T,$$

$$\textsc{subject to}$$

$$\begin{cases} & \sum_{i=1}^{n} \alpha_i \geq 1 \\ \forall i, & \alpha_i \geq 0 \\ \forall i, & \sum_{k=1}^{i} \alpha_k c_k + \alpha_i w_i \leq T \end{cases}$$

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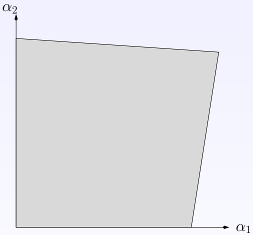

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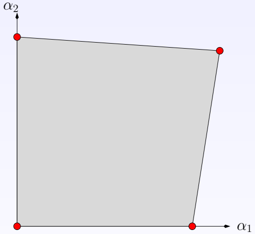

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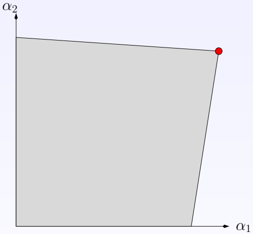

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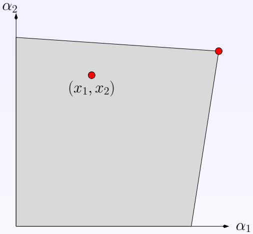

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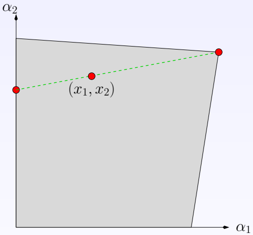


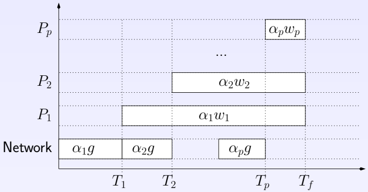
Conclusion

- The processors must be ordered by decreasing bandwidths
- All processors are working
- All processors end their work at the same time
- Formulas for the execution time and the distribution of data

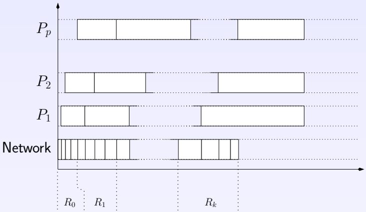
Overview

1. The context
2. Bus-like network: classical resolution
3. Bus-like network: resolution under the divisible load model
4. Star-like network
5. **Multi-round algorithms**
6. Conclusion

One round vs. multi-round

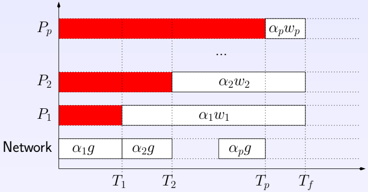


One round



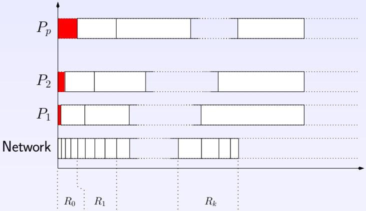
Multi-round

One round vs. multi-round



One round

⤳ long idle-times



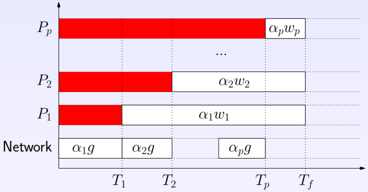
Multi-round

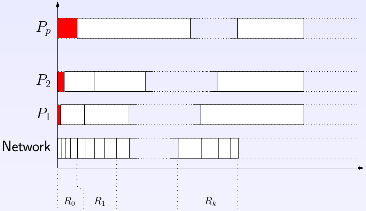
Efficient when W_{total} large

Intuition: start with small rounds, then increase chunks.

Problems:

One round vs. multi-round





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Problems:

- linear communication model leads to absurd solution
- resource selection
- number of rounds
- size of each round

Notations

- A set P_1, ..., P_p of processors
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  One-port model: P_1 sends and receives a *single* message at a time.

Complexity

Definition (One round, $\forall i, c_i = 0$)

Given W_{total}, p workers, $(P_i)_{1 \leq i \leq p}$, $(L_i)_{1 \leq i \leq p}$, and a rational number $T \geq 0$, and assuming that bandwidths are infinite, is it possible to compute all W_{total} load units within T time units?

Theorem

The problem with one-round and infinite bandwidths is NP-complete.

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Theorem

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What is the complexity of the general problem with finite bandwidths and several rounds?

The general problem is NP-hard, but does not appear to be in NP (no polynomial bound on the number of activations).

Fixed activation sequence

Hypotheses

1. Number of activations: N_{act};
2. Whether P_i is **the** processor used during activation j: $\chi_i^{(j)}$

MINIMIZE T, UNDER THE CONSTRAINTS

$$\begin{cases} \displaystyle\sum_{j=1}^{N_{\text{act}}} \sum_{i=1}^{p} \chi_i^{(j)} \alpha_i^{(j)} = W_{\text{total}} \\ \forall k \leq N_{\text{act}}, \forall l : \left(\displaystyle\sum_{j=1}^{k} \sum_{i=1}^{p} \chi_i^{(j)} (L_i + \alpha_i^{(j)} c_i) \right) + \displaystyle\sum_{j=k}^{N_{\text{act}}} \chi_l^{(j)} \alpha_l^{(j)} w_l \leq T \\ \forall i, j : \alpha_i^{(j)} \geq 0 \end{cases}$$

Can be solved in polynomial time.

Fixed number of activations

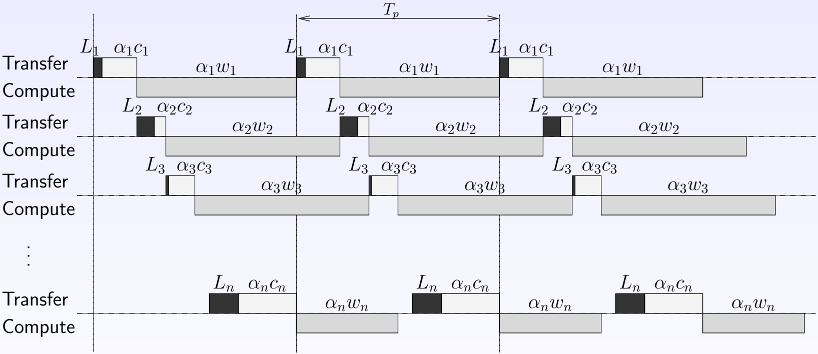
$$\textsc{Minimize } T, \textsc{ under the constraints}$$

$$\begin{cases} \displaystyle\sum_{j=1}^{N_{\mathsf{act}}} \sum_{i=1}^{p} \chi_i^{(j)} \alpha_i^{(j)} = W_{\mathsf{total}} \\ \displaystyle\forall k \leq N_{\mathsf{act}}, \forall l : \left(\sum_{j=1}^{k} \sum_{i=1}^{p} \chi_i^{(j)} (L_i + \alpha_i^{(j)} c_i) \right) + \sum_{j=k}^{N_{\mathsf{act}}} \chi_l^{(j)} \alpha_l^{(j)} w_l \leq T \\ \displaystyle\forall k \leq N_{\mathsf{act}} : \sum_{i=1}^{p} \chi_i^{(k)} \leq 1 \\ \forall i, j : \chi_i^{(j)} \in \{0, 1\} \\ \forall i, j : \alpha_i^{(j)} \geq 0 \end{cases}$$

Exact but exponential

Can lead to branch-and-bound algorithms

Periodic schedule



How to choose T_p? Which resources to select?

With no overlap (1/4)

Equations

- Divide total execution time T into k periods of duration T_p.

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- No overlap:

$$\forall i \in \mathcal{I}, \qquad L_i + \alpha_i (c_i + w_i) \leq T_p.$$

With no overlap (2/4)

Normalization

- β_i average number of tasks processed by P_i during one time unit.

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Relaxed version

$$\text{Maximize} \sum_{i=1}^{p} x_i \quad \begin{cases} \forall 1 \leq i \leq p, & x_i(c_i + w_i) \leq 1 - \frac{L_i}{T_p} \\ \sum_{i=1}^{p} x_i c_i \leq 1 - \frac{\sum_{i=1}^{p} L_i}{T_p} \end{cases}$$

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With no overlap (3/4)

Bandwidth-centric solution

- Sort: $c_1 \leq c_2 \leq \ldots \leq c_p$.
- Let q be the largest index so that $\sum_{i=1}^{q} \frac{c_i}{c_i+w_i} \leq 1$.
- If $q < p$, $\epsilon = 1 - \sum_{i=1}^{q} \frac{c_i}{c_i+w_i}$.
- Optimal solution to relaxed program:

$$\forall 1 \leq i \leq q, \quad x_i = \frac{1 - \frac{\sum_{i=1}^{p} L_i}{T_p}}{c_i + w_i}$$

and (if $q < p$):

$$x_{q+1} = \left(1 - \frac{\sum_{i=1}^{p} L_i}{T_p}\right)\left(\frac{\epsilon}{c_{q+1}}\right),$$

and $x_{q+2} = x_{q+3} = \ldots = x_p = 0$.

With no overlap (4/4)

Asymptotic optimality

- Let $T_p = \sqrt{T^*_{\max}}$ and $\alpha_i = x_i T_p$ for all i.

With no overlap (4/4)

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- Let $T_p = \sqrt{T^*_{\max}}$ and $\alpha_i = x_i T_p$ for all i.
- Then $T \leq T^*_{\max} + O(\sqrt{T^*_{\max}})$.

With no overlap (4/4)

Asymptotic optimality

- Let $T_p = \sqrt{T^*_{\max}}$ and $\alpha_i = x_i T_p$ for all i.
- Then $T \leq T^*_{\max} + O(\sqrt{T^*_{\max}})$.
- Closed-form expressions for resource selection and task assignment provided by the algorithm.

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What should be remembered?

- Underlying principle: we may not need the optimal solution; approximated solutions may be as good and far easier to achieve
- Communications costs may play a far bigger role in designing solutions than computation costs