Online scheduling

Frédéric Vivien

Frederic.Vivien@inria.fr

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Outline

1. Introduction and first results
2. Lower bound on the competitive ratio of any algorithm: the clairvoyant max-stretch case
3. The non-clairvoyant case
4. How to derive a lower bound: the max-flow case with communications

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2. Lower bound on the competitive ratio of any algorithm: the clairvoyant max-stretch case
3. The non-clairvoyant case
4. How to derive a lower bound: the max-flow case with communications

Offline vs. online algorithmics

Nature of the problem
Known

Objective function
Known

Characteristics of the instance

Known
beforehand

Offline

Offline vs. online algorithmics

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Characteristics of a job discovered
When the job is released

Offline

(Clairvoyant) Online

Offline vs. online algorithmics

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Characteristics of the instance

Known beforehand	Discovered during execution	
	Characteristics of a job discovered	
	When the job is released	When the job completes
Offline	(Clairvoyant) Online	Non-clairvoyant online

Notation and hypotheses

Notation

- Jobs J_1, ..., J_n
  Job J_j arrives in the system at the *release* date r_j
  Job J_j has a weight (or a priority) w_j
  Job J_j has an execution time p_j
  Δ is the ratio of the largest to the shortest execution time

- Completion time of job J_j: C_j
  Flow of job J_j: $F_j = C_j - r_j$ (time spent in the system)

Hypotheses

- Jobs may be preempted
- One machine (1 | *pmtn* | ???)

What should we optimize?

- Makespan: $\max_j C_j$

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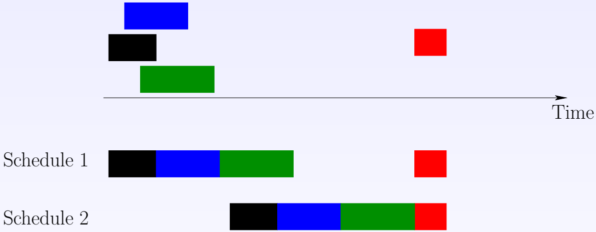
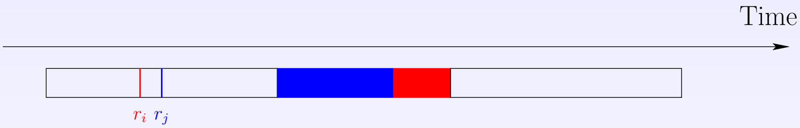


What should we optimize?

- Makespan: $\max_j C_j$
  Release dates are not taken into account
- Average flow or response time: $\sum_j (C_j - r_j)$
  Inconvenient: starvation
- Maximum flow or maximum response time: $\max_j (C_j - r_j)$
  No starvation. Favor long jobs. Worst-case optimization.
- Maximum weighted flow: $\max_j w_j (C_j - r_j)$
  Gives back some importance to short jobs.
  Particular case of the *stretch* or *slowdown*:
  w_j=1/running time of the job on empty platform.

FIFO is optimal for max-flow

Consider any instance and a schedule Θ s.t. there exists two jobs executed consecutively: J_i and J_j with $r_i < r_j$ and $C_i \geq C_j$



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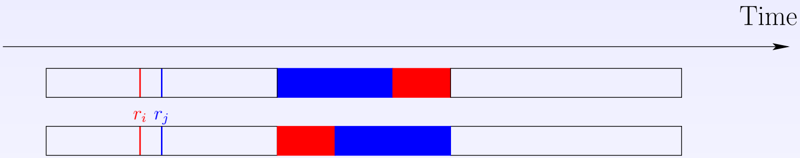


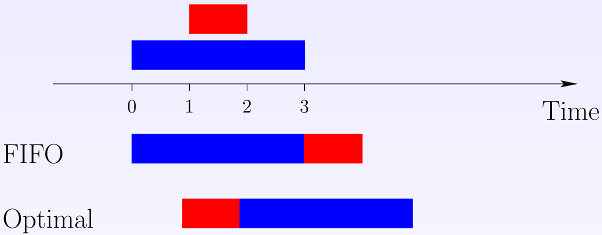
In schedule Θ' we exchange the execution order of J_i and J_j

$$\max_{1 \leq k \leq n} C'_k - r_k = \max\{\max_{\substack{1 \leq k \leq n \\ k \notin \{i,j\}}} C_k - r_k, C'_i - r_i, C'_j - r_j\}$$

$$C'_i - r_i \leq C_i - r_i \quad \text{and} \quad C'_j - r_j = C_i - r_j < C_i - r_i$$

$$\Rightarrow \qquad \max_{1 \leq k \leq n} C'_k - r_k \leq \max_{1 \leq k \leq n} C_k - r_k$$

FIFO is sub-optimal for max-stretch



Max-stretch of FIFO: $\max\{1, \frac{4-1}{1}\} = 3$.

Optimal max-stretch: $\max\{\frac{5-0}{3}, 1\} = \frac{5}{3}$.

Evaluating the quality of an online schedule

An online algorithm has a competitive factor ρ if and only if

Whatever the set of jobs J_1, ..., J_n

$$\text{Online schedule cost}(J_1, ..., J_N) \leq$$
$$\rho \times \text{Optimal off-line schedule cost}(J_1, ..., J_N)$$

The case of list schedules (1/2)

A peculiar framework: tasks are presented one by one to the scheduler that must schedule each task on a processor before seeing the next submitted task (online-list).

Theorem

Any list scheduling algorithm is $2 - \frac{1}{p}$-competitive for the online minimization of the makespan on p processors, and this bound is tight.

The case of list schedules (2/2)

Theorem

If the platform contains 2 or 3 processors (i.e., $p = 2$ or $p = 3$), then any list scheduling algorithm achieves the best possible competitive ratio for the online minimization of the makespan.

$p = 2$. We consider the instances $\mathcal{I}_1 = (1, 1)$ and $\mathcal{I}_2 = (1, 1, 2)$.

$p = 3$. We consider three instances: $\mathcal{I}_1 = (1, 1, 1)$, $\mathcal{I}_2 = (1, 1, 1, 3, 3, 3)$, and $\mathcal{I}_3 = (1, 1, 1, 3, 3, 3, 6)$.

FIFO competitiveness

Theorem

First come, first served is:

- *optimal for the online minimization of max-flow*
- Δ*-competitive for the online minimization of sum-flow*
- Δ*-competitive for the online minimization of max-stretch*
- Δ^2*-competitive for the online minimization of sum-stretch*

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FIFO competitiveness for max-stretch

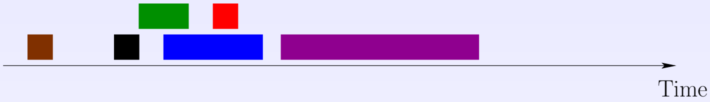
Theorem

FIFO is Δ competitive for maximum stretch minimization

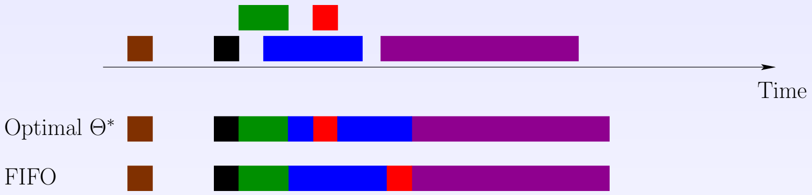
This means that

1. FIFO has a competitive factor of Δ (i.e., on no instance is FIFO's max-stretch more than Δ that of the optimal solution)
2. This bound is tight (=cannot be improved)

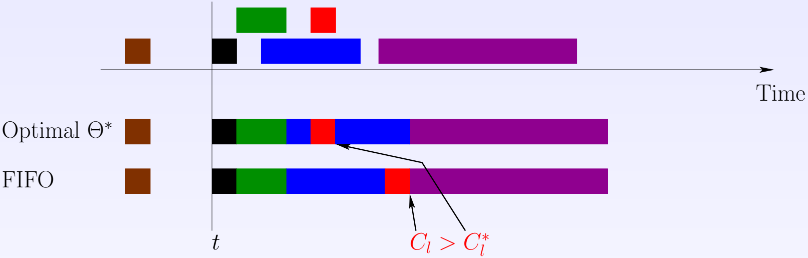
Upper bound for max-stretch



Upper bound for max-stretch



Upper bound for max-stretch

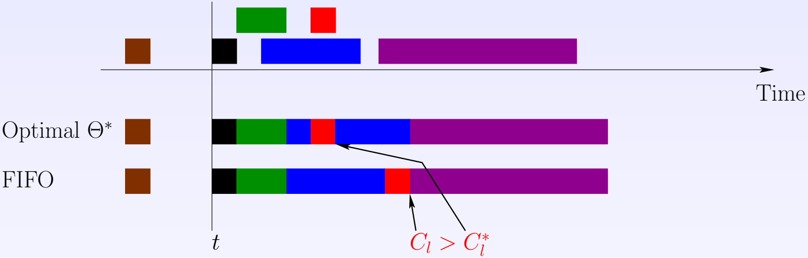


Any job J_l s.t. $\mathcal{S}_l > \mathcal{S}_l^*$ ($\Leftrightarrow C_l > C_l^*$)

t last time before C_l s.t. the processor was idle under FIFO.

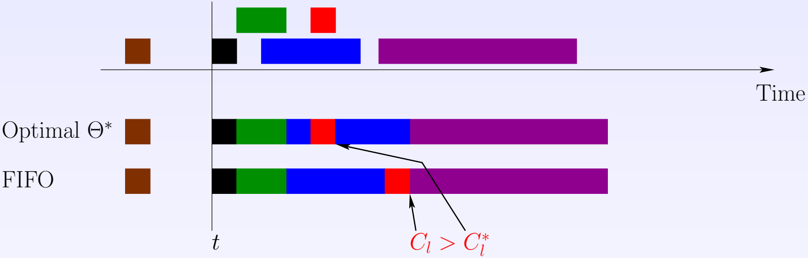
t is the release date r_i of some job J_i.

Upper bound for max-stretch



Any job J_l s.t. $\mathcal{S}_l > \mathcal{S}_l^*$ ($\Leftrightarrow C_l > C_l^*$)
During $[r_i, C_l]$, FIFO exactly executes J_i, J_{i+1}, ..., J_{l-1}, J_l.

Upper bound for max-stretch

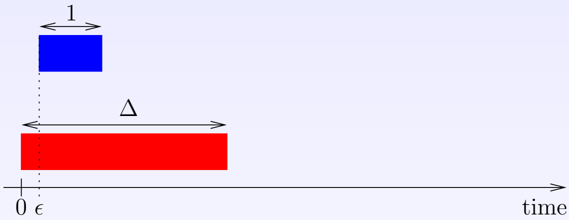


Any job J_l s.t. $\mathcal{S}_l > \mathcal{S}_l^*$ $(\Leftrightarrow C_l > C_l^*)$
During $[r_i, C_l]$, FIFO exactly executes J_i, J_{i+1}, ..., J_{l-1}, J_l.
As $C_l^* < C_l$, there is a job J_k, $i \leq k \leq l-1$ s.t. $C_k^* \geq C_l$. Then:

$$\mathcal{S}^* = \max_j \mathcal{S}_j^* \geq \mathcal{S}_k^* = \frac{C_k^* - r_k}{p_k} \geq \frac{C_l - r_l}{p_k} = \frac{C_l - r_l}{p_l}\frac{p_l}{p_k} \geq \mathcal{S}_l \times \frac{1}{\Delta}$$

$$\forall l, \mathcal{S}_l > \mathcal{S}_l^* \quad \Rightarrow \quad \Delta \times \mathcal{S}^* \geq \mathcal{S}_l$$

The bound is tight



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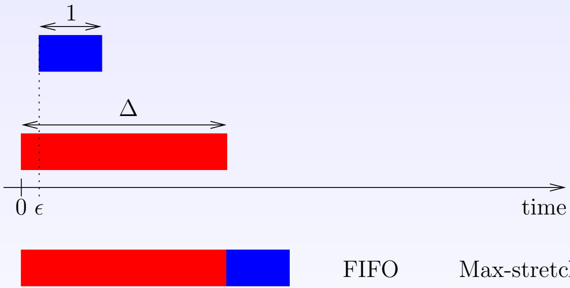
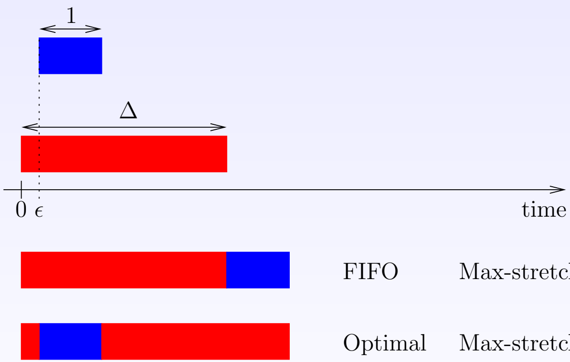


FIFO Max-stretch $= \frac{1+\Delta-\epsilon}{1}$

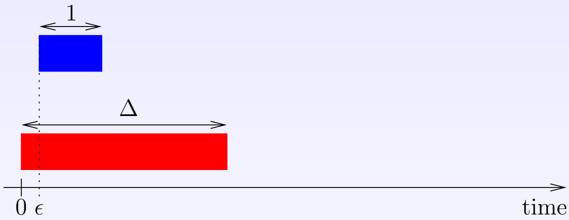
The bound is tight

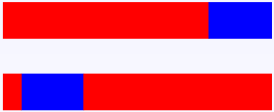


FIFO $\quad$ Max-stretch $= \frac{1+\Delta-\epsilon}{1}$

Optimal $\quad$ Max-stretch $= \frac{1+\Delta}{\Delta}$

The bound is tight





FIFO Max-stretch $= \frac{1+\Delta-\epsilon}{1}$

Optimal Max-stretch $= \frac{1+\Delta}{\Delta}$

Competitive ratio: $\frac{1+\Delta-\epsilon}{\frac{1+\Delta}{\Delta}} = \Delta \frac{1+\Delta-\epsilon}{1+\Delta} = \Delta - \epsilon \frac{\Delta}{1+\Delta} \geq \Delta - \epsilon$

Outline

Bound on the competitive ratio

Theorem

On one processor, any online scheduling algorithm with preemption minimizing the max-stretch has a competitive ratio greater than $\frac{1}{2}\Delta^{\sqrt{2}-1}$*, if the system receives at least jobs of three different sizes, and if* Δ *is the ratio between the size of the largest and the smallest job.*

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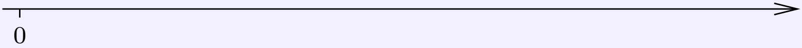
On one processor, any online scheduling algorithm with preemption minimizing the max-stretch has a competitive ratio greater than $\frac{1}{2}\Delta^{\sqrt{2}-1}$*, if the system receives at least jobs of three different sizes, and if* Δ *is the ratio between the size of the largest and the smallest job.*

Proof principle: by contradiction we assume that there exists an algorithm and we build a sequence of jobs and a scenario to make the algorithm fail.

The adversary



The adversary

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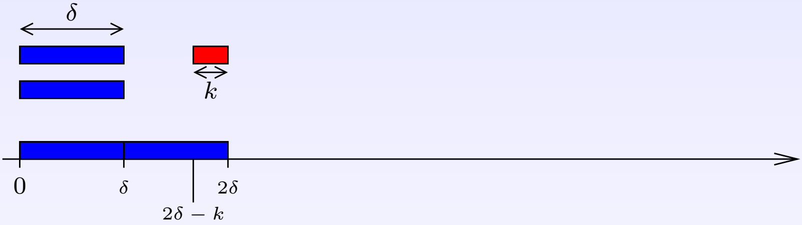
Achievable stretch: $\dfrac{2\delta - 0}{\delta} = 2.$

The adversary



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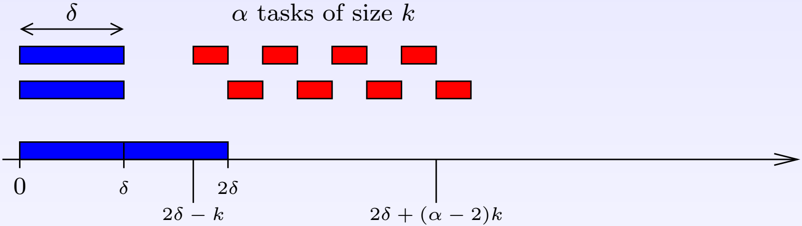
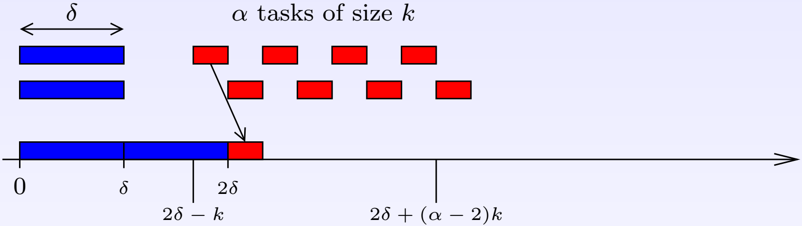


The job J_{2+j} arrives at time $2\delta + (j-2)k$.

The adversary



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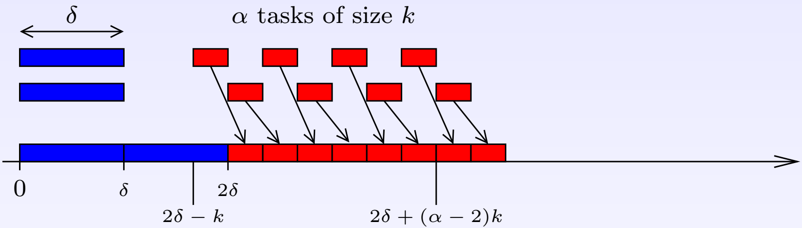
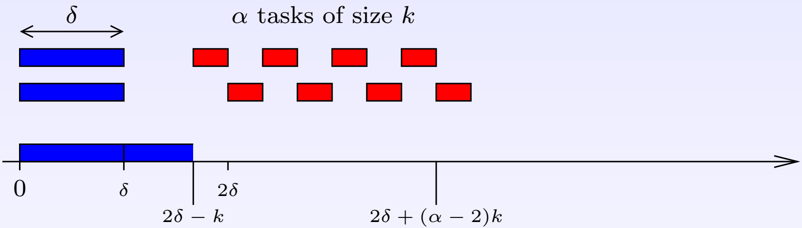


The job J_{2+j} arrives at time $2\delta + (j-2)k$.

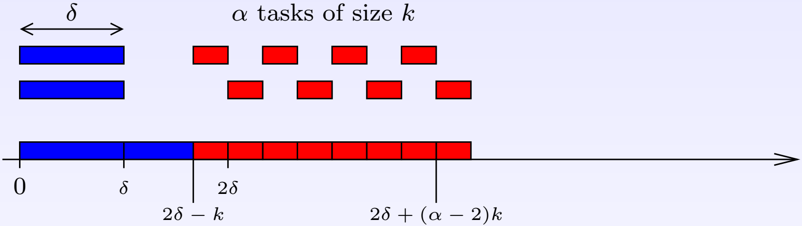
Achievable stretch: $\dfrac{(2\delta + jk) - (2\delta + (j-2)k)}{k} = 2.$

The adversary



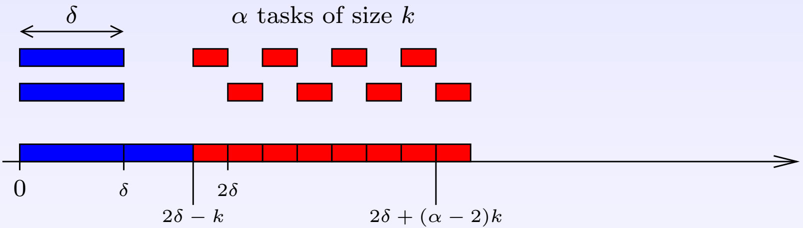
In practice: we do not know what happens after $2\delta - k$.

The adversary



We want to forbid this case (each size-k job being executed at its release date).

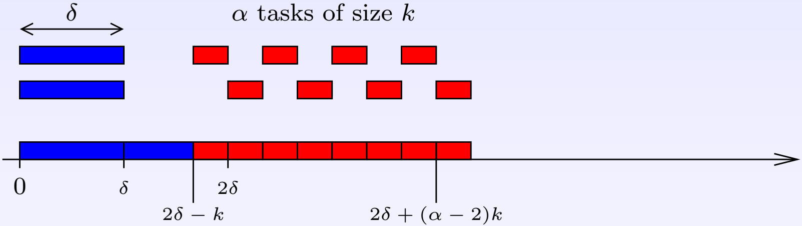
The adversary



We want to forbid this case (each size-k job being executed at its release date).

The algorithm being $\frac{1}{2}\Delta^{\sqrt{2}-1}$-competitive, J_1 and J_2 must be completed at the latest at time: $2 \cdot \frac{1}{2}\Delta^{\sqrt{2}-1} \cdot \delta = 2 \cdot \frac{1}{2}\left(\frac{\delta}{k}\right)^{\sqrt{2}-1} \cdot \delta$

The adversary



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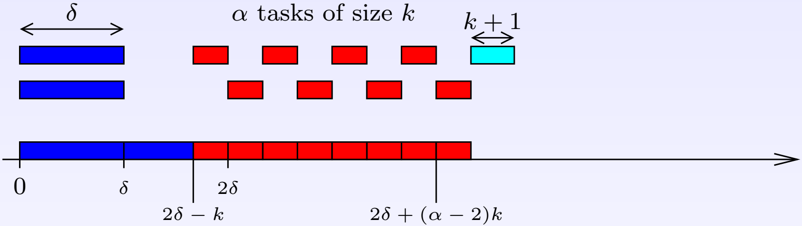
We let $\alpha = \left\lceil 1 + k - \frac{2\delta}{k} \right\rceil$ and then $2\delta + (\alpha - 1)k \geq 2 \cdot \frac{1}{2}\left(\frac{\delta}{k}\right)^{\sqrt{2}-1} \cdot \delta$.

The adversary



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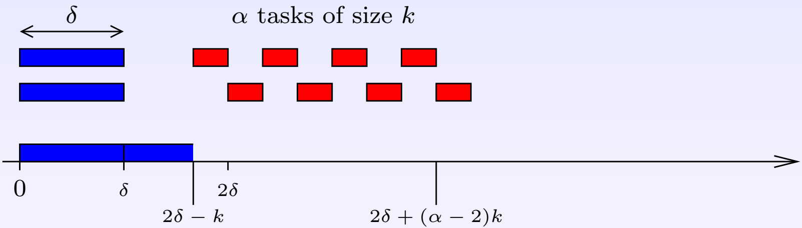

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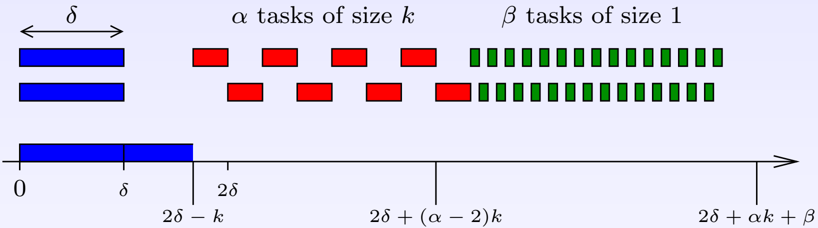
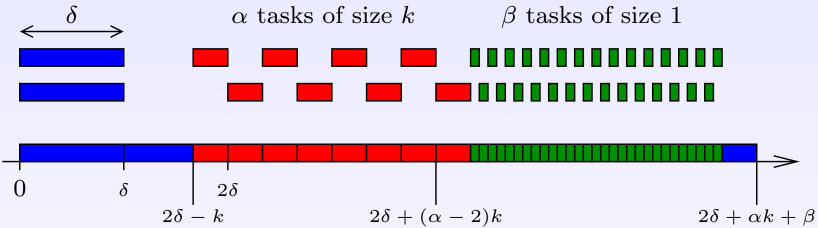


The job $J_{2+\alpha+j}$ arrives at time $2\delta + (\alpha - 1)k + (j - 1)$.

The adversary



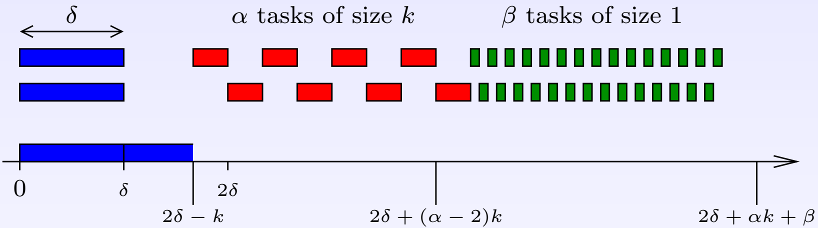
Achievable stretch (off-line)

Stretch of each job of size k or 1: 1.

Stretch of J_1 or J_2: $\dfrac{2\delta + \alpha k + \beta}{\delta}$

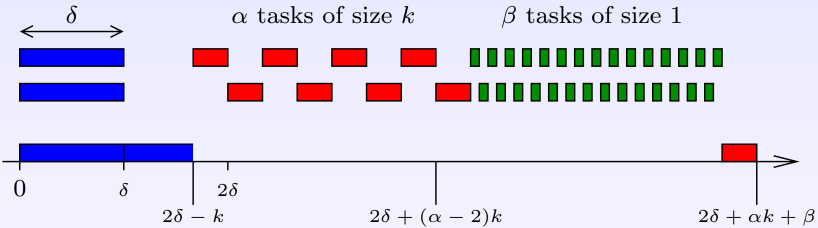
Optimal stretch $\leq \dfrac{2\delta + \alpha k + \beta}{\delta}$

The adversary



Achievable stretch (online)

The adversary

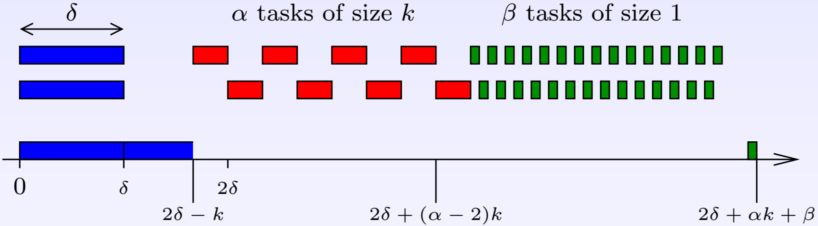


Achievable stretch (online)

The last completed job is of size k.

$$\text{Stretch} \geq \frac{(2\delta + \alpha k + \beta) - (2\delta + (\alpha - 2)k)}{k} = 2 + \frac{\beta}{k}.$$

The adversary

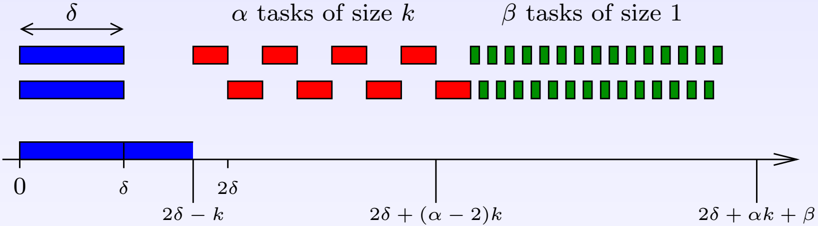


Achievable stretch (online)

The last completed job is of size 1.

$$\text{Stretch} \geq \frac{(2\delta + \alpha k + \beta) - (2\delta + (\alpha - 1)k + (\beta - 1))}{1} = k + 1.$$

The adversary



Achievable stretch (online)

$$\text{Stretch} \geq \min\left\{2 + \frac{\beta}{k}, k+1\right\}$$

We let: $\beta = \lceil k(k-1) \rceil$

Then: stretch $\geq k + 1$.

The adversary: summing things up

$$\alpha = \left\lceil 1 + k - \frac{2\delta}{k} \right\rceil$$

$$\beta = \lceil k(k-1) \rceil$$

Optimal stretch $\leq \dfrac{2\delta + \alpha k + \beta}{\delta}$

Achieved stretch $\geq k + 1$.

The adversary: summing things up

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$$\beta = \lceil k(k-1) \rceil$$

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Achieved stretch $\geq k + 1$.

We let $k = \delta^{2-\sqrt{2}}$

The adversary: summing things up

$$\alpha = \left\lceil 1 + k - \frac{2\delta}{k} \right\rceil$$

$$\beta = \lceil k(k-1) \rceil$$

Optimal stretch $\leq \dfrac{2\delta + \alpha k + \beta}{\delta}$

Achieved stretch $\geq k + 1$.

We let $k = \delta^{2-\sqrt{2}}$

Therefore $k + 1 > \left(\dfrac{1}{2}\delta^{\sqrt{2}-1}\right)\left(\dfrac{2\delta + \alpha k + \beta}{\delta}\right)$

Outline

FIFO competitiveness

Theorem

First come, first served is:

- *optimal for the online minimization of max-flow*
- Δ*-competitive for the online minimization of sum-flow*
- Δ*-competitive for the online minimization of max-stretch*
- Δ^2*-competitive for the online minimization of sum-stretch*

Lower bound as a function of n

Theorem

There is no c-competitive preemptive online algorithm minimizing the maximum stretch with $c < n$

Principle of the proof

- We suppose there exists an algorithm whose ratio $c = n - \epsilon$
- n jobs are released at time 0
- Whatever the scheduler does, no job completes before time n
- Jobs are sorted by non-decreasing cumulative computation time computed at time n: the i-th job is of size λ^{i-1}
- The maximum stretch is at least n (first job has size 1 and is not completed at n)
- Optimal: execute jobs in Shortest Processing Time first order:

$$\frac{\sum_{j=1}^{i} \lambda^{j-1}}{\lambda^{i-1}} = \frac{\lambda^i - 1}{\lambda^{i-1}(\lambda - 1)} \xrightarrow[\lambda \to +\infty]{} 1$$

EquiPartition

Theorem

EquiPartition is n-competitive for the minimization of maximum stretch.

However, EquiPartition is at best $\frac{\Delta+1}{2+\ln(\Delta)}$ competitive (when FIFO is Δ competitive)

Outline

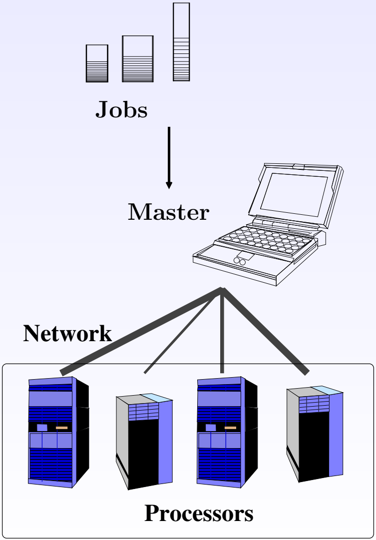
The scheduling problem

The scheduler

- Gather the jobs
- Send them to the processors

The aim

Distribute the *identical* jobs to the processors, for the jobs to be processed in the best possible way

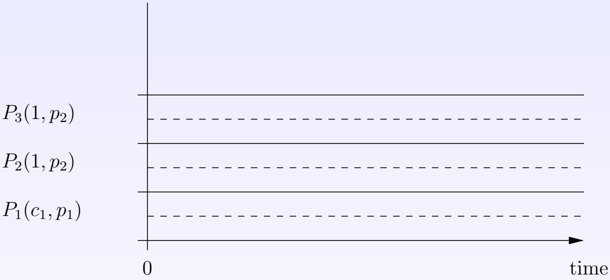


The scheduling problem

Formally

- n jobs, m processors
- p_j: processing time of a job on processor j
- c_j: time to send a job from the master to the worker j
- r_i: release date of job J_i
- C_i: completion time of job J_i
- The objective function:
  - maximal flow: $\max C_i - r_i$

Finding a lower bound on the competitiveness (1)

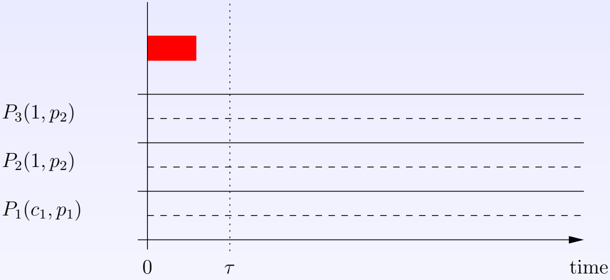


Idea:

- A fast processor with slow communications ($c_1 > 1$)
- Two identical and slow processors, with fast communications
- If only one job, one must choose the fast processor ($c_1 + p_1 < 1 + p_2$)

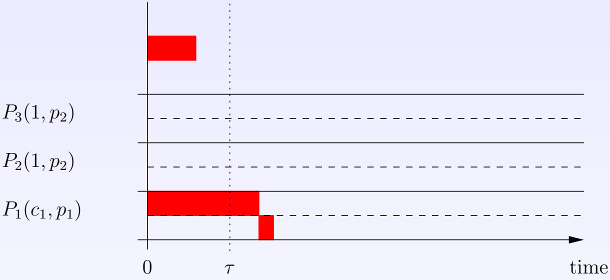
Finding a lower bound on the competitiveness (1)

Finding a lower bound on the competitiveness (1)



We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

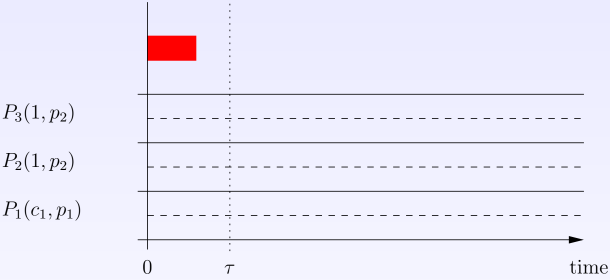
Finding a lower bound on the competitiveness (1)



We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

1. Optimal: job on P_1, max-flow $\geq c_1 + p_1$.

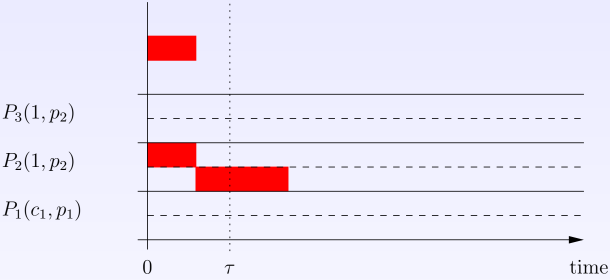
Finding a lower bound on the competitiveness (1)



We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

1. Optimal: job on P_1, max-flow $\geq c_1 + p_1$.
2. Nothing done: max-flow $\geq \tau + c_1 + p_1$, ratio $\geq \frac{\tau + c_1 + p_1}{c_1 + p_1}$.

Finding a lower bound on the competitiveness (1)

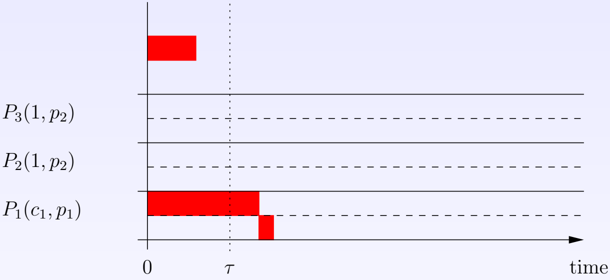


We look at time $\tau \geq 1$ to see what has happened. Three possibilities:

1. Optimal: job on P_1, max-flow $\geq c_1 + p_1$.
2. Nothing done: max-flow $\geq \tau + c_1 + p_1$, ratio $\geq \frac{\tau + c_1 + p_1}{c_1 + p_1}$.
3. Job sent to P_2, max-flow $\geq 1 + p_2$. Ratio $\geq \frac{1 + p_2}{c_1 + p_1}$.

We want to force the algorithm to process the first job on P_1.

Finding a lower bound on the competitiveness (1)

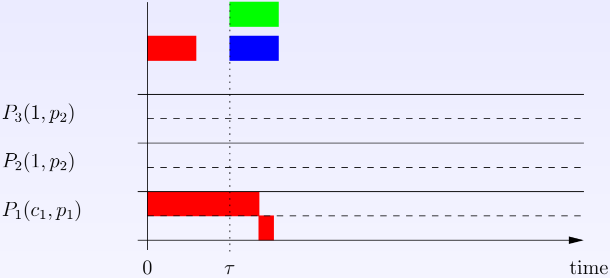


We look at time $\tau \geq 1$ to see what has happened. If the scheduler did not pick the first possibility, the adversary sends no more jobs. Later we will choose τ, c_1, p_1 and p_2 such that the ratio achieved,

$$\min\left\{\frac{1+p_2}{c_1+p_1}, \frac{\tau+c_1+p_1}{c_1+p_1}\right\},$$

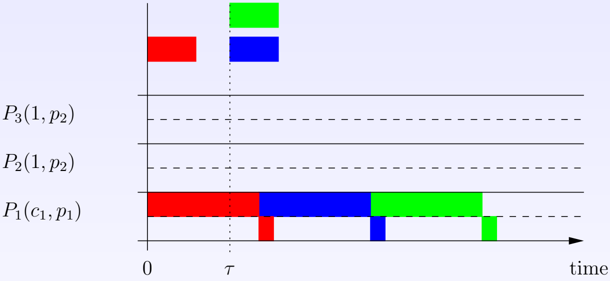
is as large as possible.

Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
We consider all the possible cases.

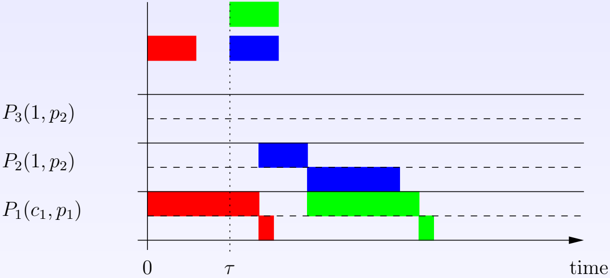
Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
The two jobs are executed on P_1:

$$\max\{c_1 + p_1, \\ \max\{\max\{c_1, \tau\} + c_1 + p_1, c_1 + 2p_1\} - \tau, \\ \max\{\max\{c_1, \tau\} + c_1 + p_1 + \max\{c_1, p_1\}, c_1 + 3p_1\} - \tau\}$$

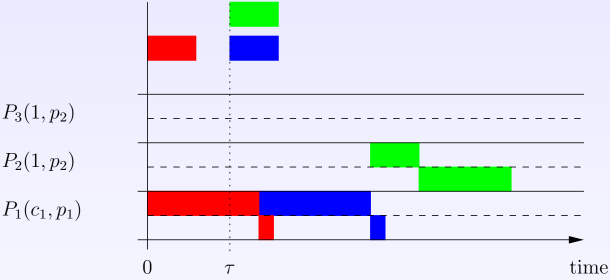
Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
The first of the two jobs is executed on P_2 (or P_3), and the other one on P_1.

$$\max\{c_1 + p_1,$$
$$(\max\{c_1, \tau\} + c_2 + p_2) - \tau,$$
$$\max\{\max\{c_1, \tau\} + c_2 + c_1 + p_1, c_1 + 2p_1\} - \tau\}$$

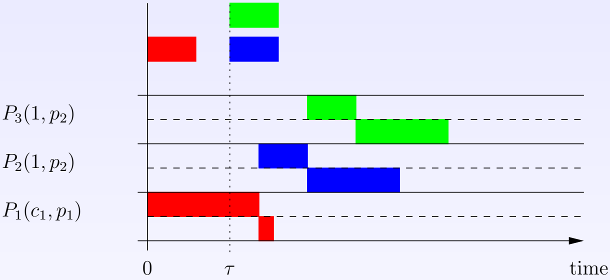
Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
The first of the two jobs is executed on P_1, and the other one on P_2 (or P_3).

$$\begin{aligned}\max\{&c_1 + p_1,\\ &\qquad \max\{\max\{c_1, \tau\} + c_1 + p_1, c_1 + 2p_1\} - \tau,\\ &\qquad\qquad (\max\{c_1, \tau\} + c_1 + c_2 + p_2) - \tau\}\end{aligned}$$

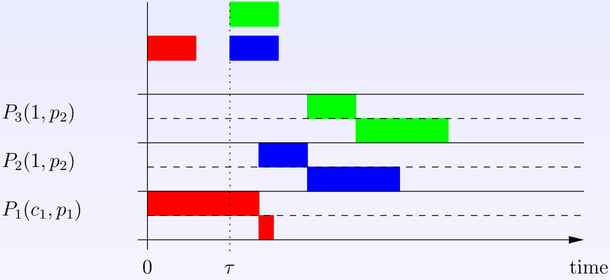
Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
One of the two jobs is executed on P_2 and the other one on P_3.

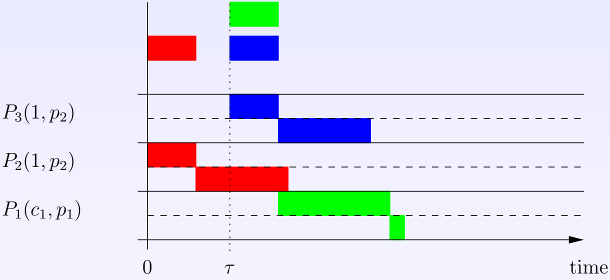
$$\max\{c_1+p_1, (\max\{c_1,\tau\}+c_2+p_2)-\tau, (\max\{c_1,\tau\}+c_2+c_2+p_2)-\tau\}$$

Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
The case where both jobs are executed on P_2 (or both on P_3) is worse than the previous one, therefore, we do not need to study it.

Finding a lower bound on the competitiveness (1)



At time τ we send two new jobs.
The (desired) optimal: the first job on P_2, the second on P_3, and the third on P_1.

$$\max\{c_2+p_2, (\max\{c_2, \tau\}+c_2+p_2)-\tau, (\max\{c_2, \tau\}+c_2+c_1+p_1)-\tau\}$$

Finding a lower bound on the competitiveness (2)

Lower bound on the competitiveness of any online algorithm:

$$
\min \begin{cases}
\frac{\tau + c_1 + p_1}{c_1 + p_1}, \\[2ex]
\frac{1 + p_2}{c_1 + p_1}, \\[2ex]
\dfrac{\min \begin{cases}
\max\{c_1 + p_1, \max\{\max\{c_1, \tau\} + c_1 + p_1, c_1 + 2p_1\} - \tau, \\
\qquad \max\{\max\{c_1, \tau\} + c_1 + p_1 + \max\{c_1, p_1\}, c_1 + 3p_1\} - \tau\} \\
\max\{c_1 + p_1, (\max\{c_1, \tau\} + c_2 + p_2) - \tau, \max\{\max\{c_1, \tau\} + c_2 + c_1 + p_1, c_1 + 2p_1\} - \tau\} \\
\max\{c_1 + p_1, \max\{\max\{c_1, \tau\} + c_1 + p_1, c_1 + 2p_1\} - \tau, (\max\{c_1, \tau\} + c_1 + c_2 + p_2) - \tau\} \\
\max\{c_1 + p_1, (\max\{c_1, \tau\} + c_2 + p_2) - \tau, (\max\{c_1, \tau\} + c_2 + c_2 + p_2) - \tau\}
\end{cases}}{\max\{c_2 + p_2, (\max\{c_2, \tau\} + c_2 + p_2) - \tau, (\max\{c_2, \tau\} + c_2 + c_1 + p_1) - \tau\}}
\end{cases}
$$

Problem: to find τ, c_1, p_1, and p_2 (as $c_2 = 1$) which maximizes this lower bound.

Constraints: $c_1 + p_1 < 1 + p_2$.

Finding a lower bound on the competitiveness (3)

1. Numeric resolution

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2. Characterization of the shape of the optimal: $\tau < c_1$, $p_1 = 0$, etc.

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2. Characterization of the shape of the optimal: $\tau < c_1$, $p_1 = 0$, etc.
3. New system:

$$\min \begin{cases} \frac{\tau+c_1}{c_1} \\ \frac{1+p_2}{c_1} \\ \frac{\min \begin{cases} 3c_1 - \tau \\ c_1 + 1 - \tau + p_2 \\ 2c_1 - \tau + 1 + p_2 \\ c_1 + 2 + p_2 - \tau \end{cases}}{1+p_2} \end{cases} = \min \begin{cases} \dfrac{\tau + c_1}{c_1} \\ \dfrac{1 + p_2}{c_1} \\ \dfrac{c_1 + 1 - \tau + p_2}{1 + p_2} \end{cases}$$

Finding a lower bound on the competitiveness (3)

1. Numeric resolution
2. Characterization of the shape of the optimal: $\tau < c_1$, $p_1 = 0$, etc.
3. New system:

$$\min \begin{cases} \frac{\tau+c_1}{c_1} \\ \frac{1+p_2}{c_1} \\ \frac{\min\begin{cases} 3c_1 - \tau \\ c_1 + 1 - \tau + p_2 \\ 2c_1 - \tau + 1 + p_2 \\ c_1 + 2 + p_2 - \tau \end{cases}}{1+p_2} \end{cases} = \min \begin{cases} \dfrac{\tau + c_1}{c_1} \\ \dfrac{1 + p_2}{c_1} \\ \dfrac{c_1 + 1 - \tau + p_2}{1 + p_2} \end{cases}$$

4. Solution: $c_1 = 2(1+\sqrt{2})$, $p_2 = \sqrt{2}c_1 - 1$, $\tau = 2$, $\rho = \sqrt{2}$.