# On Orbit Equivalence of Measure Preserving Actions

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**Abstract** We give a brief survey of some classification results on orbit equivalence of probability measure preserving actions of countable groups. The notion of $\ell^2$ Betti numbers for groups is gently introduced. An account of orbit equivalence invariance for $\ell^2$ Betti numbers is presented together with a description of the theory of equivalence relation actions on simplicial complexes. We relate orbit equivalence to a measure theoretic analogue of commensurability and quasi-isometry of groups: measure equivalence. Rather than a complete description of these subjects, a lot of examples are provided.

## 1 Equivalence Relations

### 1.1 Equivalence Relation Defined by an Action

Let $(X, \mu)$ be a standard Borel space, where $\mu$ is a probability measure without atoms. Recall that $(X, \mu)$ is Borel isomorphic to the unit interval of the reals, with the Lebesgue measure.

Let $\Gamma$ be a countable group and $\alpha$ an action of $\Gamma$ on $(X, \mu)$ by $\mu$-preserving Borel automorphisms. Consider the *orbit equivalence relation* on $X$:

$$\mathcal{R}_\alpha = \{(x, \gamma.x) : x \in X,\ \gamma \in \Gamma\}\,.$$

As a subset of $X \times X$, this equivalence relation is just the union of the graphs of the $\gamma \in \Gamma$. In this measured context, null sets are neglected. Thus the action is *free* if the only element of $\Gamma$ with a fixed-point set of positive measure is the identity element.

*Example 1.1.* The first examples to keep in mind are the following:

(1) The action of $\mathbb{Z}^n$ on the circle $\mathbb{S}^1$ by rationally independent rotations (belongs to the next two families). This gives an idea of the wildness of the quotient space $\Gamma\backslash X$.
(2) Free volume-preserving group actions on finite volume manifolds.
(3) A compact group $K$, its Haar measure $\mu$ and the action of a countable subgroup $\Gamma$ by left multiplication on $K$.

(4) The shift action of $\Gamma$ on the space $X = \{0,1\}^{\Gamma}$ of sequences of 0's and 1's indexed by $\Gamma$, with any invariant probability measure – for example the product of equiprobability measures on $\{0,1\}$. This action is free when $\Gamma$ is infinite. Thus, every countable group admits at least one free probability measure preserving (p.m.p.) action.

*Question:* Which properties of the group $\Gamma$ are determined by the orbit equivalence relation $\mathcal{R}_{\alpha}$?

## 1.2 Standard Countable Borel Equivalence Relations

The orbit equivalence relation $\mathcal{R} = \mathcal{R}_{\alpha}$ enjoys the following properties:

1. the equivalence classes (or *orbits*) of $\mathcal{R}$ are countable;
2. $\mathcal{R}$ is a Borel subset of $X \times X$;
3. every partial isomorphism $\varphi : A \to B$ whose graph is contained in $\mathcal{R}$ preserves the measure ($\mathcal{R}$ preserves $\mu$).

A *partial isomorphism* $\varphi : A \to B$ is a Borel isomorphism between two Borel subsets $A$ and $B$ of $X$. Observe that if its graph $\{(x, \varphi(x)) : x \in A\}$ is contained in $\mathcal{R}$ then $A$ admits a partition $A = \coprod_{\gamma \in \Gamma} A_{\gamma}$ where $x \in A_{\gamma} \Rightarrow \varphi(x) = \alpha(\gamma)(x)$ (soon replaced by the notation $\gamma.x$). The third item is now obvious.

A *standard countable measure preserving* equivalence relation $\mathcal{R}$ on $(X, \mu)$ is an equivalence relation that satisfies 1–3. This more general notion was introduced by J. Feldman and C. Moore who immediately observed [FM77a, Theorem 1] that every such equivalence relation is in fact the orbit equivalence relation of some group $\Gamma$ acting by $\mu$-preserving Borel automorphisms of $X$. The question of finding a freely acting $\Gamma$ remained open until A. Furman's work [Fur99b, Theorem D], exhibiting lots of examples where it is impossible.

*Example 1.2.* There are at least two kinds of examples – where an underlying $\Gamma$ is not obvious – which motivates this generalization:

(1) Let $Y \subset X$ be a Borel subset which meets all orbits of $\mathcal{R}$. The *induced equivalence relation* $\mathcal{R}_Y := \mathcal{R} \cap (Y \times Y)$ on $Y$, whose classes are restrictions of classes to $Y$, preserves the normalized probability measure $\mu_Y = \mu/\mu(Y)$.
(2) When looking at a minimal measured lamination, choose a total transversal $X$ of finite measure. The holonomy pseudogroup gives rise to such a general equivalence relation on $X$, with the normalized transverse measure. It is generated by the "return maps". Two points of the transversal are in the same class iff they belong to the same leaf of the lamination.

### 1.3 Orbit Equivalence

**Definition 1.3** *Two actions $\alpha_i$ of groups $\Gamma_i$ on $(X_i, \mu_i)$, $i = 1, 2$ are said to be* Orbit Equivalent (OE) *if they define the* same equivalence relation*, i.e. if there exists a Borel isomorphism $f : X_1 \to X_2$ such that $f_*(\mu_1) = \mu_2$ and for $\mu_1$-almost all $x \in X_1$,*

$$f(\Gamma_1.x) = \Gamma_2.f(x)\,.$$

An orbit equivalence between free actions gives rise to a cocycle (see R. Feres' contribution to the present volume [Fer01]) $\sigma : \Gamma_1 \times X_1 \to \Gamma_2$, where $\sigma(\gamma, x)$ is the unique element $\lambda \in \Gamma_2$ s.t. $\lambda.x = \gamma.x$. The cocycle identity is easily checked: $\sigma(\gamma_1\gamma_2, x) = \sigma(\gamma_1, \gamma_2.x)\sigma(\gamma_2, x)$.

The coarseness of this notion may be really distressing for certain ergodic theorists: H. Dye showed that (for $\Gamma_1 = \Gamma_2 = \mathbb{Z}$) any two ergodic probability measure preserving $\mathbb{Z}$-actions are orbit equivalent [Dye59, Theorem 1, p. 143 and Theorem. 5, p. 154]. A. Vershik obtained the same result at about the same time.

Later on, in [Dye63, Theorem 1, p. 560], Dye showed the same result for any group $\Gamma$ of polynomial growth or $\Gamma$ infinite abelian: any free ergodic probability measure preserving $\Gamma$-action is orbit equivalent to an ergodic $\mathbb{Z}$-action. All these actions thus define THE same equivalence relation $\mathcal{R}_\alpha$. The sadness of those people I just mentioned is far from decreasing.

R. Zimmer [Zim78] introduced the notion of amenability for an equivalence relation and showed, in particular, that if $\Gamma$ has a free p.m.p. action which is orbit equivalent to a $\mathbb{Z}$-action then $\Gamma$ is amenable. The natural conjecture was that any p.m.p. ergodic action of an amenable group is orbit equivalent to THE relation $\mathcal{R}_\alpha$, and this was proved by Ornstein–Weiss [OW80], and by Connes–Feldman–Weiss [CFW81], in the context of general non-singular (rather than p.m.p.) amenable equivalence relations.

This p.m.p. equivalence relation $\mathcal{R}_\alpha$ is characterized as ergodic and *hyperfinite*: it is an increasing union of equivalence relations with finite classes.

Recall that abelian, nilpotent, and solvable groups are amenable groups. By contrast, the free groups $\mathbf{F}_n, n \geq 2$ are not, as well as any group containing $\mathbf{F}_2$, and it is not easy to produce a non-amenable group not containing $\mathbf{F}_2$.

### 1.4 Stable Orbit Equivalence

It is important to realize that when the classes are infinite (in all non-trivial situations), the equivalence relation does not admit any *Borel fundamental domain*, i.e. a Borel subset $D \subset X$ that meets once and only once the orbit of $\mu$-almost all $x \in X$. In other words it is not possible to measurably pick one point in each orbit, or the space of orbits is ugly. This is obvious in the context of free actions, where the iterates $\gamma.D$ of a hypothetical fundamental domain would provide a partition of almost all of $X$ (finite measure) with infinitely many pieces of equal measure. And it is almost that obvious in general.

Nevertheless, the following notion gives a meaning to the idea of two equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$ having the "same space of orbits": First, observe that if a Borel subset $Y \subset X$ meets almost all orbits of $\mathcal{R}$, then $\mathcal{R}$ and the induced equivalence relation $\mathcal{R}_Y$ (Example 1.2.(1)) have the same space of orbits. Now comes the:

**Definition 1.4** *Two equivalence relations* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *on* $(X_i, \mu_i)$*,* $i = 1, 2$ *are said to be* Stably Orbit Equivalent (SOE) *if there exist Borel subsets* $Y_i \subset X_i$*,* $i = 1, 2$ *which meet almost all orbits of* $\mathcal{R}_i$ *on which the induced equivalence relations are the same* $\mathcal{R}_{1Y_1} \simeq \mathcal{R}_{2Y_2}$*, via an isomorphism* $f : Y_1 \to Y_2$ *which preserves the normalized measures* $f_*(\mu_1/\mu_1(Y_1)) = \mu_2/\mu_2(Y_2)$*. The number* $i(\mathcal{R}_1, \mathcal{R}_2) = \mu_2(Y_2)/\mu_1(Y_1)$ *is called the* index *of* $\mathcal{R}_2$ *in* $\mathcal{R}_1$ *(given by this SOE).*

In Example 1.2.(2), the choice of various transversal leads to SOE equivalence relations. Notice that for a given $\mathcal{R}$, the set of possible $i(\mathcal{R}, \mathcal{R})$ forms a subgroup of $\mathbb{R}_+^*$, called the *fundamental group of* $\mathcal{R}$.

*Remark 1.5.* Why is it called *stable*? The *countable stabilization* of an equivalence relation $(X, \mu, \mathcal{R})$ is the equivalence relation $\mathcal{R}'$ on the space $(X \times C, \mu')$, where $C$ is any infinite discrete countable set and $\mu' = \mu \times$counting measure, defined by $(x_1, c_1) \sim (x_2, c_2)$ iff $(x_1, x_2) \in \mathcal{R}$. It can be shown that SOE equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$ have isomorphic countable stabilizations. More precisely, there exists a Borel isomorphism $X_1 \times C_1 \xrightarrow{f} X_2 \times C_2$ sending $\mu_1'$ to $\lambda \cdot \mu_2'$, with $\lambda^{-1} = i(\mathcal{R}_1, \mathcal{R}_2)$, and sending almost every $\mathcal{R}_1'$-class onto an $\mathcal{R}_2'$-class.

We refer to [FM77a], [Moo82], [Sch87] and [Zim84] for more material on this section.

## 2 Measure Equivalence

Stable orbit equivalence must be related to another notion introduced by M. Gromov and developed by A. Furman [Fur99a], namely measure equivalence (ME) between countable groups, which is a measure theoretical analogue of quasi-isometry.

**Criterion for quasi-isometry ([Gro93, 0.2.C$_2'$]).** Two finitely generated groups $\Gamma_1$ and $\Gamma_2$ are *quasi-isometric (QI)* iff there exist commuting, continuous actions of $\Gamma_1$ and $\Gamma_2$ on some locally compact space $M$, such that the action of each of the groups is properly discontinuous and has a *compact* fundamental domain. Similarly:

**Definition 2.1 ([Gro93, 0.5.E])** *Two countable groups $\Gamma_1$ and $\Gamma_2$ are* Measure Equivalent (ME) *iff there exist commuting, measure preserving, free actions of $\Gamma_1$ and $\Gamma_2$ on some Lebesgue measure space $(\Omega, m)$ such that the action of each of the groups admits a* finite measure *fundamental domain ($D_i$ for $\Gamma_i$).*

In this case we say that $\Gamma_1$ is ME to $\Gamma_2$ with *index*

$$i_\Omega(\Gamma_1, \Gamma_2) = m(D_2)/m(D_1)\,,$$

and, following [Fur99a], we say that $(\Omega, m)$ (together with the actions) is a *coupling* between $\Gamma_1$ and $\Gamma_2$.

*Example 2.2.* (1) The basic example of QI groups are *cocompact* lattices in the same locally compact second countable group $G$ (applying the criterion to $M = G$ and $\Gamma_1$ and $\Gamma_2$ acting by left and right multiplication).

(2) The basic example of ME groups are general lattices $\Gamma_1$ and $\Gamma_2$ in the same locally compact second countable group $G$. The existence of a lattice (that is a discrete, finite covolume subgroup) forces $G$ to be unimodular: the Haar measure is invariant under the commuting actions by left and right multiplication of $\Gamma_1$ and $\Gamma_2$.
Thus, $(G, \mathrm{Haar})$ gives a measure equivalence between $\Gamma_1$ and $\Gamma_2$ which are ME with index $i_G(\Gamma_1, \Gamma_2) = \mathrm{Vol}(G/\Gamma_2)/\mathrm{Vol}(\Gamma_1\backslash G)$.

(3) Notice that the definition also makes sense when $\Gamma_1 \simeq \Gamma_2$ and that the values $i_\Omega(\Gamma_1, \Gamma_2)$ for various couplings may a priori assume any value in $\mathbb{R}_+^*$ as shown by the example of $\Gamma_1 = \mathbb{Z}$, $\Gamma_2 = \alpha\mathbb{Z}$ acting on $\mathbb{R}$ by translations which produces the values $i_{\mathbb{R}}(\mathbb{Z}, \mathbb{Z}) = |\alpha|$ (see Sect. 2.2).

### 2.1 Comparison SOE–ME

ME is an equivalence relation on countable groups and is strongly related to SOE (as was already noticed by A. Furman [Fur99b]).

**Theorem 2.3** *The groups $\Gamma_1$ and $\Gamma_2$ are measure equivalent (with index $i$) iff they admit stably orbit equivalent free actions (with index $i$).*

*Proof.* ($\Leftarrow$) Suppose, that the free actions of $\Gamma_i, i = 1, 2$ are in fact orbit equivalent via a measure preserving isomorphism $f : X_1 \to X_2$ (see Definition 1.3). Then one gets an identification of the equivalence relations $\mathcal{R}_{\Gamma_1} = \mathcal{R}_{\Gamma_2}$, as subsets of $X_1 \times X_2$. The product action of the group $\Gamma_1 \times \Gamma_2$ on $X_1 \times X_2$ by $(\gamma_1, \gamma_2).(x, y) = (\gamma_1.x, \gamma_2.y)$ restricts to an action on $\mathcal{R}_{\Gamma_1} = \mathcal{R}_{\Gamma_2}$, which preserves the natural measure $\nu$ (cf. Example 5.1). Each group $\Gamma_i$ acts freely and admits as a finite measure fundamental domain the diagonal of the relation $\{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = x_2\}$. We have constructed the desired ME.

Let now $\mathcal{R}_{\Gamma_i}$, $i = 1, 2$, be equivalence relations given by SOE free actions. There is an isomorphism $f : X_1 \times \mathbb{N} \to X_2 \times \mathbb{N}$ which rescales the measure and allows the identification of the countable stabilizations $\mathcal{R}'_1 = \mathcal{R}'_2$ as subsets of $(X_1 \times \mathbb{N}) \times (X_2 \times \mathbb{N})$ (cf. Remark 1.5). The two possible measures on it are proportional: choose, say, the one given by $\mathcal{R}'_2$, and denote it by $\nu$. The same as above may now be done on the intersection $\mathcal{R}_2{}' \cap (X_1 \times \{0\}) \times (X_2 \times \{0\})$ leading to a free action of $\Gamma_1 \times \Gamma_2$. Any Borel section of $X_2 \times \{0\}$ (resp. $X_1 \times \{0\}$) is a fundamental domain for the $\Gamma_1$-action (resp. $\Gamma_2$-action) with $\nu$-measures equal to 1 (resp. $i(\mathcal{R}_{\Gamma_1}, \mathcal{R}_{\Gamma_2})$).

($\Rightarrow$) Conversely, start from a coupling $(\Omega, m)$ and finite measure fundamental domains $D_i \subset \Omega$ for $\Gamma_i$, $i = 1, 2$. By commutativity, one gets (extending the left–right notations of Example 2.2 (2) to the commuting actions) an action of $\Gamma_1$ on $\Omega/\Gamma_2 \simeq D_2$ and an action of $\Gamma_2$ on $\Gamma_1\backslash\Omega \simeq D_1$.

These actions are SOE. In fact, the corresponding equivalence relations $\mathcal{R}_{\Gamma_1}$ and $\mathcal{R}_{\Gamma_2}$ have countable stabilizations on $\Omega = \Omega/\Gamma_2 \times \Gamma_2$ and on $\Omega = \Gamma_1\backslash\Omega \times \Gamma_1$ which can both be identified with the orbit equivalence relation of the $(\Gamma_1 \times \Gamma_2)$-action on $\Omega$.

Recall that the measure on $\Omega/\Gamma_2$ and $\Gamma_1\backslash\Omega$ must be normalized. The corresponding measures on $\Omega$ are $\mu'_1 = (m(D_2))^{-1}m$ and $\mu'_2 = (m(D_1))^{-1}m$. Following Remark 1.5, the SOE index $i(\mathcal{R}_{\Gamma_1}, \mathcal{R}_{\Gamma_2})$ equals $\lambda^{-1}$, where $\lambda$ satisfies $\mu'_1 = \lambda\mu'_2$, i.e. $\lambda^{-1} = m(D_2)/m(D_1)$.

If the actions are not free, just choose any free probability measure preserving action of $\Gamma_1$ on a standard Borel space $(X, \mu)$, let $\Gamma_2$ act trivially ($\gamma_2 x = x$) and replace $(\Omega, m)$ by $(\Omega \times X, m \times \mu)$ with the diagonal $\Gamma_1 \times \Gamma_2$-action, which turns out to be free. This new coupling leads to free actions. □

### 2.2 Values of Indices

Remark that two groups $\Gamma_1$ and $\Gamma_2$ are commensurable (i.e. there exists a group $\Lambda$ which is isomorphic to a finite index subgroup in both of them) iff they are ME for a countable coupling $\Omega$ (with counting measure) and in this case $i_\Omega(\Gamma_1, \Gamma_2) = [\Gamma_1 : \Lambda]/[\Gamma_2 : \Lambda]$. Given $\Gamma$, the set of values $\{i_\Omega(\Gamma, \Gamma)\}$ for all couplings $\Omega$ between $\Gamma$ and itself is denoted by $I_{ME}(\Gamma)$. Notice that it is a ME invariant.

The condition $I_{ME}(\Gamma) = \{1\}$ means that for each free p.m.p. action of $\Gamma$, the induced equivalence relation $(\mathcal{R}_\Gamma)_Y$ (Example 1.2) cannot be generated by any free action of $\Gamma$, when $Y$ is a proper ($\mu(Y) < \mu(X)$) Borel subset which meets all orbits. This implies, in particular the following:

[$*$] for each free p.m.p. action of $\Gamma$, the equivalence relation $\mathcal{R}_\Gamma$
has trivial fundamental group (for every self SOE, $i(\mathcal{R}_\Gamma, \mathcal{R}_\Gamma) \equiv 1$).

The condition $I_{ME}(\Gamma) = \{1\}$ also implies that $i_\Omega(\Gamma, \Gamma_1)$ only depends on $\Gamma_1$ in the ME class of $\Gamma$ and not on the particular coupling $\Omega$.

It follows from Theorem 4.1 below, that if one of the $\ell^2$ Betti numbers of $\Gamma$ is $\neq 0, \infty$ (see Sect. 4), then $I_{ME}(\Gamma) = \{1\}$. Also lattices in higher rank connected simple Lie groups satisfy $I_{ME}(\Gamma) = \{1\}$ ([GG88, Corollary B.3], [Fur99a], [Fur99b]). On the other hand, amenable groups do not have [$*$], and neither do groups $\Lambda$ which are ME to the direct product of an infinite amenable group by any group. For such a $\Lambda$, one has $I_{ME}(\Lambda) = I_{ME}(\mathbb{Z}) = \mathbb{R}_+^*$.

*Question:* Do there exist groups $\Gamma$ with an infinite discrete $I_{ME}(\Gamma)$ ? For this question to make sense, one should restrict the attention to ergodic situations.

### 2.3 Comparison QI–ME

Amenability is both a QI and a ME invariant. In fact, Ornstein–Weiss' result (cf. Sect. 1.3) implies that all infinite (countable) amenable groups belong to the same ME class. This single class splits into many QI classes (distinguished for example by growth).

This could suggest that QI implies ME and that the latter in just a coarser equivalence relation. The QI classification of lattices in semi-simple Lie groups (where lattices in a given $G$ split into several QI classes, but remain in the same ME class – Example 2.2) also support such an idea.

However, this turns out to be wrong. In fact, A. Furman proved that Kazhdan property (T) is a ME invariant [Fur99a, Theorem 8.2]. But Kazhdan property (T) is not a QI invariant: if $\Gamma$ has property (T) and admits a non-trivial $\mathbb{Z}$-valued bounded 2-cocycle (many property (T) hyperbolic groups admit such cocycles, e.g. cocompact lattices in $Sp(1,n)$, [Li92]) then the direct product $\Gamma_1 = \Gamma \times \mathbb{Z}$ does not have property (T), but is QI to a non-trivial central extension of $\Gamma$ by $\mathbb{Z}$ which has property (T). I am grateful to A. Furman and N. Monod for discussions on this example.

For ME groups, the Euler characteristics are positively proportional (Theorem 4.1 below). This is not the case for QI. In fact, by forming the free product $\Gamma * \mathbf{F}_p$ of a group $\Gamma$ with various free groups ($p \geq 2$), one gets QI groups (Gromov hyperbolic if $\Gamma$ was). This is because the $\mathbf{F}_p$ are in fact Lipschitz equivalent according to a theorem due to P. Papasoglu (see P. De la Harpe's book [Har00, IV-B.46] for further results and references). Now, if $\chi(\Gamma) \geq 2$, the Euler characteristics $\chi(\Gamma * \mathbf{F}_p) = \chi(\Gamma) - p$ do not all have the same sign. In fact, Theorem 4.1 states that ME groups have proportional $\ell^2$ Betti numbers $\beta_n$ while P. Pansu showed that for QI groups $\Gamma_1$ and $\Gamma_2$, one has $\beta_n(\Gamma_1) = 0$ iff $\beta_n(\Gamma_2) = 0$.

To complete the picture, let us come back to lattices in Lie groups. A. Furman, improving R. Zimmer's super-rigidity for cocycles, showed [Fur99a, Theorem 3.2] that for a higher rank simple Lie group $G$, the collection of all its lattices (up to finite groups) forms a single ME class. More precisely, let $\Gamma$ be a lattice in a simple, connected Lie group $G$ with finite center and $\mathbb{R}$-rank $\geq 2$. Furman proved that if a countable group $\Lambda$ is measure equivalent to

$\Gamma$, then $\Lambda$ has a finite index subgroup which maps with finite kernel onto a lattice of $Ad(G)$.

Finally, the ME class of the free group $\mathbf{F}_2$ on two generators, in addition to all finitely generated (non-cyclic) free groups and compact surface groups, contains all free products of a finite number of amenable groups, with the exception of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

## 3 Cost of an Equivalence Relation

### 3.1 Graphings

To any group $\Gamma$, one can associate the minimal number $n(\Gamma)$ of generators. The cost of an equivalence relation is a similar quantity. It was first introduced by G. Levitt [Lev95].

One way to produce an equivalence relation is by considering a *graphing*, i.e. a countable family $\Phi = (\varphi_i : A_i \to B_i)_{i \in I}$ of $\mu$-preserving partial isomorphisms between Borel subsets $A_i, B_i \in X$. A graphing *generates* a standard countable $\mu$-preserving Borel equivalence relation $\mathcal{R}_\Phi$, namely the smallest equivalence relation that satisfies

$$x \sim \varphi_i(x) \text{ for all } i \in I \text{ and all } x \in A_i\,.$$

In other words, $(x, y) \in \mathcal{R}_\phi$ iff there exists a $\Phi^{\pm 1}$-word (i.e. a word in the $\varphi_i$'s and their inverses) whose associated partial isomorphism is defined at $x$ and sends $x$ to $y$.

The *cost* of the generating system $\Phi$ is the number of elements in $\Phi$, weighted by the measure of their domain: $\mathcal{C}(\Phi) = \sum_{i \in I} \mu(A_i)$.

The *cost* of the equivalence relation is the infimum of the costs over all the generating graphings: $\mathcal{C}(\mathcal{R}) := \inf\{\mathcal{C}(\Phi) : \mathcal{R}_\Phi = \mathcal{R}\}$. This is by definition an invariant of the equivalence relation (of OE), and the difficulty is now the computation. If all classes are infinite, then $\mathcal{C}(\mathcal{R}) \geq 1$. The costs of SOE equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are related as follows : $\mu_2(Y_2)(\mathcal{C}(\mathcal{R}_1) - 1) = \mu_1(Y_1)(\mathcal{C}(\mathcal{R}_2) - 1)$, where $\mu_1, \mu_2$ are probability measures on $X_1, X_2$ ([Gab00a, Corollaire II.12.]).

A generating graphing $\Phi$ associates in a Borel varying way a Cayley graph like structure to each orbit (this explains the terminology). Precisely, for each $x$ in $X$, its orbit under $\mathcal{R}$ is the vertex set of a graph $\Phi_x$, where two points are neighbors if one of the $\varphi_i$'s sends one to the other. The generating condition implies the connectedness of these graphs. Graphings appeared in a paper of S. Adams [Ada90].

*Example 3.1.* (1) For a free action $\alpha$ of a group $\Gamma$, given a generating family $(\gamma_1, \gamma_2, \dots)$ of $\Gamma$, one has a particular graphing $\Phi$ where each $\varphi_i$ is the automorphism $\alpha(\gamma_i)$, defined on all of $X$. Each orbit graph for that graphing is then isomorphic to the Cayley graph of $\Gamma$.

(2) Ornstein–Weiss' theorem (cf. Sect. 1.3) implies that almost all orbits of a free action of an infinite amenable group can be given a line structure and thus the corresponding orbit relation has cost 1.

A *treeing* is a graphing where the graphs associated to almost all $x \in X$ are trees. If $\Phi$ is not a treeing, there is a $\Phi^{\pm 1}$-word with a non-trivial fixed point set. One can then remove a subset from the domain of one of its letters (opening cycles in the graphs $\Phi_x$), in order to define a new generating graphing with smaller cost (when finite). Conversely:

**Theorem 3.2 ([Gab98], [Gab00a, Théorème 2])** *If $\Phi$ is a treeing of the equivalence relation $\mathcal{R}$, then it realizes the infimum of the costs:* $\mathcal{C}(\mathcal{R}) = \mathcal{C}(\Phi)$.

While the general result involving free products with amalgamation over amenable groups can be found in [Gab00a, Théorème 2, Théorème IV.15], the reader interested in the proof of that restricted statement is advised to look at the simplified proof in the announcement [Gab98].

As a corollary, any free action of the free group $\mathbf{F}_n$ has cost $n$ (see Example 3.1 with a basis of $\mathbf{F}_n$ as generating set), thus free groups of different ranks cannot have orbit equivalent actions. Now comes the natural:

*Question:* How many non-orbit equivalent p.m.p. free actions does the free group $\mathbf{F}_2$ admit?

Only finitely many of them are known (three following a personal communication of S. Popa). Note that S. Gefter and V. Golodets [GG88, Corollaire A.9, p. 843] showed that non-cocompact lattices in a simple, connected Lie group with finite center and $\mathbb{R}$-rank $\geq 2$, have a continuum of non-OE free p.m.p. ergodic actions.

Costs of actions are computed for lots of groups $\Gamma$ in [Gab00a]. For example, if $\Gamma$ is an amalgamated product $A *_C B$ of finite groups, then cost$= 1-(1/|A|+1/|B|-1/|C|)$ (it is always treeable); if $\Gamma$ is the fundamental group $\pi_1(S_g)$ of a compact orientable surface of genus $g$, then cost$= 2g-1$.

*Question:* Does there exist a group with two actions of different costs?

### 3.2 Non-Treeability

But not every equivalence relation admits a treeing. Countable groups with Kazhdan property (T) are not tree-friendly. For example, any simplicial action of such a group on a tree fixes a point. Adams and Spatzier [AS90, Theorem 1.8, Lemma 2.1] showed that probability measure preserving ergodic free actions of (infinite) Kazhdan property (T) groups do not admit any treeing. The analogy between these two results will be made more transparent in Example 6.4, Sect. 6.

The difficulty in showing that the cost is a non-trivial invariant comes from the fact that many groups have only cost 1 actions (like direct products of infinite groups, see [Gab00a, VI-D] for other examples). Thus, the following statement gives many groups with no treeable free action: the only groups which admit both a cost 1 and a treeable free action are amenable [Gab00a, Corollaire VI.22].

*Question:* Does the surface group $\pi_1(S_g)$ admit a non-treeable action?

### 3.3 An Application

The theory of graphings is useful for a classification of equivalence relations. It can also be applied to group theory in order to show a result similar to Schreier's theorem for normal subgroups of the free group:

**Theorem 3.3 ([Gab01])** *If the first $\ell^2$ Betti number of $\Gamma$ is not zero, then any finitely generated normal subgroup $N$ of $\Gamma$ is either finite or has finite index.*

Notice that finite generation may be replaced by finiteness of $\beta_1(N)$, as it was proven by W. Lück, under additional hypotheses on the quotient $\Gamma/N$, like containing an infinite order element or arbitrarily large finite groups [Lüc94] and [Lüc98b, Theorem 3.3]. Let us show the following weaker statement:

**Theorem 3.4** *If $\Gamma$ has no cost 1 free action, then any finitely generated normal subgroup of $\Gamma$ is either finite or has finite index.*

*Proof.* Consider an extension $1 \to N \to \Gamma \xrightarrow{f} Q \to 1$. Suppose that $N$ and $Q$ are infinite and that $N$ is finitely generated. Let us show that $\Gamma$ has a cost 1 free action, by exhibiting generating graphings of small costs.

Consider a probability measure preserving ergodic free action of $\Gamma$ (resp. $Q$) on $X$ (resp. $Y$); then the diagonal $\Gamma$-action $\gamma.(x,y) = (\gamma.x, f(\gamma).y)$ on $Z = X \times Y$ is free and preserves the product measure. It generates an equivalence relation $\mathcal{R} = \mathcal{R}_\Gamma$. The fibers of the projection $\pi : Z \to Y$ are preserved by $N$.

Fix $\epsilon > 0$. The relation $\mathcal{R}_Q$, as any equivalence relation whose classes are infinite, contains a *hyperfinite* (see Sect. 1.3) subrelation $\mathcal{S}$, whose classes are all infinite. The latter is generated by an automorphism $t$ of $Y$ and $Y$ admits a partition $Y = \coprod_{q \in Q} Y_q$ where $x \in Y_q$ iff $q(x) = t(x)$. Define $Z_q = \pi^{-1}(Y_q)$, choose a pullback $\gamma_q \in f^{-1}(q)$ and denote by $\varphi_q$ the partial isomorphism of $Z$ defined by restricting $\gamma_q$ to $Z_q$. As the $Z_q$'s form a partition of $Z$, the cost of $\Phi_1 := (\varphi_q)$ equals 1.

Now let $A \subset Z$ be a $\pi$-saturated Borel subset of measure $\epsilon$ and denote by $\varphi'_j$ the restriction of $n_j$ to $A$ where $n_1, \ldots, n_r$ is a generating set for $N$. The graphing $\Phi_2 := (\varphi'_j)_{j=1,\ldots,r}$ generates the restriction of $\mathcal{R}_N$ to $A$ and its cost equals $(r\epsilon)$.

*Claim:* The relation generated by $\Phi_1 \cup \Phi_2$ contains $\mathcal{R}_N$. In fact, for each $(x,y) \in Z$ there exists a $\Phi_1^{\pm 1}$-word $m_1$ defined at $(x,y)$ and sending $(x,y)$

into $A$. Let $\omega$ be the corresponding element of $\Gamma$. If $n \in N$, the point $n.(x,y)$ lies in the same $\pi$-fiber as $(x,y)$, so that $\omega$ is also defined at $n.(x,y)$. By normality, $\omega n \omega^{-1}$ belongs to $N$, thus $\omega(x,y)$ and $\omega n(x,y)$ are connected by a $\Phi_2^{\pm 1}$-word $m_2$. The word $m_1^{-1} m_2 m_1$ connects $(x,y)$ to $n.(x,y)$.

Choose a sequence of positive numbers $(\eta_\gamma)_{\gamma \in \Gamma}$ with sum $\leq \epsilon$ and Borel subsets $A_\gamma \subset Z$ of measure $\leq \eta_\gamma$ which meet every $N$-orbit – since $N$ is infinite, this is possible; one could also have supposed that the $N$-action is ergodic on $X$. Denote by $\varphi_\gamma$ the restriction of $\gamma$ to $A_\gamma$. The graphing $\Phi_3 = (\varphi_\gamma)_{\gamma \in \Gamma}$ has cost $\leq \epsilon$.

*Claim:* $\Phi_1 \cup \Phi_2 \cup \Phi_3$ generates $\mathcal{R}_\Gamma$. In fact, for each $(x,y) \in Z$, there is an element of $N$ sending it into $A_\gamma$, and also a $(\Phi_1 \cup \Phi_2)^{\pm 1}$-word $m$. By normality, $\gamma.(x,y)$ and $\varphi_\gamma m(x,y)$ are $N$-equivalent, thus they are connected by a $(\Phi_1 \cup \Phi_2)^{\pm 1}$-word $m'$. Now $m' \varphi_\gamma m$ connects $(x,y)$ to $\gamma.(x,y)$.

Since $\Phi_1 \cup \Phi_2 \cup \Phi_3$ has cost $1 + r\epsilon + \epsilon$, one concludes that $\mathcal{R}_\Gamma$ has cost 1. □

## 4 $\ell^2$ Betti Numbers for Groups

To each countable group $\Gamma$ is associated a sequence of numbers $\in [0, \infty]$ called its $\ell^2$ *Betti numbers* $(\beta_n(\Gamma))_{n \in \mathbb{N}}$ that are defined using the $\ell^2$ chains of CW-complexes on which $\Gamma$ acts. They were defined in this generality in [CG86].

**Theorem 4.1 ([Gab01])** *If $\Gamma_1$ and $\Gamma_2$ are measurably equivalent, then they have proportional $\ell^2$ Betti numbers.*

*More precisely if $(\Omega, m)$ is a coupling between them with index $i_\Omega(\Gamma_1, \Gamma_2)$, then for all $n \in \mathbb{N}$, one has $\beta_n(\Gamma_2) = i_\Omega(\Gamma_1, \Gamma_2) \cdot \beta_n(\Gamma_1)$.*

In particular, ME groups have positively proportional Euler characteristics (when defined) $\chi(\Gamma) = \chi^{(2)}(\Gamma) = \sum_n (-1)^n \beta_n(\Gamma)$.

**Corollary 4.2** *Lattices in the same locally compact second countable group have proportional $\ell^2$ Betti numbers with constant of proportionality equal to the ratio of the covolumes.*

### 4.1 $\ell^2$ Homology

In this section, we give some indications about $\ell^2$ homology and $\ell^2$ Betti numbers for cocompact free actions on simplicial complexes. We refer to W. Lück's and B. Eckmann's surveys ([Lüc98], [Eck00]) for a general exposition of these ideas, or to Lück's forthcoming book [Lück].

Let $K$ be a simplicial complex on which $\Gamma$ acts freely and simplicially. The space of $n$-chains is the free $\mathbb{Z}$-module $C_n(K, \mathbb{Z})$ with the family $S_n$ of $n$-simplices as a basis. It is the space of (finite) formal integer linear combinations of elements of $S_n$. The boundary map $\partial_n$ sends an $n$-simplex to the obvious $(n-1)$-chain – one has at some point to order simplices, let us forget about it – and is extended to $n$-chains by linearity.

If now $S_n$ is considered as a Hilbert orthonormal base, one gets the space of $\ell^2$ $n$-chains $C_n^{(2)}(K)$, i.e. the space of (infinite) formal linear combinations of elements of $S_n$ with square summable coefficients.

Once a representative is chosen in each orbit of $S_n$, there are natural $\Gamma$-equivariant identifications of $S_n$ with $\alpha_n$ copies of $\Gamma$, where $\alpha_n$ is the number of $\Gamma$-orbits in $S_n$. These induce isomorphisms $C_n(K,\mathbb{Z}) \simeq \oplus_{i=1}^{\alpha_n}\mathbb{Z}\Gamma$ and $C_n^{(2)}(K) \simeq \oplus_{i=1}^{\alpha_n}\ell^2(\Gamma)$ where $\Gamma$ acts by left translations, and where the latter is a Hilbert sum. The spaces of chains thus get a $\Gamma$-module structure.

The $\partial_n$'s, defined on the Hilbert basis, require some finiteness condition to extend to a well defined and bounded (hence continuous) operator on $\ell^2$ chains. For example, suppose that a vertex $v$ is the endpoint of infinitely many edges $e_1, e_2, \ldots$ of $K$. Since $\sum_{j\in\mathbb{N}}(1/j^2)$ converges, the chain $\sum_{j\in\mathbb{N}}(1/j)e_j$ is $\ell^2$, i.e. belongs to $C_1^{(2)}(K)$, but by applying the boundary operator $\partial_1$ formally, the vertex $v$ would receive the coefficient $\sum_{j\in\mathbb{N}}(1/j)$ !! More precisely, the sequence $\partial_1(\sum_{j=1}^{p}(1/j)e_j)$ is not bounded when $p\to\infty$ since, when expressed in the vertices Hilbert base, its $v$-coefficient is $\sum_{j=1}^{p}(1/j)$. But if $K$ is cocompact, then the boundary maps extend to bounded operators still denoted $\partial_n$, and give a chain complex $(\partial\circ\partial=0)$

$$0 \xleftarrow{\partial_0} C_0^{(2)}(K) \xleftarrow{\partial_1} C_1^{(2)}(K) \xleftarrow{\partial_2} C_2^{(2)}(K) \xleftarrow{\partial_3} C_3^{(2)}(K) \xleftarrow{\partial_4} \cdots \quad (1)$$

As usual, consider its homology $H_n^{(2)}(K) := \mathrm{Ker}\ \partial_n/\mathrm{Im}\ \partial_{n+1}$. If one wants to keep dealing with Hilbert spaces one has to divide out by a closed subspace, i.e. to consider the closure $\overline{\mathrm{Im}\ \partial_{n+1}}$ of $\mathrm{Im}\ \partial_{n+1}$, thus considering the so called *reduced $\ell^2$ homology* $\overline{H}_n^{(2)}(K) := \mathrm{Ker}\ \partial_n/\overline{\mathrm{Im}\ \partial_{n+1}}$.

## 4.2 $\ell^2$ Betti Numbers

In fact, all of this is $\Gamma$-equivariant, so that the reduced $\ell^2$ homology has the additional structure of a *Hilbert $\Gamma$-module.* It is a Hilbert space, with a $\Gamma$ representation and it admits an isometric equivariant embedding into an orthogonal sum $\mathcal{L}_r = \oplus_{i=1}^{r}\ell^2(\Gamma)$, where $\Gamma$ acts by $\lambda_r$, the sum of its left regular representations $\lambda$ on each $\ell^2(\Gamma)$. Namely, $\overline{H}_n^{(2)}(K)$ embeds into $C_n^{(2)}(K) \simeq \oplus_{i=1}^{\alpha_n}\ell^2(\Gamma)$ as the space $\mathcal{H}_n(K)$ of *harmonic $n$-chains*, the orthocomplement of $\overline{\mathrm{Im}\ \partial_{n+1}}$ in $\mathrm{Ker}\ \partial_n$.

It thus gets a $\Gamma$-dimension (see Sect. 4.4), in some sense a dimension modulo the $\Gamma$-action (cf. 1–2, Sect. 4.4).

The $\ell^2$ *Betti numbers* for the action of $\Gamma$ on $K$ are by definition:

$$\beta_n(K,\Gamma) := \dim_\Gamma \overline{H}_n^{(2)}(K)\,.$$

These numbers were first introduced by M. Atiyah in [Ati76], and Cheeger–Gromov have extended the notion to not necessarily cocompact simplicial actions [CG86], and even in the much more general context of singular $\ell^2$ cohomology. A very nice alternative theory was developed by W. Lück ([Lüc98a], [Lüc98b]).

The $\ell^2$ *Betti numbers* of the group $\Gamma$ are those of any free $\Gamma$-action on a contractible simplicial complex $E\Gamma$:

$$\beta_n(\Gamma) = \beta_n(E\Gamma, \Gamma)\,,$$

and are shown to depend only on $\Gamma$ [CG86].

### 4.3 Example

Let us consider the example of the free group $\Gamma = \mathbf{F}_n$ acting on the regular tree $T$ which is the universal cover of a bouquet of $n$ circles with one vertex and $n$ oriented edges $e_1, e_2, \ldots, e_n$. The tree is the Cayley graph of $\mathbf{F}_n$ for a generating basis. Once a vertex $\overline{x}$ and an edge $\overline{e}_i$ in each of the $n$ orbits of edges are chosen, $C_0(T, \mathbb{Z}) = \mathbb{Z}\Gamma.\overline{x}$ and $C_1(T, \mathbb{Z}) = \oplus_{i=1}^n \mathbb{Z}\Gamma.\overline{e}_i$ give a chain complex $0 \longleftarrow \mathbb{Z}\Gamma \xleftarrow{\partial_1} \oplus_{i=1}^n \mathbb{Z}\Gamma \longleftarrow 0$ where $\partial_1$ is injective and not surjective – the image 0-chains are those for which the sum of the coefficients vanishes – leading to a 1-dimensional $H_0$ and trivial $H_1$ which is not very interesting (just related to the fact that $T$ is contractible). To get something interesting one has to "divide out first" by $\Gamma$.

By contrast, at the $\ell^2$ level and for $n \geq 2$, in the associated chain complex $0 \longleftarrow \ell^2(\Gamma) \xleftarrow{\partial_1} \oplus_{i=1}^n \ell^2(\Gamma) \longleftarrow 0$ $\partial_1$ becomes surjective – as Kesten's theorem shows, since $\Gamma$ is non-amenable – and far from injective since $\dim_\Gamma \mathrm{Ker}\ \partial_1 = \dim_\Gamma \oplus_{i=1}^n \ell^2(\Gamma) - \dim_\Gamma \overline{\mathrm{Im}\ \partial_1} = n-1$ (Item 7, Sect. 4.4). Thus $H_0^{(2)}(T) = 0$ while $H_1^{(2)}(T) \simeq \oplus_{i=1}^{n-1} \ell^2(\Gamma)$. Taking the von Neumann dimension amounts in some sense to "dividing out afterwards".

An example of a harmonic 1-chain is obtained in the most obvious manner: put the coefficient $+1$ on some oriented edge $e = [a, b]$. Tail $a$ and head $b$ of $e$ receive $-1$, resp. $+1$, by $\partial_1$. Compensate this by putting the coefficient $(2n-1)^{-1}$ on each edge with head $a$ or with tail $b$, and so on, put $(2n-1)^{-d}$ on edges at distance $d$ oriented towards $a$, or oriented away from $b$. Since $n \geq 2$, this is an $\ell^2$ chain.

### 4.4 Von Neumann Dimension

Hilbert $\Gamma$-modules $M$ possess a well defined generalized dimension $\dim_\Gamma M$. It enjoys the following properties:

1. $\dim_\Gamma \ell^2(\Gamma) = 1$;
2. if $\Gamma$ is finite, $\dim_\Gamma M = \frac{1}{|\Gamma|} \dim M$ (usual vector space dimension);

3. $\dim_\Gamma M \geq 0$;
4. $\dim_\Gamma M = 0 \Longleftrightarrow M = 0$;
5. $M \subset N \Rightarrow \dim_\Gamma M \leq \dim_\Gamma N$;
6. $\dim_\Gamma M \oplus N = \dim_\Gamma M + \dim_\Gamma N$;
7. $\dim_\Gamma \mathrm{Ker}\, f + \dim_\Gamma \overline{\mathrm{Im}\, f} = \dim_\Gamma M_1$, when $f : M_1 \to M_2$ is a $\Gamma$-equivariant bounded operator between Hilbert $\Gamma$-modules .

The $\Gamma$*-dimension* is defined as follows: consider first a closed $\Gamma$-invariant subspace $M$ of $\ell^2(\Gamma) = \mathcal{L}_1$ and let $p$ be the orthogonal projection onto $M$. By $\Gamma$-invariance, the operator $p$ belongs to the von Neumann algebra of the group $\Gamma$, the algebra $N(\Gamma)$ of those operators that commute with all the unitary operators $\lambda(\gamma)$, $\gamma \in \Gamma$. The crucial feature is the existence of a finite trace $\tau$ on $N(\Gamma)$: for $a \in N(\Gamma)$, $\tau(a) = \langle a(\delta_e)|\delta_e\rangle$ (where $\delta_e$ is the characteristic function of the identity element of $\Gamma$). Now the von Neumann dimension of $M$ is the trace of its projector $\dim_\Gamma M = \tau(p)$.

If $M$ is a closed $\Gamma$-invariant subspace of $\mathcal{L}_r$, its projection belongs to the commuting algebra of $\lambda_r(\Gamma)$ and admits a bloc decomposition as an $r \times r$ matrix with coefficients $p_{i,j}$ in $N(\Gamma)$. By definition $\dim_\Gamma M = \sum_{i=1}^{r} \tau(p_{i,i})$.

The *trace property* ($\tau(ab) = \tau(ba)$) ensures that any two $\Gamma$-equivariant isometric embeddings in $\mathcal{L}_r$ of a Hilbert $\Gamma$-module have indeed the same dimension, giving rise to the well defined notion of $\Gamma$*-dimension* for such a module. This definition has a natural extension when $r = \infty$.

## 5 Simplicial Actions of an Equivalence Relation

### 5.1 Fibered Spaces

**Fundamental Example.** Consider a free measure preserving action of a countable group $\Gamma$ on $(X, \mu)$ and the orbit equivalence relation $\mathcal{R}_\Gamma$. Consider moreover an honest free simplicial action of $\Gamma$ on a countable simplicial complex $K$. The space $\Sigma_K := X \times K$ is equipped with the diagonal action of $\Gamma$: $\gamma.(x, \tau) = (\gamma.x, \gamma.\tau)$. It is fibered over $X$.

Choose a fundamental domain $D^{(0)} := \{v_1, v_2, \dots, \}$ , that is a set of $\Gamma$-orbit representatives for the $\Gamma$-action on the 0-skeleton $K^{(0)}$ of $K$. The set $X \times D^{(0)}$ is a fundamental domain for the diagonal action on $X \times K^{(0)}$ which permits us to identify the latter with countably many copies of $\mathcal{R}$ (one for each vertex $v_j \in D^{(0)}$) by $\Theta_j : (x, \gamma.v_j) \mapsto (x, \gamma^{-1}.x)$. Notice that the $\Theta_j$ are equivariant when $\mathcal{R}$ is given the $\Gamma$-action on the left coordinates: $\Theta_j(\gamma'.(x, \gamma.v_j)) = (\gamma'.x, \gamma^{-1}.x) = \gamma'.\Theta_j(x, \gamma.v_j)$.

An analogous construction can be done in each dimension, by choosing a fundamental domain $D^{(n)}$ for the $\Gamma$-action on the set of $n$-simplices $K^{(n)}$ of $K$. This permits us to identify the space $X \times K^{(n)}$ with countably many copies of $\mathcal{R}$. Notice that an $n$-simplex in $X \times K^{(n)}$ is made of an $(n+1)$-tuple of points in $X \times K^{(0)}$, with the same projection on $X$.

**Fibered Spaces.** A *standard Borel fibered space over* $X$ is a standard Borel space $U$ together with a specified Borel map (*projection map*), with countable fibers $p : U \to X$.

The *natural measure* $\nu_U$ on $U$ is defined as the product of $\mu$ (the measure on $X$) with the counting measure in the fibers of $p$. Thus, the measure of a Borel subset $V \subset U$ is obtained by integrating over $(X, \mu)$ the function $x \mapsto$ number of points in the intersection of $V$ with the fiber of $x$. Alternatively, $\nu_U$ is built by considering any countable Borel partition $U = \coprod U_i$ such that the restriction of $p$ to each $U_i$ is injective and by putting on each $U_i$ the pull back of $\mu$.

When $U$ and $V$ are standard Borel fibered spaces over $X$, via projection maps $p$ and $q$, their *fibered product*

$$U * V = \{(u, v) \in U \times V/\ p(u) = q(v)\}$$

is a standard Borel fibered space over $X$.

*Example 5.1.* The equivalence relation $\mathcal{R}$ has two natural fiberings over $X$, given by the projection maps $p_l :\ (x, y) \mapsto x$ and $p_r :\ (x, y) \mapsto y$, where the $r$ in $p_r$ stands for right while $p_l$ is the range map and $p_r$ is the source map for the groupoid $\mathcal{R}$. Due to the invariance of $\mu$ for $\mathcal{R}$, the natural measures defined by these two fiberings coincide; just denote them by $\nu$.

### 5.2 Groupoid Actions

**Space with Standard Left $\mathcal{R}$-Action.** A *standard left* $\mathcal{R}$*-space* or *space with standard left* $\mathcal{R}$*-action* consists of a (standard Borel) fibered space $U$ over $X$ and a map called the *action map* defined on the fibered product, where $\mathcal{R}$ fibers via $p_r$,

$$\mathcal{R} * U \to U,\ ((y, z), u) \mapsto (y, z).u\,,$$

such that $(x, y).[(y, z).u] = (x, z).u$ and $(z, z).u = u$. In particular, $z = p(u)$ and $y = p((y, z).u)$. The space $X$ itself is a standard left $\mathcal{R}$-space. The projection map $p$ of a left $\mathcal{R}$-space is $\mathcal{R}$-equivariant: $p((y, z).u) = (y, z).p(u)$. The *orbit* of $u$ is the set $\mathcal{R}.u := \{(y, z).u : (y, z) \in \mathcal{R}, z = p(u)\}$ and the *saturation* $\mathcal{R}.B$ of a Borel subset $B \subset U$ is the union of the orbits that meet $B$.

**Discrete Actions.** A standard left $\mathcal{R}$-space $U$ is *discrete* if the action admits a Borel fundamental domain $D$, i.e. a Borel subset $D \subset U$ that meets once and only once the orbit of $\nu_U$-almost all $u \in U$.

*Example 5.2.* An obvious instance of such a left $\mathcal{R}$-space is the fibered space $(U, p) = (\mathcal{R}, p_l)$ itself, with the action map $\mathcal{R} * U \to U$, $((x, y), (y, z)) \mapsto (x, z)$ and the diagonal of $\mathcal{R} = U$ as a fundamental domain.

Given a discrete standard left $\mathcal{R}$-space $U$, choose any countable Borel partition of a fundamental domain $D = \coprod_{i \in I} D_i$, such that, on each $D_i$, $p$ restricts to bijections $D_i \xrightarrow{p} p(D_i) \subset X$. The natural identification of $D_i$ with the diagonal subset $\Delta_i = \{(z,z) : z \in p(D_i)\} \subset \mathcal{R}$ extends by $\mathcal{R}$-equivariance to an identification of the saturation $\mathcal{R}.D_i$ with the saturation $\mathcal{R}.\Delta_i = p_r^{-1}(p(D_i))$. One thus gets an isomorphism of discrete standard left $\mathcal{R}$-spaces between $U$ and the disjoint union $\coprod_{i \in I} \mathcal{R}.\Delta_i$.

If $U$ is a discrete left $\mathcal{R}$-space (with fundamental domain $D$), then also $U * U * \cdots * U$ is a discrete left $\mathcal{R}$-space (with fundamental domain equal to $D * U * \cdots * U$).

## 6 Actions of the Equivalence Relation on a Simplicial Complex

**Definition 6.1** *A* simplicial complex with standard left $\mathcal{R}$-action *or more briefly an* $\mathcal{R}$-simplicial complex $\Sigma$ *consists of the following data:*

- *a discrete left $\mathcal{R}$-space $\Sigma^{(0)} \xrightarrow{p} X$ (space of vertices);*
- *for each $n \in \mathbb{N}$, a Borel subset $\Sigma^{(n)} \subset \underbrace{\Sigma^{(0)} * \cdots * \Sigma^{(0)}}_{n+1 \text{ times}}$, called the space of ordered $n$-simplexes (possibly empty for large $n$'s), satisfying four conditions:*
  1. *(permutations) $\Sigma^{(n)}$ is invariant under permutation of the coordinates;*
  2. *(non-degeneracy) if $(v_0, v_1, \cdots, v_n) \in \Sigma^{(n)}$, then $v_0 \neq v_1$, and:*
  3. *(boundary condition) $(v_1, \cdots, v_n) \in \Sigma^{(n-1)}$;*
  4. *(invariance) $\mathcal{R}.\Sigma^{(n)} = \Sigma^{(n)}$.*

The data in the fiber of each $x \in X$ is just an ordinary (countable) simplicial complex, denoted by $\Sigma_x$. Notice that the first two conditions could be slightly modified according to your favorite definition of a simplicial complex.

The $\mathcal{R}$-simplicial complex $\Sigma$ is *$n$-connected*, resp. *contractible*, resp. *$n$-dimensional* if for almost all $x$ in $X$, the simplicial complex $\Sigma_x$ has the corresponding property.

*Example 6.2.* (1) The basic and motivating example for this is the fundamental example in Sect. 5.1, where each $\Sigma_x$ is identified with a copy of $K$.

(2) A graphing $\Phi = (\varphi_i)_{i \in I}$ of $\mathcal{R}$ (cf. Sect. 3.1), with no loops or double edges, gives a 1-dimensional connected $\mathcal{R}$-simplicial complex. As left $\mathcal{R}$-spaces (cf. Example 5.2) $\Sigma^{(0)} = \mathcal{R}$ and $\Sigma^{(1)} = \{((x,y),(x,z)) \in \mathcal{R} * \mathcal{R} : y = \varphi_i^{\pm 1}(z) \text{ for some } i \in I\}$.

(3) The *complete structure* where each $\Sigma^{(n)} = \mathcal{R} * \cdots * \mathcal{R}$ with all diagonals removed (for non-degeneracy) is a contractible infinite dimensional example.

**Definition 6.3** *The* geometric dimension *of an equivalence relation* $\mathcal{R}$ *is the smallest dimension of a contractible* $\mathcal{R}$*-simplicial complex.*

*The* ergodic dimension *of a countable group* $\Gamma$ *is the smallest dimension of the equivalence relations* $\mathcal{R}_\Gamma$ *produced by free actions of* $\Gamma$.

*Example 6.4.* The already mentioned result of Adams and Spatzier (Sect. 3.2) about non-treeability for ergodic free actions of Kazhdan property (T) groups $\Gamma$ can now be rephrased: $\mathcal{R}_\Gamma$ does not admit any 1-dimensional contractible $\mathcal{R}_\Gamma$-simplicial complex and Kazhdan property (T) groups have ergodic dimension $> 1$.

One easily concludes that measure equivalent groups have the same ergodic dimension and that geometric (resp. ergodic) dimensions decrease when passing to subrelations (resp. subgroups).

# 7 $\ell^2$ Betti Numbers for Equivalence Relations and Their Actions

## 7.1 $\ell^2$ homology of $\Sigma$

An $\mathcal{R}$-simplicial complex $\Sigma$ defines a *Borel field of simplicial complexes* $x \mapsto \Sigma_x$ and, for each $n \in \mathbb{N}$, a *Borel field of Hilbert spaces* $x \mapsto C_n^{(2)}(\Sigma_x)$.

What is a *Borel vector field*? Notice that a Borel section $s$ of the fibering $\Sigma^{(n)} \to X$ leads to a $C_n(\Sigma_x)$-valued vector field $x \mapsto s(x)$. A vector field $x \mapsto \sigma(x) \in C_n^{(2)}(\Sigma_x)$ is Borel if $x \mapsto \langle \sigma(x) | s(x) \rangle_x$ is a Borel function for every Borel section $s$.

The Hilbert integral of the $\ell^2$ $n$-chains of $\Sigma_x$ is called the *space of $n$-dimensional $\ell^2$ chains* of $\Sigma$ and denoted by:

$$C_n^{(2)}(\Sigma) := \int_X^{\oplus} C_n^{(2)}(\Sigma_x) d\mu(x) \, .$$

It is the Hilbert space of those Borel vector fields that are square integrable, that is for which $x \mapsto ||\sigma(x)||_x \in L^2(X, \mu)$.

If the $\Sigma_x$ are uniformly locally bounded, then the fields of the boundary operators lead by integration to bounded operators $\partial_n : C_n^{(2)}(\Sigma) \to C_{n-1}^{(2)}(\Sigma)$ still satisfying $\partial \circ \partial = 0$. The reduced homology $\overline{H}_n^{(2)}(\Sigma)$ of the associated chain complex is called the *reduced $\ell^2$ homology of* $\Sigma$. Like in the group case (Sect. 4), these spaces have additional structure: they are Hilbert modules – the von Neumann algebra $\mathcal{M}$ involved is that of the equivalence relation $\mathcal{R}$ (see Sect. 7.2) – giving them a von Neumann dimension, "dimension modulo the groupoid action", which is called $\ell^2$ *Betti number* of $\Sigma$ and denoted:

$$\beta_n(\Sigma, \mathcal{R}) := \dim_{\mathcal{M}} \overline{H}_n^{(2)}(\Sigma) \, .$$

By using some ideas from [CG86], this definition is extended from uniformly locally bounded $\Sigma$'s to general $\Sigma$'s (see [Gab01]). A major difficulty arises here from the fact that operators appearing in homotopy equivalences do not lead to bounded operators (see [Gab00b]). The next result must be compared to A. Connes' notion of Betti numbers of foliations [Co82, p. 549] and to the claim in [Gro93, 8.A$_4$, p. 233] where the hypothesis forces boundedness of certain operators.

**Theorem 7.1 ([Gab01])** *Any two contractible $\mathcal{R}$-simplicial complexes have the same $\ell^2$ Betti numbers, called the $\ell^2$ Betti numbers of the equivalence relation $\mathcal{R}$ and denoted by $\beta_n(\mathcal{R})$. If $\mathcal{R}$ is defined by a free measure preserving action of a countable group $\Gamma$, then for all $n \in \mathbb{N}$, $\beta_n(\mathcal{R}) = \beta_n(\Gamma)$.*

If $\beta_p(\mathcal{R}) > 0$ then the geometric dimension of $\mathcal{R}$ is $\geq p$. One deduce the existence of equivalence relations of any geometric dimension: consider a free action of $\mathbf{F}_2 \times \mathbf{F}_2 \times \cdots \times \mathbf{F}_2$, $p$ times, to get a relation of dimension exactly $p$.

By the Morse inequalities, one gets a relation between $\ell^2$ Betti numbers and cost: $\mathrm{cost}(\mathcal{R}) - 1 \geq \beta_1(\mathcal{R}) - \beta_0(\mathcal{R})$, with no known example of strict inequality.

*Question:* Is it always the case, that $\mathrm{cost}(\mathcal{R}) - 1 = \beta_1(\mathcal{R}) - \beta_0(\mathcal{R})$?

### 7.2 Von Neumann Algebra of an Equivalence Relation

Consider the Hilbert space $L^2(\mathcal{R}, \nu)$ arising from the Borel space $\mathcal{R}$ with its measure $\nu$ (see Example 5.1).

For each partial isomorphism $\varphi : A \to B$ whose graph is contained in $\mathcal{R}$ (see Sect. 1.2) denote by $L_\varphi$ the operator on $L^2(\mathcal{R}, \nu)$ defined by $L_\varphi(\eta(x, y)) = \eta(\varphi^{-1}(x), y)$ if $x \in B$ and 0 otherwise. The *von Neumann algebra* $\mathcal{M}$ *of* $\mathcal{R}$ is the algebra of those bounded operators that commute with the family $\mathcal{L}$ of all the $L_\varphi$'s (see [FM77b] and [Moo82]). Notice that adjoining to $\mathcal{L}$ the $L_\psi$'s defined for $\psi \in L^\infty(X, \mu)$ by $L_\psi(\eta(x, y)) = \psi(x)\eta(x, y)$ would not change $\mathcal{M}$.

If $\mathcal{R} = \mathcal{R}_\Gamma$, then by the observation in Sect. 1.2, one could replace $\mathcal{L}$ by the $L_\gamma$'s, $\gamma \in \Gamma$, together with the $L_\psi$'s. In the free action case, $\mathcal{M}$ is described as a " cross product" of $L^\infty(X, \mu)$ with $\Gamma$ (see [MvN36, part IV, p. 192–209]).

The algebra $\mathcal{M}$ has a trace $\tau(a) = \langle a(\chi_\Delta) | \chi_\Delta \rangle$, where $\chi_\Delta$ is the characteristic function of the diagonal of $\mathcal{R}$. This allows us to define, as in Sect. 4.4, the von Neumann dimension of closed $\mathcal{L}$-invariant subspaces of $\oplus L^2(\mathcal{R}, \nu)$.

The embedding of $C_n^{(2)}(\Sigma)$ into a sum of $L^2(\mathcal{R}, \nu)$ is obtained from the identification at the end of Sect. 5.2.

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