AN INDEX FOR COUNTING FIXED POINTS OF AUTOMORPHISMS OF FREE GROUPS

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Abstract. Let α be an automorphism of a free group F of rank n. The Scott conjecture, proved by Bestvina-Handel, states that the fixed subgroup $\operatorname{Fix}\alpha = \{g \in F \mid \alpha(g) = g\}$ has rank at most n. Using $\mathbf{R}$-trees, we show the stronger inequality $\operatorname{rk}\operatorname{Fix}\alpha + \frac{1}{2}a(\alpha) \leq n$, where $a(\alpha)$ is the number of $\operatorname{Fix}\alpha$-orbits of attracting fixed points for the action of α on the boundary of F.

Introduction

Let α be an automorphism of $F = F_n$, the free group of rank n. The Scott conjecture, proved by Bestvina-Handel [BH], states that the fixed subgroup $\operatorname{Fix}\alpha = \{g \in F \mid \alpha(g) = g\}$ has rank at most n.

We shall improve this result by showing:

Theorem 1. *If α is any automorphism of F_n, then* $\operatorname{rk}\operatorname{Fix}\alpha + \dfrac{1}{2}a(\alpha) \leq n.$

Here $a(\alpha)$ is the number of *equivalence classes of attracting fixed points* for the action of α on the boundary of F (defined below). This answers positively a conjecture of Cooper's ([Co, p. 455]).

If $\operatorname{Fix}\alpha$ is trivial, our result specializes to:

Corollary. *An automorphism α of F_n with* $\operatorname{Fix}\alpha = \{1\}$ *fixes at most $4n$ ends of F_n.*

To define $a(\alpha)$ in general, we consider the boundary δF of F (see Part I), the Cantor set of ends of F if $n \geq 2$. If we choose a free basis $g_1, \dots, g_n$, it may be viewed as the set of all infinite reduced words $X = x_1 \cdots x_i \cdots$ in the letters $g_j^{\pm 1}$. The action of α on F extends to a continuous action of α on δF. The boundary of the subgroup $\operatorname{Fix}\alpha$ naturally embeds in δF, and α acts on $\delta(\operatorname{Fix}\alpha)$ as the identity.

We consider fixed points of α in δF. It turns out (Proposition I.1) that such a fixed point X either belongs to $\delta(\operatorname{Fix}\alpha)$, or is attracting, or is repelling (i.e. attracting for α^{-1}). Here *attracting* may be understood in the topological sense ($\lim_{p \to +\infty} \alpha^p(X') = X$ for X' close to X in $F \cup \delta F$), or in the algebraic sense of [CL 1, 1.4]. As in [CL 1], we say that two fixed points $X_1, X_2 \in \delta F$ are *equivalent* if there exists $g \in \operatorname{Fix}\alpha$ such that $X_2 = gX_1$. Note that any point equivalent to an attracting fixed point of α is itself an attracting fixed point of α.

We let $\mathcal{A}(\alpha)$ be the set of equivalence classes of attracting fixed points of α, and we denote $a(\alpha)$ the cardinality of $\mathcal{A}(\alpha)$. Finiteness of $a(\alpha)$ follows from [Co] (or [CL 1]).

Theorem 1 may be illustrated by the following example from [CL 1]. Let $\alpha : F_2 \to F_2$ be given by $\alpha(a) = aba$, $\alpha(b) = ba$. The fixed subgroup has rank 1, it is generated by $aba^{-1}b^{-1}$. One obtains two inequivalent fixed words $X_1 = ababaaba\cdots$ and $X_2 = a^{-1}b^{-1}a^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}\cdots$ by taking the limit as p goes to $+\infty$ of $\alpha^p(a)$ and $\alpha^p(a^{-1})$ respectively. Note that $X_3 = baabaaba\cdots = \lim_{p\to\infty}\alpha^p(b)$ is equivalent to X_1. The automorphism α is induced by a pseudo-Anosov homeomorphism φ of a punctured torus M. The fixed subgroup corresponds to the boundary of M, while the equivalence classes of attracting fixed points correspond to the separatrices of the unstable foliation of φ.

This paper elaborates on the note [GLL], which contains a proof of the Scott conjecture based on **R**-trees. As in [GLL], the two main ingredients in this paper are the existence of a certain α-invariant **R**-tree, and an inequality by Gaboriau-Levitt [GL] about stabilizers of branch points in **R**-trees.

Given an automorphism α of F, there is an action of F on an **R**-tree T, which is α-invariant in the following sense: its length function ℓ satisfies $\ell \circ \alpha = \lambda\ell$ for some number $\lambda \geq 1$ (this action represents a fixed point for the action of α on the boundary of Culler-Vogtmann's outer space). Equivalently, there exists a homothety $H : T \to T$ with stretching factor $\lambda \geq 1$ (i.e. $d(Hx, Hy) = \lambda d(x, y)$ for $x, y \in T$), such that $\alpha(w)H = Hw$ for all $w \in F$ (we identify an element of F and the associated isometry of T).

The existence of such an invariant tree, with a very small action of F, is now well-known (Bestvina-Handel, Skora, Lustig [Lu], Paulin [Pa 3]). In Part II we shall construct T and derive additional properties. In particular we prove the fact (due to Lustig [Lu]) that T may be assumed to have *trivial arc stabilizers.* We also prove that H has a fixed point $Q \in T$ whenever $\operatorname{rk}\operatorname{Fix}\alpha + \frac{1}{2}a(\alpha) > 0$.

In Part IV we analyze the invariant tree T, relating properties of α to geometric properties of T. This may be viewed as the heart of the paper. We distinguish several cases. Here we mention only the most interesting one, when $\lambda > 1$. In this case we use the fact (proved in Part III using an estimate by Bestvina-Feighn-Handel for maps between metric graphs) that every **R**-tree with a very small action of F has the following Bounded BackTracking property (BBT): *Given $Q \in T$, there exists $C > 0$ such that, if v, w are reduced words in F with no cancellation in the product vw, then $d(vQ, [Q, vwQ]) \leq C$, where $[Q, vwQ]$ denotes the segment between Q and vwQ.*

Consider an attracting fixed point of α, represented by a fixed infinite word $X = x_1\cdots x_i\cdots$. We write $X_p = x_1\cdots x_p$. Applying (BBT) to the fixed point Q of H we show that either $X \in \delta(\operatorname{Stab} Q) \subset \delta F$ (where $\operatorname{Stab} Q \subset F$ is the stabilizer of Q), or as p goes to $+\infty$ the sequence X_pQ goes to infinity in T, staying at a bounded distance from an H-invariant infinite ray ρ starting at Q.

This leads to a *injection from the set of equivalence classes of attracting fixed points of α not contained in $\delta(\operatorname{Stab} Q)$, to the set of orbits of the action of* $\operatorname{Stab} Q$

on $\pi_0(T \setminus \{Q\})$.

We now use Theorem III.2 of [GL], which in the special case when arc stabilizers are trivial may be stated as follows:

Theorem 2 [GL]. *Let T be an $\mathbf{R}$-tree with a minimal F_n-action whose arc stabilizers are all trivial. Given $Q_1, \dots, Q_q \in T$ belonging to distinct orbits, we have*

$$\sum_{\ell=1}^{q} (\operatorname{rk} \operatorname{Stab} Q_\ell + \frac{1}{2} v(Q_\ell) - 1) \leq n - 1,$$

where $v(Q_\ell) \geq 1$ is the number of orbits of the action of $\operatorname{Stab} Q_\ell$ *on* $\pi_0(T \setminus \{Q_\ell\})$. *In particular,* $\operatorname{rk} \operatorname{Stab} Q \leq n-1$ *and* $v(Q) \leq 2n$ *for every* $Q \in T$.

Using Theorem 2, we prove Theorem 1 by induction on n in Part V. As in [GLL], we need to consider several automorphisms simultaneously to make the induction work.

Two automorphisms α, β of F represent the same outer automorphism $\Phi \in \operatorname{Out}(F)$ if there exists $m \in F$ such that $\beta = i_m \circ \alpha$, with $i_m(g) = mgm^{-1}$. As in [GLL], we say that α and β are *similar* if m can be written $m = c\alpha(c^{-1})$ for some $c \in F$, or equivalently if $\beta = i_c \circ \alpha \circ (i_c)^{-1}$. Similar automorphisms represent the same automorphism, up to a change of basis in F. In particular the rank of $\operatorname{Fix} \alpha$, and the number $a(\alpha)$, are similarity invariants.

What we actually prove by induction on n in Part V is the following statement:

Theorem 1'. *Let $\alpha_0, \dots, \alpha_k$ be automorphisms of F_n representing the same outer automorphism and belonging to distinct similarity classes. Then*

$$\sum_{i=0}^{k} (\operatorname{rk} \operatorname{Fix} \alpha_i + \frac{1}{2} a(\alpha_i) - 1) \leq n - 1.$$

Equivalently: $\sum_{\alpha \in \mathcal{S}(\Phi)} \max(0, \operatorname{rk} \operatorname{Fix} \alpha + \frac{1}{2} a(\alpha) - 1) \leq n - 1$, *where $\mathcal{S}(\Phi)$ is the set of similarity classes of automorphisms representing a given outer automorphism* $\Phi \in \operatorname{Out}(F)$.

(This theorem is only superficially stronger than Theorem 1. It follows from Theorem 1 by applying it to the automorphism of $F_n * F_k$ equal to α_0 on F_n and sending the i-th generator t_i of F_k to $t_i u_i$ where $\alpha_i = i_{u_i} \circ \alpha_0$.)

As an example, let M be a compact orientable surface of genus 2 with one boundary component. Let φ be a pseudo-Anosov homeomorphism of M whose unstable foliation has two singularities, a 7-prong saddle x in the interior of M and a tripod y on the boundary (see [MaSm, Theorem 2] for a proof of existence). The homeomorphism φ induces two non-similar automorphisms α_x and α_y of $F_4 \simeq \pi_1 M$, depending on whether the basepoint is set at x or y (note that x and y are fixed by φ but belong to different Nielsen classes). One has $\operatorname{Fix} \alpha_y \simeq \mathbf{Z}$ and $a(\alpha_y) = 1$ (corresponding to the infinite separatrix of y). On the other hand $\operatorname{Fix} \alpha_x$ is trivial and $a(\alpha_x) = 7$. In this example, equality holds in Theorem 1'.

In Part VI, written rather informally, we discuss and interpret the *index*

$$\mathrm{ind}\,(\alpha) := \mathrm{rk}\,\mathrm{Fix}\,\alpha + \frac{1}{2}a(\alpha) - 1$$

of an automorphism of F, as well as the index

$$\mathrm{ind}\,(\Phi) := \sum_{\alpha\in\mathcal{S}(\Phi)} \max(0, \mathrm{rk}\,\mathrm{Fix}\,\alpha + \frac{1}{2}a(\alpha) - 1)$$

of $\Phi \in \mathrm{Out}\,(F)$ that appears in Theorem 1'. It turns out that $\mathrm{ind}\,(\Phi)$ is strongly related to important structural properties of Φ. In particular we will consider automorphisms of maximal index $n-1$.

I. The action of α on δF

Let F be a free group of rank n and $\overline{F}$ its compactification (end completion as in [Co], or compactification as a hyperbolic group, see e.g. [Sho 1, ch. 4]). The boundary $\delta F = \overline{F} \setminus F$ is a compact space, homeomorphic to a Cantor set if $n \geq 2$. The natural actions of F and $\mathrm{Aut}\,(F)$ on F extend continuously to $\overline{F}$. Any finitely generated subgroup $F' \subset F$ is quasiconvex [Sho 2], hence inclusion induces a natural embedding $\delta F' \to \delta F$ (see [CDP, p. 115]).

Once we fix a free generating system $g_1, \ldots, g_n$ for F, we view F as the set of reduced words in the letters $g_j^{\pm 1}$, and δF as the set of infinite reduced words $X = x_1 \cdots x_i \cdots$. We denote $X_i = x_1 \cdots x_i$.

Given two reduced words X, X', finite or infinite, we let $X \wedge X'$ be their common initial segment (the empty word if $x_1 \neq x_1'$) and we denote $c_{X'}(X)$ the length of $X \wedge X'$. A sequence of reduced words $X_p \in \overline{F}$ converges to $X \in \delta F$ if and only if $\lim_{p\to+\infty} c_X(X_p) = +\infty$.

Now let α be an automorphism of F. Recall ([Co], see Part III) that there is a *cancellation bound* for α, that is a number B such that

$$|\alpha(vw)| \geq |\alpha(v)| + |\alpha(w)| - 2B$$

whenever v, w, vw are finite reduced words with $|vw| = |v| + |w|$ ($|\;\;|$ denotes word length).

Let $X = x_1 \cdots x_i \cdots$ be fixed by α. We write $\alpha(X_i) = X_{k(i)} Z_i$ with $k(i) = c_X(\alpha(X_i))$. Since X is fixed by α, the sequence $k(i)$ goes to $+\infty$ as i increases. Bounded cancellation implies $|Z_i| \leq B$. Also note that $|k(i+1) - k(i)|$ is bounded by a constant depending only on α (namely $\max |\alpha(g_i)|$).

As in [CL 1], we say that X is an *attracting fixed word* of α if

$$\lim_{i\to+\infty} (k(i) - i) = +\infty.$$

Note that there exists i_0 such that for all $i \geq i_0$ one has $k(i) \geq i + B + 1$ and hence

$$c_X(X') \geq i_0 \implies c_X(\alpha(X')) > c_X(X'). \tag{$*$}$$

We say that X is an *attracting fixed point* of α if there exists a neighborhood U of X in $\overline{F}$ such that

$$X' \in U \implies \lim_{p \to +\infty} \alpha^p(X') = X.$$

The following proposition shows that the two notions of attraction are the same. We say that X is *repelling* for α if it is attracting for α^{-1}.

Proposition I.1. *Let $X \in \delta F$ be an infinite fixed word of α. The following are equivalent:*

(1) *X is an attracting or repelling fixed word.*
(2) *X is an attracting or repelling fixed point for the action of α on $\overline{F}$.*
(3) *$X \notin \delta(\mathrm{Fix}\, \alpha)$.*

(of course the subcases attracting or repelling in (1) and (2) coincide).

Proof. (1) $\implies$ (2): Suppose X is an attracting fixed word. Choose i_0 as in $(*)$. If $c_X(X') \geq i_0$, we get $c_X(\alpha(X')) > c_X(X')$, hence $\lim_{p\to+\infty} c_X(\alpha^p(X')) = +\infty$ and $\lim_{p\to+\infty} \alpha^p(X') = X$.

(2) $\implies$ (3): this is clear since α acts as the identity on $\overline{\mathrm{Fix}\, \alpha}$ and no point $X \in \delta(\mathrm{Fix}\, \alpha)$ can be isolated in $\overline{\mathrm{Fix}\, \alpha}$.

To prove (3) $\implies$ (1), let X be any infinite fixed word. As in [Co], we consider the words $w_i = X_i^{-1}\alpha(X_i)$. We note that

$$w_i = w_p \implies X_p X_i^{-1} \in \mathrm{Fix}\, \alpha.$$

If the sequence of words w_i takes the same value infinitely often we get $X \in \overline{\mathrm{Fix}\, \alpha}$ since for fixed i we have $X = \lim_{p\to+\infty} X_p X_i^{-1}$.

Otherwise $|w_i|$ goes to infinity. Recall that $\alpha(X_i) = X_{k(i)} Z_i$ with $|Z_i| \leq B$ and $|k(i+1) - k(i)|$ bounded. Since $|w_i|$ is comparable to $|k(i) - i|$ (the difference is bounded by B) we see that $k(i) - i$ goes to either $+\infty$ or $-\infty$ as i increases. If the limit is $+\infty$, then X is an attracting fixed word. If it is $-\infty$, we repeat the argument for α^{-1}. Define $\overline{k}(i)$ and $\overline{Z}_i$ analogously to $k(i)$ and Z_i. Writing $X_i = \alpha^{-1}(X_{k(i)})\alpha^{-1}(Z_i)$ we see that $\overline{k}(k(i)) - i$ stays bounded. Then $\overline{k}(k(i)) - k(i) = (\overline{k}(k(i)) - i) - (k(i) - i)$ goes to $+\infty$, and so does $\overline{k}(i) - i$ (recall that $k(i)$ goes to $+\infty$). This means that X is a repelling fixed word. ◻

Note that the corollary stated in the introduction is an immediate consequence of Theorem 1: if $\mathrm{Fix}\, \alpha = \{1\}$, then there are at most $2n$ attracting fixed points and $2n$ repelling ones.

II. The invariant tree

A. Statement of the result.

For the convenience of the reader we state some basic definitions about **R**-trees. For more detailed information see e.g. [MoSh], [CM] or the survey articles [Sha 1], [Sha 2].

An $\mathbf{R}$-tree is a non-empty metric space in which any two distinct points x, y are joined by a unique arc $[x, y]$, and in which every such arc is isometric to a closed interval of $\mathbf{R}$ (recall that a (nondegenerate) *arc* is a space homeomorphic to $[0, 1]$).

Alternatively, an $\mathbf{R}$-tree is a path-connected metric space (X, d) which satisfies the 0-hyperbolicity condition

$$d(x, z) + d(y, w) \leq \max\{d(x, y) + d(z, w), d(x, w) + d(z, y)\},$$

see [GH].

We consider $\mathbf{R}$-trees equipped with a left action of a group G by isometries. The *stabilizer* $\mathrm{Stab}\,(x)$ of a point $x \in T$ is the subgroup of G consisting of elements fixing x. Similarly, the stabilizer of an arc $[x, y]$ is the subgroup consisting of elements fixing $[x, y]$ pointwise.

The G-action on T (or sometimes just T) is called

– *trivial* $\iff \exists\, x \in T$ with $\mathrm{Stab}\,(x) = G$,

– *simplicial* $\iff T$ arises from a simplicial tree by setting each edge length equal to some positive value (always 1 in this paper),

– *minimal* $\iff$ there is no proper G-invariant subtree,

– *free* $\iff$ the stabilizer of every point is trivial,

– *with trivial arc stabilizers* $\iff$ the stabilizer of every arc $[x, y]$ is trivial,

– *small* $\iff$ no arc stabilizer contains a free group of rank 2.

Let α be an automorphism of F. Recall that the *index* of α is the quantity

$$\mathrm{ind}\,(\alpha) = \mathrm{rk}\,\mathrm{Fix}\,\alpha + \frac{1}{2}a(\alpha) - 1$$

that appears in Theorem 1'. It is an integer, or a half-integer, with $\mathrm{ind}\,(\alpha) \geq -1$. Our interest will be in automorphisms with positive index.

We now state the main result of this section which is proved in subsections B to E. A *homothety* of a metric space (X, d) is a map $H : X \to X$ satisfying $d(Hx, Hy) = \lambda d(x, y)$ for some fixed $\lambda > 0$ called the *stretching factor*.

Theorem II.1. *For every automorphism α of F there exists an $\mathbf{R}$-tree T such that:*

(1) *F acts on T non-trivially, minimally, with trivial arc stabilizers.*

(2) *There exist $\lambda \geq 1$ and a homothety H: $T \to T$ with stretching factor λ such that*

$$\alpha(w)H = Hw\colon\ T \to T$$

for all $w \in F$. If $\lambda = 1$, then T is simplicial.

(3) *If $\mathrm{ind}\,(\alpha) > 0$, then H has at least one fixed point $Q \in T$. More generally, if $\beta = i_m \circ \alpha$ satisfies $\mathrm{ind}\,(\beta) > 0$, then $H_\beta = mH$ has a fixed point (recall that $i_m(g) = mgm^{-1}$).*

Remark II.2. It is easy to see that, given α and T, the equation $\alpha(w)H = Hw$ uniquely determines H (and λ). The second part of Assertion 3 will be used when we study several automorphisms simultaneously (see Part V). Note that $\beta(w)H_\beta = H_\beta w$ for all $w \in F$, since

$$\beta(w)H_\beta = m\alpha(w)m^{-1}mH = m\alpha(w)H = mHw = H_\beta w.$$

Thus we may use the same tree T to study all automorphisms representing a given outer automorphism Φ. Indeed, Assertion 2 is equivalent to the equation $\ell \circ \Phi = \lambda\ell$, where ℓ is the length function of the action of F on T.

B. A criterion for a fixed point.

Let τ be a finite connected graph, with $\pi_1\tau$ free of rank $n \geq 2$. We fix a universal covering $\tilde{\tau} \to \tau$, and an isomorphism from the group of covering transformations to F (this is a way of identifying $\pi_1\tau$ with F without having to choose a basepoint; of course this identification is only defined up to a conjugacy in F).

Let $f : \tau \to \tau$ be a homotopy equivalence. It induces a well-defined outer automorphism Φ of F, and the formula

$$\alpha(w)\tilde{f} = \tilde{f}w \tag{**}$$

defines a 1-to-1 correspondence between the set of lifts $\tilde{f} : \tilde{\tau} \to \tilde{\tau}$ and the set of automorphisms α of F representing Φ. We fix an automorphism α and the corresponding $\tilde{f}$.

The following result generalizes Lemma 2.1 of [BH].

Proposition II.3. *Let $\tilde{f} : \tilde{\tau} \to \tilde{\tau}$ be associated with the automorphism α of F by (**). If $\mathrm{ind}(\alpha) > 0$, then $\tilde{f}$ has a fixed point.*

Proof. As in [BH, p. 19], we shall use the following geometric fixpoint criterion for $\tilde{f}$: *Let $x, y \in \tilde{\tau}$ be distinct points with the property that $\tilde{f}(x)$ is distinct from x and is not contained in the same connected component of $\tilde{\tau} - \{x\}$ as y, and conversely. Then $\tilde{f}$ has a fixed point on $[x, y]$.*

Make τ into a metric graph by declaring that every edge has length 1, and equip $\tilde{\tau}$ with the lifted metric. For any point $P \in \tilde{\tau}$, the map $j : w \mapsto wP$ gives a quasi-isometric embedding $F \to \tilde{\tau}$. This induces a homeomorphism between δF and the space $\delta\tilde{\tau}$ of ends of $\tilde{\tau}$, which is independent of the choice of P. Furthermore, the distance in $\tilde{\tau}$ between $\tilde{f}(j(w)) = \alpha(w)\tilde{f}(P)$ and $j(\alpha(w)) = \alpha(w)P$ is bounded by the distance between P and $\tilde{f}(P)$, independently of $w \in F$. It follows that the extension of $\tilde{f}$ to $\delta\tilde{\tau}$ agrees with the extension of α to δF, and an attracting fixed point of α in δF defines an attracting fixed point for $\tilde{f}$ in $\delta\tilde{\tau}$.

In the situation of Proposition II.3, first assume that α has at least two distinct (possibly equivalent) attracting fixed points X_1, X_2 in δF. Then any two points $x, y \in \tilde{\tau}$ which are sufficiently close to the corresponding fixed points on $\delta\tilde{\tau}$ satisfy the hypothesis of the above criterion and $\tilde{f}$ has a fixed point.

Such points X_1, X_2 exist if $a(\alpha) \geq 2$. They also exist if there exist both an attracting fixed point X_1 in δF and a nontrivial α-fixed $u \in F$, as we can then take $X_2 = uX_1$. Since we assume $\mathrm{ind}\,(\alpha) > 0$, the only remaining case is when $\mathrm{rk}\,(\mathrm{Fix}\,\alpha) \geq 2$. But then Lemma 2.1 of [BH] applies. $\square$

C. Partial train track maps.

There have been various independent attempts (see [BH], [Lo], [Lu]) to carry Thurston's concept of train tracks for surface homeomorphisms over to automorphisms of free groups. Our notion of partial train track maps in Definition II.4 is close to but weaker than the "relative train track maps" in [BH, ch.5]. Below we crucially use the existence of such a relative train track map for every outer automorphism of F, as shown in [BH, Theorem 5.12]. Alternatively one can use §5 of [LuOe].

Given $f : \tau \to \tau$ as above, we say that a locally injective path $c : [0,1] \to \tau$ from a point p to a point q is *f-backtracking* if $f(p) = f(q)$ and the loop $f \circ c$ is null-homotopic in τ. Equivalently, any lift of c to the universal covering $\tilde{\tau}$ is a segment $[\tilde{p}, \tilde{q}]$ with $\tilde{f}(\tilde{p}) = \tilde{f}(\tilde{q})$.

Definition II.4. A continuous map $f\colon \tau \to \tau$ is called a *partial train track map* relative to $\tau' \subset \tau$ if the following conditions are satisfied:

(1) τ is a finite connected graph with no vertices of valence 1.
(2) f is a homotopy equivalence.
(3) f preserves the set τ^0 of vertices of τ : $f(\tau^0) \subset \tau^0$.
(4) τ' is a (not necessarily connected) subgraph of τ which satisfies:
   (a) $\tau' \neq \tau$
   (b) $f(\tau') \subset \tau'$
   (c) $\tau' \cup \tau^0$ is maximal with respect to (a) and (b).
(5) For any $k \geq 1$, all f^k-backtracking paths which are contained in $\tau - (\tau' \cup \tau^0)$ are mapped by f^k to τ'.

Lemma II.5. *Let $f : \tau \to \tau$ be a relative train track map in the sense of* [BH]*, with f-invariant maximal filtration $\tau_0 \subset \ldots \subset \tau_m$. If τ has no vertices of valence 1, then f is a partial train track map relative to $\tau' = \tau_{m-1}$.*

Proof. Condition (1) of Definition II.4 holds by assumption. Conditions (2) and (3) hold, as f is a topological representative, see [BH, pp. 4-5]. Condition (4) follows from our definition of τ'. If the m-th stratum of τ is exponentially growing, then Condition (5) follows from Lemma 5.8 of [BH]. If this stratum is not exponentially growing, then the transition matrix $M(f)$ is a permutation matrix and hence, for every $k \geq 1$, the f^k-image of any edge $e \in \tau - \tau'$ contains only one edge of $\tau - \tau'$. This implies Condition (5). $\square$

Proposition II.6. *Given an outer automorphism Φ of F, there exists a partial train track map $f\colon \tau \to \tau$ which induces Φ.*

Proof. By Theorem 5.12 of [BH], there exists a relative train track map $f\colon \tau \to \tau$ for Φ. By Lemma 5.2 of [BH] we can modify

f and τ so that τ doesn't contain any vertex of valence one. Now apply Lemma II.5. $\square$

Now let $f : \tau \to \tau$ be a partial train track map. Let $e_1, \ldots, e_p$ be the (unoriented) edges of $\tau - \tau'$ (or more precisely of the closure of $\tau - \tau'$). Conditions (3) and (5) of Definition II.4 imply that for any $k \geq 1$ the f^k-image of any edge $e_i \subset \tau - \tau'$ crosses properly over the edges of $\tau - \tau'$. Hence f determines a $p \times p$ transition matrix $M(f)$ with non-negative integer entries, where the i-th row records the number of times the f-image of e_i crosses the edges $e_1, \ldots, e_p$, disregarding the orientation.

¿From condition (4c) it follows that $M(f)$ is either zero, or irreducible with Perron-Frobenius eigenvalue $\lambda \geq 1$ (for definitions and the Perron-Frobenius Theorem, see [Ga] or [DV]). As Conditions (1) and (2) exclude the possibility of $M(f)$ being zero, we have $\lambda \geq 1$.

Let $v = (v_i)$ be a strictly positive eigenvector v associated to the eigenvalue λ (it exists and is unique up to scaling). We define the *PF-length* of an edge e_i of $\tau - \tau'$ as $L(e_i) = v_i$ (and we set $L(e) = 0$ for all edges $e \subset \tau'$). Note that the image by f of any edge e is a path of PF-length $\lambda L(e)$.

Let $\bar{\tau}$ be the tree obtained from the universal covering $\tilde{\tau}$ by collapsing every connected component of the preimage of τ' to a point. The action of F on $\tilde{\tau}$ induces an action on $\bar{\tau}$ which is not necessarily free but which has trivial arc stabilizers. Make $\bar{\tau}$ into a metric tree by lifting the PF-length L to the edges of $\bar{\tau}$. We get an F-invariant distance function d: if $x, y \in \bar{\tau}$, then $d(x, y)$ is the PF-length of the segment $[x, y]$.

Remark. If the automorphism α is irreducible (in the sense of [BH]), then there exists an absolute train track representative for α, i.e. a train track representative $f\colon \tau \to \tau$ with only one stratum. In this important special case f is a partial train track map relative to $\tau' = \emptyset$, and $\bar{\tau} = \tilde{\tau}$.

Given $\alpha \in \mathrm{Aut}\,(F)$, let f be a partial train track map inducing the outer automorphism determined by α. Let $\tilde{f} : \tilde{\tau} \to \tilde{\tau}$ be the lift of f associated to α by $(**)$ (see the beginning of II.B).

Since $f(\tau') \subset \tau'$, the map $\tilde{f}$ induces $\bar{f} : \bar{\tau} \to \bar{\tau}$. By Condition (5) of Definition II.4, the image by $\bar{f}$ of an edge e is a segment of length $\lambda L(e)$. We may then redefine $\bar{f}$ on the interior of e, so that $\bar{f}|e$ expands uniformly by λ. We do this equivariantly, thus making sure that the relation $\alpha(w)\bar{f} = \bar{f}w$ still holds. The new map (also denoted $\bar{f}$)

satisfies $d(\bar{f}(x), \bar{f}(y)) \leq \lambda d(x, y)$, and equality holds if x, y belong to the same edge.

D. The case $\lambda = 1$.

If $\lambda = 1$, then $M(f)$ is a permutation matrix, and we can choose the eigenvector v to have all entries $v_i = 1$. Since f permutes the edges of $\tau - \tau'$, the map $\bar{f} : \bar{\tau} \to \bar{\tau}$ is a homeomorphism and therefore a global isometry (no folding occurs).

We claim that $T = \bar{\tau}$ satisfies the conclusions of Theorem II.1. Conditions (1) and (2) obviously hold, with $H = \bar{f}$. If $\mathrm{ind}\,(\alpha) > 0$, the map $\tilde{f}$ has a fixed point

$q \in \tilde{\tau}$ by Proposition II.3. The image Q of q in T is a fixed point of H. To get a fixed point for $H_\beta = mH$, we simply apply this argument to the map $m\tilde{f}$, noting that it satisfies $(**)$ with respect to β.

E. The case $\lambda > 1$.

The rest of Part II will be devoted to the proof of Theorem II.1 when $\lambda > 1$. In this case we first have to replace the distance d by a pseudo-distance d_∞ for which $\bar{f}$ acts as a homothety (recall that $\bar{f}$ only satisfies the inequality $d(\bar{f}(x), \bar{f}(y)) \leq \lambda d(x, y)$). We present this construction with a fairly high degree of generality.

A *pseudo-distance* on a set Z is a function $\theta \colon Z \times Z \to \mathbf{R}^+$ which satisfies the axioms of a distance, except that there may be distinct points of Z which have distance 0. The function θ induces a genuine distance on the set $Z(\theta)$ obtained from Z by identifying x, y whenever $\theta(x, y) = 0$. We will denote $\psi : Z \to Z(\theta)$ the canonical quotient map.

Let Z be a set equipped with a distance d (a pseudo-distance would in fact be enough). Let $\lambda > 0$, and let $h\colon Z \to Z$ be any map which satisfies a Lipschitz condition $d(h(x), h(y)) \leq \lambda d(x, y)$ for all $x, y \in Z$.

Let d_∞ be the limit of the non-increasing sequence of pseudo-distances

$$d_k(x, y) = \frac{d(h^k(x), h^k(y))}{\lambda^k}.$$

It obviously satisfies $d_\infty(x, y) \leq d(x, y)$ and $d_\infty(h(x), h(y)) = \lambda\, d_\infty(x, y)$ for all $x, y \in Z$.

The map H induced by h on the associated metric space $Z(d_\infty)$ is thus a homothety with stretching factor λ.

Now assume that a group G acts on Z isometrically with respect to d, and that there exists a map $\Delta \colon G \to G$ (usually an automorphism) which "commutes" with h in the sense that $h(g(x)) = \Delta(g)(h(x))$ for all $g \in G,\ x \in Z$.

Then G preserves the pseudo-distance d_∞ as well. In other words there is an induced isometric action of G on $Z(d_\infty)$, for which the natural surjection $\psi : Z \to Z(d_\infty)$ is G-equivariant. Furthermore this action commutes with H in the sense that $\Delta(g)H = Hg$ for all $g \in G$. We shall also denote d_∞ the induced distance on $Z(d_\infty)$.

In particular, suppose that Z is a path-connected metric space and d satisfies a δ-hyperbolicity inequality

$$d(x, z) + d(y, w) \leq \max\{d(x, y) + d(z, w), d(x, w) + d(z, y)\} + 2\delta.$$

If $\lambda > 1$, then $T = Z(d_\infty)$ is 0-hyperbolic. It is an $\mathbf{R}$-tree, equipped with an isometric action of G and a homothety $H\colon T \to T$ with stretching factor λ (T is path-connected because $\psi : Z \to T$ is continuous as $d_\infty \leq d$).

Remark. In general, it is quite possible that all points of Z have d_∞-distance 0 from each other, so that T consists of a single point only. In particular this is true if λ is strictly larger than the infimum of all numbers λ such that $d(h(x), h(y)) \leq \lambda d(x, y)$.

Applying the above construction to the metric space $Z = (\bar{\tau}, d)$ and the map $h = \bar{f}$, we obtain an $\mathbf{R}$-tree $T = \bar{\tau}(d_\infty)$ equipped with an isometric action of F and a homothety H satisfying Condition (2) of Theorem II.1. Condition (3) holds for the same reasons as in the case $\lambda = 1$ (of course any homothety with $\lambda > 1$ has a unique fixed point in the metric completion of T, but we insist that the fixed point be in T).

There remains to check Condition (1).

Lemma II.7. *The action of F on T is non-trivial and minimal.*

Proof. For each edge e_i of $\tau - \tau'$, choose a lift $\bar{e}_i = [a, b]$ to $\bar{\tau}$. By Condition (5) of Definition II.4, all the backtracking in $f^k(e_i)$ takes place in τ'. It follows that $d_k(a, b) = \frac{d(h^k(a), h^k(b))}{\lambda^k}$ equals $d(a, b)$ for all k, and the image of $\bar{e}_i$ in T is a segment f_i of length $L(e_i)$.

Note that the F-orbit of any $x \in T$ meets some f_i. In particular, if $T' \subset T$ is an F-invariant subtree, the Hausdorff distance $D(T', T) = \sup_{x \in T} d_\infty(x, T')$ is finite: it is bounded by the diameter of any subtree containing all of the f_i's. If furthermore T' is invariant under H, then $D(T', T)$ has to be 0 because $\lambda D(T', T) = D(T', T)$.

Now we can prove that the action is not trivial. Assume it is. The fixed subtree $T_0 = \{x \in T \mid gx = x \ \forall g \in F\}$ is compact (it is contained in the union of the f_i's because F acts as the identity on $T_0 \cap f_i$ and every orbit meets some f_i). Since it is H-invariant, it has to be a point. This forces T to be a point because $D(T_0, T) = 0$, a contradiction since $f_i \subset T$.

As the action is nontrivial, there is a unique minimal F-invariant subtree T_1 (see [CM]), which is H-invariant. We want to show $T_1 = T$. We know that $D(T_1, T) = 0$, in other words $T - T_1$ may contain only endpoints of T (points x with $T - \{x\}$ connected). It follows that the interior of each segment f_i is contained in T_1. Irreducibility of $M(f)$ and $\lambda > 1$ imply that $\bar{e}_i$ occurs in the interior of $\bar{f}^k(\bar{e})$ for some edge e of $\bar{\tau}$ and some $k \geq 1$. This proves that f_i is contained in $H^k T_1 = T_1$, and therefore $T_1 = T$. ◻

Lemma II.8. *The F-action on T has trivial arc stabilizers.*

Proof. The action of F on $T = \bar{\tau}(d_\infty)$ is the limit of the sequence of actions $(F, \bar{\tau}(d_k))$. Each of these actions is small because it has trivial arc stabilizers. Therefore the action on T is small, as a limit of small actions (see [CM], [Pa 1]).

To prove that T has trivial arc stabilizers, assume that $c \subset T$ is a nondegenerate arc fixed by some nontrivial $w \in F$. Let $p \geq 1$ be the largest integer such that w is a p-th power (in fact $p = 1$ because the action is very small in the sense of [CL 2]). Recall that the F-orbit of any $x \in T$ meets some f_i. Since the length of $H^k(c)$ grows arbitrary large with k, we can find, for sufficiently large k, disjoint non-degenerate subarcs $c_0, \ldots, c_p$ of $H^k(c)$ such that $c_i = v_i c_0$ for some $v_i \in F$ $(i = 1, \ldots, p)$.

The element $w' = \alpha^k(w)$ fixes $H^k(c)$. Since $w', v_1, \ldots, v_p$ all have different actions on c_0, there exists i such that v_i and w' do not generate a cyclic subgroup of F. Then w' and $v_i^{-1}w'v_i$ generate a free subgroup of rank 2 which fixes c_i pointwise, contrary to the fact that the action is small. $\square$

Remark. The last argument may be extended to small actions of hyperbolic groups such that there exists a homothety with $\lambda > 1$ which commutes with an automorphism α (in the sense of Assertion (2) of Theorem II.1). This applies in particular to the actions constructed by Paulin [Pa 3].

III. Bounded backtracking

Let $f : T_1 \to T_2$ be a continuous map between **R**-trees. As in Part II, a path c: $[0,1] \to T_1$ is called an *f-backtracking path* if c is injective and $f(c(0)) = f(c(1))$. We say that the map f has *bounded backtracking* if the image $(f \circ c)([0,1])$ of any f-backtracking path c has diameter bounded independently of c.

Proposition III.1. *Let T be an **R**-tree with a minimal action of F. The following properties are equivalent:*

(BBT1) *Given $Q \in T$, there exists $C > 0$ such that, if v, w, $vw \in F$ have word length satisfying $|vw| = |v| + |w|$, then $d(vQ, [Q, vwQ]) \leq C$.*

(BBT2) *Given $Q \in T$, there exists $C > 0$ such that $d((v_1 \wedge v_2)Q, [v_1Q, v_2Q]) \leq C$ if v_1, v_2 are reduced (recall that $\wedge$ denotes the common initial subword).*

(BBT3) *Every F-equivariant map f: $\tilde{\Gamma} \to T$, where $\tilde{\Gamma}$ is a simplicial **R**-tree with a free minimal F-action, has bounded backtracking.*

(BBT4) *For every F-equivariant map f: $\tilde{\Gamma} \to T$, where $\tilde{\Gamma}$ is a simplicial **R**-tree with a free minimal F-action, there exists a constant $\delta \geq 0$ such that for all $x, y \in \tilde{\Gamma}$ one has $f([x,y]) \subset \mathcal{N}_\delta([f(x), f(y)])$ (the δ-neighborhood of $[f(x), f(y)]$).*

Notice that (BBT4) admits direct generalizations to actions of arbitrary groups G on **R**-trees, through replacing $\tilde{\Gamma}$

by a Cayley graph of G. Notice also that in this view BBT appears to be a kind of "one-sided quasi-isometry".

We shall not prove Proposition III.1, as we only need the easy implication (BBT1) $\Longrightarrow$ (BBT2) (which is proved by setting v and w equal to the reduced words representing $v_1^{-1}(v_1 \wedge v_2)$ and $(v_1 \wedge v_2)^{-1}v_2$ respectively). We will need, however, the fact that the tree T provided by Theorem II.1 satisfies (BBT1) when $\lambda > 1$. This may be derived directly from the construction of T in II.E, but we give a general argument.

Note that every free minimal action of F on a simplicial **R**-tree T satisfies (BBT3), as every map f as in (BBT3) is a quasi-isometry. Also note that Cooper's cancellation bound ([Co], see Part I), is a special case of this. However, we need the following stronger version, due to Bestvina-Feighn-Handel (see [DV, Lemma II.2.4]).

Lemma III.2. *If a minimal action of F on an **R**-tree T is free and simplicial, then it satisfies (BBT1) with $C = \sum_{i=1}^{n} d(Q, g_iQ)$.*

Proof. Let Z be the quotient graph of T by the action of F. Let Y be a wedge of n circles, the i-th circle having length $d(Q, g_i Q)$. Let $\gamma : Y \to Z$ be the natural map sending the i-th circle to the projection of the segment $[Q, g_i Q]$. Then apply Lemma II.2.4 of [DV]. $\square$

The action given by Theorem II.1 has trivial arc stabilizers, hence it is very small in the sense of Cohen-Lustig [CL 2].

Corollary III.3. *Every very small action of F on an $\mathbf{R}$-tree T has BBT.*

We do not know whether this can be extended to more general actions, such as small actions of hyperbolic groups.

Proof of Corollary III.3. Every very small F-action on T is a limit of free simplicial F-actions on $\mathbf{R}$-trees T_i by [BF]. This limit may be understood as convergence of length functions, or equivalently [Pa 2] as convergence in the equivariant Gromov-Hausdorff topology: given $Q \in T$, a finite subset $F_0 \subset F$, and $\varepsilon > 0$, then for i large there exists $Q_i \in T_i$ such that the distance between Q_i and gQ_i in T_i is ε-close to $d(Q, gQ)$ for $g \in F_0$. It follows easily that Lemma III.2 is also valid for very small actions. $\square$

Now let T be an $\mathbf{R}$-tree with a minimal action of F satisfying (BBT1). A *ray* $\rho \subset T$, with origin a point $Q \in T$, is the image of an isometric map $(0, \infty) \to T$, with closure $\bar{\rho} = \rho \cup \{Q\}$. Two rays (with different origins) are *equivalent* if their intersection has infinite length. We denote δT the set of equivalence classes. The action of F on T induces a natural action of F on δT.

Suppose ρ is a ray, R, S are points of T, and $v_n \in F$ is a sequence such that the length of $[R, v_n S] \cap \rho$ goes to infinity as $n \to \infty$. Then clearly the length of $[R', v_n S'] \cap \rho'$ goes to infinity for any $R', S' \in T$ and ρ' equivalent to ρ.

Lemma III.4. *Suppose the length of $[R, v_n S] \cap \rho$ goes to infinity as $n \to \infty$. Then v_n converges to some $X \in \delta F$ depending only on ρ. Furthermore the length of $[R, w_n S] \cap \rho$ goes to infinity for every sequence $w_n \to X$.*

Proof. We may assume $R = S = Q$, the origin of ρ. First note that $|v_n|$ goes to infinity with n. Thus some subsequence converges to a point of δF. To prove uniqueness of the limit, we suppose v_n, v'_n are two sequences as in the lemma, converging to $X, X' \in \delta F$ respectively, and we show $X = X'$. Indeed, with the notations of Part I, the

distance between Q and $(v_n \wedge v'_n)Q$ goes to infinity as $n \to \infty$ by (BBT2). This implies that $|v_n \wedge v'_n|$ goes to infinity, i.e. $X = X'$.

Denoting $X_{t(p)} = v_p \wedge X = \lim_{i \to \infty}(v_p \wedge v_i)$, we note that the length of $[Q, X_{t(p)}Q] \cap \rho$ goes to infinity with p by (BBT2). Property (BBT1) then implies that the length of $[Q, w_n Q] \cap \rho$ goes to infinity for any $w_n \to X$. $\square$

Given $R, S \in T$ and a ray ρ, minimality of the action of F on T provides a sequence v_n as in Lemma III.4. We define $j(\rho) \in \delta F$ as the limit X of v_n. Of course $j(\rho)$ depends only on the equivalence class of ρ.

Lemma III.5. *This defines an F-equivariant injection $j : \delta T \to \delta F$. If H is a homothety of T as in Theorem II.1, then $\alpha(j(\rho)) = j(H(\rho))$ for any ray ρ. The image of j is disjoint from every $\delta(\operatorname{Stab} Q)$.*

Proof. Equivariance and injectivity are clear. For the second assertion, simply note that

$$H([R, v_n S] \cap \rho) = [R', \alpha(v_n) S'] \cap H(\rho)$$

with $R' = H(R)$ and $S' = H(S)$. Since the length of $[Q, v_n Q] \cap \rho$ is 0 if $v_n \in \operatorname{Stab} Q$, we have $j(\rho) \notin \delta(\operatorname{Stab} Q)$. ☐

IV. Analyzing the tree

Let α be an automorphism of F with positive index

$$\operatorname{ind}(\alpha) = \operatorname{rk} \operatorname{Fix} \alpha + \frac{1}{2} a(\alpha) - 1 > 0,$$

and let T, H be given by Theorem II.1. Recall that H has at least one fixed point.

Lemma IV.1.

(1) *If Q is fixed by H, then* $\operatorname{Stab} Q$ *is α-invariant. We denote $\alpha^Q = \alpha|_{\operatorname{Stab} Q}$.*
(2) *If Q is the only fixed point of H, then* $\operatorname{Stab} Q$ *contains* $\operatorname{Fix} \alpha$ *(i.e.* $\operatorname{Fix} \alpha = \operatorname{Fix} \alpha^Q$*).*

Proof. (1) From $wQ = Q$ it follows $\alpha(w)Q = \alpha(w)HQ = HwQ = Q$.
(2) From $\alpha(w) = w$ it follows $HwQ = \alpha(w)HQ = wQ$, so that $wQ = Q$ if H has only one fixed point. ☐

Our main goal will be to relate $\operatorname{ind}(\alpha)$ to data coming from T and automorphisms of groups of smaller rank (such as $\operatorname{Stab} Q$, which has rank $< n$ by Theorem 2). In particular, we will map subsets of $\mathcal{A}(\alpha)$ injectively into $\mathcal{A}(\alpha^Q)$ thanks to the following lemma (recall that $\mathcal{A}(\alpha)$ is the set of equivalence classes of attracting fixed points of α).

Lemma IV.2. *Let $Q \in T$ be fixed by H. Let $\mathcal{A}_{\operatorname{Stab} Q}(\alpha) \subset \mathcal{A}(\alpha)$ be the set of equivalence classes of attracting fixed points having nonempty intersection with $\delta(\operatorname{Stab} Q)$.*

(1) *There is an injection $\theta_Q : \mathcal{A}_{\operatorname{Stab} Q}(\alpha) \hookrightarrow \mathcal{A}(\alpha^Q)$.*
(2) *If Q is the only fixed point of H, then θ_Q is a bijection.*

Proof. (1) Given a class in $\mathcal{A}_{\operatorname{Stab} Q}(\alpha)$, represent it by some $X \in \delta(\operatorname{Stab} Q)$. Proposition I.1 implies that X is an attracting fixed point of α^Q, and we map the class of X in $\mathcal{A}(\alpha)$ to the class of X in $\mathcal{A}(\alpha^Q)$. Since fixed points $X, X' \in \delta(\operatorname{Stab} Q)$ that are inequivalent for α are inequivalent for α^Q, we get the required (non-canonical) injection.

(2) If Q is the only fixed point of H, then $\operatorname{Fix} \alpha = \operatorname{Fix} \alpha^Q \subset \operatorname{Stab} Q$. This implies that any X' equivalent to $X \in \delta(\operatorname{Stab} Q)$ also belongs to $\delta(\operatorname{Stab} Q)$ and is

equivalent to X as a fixed point of α^Q. It follows that θ_Q is a bijection: the class of an attracting $X \in \delta(\operatorname{Stab} Q)$ is the image of the class of X in $\mathcal{A}_{\operatorname{Stab} Q}(\alpha)$ (note that X is attracting for α by Proposition I.1). $\square$

Now we distinguish several cases.

Case A: $\lambda > 1$.

In this case the map H has a unique fixed point Q. Note that T has BBT by Corollary III.3. We will use properties (BBT1) and (BBT2) relative to Q.

If $a, b \in T$, we say that the segments $[Q, a]$ and $[Q, b]$ *overlap* if their intersection is a (non-degenerate) arc. This is equivalent to saying that a and b belong to the same component of $T \setminus \{Q\}$.

If H sends a component A of $T \setminus \{Q\}$ to itself, then A contains a unique H-*invariant* ray ρ at Q , i.e. $\rho(\lambda x) = H\rho(x)$ for $x > 0$ (see [Li]). Such rays will be called *eigenrays of* H. Notice that $\overline{\rho}$ contains $[Q, a] \cap [Q, Ha]$ for every $a \in A$. (As noted in [Li], the restriction of H to A is topologically conjugate to a hyperbolic isometry of an $\mathbf{R}$-tree, and ρ corresponds to the translation axis).

Proposition IV.3. *If $X \in \delta F$ is attracting, then either $X \in \delta(\operatorname{Stab} Q)$, or there exists an eigenray ρ of H such that $X = j(\rho)$.*

Remark. The conclusions are mutually exclusive by Lemma III.5.

Proof. We shall use repeatedly equations such as $H^p(X_iQ) = \alpha^p(X_i)Q$, obtained by combining $\alpha(w)H = Hw$ and $HQ = Q$.

Let $X = x_1 \cdots x_i \cdots$ be an attracting fixed word (in a given free basis $g_1, \dots, g_n$ of F). If $X_iQ = Q$ for infinitely many i, we have $X \in \delta(\operatorname{Stab} Q)$. Otherwise we fix an integer i with $X_iQ \neq Q$ such that the sequence $c_X(\alpha^p X_i)$ is strictly increasing (see $(*)$ in Part I).

Recall (Theorem 2) that the action of $\operatorname{Stab} Q$ on $\pi_0(T \setminus \{Q\})$ has finitely many orbits. Thus there exist $d \geq 1$ and $w \in \operatorname{Stab} Q$ such that $[Q, wX_iQ]$ and $[Q, H^d(X_iQ)]$ overlap. We distinguish two cases, depending on whether w is trivial or not.

- First assume that w is trivial. Let A be the component of $T \setminus \{Q\}$ containing X_iQ, and ρ the corresponding eigenray of H^d.

By assumption the sequence $\theta(p) = c_X(\alpha^{pd}X_i)$ is strictly increasing. In particular notice $X_{\theta(p)} = (\alpha^{pd}X_i) \wedge (\alpha^{(p+1)d}X_i)$. By (BBT2), the distance from $X_{\theta(p)}Q$ to $[(\alpha^{pd}X_i)Q, (\alpha^{(p+1)d}X_i)Q] = [H^{pd}(X_iQ), H^{(p+1)d}(X_iQ)]$ is bounded by C. On the other hand the intersection between $[Q, H^{pd}(X_iQ)]$ and $[Q, H^{(p+1)d}(X_iQ)] = [Q, H^d(H^{pd}(X_iQ))]$ is contained in the H^d-eigenray $\overline{\rho}$. Furthermore the length of this intersection is a constant multiple of λ^{pd}. This implies that the length of $[Q, X_{\theta(p)}Q] \cap \rho$ goes to infinity with p, i.e. $X = j(\rho)$ (see Lemma III.4).

There remains to show that ρ is invariant under H, not only under H^d. This follows from Lemma III.5 since $j(\rho) = \alpha(j(\rho)) = j(H(\rho))$ and hence $\rho = H(\rho)$.

- Now we assume that the element $w \in \operatorname{Stab} Q$ is not trivial. In this case we show $X \in \delta(\operatorname{Stab} Q)$.

The length of the intersection between $[Q, \alpha^p(wX_i)Q] = [Q, H^p(wX_iQ)]$ and $[Q, (\alpha^{p+d}(X_i))Q] = [Q, H^p(H^d(X_iQ))]$ goes to infinity as p goes to ∞. By (BBT2) this implies that $d(Q, w_pQ)$ goes to infinity, with $w_p = \alpha^p(wX_i) \wedge \alpha^{p+d}(X_i)$. This in turn implies that the length of the word w_p goes to infinity.

Write $w_p = \alpha^p(w)\alpha^p(X_i) \wedge \alpha^{p+d}(X_i)$ and recall that $c_X(\alpha^p(X_i))$ goes to infinity with p. If the amount of cancellation between $\alpha^p(w)$ and $\alpha^p(X_i)$ does not remain bounded, then X is a limit point of the sequence $\alpha^p(w^{-1})$ and belongs to $\delta(\mathrm{Stab}\, Q)$. Similarly, if the length of $\alpha^p(w) \wedge \alpha^{p+d}(X_i)$ is unbounded, then X is a limit point of $\alpha^p(w)$ and belongs to $\delta(\mathrm{Stab}\, Q)$.

Since $|w_p|$ goes to infinity, the only remaining possibility is that $|\alpha^p(w)|$ is bounded. Choose a subsequence p_k such that $\alpha^{p_k}(w)$ is a nontrivial word $v \in \mathrm{Stab}\, Q$ independent of k. We then get $vX = X$ since $\alpha^{p_k}(X_i)$ and $\alpha^{p_k+d}(X_i)$ converge to X as p goes to ∞. This implies $X \in \delta(\mathrm{Stab}\, Q)$ since X is a limit point of one of the sequences v^p or v^{-p}. $\square$

Let $\mathcal{V}(Q)$ be the set of orbits of the action of $\mathrm{Stab}\, Q$ on components of $T \setminus Q$, and let $\mathcal{V}^H(Q) \subset \mathcal{V}(Q)$ be the subset consisting of orbits containing a component fixed by H.

Proposition IV.4.

(1) *There exists an injection $\tau : \mathcal{A}(\alpha) \hookrightarrow \mathcal{A}(\alpha^Q) \cup \mathcal{V}(Q)$ (disjoint union).*
(2) *The image of τ is $\mathcal{A}(\alpha^Q) \cup \mathcal{V}^H(Q)$.*

Remark. Assertion (2) is not needed for the proof of Theorem 1.

Recalling that $\mathrm{Fix}\, \alpha = \mathrm{Fix}\, \alpha^Q$ and that $\mathrm{ind}\,(\alpha) = \mathrm{rk}\, \mathrm{Fix}\, \alpha + \frac{1}{2}a(\alpha) - 1$, we get from Assertion (1):

Corollary. $\mathrm{ind}\,(\alpha) \leq \mathrm{ind}\,(\alpha^Q) + \frac{1}{2}v(Q)$. $\square$

Proof of Proposition IV.4. (1) If an attracting fixed point X of α belongs to $\delta(\mathrm{Stab}\, Q)$, so does every X' equivalent to X since $\mathrm{Stab}\, Q$ contains $\mathrm{Fix}\, \alpha$. We map the corresponding set of classes $\mathcal{A}_{\mathrm{Stab}\, Q}(\alpha)$ bijectively onto $\mathcal{A}(\alpha^Q)$ using Lemma IV.2.

If an attracting fixed point X does not belong to $\delta(\mathrm{Stab}\, Q)$, we map the class of X to the orbit under $\mathrm{Stab}\, Q$ of the component A_X of $T \setminus \{Q\}$ containing the points X_pQ for p large and the associated eigenray $\rho = \rho_X$. To prove that τ is well-defined and injective on $\mathcal{A}(\alpha) \setminus \mathcal{A}_{\mathrm{Stab}\, Q}(\alpha)$, we need to check that A_X and $A_{X'}$ belong to the

same $\mathrm{Stab}\, Q$-orbit if and only if X and X' are equivalent.

If X and X' are equivalent, then by definition there exists $w \in \mathrm{Fix}\, \alpha$ with $X' = wX$. Equivariance of j (see Lemma III.5) gives $j(w\rho_X) = wj(\rho_X) = X' = j(\rho_{X'})$ and thus $w\rho_X = \rho_{X'}$. In particular $wA_X = A_{X'}$.

Conversely, assume that some $h \in \mathrm{Stab}\, Q$ maps A_X to $A_{X'}$. The intersection of the rays ρ_X and $h^{-1}\rho_{X'}$ is a nondegenerate segment $(Q, Q']$. For $a \in (Q, Q']$ close enough to Q we have $Ha \in (Q, Q']$ and therefore $hHa = Hha$. But we also have $Hha = \alpha(h)Ha$. This implies that $h^{-1}\alpha(h)$ fixes the nondegenerate subsegment $[Q, Ha]$ of $[Q, Q']$. As by Theorem II.1 the group action on T has trivial arc stabilizers, it follows that $\alpha(h) = h$. In particular the isometry defined by h commutes with H and therefore the fact that ρ_X and $h^{-1}\rho_{X'}$ overlap non-trivially implies $\rho_X = h^{-1}\rho_{X'}$. We get $X = j(\rho_X) = j(h^{-1}\rho_{X'}) = h^{-1}X'$, and hence X and X' are equivalent.

(2) To prove Assertion (2), it suffices to consider an H-invariant component A of $T \setminus Q$, and to show that its orbit

under $\operatorname{Stab} Q$ belongs to the image of τ. Let ρ be the H-eigenray contained in A, and $X = j(\rho) \in \delta F$. We now show that X is an attracting fixed point of α. This will complete the proof since the class of X maps to the $\operatorname{Stab} Q$-orbit of A.

From Lemma III.5 it follows $\alpha(X) = \alpha(j(\rho)) = j(H(\rho)) = j(\rho) = X$. By way of contradiction, assume that X is not attracting. Writing $\alpha(X_i) = X_{k(i)}Z_i$ with $|Z_i| \leq B$ as in Part I, there exist arbitrarily large i such that $k(i) \leq i + p$ for some fixed integer p. For these values of i, we claim that $d(\alpha(X_i)Q, [Q, X_iQ])$ is bounded by a constant independent of i. Indeed

$$d(\alpha(X_i)Q, X_{k(i)-p}Q) \leq (B + p) \max_j d(Q, g_jQ),$$

and $d(X_{k(i)-p}Q, [Q, X_iQ]) \leq C$ by (BBT1) since $k(i) - p \leq i$.

On the other hand $[Q, \alpha(X_i)Q] \cap \rho = [Q, H(X_iQ)] \cap \rho = H([Q, X_iQ] \cap \rho)$ has length λ times bigger than $[Q, X_iQ] \cap \rho$. As the length of $[Q, X_iQ] \cap \rho$ goes to infinity as $i \to \infty$, this contradicts the above observation. □

For later purposes we also need the following fact:

Lemma IV.5. *Let A, A' be components of $T \setminus \{Q\}$ belonging to the same* $\operatorname{Stab} Q$*-orbit. Suppose A is invariant under H and A' is invariant under mH for some $m \in \operatorname{Stab} Q$. Then there exists $c \in \operatorname{Stab} Q$ such that $m = c\alpha(c^{-1})$ (hence $i_m \circ \alpha$ is similar to α).*

Proof. Let c be an element of $\operatorname{Stab} Q$ mapping A to A'. For $a \in A$ close enough to Q on the H-invariant ray we write $cHa = (mH)ca = m\alpha(c)Ha$. Thus $c^{-1}m\alpha(c)$ fixes a nondegenerate segment. Since arc stabilizers are trivial this implies $m = c\alpha(c^{-1})$. □

Case B: $\lambda = 1$.

If $\lambda = 1$, then T is simplicial. In this case we choose T so as to minimize the number of edges of the quotient graph $\Gamma = T/F$.

Lemma IV.6. *If H has more than one fixed point, then $\Gamma = T/F$ has only one edge.*

Proof. Let e be an edge fixed (pointwise) by H and let T' be the tree obtained by collapsing each component of the orbit of e to a point. This orbit is preserved by H, so that H induces an isometry H' of T' satisfying the commutation equation $\alpha(w)H' = H'w$ with the induced F-action on T'. If Γ has more than one edge, the action of F on T' is nontrivial. This contradicts the choice of T. □

Hence for $\lambda = 1$ we need only consider three more cases:

(1) H has exactly one fixed point Q.
(2) H has more than one fixed point and Γ is a segment.
(3) H has more than one fixed point and Γ is a loop.

Case B1: H has exactly one fixed point Q.

Proposition IV.7. *There is an injection $\tau : \mathcal{A}(\alpha) \hookrightarrow \mathcal{A}(\alpha^Q)$.*

Since $\operatorname{Fix} \alpha = \operatorname{Fix} \alpha^Q$ by Lemma IV.1 we get:

Corollary. $\operatorname{ind}(\alpha) \leq \operatorname{ind}(\alpha^Q)$. □

Proof of Proposition IV.7. By Lemma IV.2, it suffices to prove that every attracting fixed point of α belongs to $\delta(\operatorname{Stab} Q)$. Recall that the action of F on T is simplicial with trivial edge stabilizers. Bass-Serre theory expresses F as a free product whose factors are vertex groups and $\pi_1 \Gamma$ (see [Ser, I.5.1]). We use this to get a preferred free basis of F as follows.

Let $T_0 \subset T$ be a finite subtree containing Q and projecting isomorphically onto a maximal subtree $\Gamma_0 \subset \Gamma$. Orient the edges of $\Gamma \setminus \Gamma_0$. The basis we construct consists of the element of F naturally associated to each edge of $\Gamma \setminus \Gamma_0$, together with bases of the groups $\operatorname{Stab} V$, for V a vertex of T_0.

Such a basis has the following property. Suppose $w \in F \backslash \operatorname{Stab} Q$, and w' contains w as an initial subword. Then the segments $[Q, wQ]$ and $[Q, w'Q]$ intersect in a nondegenerate segment.

Now let X be an attracting fixed word of α (in a preferred basis of F). We show $X \in \delta(\operatorname{Stab} Q)$. Fix an integer p. Since H is an isometry having a fixed point, the midpoint of the segment between $X_p Q$ and $H(X_p Q) = \alpha(X_p)Q$ is equal to Q because it is fixed by H (see e.g. [MoSh, Lemma II.2.16]). The segments $[Q, X_p Q]$ and $[Q, \alpha(X_p)Q]$ thus do not overlap. For p large, the word X_p is an initial subword of $\alpha(X_p)$. Our choice for the basis of F then implies $X_p \in \operatorname{Stab} Q$. Hence $X \in \delta(\operatorname{Stab} Q)$ as required. □

Case B2: H has more than one fixed point and Γ is a segment.

Let $e = [Q, R]$ be an edge of T fixed (pointwise) by H. Bass-Serre theory gives a nontrivial decomposition $F = \operatorname{Stab} Q * \operatorname{Stab} R$. This decomposition is α-invariant by Lemma IV.1.

Proposition IV.8.

(1) $\operatorname{Fix} \alpha = \operatorname{Fix} \alpha^Q * \operatorname{Fix} \alpha^R$.
(2) *There is an injection $\tau : \mathcal{A}(\alpha) \hookrightarrow \mathcal{A}(\alpha^Q) \cup \mathcal{A}(\alpha^R)$.*

Corollary. $\operatorname{ind}(\alpha) \leq \operatorname{ind}(\alpha^Q) + \operatorname{ind}(\alpha^R) + 1$. □

Proof of Proposition IV.8. The proof we give is purely algebraic, using only the α-invariant decomposition of F. We choose a free basis of F consisting of a basis of $\operatorname{Stab} Q$ together with a basis of $\operatorname{Stab} R$.

Write any finite word $v \in F$ as a product of subwords belonging alternatively to both factors. If v is fixed by α, then each subword has to be fixed. This shows the first assertion.

Now suppose X is any attracting fixed infinite word. If X contains infinitely many letters from both factors, then it is an infinite product of subwords belonging alternatively to $\mathrm{Fix}\,\alpha^Q$ and $\mathrm{Fix}\,\alpha^R$, and $X \in \delta(\mathrm{Fix}\,\alpha)$. This is impossible since X is attracting. Thus X is equivalent to an infinite word contained in one of the factors, and we get τ using Lemma IV.2. $\square$

Case B3: H has more than one fixed point and Γ is a loop.

Let again $e = [Q, R]$ be an edge of T fixed (pointwise) by H. We now have (by Bass-Serre theory) $F = (\mathrm{Stab}\,Q) * < t >$, where t is any element such that $t(Q) = R$. Note that $\alpha(t)Q = \alpha(t)HQ = HtQ = R$, so that $\alpha(t) = tu$ with $u \in \mathrm{Stab}\,Q$.

If there exists t' with $t'(Q) = R$ and $\alpha(t') = t'$, we have an α-invariant decomposition as before and the previous analysis yields $\mathrm{ind}\,(\alpha) \leq \mathrm{ind}\,(\alpha^Q) + 1$. We assume therefore that there is no such t'. This implies that u cannot be written $u = v\alpha(v^{-1})$ with $v \in \mathrm{Stab}\,Q$, since otherwise we can set $t' = tv$.

Proposition IV.9.

(1) $\mathrm{Fix}\,\alpha = \mathrm{Fix}\,\alpha^Q * t\,\mathrm{Fix}\,(i_u \circ \alpha^Q)\,t^{-1}$.
(2) *There is an injection* $\tau : \mathcal{A}(\alpha) \hookrightarrow \mathcal{A}(\alpha^Q) \cup \mathcal{A}(i_u \circ \alpha^Q)$.

Corollary. $\mathrm{ind}\,(\alpha) \leq \mathrm{ind}\,(\alpha^Q) + \mathrm{ind}\,(i_u \circ \alpha^Q) + 1$. $\square$

Proof of Proposition IV.9. Choose a free basis of F consisting of t together with a basis of $\mathrm{Stab}\,Q$. Any element of F has a unique reduced expression

$$w = v_0 t^{\varepsilon_1} v_1 t^{\varepsilon_2} \dots v_{p-1} t^{\varepsilon_p} v_p$$

where $\varepsilon_i = \pm 1$ and v_i is a (possibly trivial) word not containing $t^{\pm 1}$. We study the word $\alpha(w)$, paying special attention to the letters $t^{\pm 1}$.

Recall that $\alpha(v_i)$ does not contain $t^{\pm 1}$ and that $\alpha(t) = tu$. Thus no new letters $t^{\pm 1}$ appear in $\alpha(w)$. Also note that there can be no cancellation between the p letters t^{ε_i} in $\alpha(w)$, since the image of a subword tvt^{-1} is $tu\alpha(v)u^{-1}t^{-1}$ (similarly $\alpha(t^{-1}vt) = u^{-1}t^{-1}\alpha(v)tu$).

Now assume that w is fixed by α. This forces the words v_i $(1 \leq i \leq p-1)$ to satisfy equations, whose form depends on ε_i and ε_{i+1}:

$$\begin{aligned} v_i = \alpha(v_i) &\quad \text{if } w \text{ contains } t^{-1}v_i t \\ v_i = u\alpha(v_i)u^{-1} &\quad \text{if } w \text{ contains } tv_i t^{-1} \\ v_i = u\alpha(v_i) &\quad \text{if } w \text{ contains } tv_i t \\ v_i = \alpha(v_i)u^{-1} &\quad \text{if } w \text{ contains } t^{-1}v_i t^{-1}. \end{aligned}$$

The first two equations express that v_i should belong to $\mathrm{Fix}\,(\alpha^Q)$ or $\mathrm{Fix}\,(i_u \circ \alpha^Q)$. On the other hand the assumption that u cannot be written $u = v\alpha(v^{-1})$ prevents the other two equations from being satisfied. Thus the letters t and t^{-1} *alternate* in w. Furthermore the relation $\alpha(w) = w$ also implies $v_0 \in \mathrm{Fix}\,\alpha^Q$, $\varepsilon_1 = 1$, and $v_p \in \mathrm{Fix}\,\alpha^Q$, $\varepsilon_p = -1$.

Summing up, the words invariant under α are precisely those of the form $v_0 t v_1 t^{-1} v_2 t \dots v_{2q}$ with $v_i \in \mathrm{Fix}\,\alpha^Q$ for i even and $v_i \in \mathrm{Fix}\,(i_u \circ \alpha^Q)$ for i odd. We have shown Assertion 1.

Proving Assertion 2 is now easy. If an attracting fixed word X contains $t^{\pm 1}$ infinitely often, then X is an infinite product $v_0 t v_1 t^{-1} v_2 t v_3 t^{-1} \cdots$ with the v_i's as above. This implies $X \in \delta(\mathrm{Fix}\,\alpha)$, a contradiction. Thus X is equivalent to a word X_0 not

containing $t^{\pm 1}$, or to a word tX_0 with no letter $t^{\pm 1}$ in X_0. The word X_0 is an attracting fixed word of α^Q in the first case, of $i_u \circ \alpha^Q$ in the second: if $\alpha(tX_0) = tX_0$, then $tu\alpha^Q(X_0) = tX_0$ and therefore $X_0 = u\alpha^Q(X_0) = (i_u \circ \alpha^Q)(X_0)$.

Once again we obtain τ using Lemma IV.2. ◻

Notice that results analogous to Propositions IV.8 and IV.9 are true if T has more than one orbit of edges, provided H fixes each of them.

V. The induction

Let $\alpha_0, \dots, \alpha_k$ be automorphisms of F representing the same outer automorphism and belonging to distinct similarity classes. We prove the following inequality by induction on $n = \mathrm{rk}\,F$:

$$\sum_{i=0}^{k} \mathrm{ind}\,(\alpha_i) \leq n - 1.$$

It is clear if $n = 1$, so we assume $n \geq 2$. We also assume $\mathrm{ind}\,(\alpha_i) > 0\ \forall i$. Since the index is a similarity invariant, we will be free to replace each α_i by a similar automorphism when needed.

Apply Theorem II.1 to $\alpha = \alpha_0$. We get a tree T and a homothety $H : T \to T$ with stretching factor λ. If $\beta = i_m \circ \alpha$ with $m \in F$, we associate to β the map $H_\beta = mH$. It satisfies $\beta(w)H_\beta = H_\beta w$ for all $w \in F$ (see Remark II.2). If $\beta = i_c \circ \alpha \circ (i_c)^{-1} = i_{c\alpha(c^{-1})} \circ \alpha$ is similar to α, we get $H_\beta = c\alpha(c^{-1})H_\alpha = cH_\alpha c^{-1}$. In particular the fixed point sets satisfy $\mathrm{Fix}\,H_\beta = c\mathrm{Fix}\,H_\alpha$.

For simplicity we write H_i for H_{α_i}. We may assume as before that the quotient graph $\Gamma = T/F$ has only one edge if $\lambda = 1$ and some H_i has more than one fixed point (see Remark II.2 and Lemma IV.6).

By Assertion 3 of Theorem II.1, each H_i has at least one fixed point. If Q is fixed by H_i, then $\mathrm{Stab}\,Q$ is α_i-invariant (Lemma IV.1). We denote α_i^Q the induced automorphism.

Lemma V.1. *Suppose $Q \in T$ is fixed by both H_i and H_j $(i \neq j)$ and* $\mathrm{rk}\,\mathrm{Stab}\,Q \geq 2$*. Then α_i^Q, α_j^Q represent the same outer automorphism of* $\mathrm{Stab}\,Q$ *and belong to distinct similarity classes in* $\mathrm{Aut}\,(\mathrm{Stab}\,Q)$.

Proof. If $\alpha_j = i_h \circ \alpha_i$, we have $h \in \operatorname{Stab} Q$ because $H_j = hH_i$. Thus α_i^Q and α_j^Q represent the same outer automorphism of $\operatorname{Stab} Q$.

Now suppose there exists $v \in \operatorname{Stab} Q$ such that $\alpha_j(g) = v\alpha_i(v^{-1}gv)v^{-1}$ for all $g \in \operatorname{Stab} Q$. Then

$$h\alpha_i(g)h^{-1} = \alpha_j(g) = v\alpha_i(v^{-1})\alpha_i(g)\alpha_i(v)v^{-1}$$

for $g \in \operatorname{Stab} Q$. Since $\operatorname{Stab} Q$ has rank ≥ 2, we deduce $h = v\alpha_i(v^{-1})$, so that $\alpha_j(g) = v\alpha_i(v^{-1}gv)v^{-1}$ holds for every $g \in F$. This is a contradiction since α_i and α_j are not similar. $\square$

As in Part IV, we now distinguish several cases.

Case A: $\lambda > 1$.

Each H_i has exactly one fixed point Q_i. Recall that $\operatorname{Fix} H_\beta = c\operatorname{Fix} H_\alpha$ if $\beta = i_c \circ \alpha \circ (i_c)^{-1}$ is similar to α. Replacing each α_i by a similar automorphism, we may then assume that for $i \neq j$ either $Q_i = Q_j$, or Q_i and Q_j belong to different F-orbits. Let $\mathcal{Q} \subset T$ be the set of all points Q_i, and $\pi : \{0, \dots, k\} \to \mathcal{Q}$ the map taking i to Q_i.

Proposition IV.4 yields injections $\tau_i : \mathcal{A}(\alpha_i) \to \mathcal{A}(\alpha_i^{Q_i}) \cup \mathcal{V}(Q_i)$. We denote $v_i(Q_i)$ the cardinality of $\tau_i(\mathcal{A}(\alpha_i)) \cap \mathcal{V}(Q_i)$. Using Proposition IV.4 and its corollary we write

$$\sum_{i=0}^{k} \operatorname{ind}(\alpha_i) \leq \sum_{i=0}^{k} (\operatorname{ind}(\alpha_i^{Q_i}) + \frac{1}{2} v_i(Q_i)) = \sum_{Q \in \mathcal{Q}} \sum_{i \in \pi^{-1}(Q)} (\operatorname{ind}(\alpha_i^{Q_i}) + \frac{1}{2} v_i(Q_i)).$$

We then have

$$\sum_{i \in \pi^{-1}(Q)} \operatorname{ind}(\alpha_i^{Q_i}) \leq \operatorname{rk} \operatorname{Stab} Q - 1.$$

This is clear for points Q with $\operatorname{rk} \operatorname{Stab} Q \leq 1$. For other points it follows from Lemma V.1 and the induction hypothesis, as $\operatorname{rk} \operatorname{Stab} Q \leq n - 1$ by Theorem 2.

On the other hand

$$\sum_{i \in \pi^{-1}(Q)} \frac{1}{2} v_i(Q_i) \leq \frac{1}{2} v(Q)$$

because by Lemma IV.5 two components of $T \setminus \{Q\}$ containing rays invariant under H_i and H_j respectively cannot be in the same $\operatorname{Stab} Q$-orbit if $i \neq j$.

As a result we obtain

$$\begin{aligned}
\sum_{i=0}^{k} \operatorname{ind}(\alpha_i) &\leq \sum_{Q \in \mathcal{Q}} \sum_{i \in \pi^{-1}(Q)} (\operatorname{ind}(\alpha_i^{Q_i}) + \frac{1}{2} v_i(Q_i)) \\
&\leq \sum_{Q \in \mathcal{Q}} (\operatorname{rk} \operatorname{Stab} Q - 1 + \frac{1}{2} v(Q)) \\
&\leq n - 1.
\end{aligned}$$

The third inequality follows from Theorem 2 since different points of $\mathcal{Q}$ are in different F-orbits.

Case B1: $\lambda = 1$ and each H_i has exactly one fixed point.

The proof is the same, using the corollary to Proposition IV.7. One does not need the v terms.

Now we suppose some H_i (say H_0) fixes (pointwise) an edge $e = [Q, R]$. Recall that we have assumed the graph $\Gamma = T/F$ has only one edge.

Case B2: H_0 has more than one fixed point, and Γ is a segment.

Note that for $i > 0$ the map H_i has only one fixed point Q_i. Otherwise we could replace α_i by a similar automorphism and get H_i to fix e (recall that there is only one F-orbit of edges). This would contradict triviality of edge stabilizers since $H_i = m_i H$ for some nontrivial $m_i \in F$.

The fixed point Q_i cannot be the midpoint of an edge because $\operatorname{Stab} Q_i$ would be trivial (F acts without inversions) and Proposition IV.7 would imply $\operatorname{ind}(\alpha_i) < 0$. Thus Q_i belongs to the orbit of either Q or R. Changing α_i within its similarity class if needed, we may assume $Q_i = Q$ or R.

Taking π to be the obvious map from $\{1, \ldots, k\}$ to $\{Q, R\}$, we get by the corollaries to Propositions IV.8 and IV.7

$$\begin{aligned}
\sum_{i=0}^{k} \operatorname{ind}(\alpha_i) &\leq \operatorname{ind}(\alpha_0^Q) + \operatorname{ind}(\alpha_0^R) + 1 + \sum_{i \in \pi^{-1}(Q)} \operatorname{ind}(\alpha_i^Q) + \sum_{i \in \pi^{-1}(R)} \operatorname{ind}(\alpha_i^R) \\
&\leq 1 + \left(\operatorname{ind}(\alpha_0^Q) + \sum_{i \in \pi^{-1}(Q)} \operatorname{ind}(\alpha_i^Q) \right) + \left(\operatorname{ind}(\alpha_0^R) + \sum_{i \in \pi^{-1}(R)} \operatorname{ind}(\alpha_i^R) \right) \\
&\leq 1 + (\operatorname{rk} \operatorname{Stab} Q - 1) + (\operatorname{rk} \operatorname{Stab} R - 1) \\
&\leq n - 1.
\end{aligned}$$

The third inequality comes from Lemma V.1 and the induction hypothesis, as F is isomorphic to the nontrivial free product $\operatorname{Stab} Q * \operatorname{Stab} R$ (see the discussion in Part IV).

Case B3: H_0 has more than one fixed point, and Γ is a loop.

Arguing as in the beginning of case B2, we may assume that H_0 fixes an edge $e = [Q, R]$ and that Q is the only fixed point of H_i for $i > 0$ (now there is only one F-orbit of vertices).

With the notations of Part IV we have $F = (\operatorname{Stab} Q) * < t >$ with $\alpha_0(t) = tu$, $u \in \operatorname{Stab} Q$. If t may be chosen with $\alpha_0(t) = t$, we simply write

$$\sum_{i=0}^{k} \operatorname{ind}(\alpha_i) \leq \operatorname{ind}(\alpha_0^Q) + 1 + \sum_{i=1}^{k} \operatorname{ind}(\alpha_i^Q) \leq \operatorname{rk} \operatorname{Stab} Q = n - 1.$$

Otherwise we recall that u cannot be written $u = v\alpha_0(v^{-1})$ with $v \in \operatorname{Stab} Q$ and we apply the corollaries to Propositions IV.7 and IV.9. We get

$$\sum_{i=0}^{k} \operatorname{ind}(\alpha_i) \leq \sum_{i=0}^{k} \operatorname{ind}(\alpha_i^Q) + \operatorname{ind}(i_u \circ \alpha_0^Q) + 1.$$

There is nothing to prove if $\mathrm{Stab}\, Q$ has rank 1. Otherwise we argue as follows. By Lemma V.1, the automorphisms $\alpha_0^Q, \alpha_1^Q, \ldots, \alpha_k^Q, i_u \circ \alpha_0^Q$ represent the same outer automorphism of $\mathrm{Stab}\, Q$. If we show that no two of them are similar, the inductive proof will be complete since we can write

$$\sum_{i=0}^{k} \mathrm{ind}\,(\alpha_i) \leq \mathrm{rk}\, \mathrm{Stab}\, Q - 1 + 1 = n - 1.$$

By Lemma V.1 we need only check that $i_u \circ \alpha_0^Q$ is not similar to any of the others. Arguing as in the proof of Lemma V.1 we first see that it is not similar to α_0^Q since

$$i_u \circ \alpha_0^Q = i_v \circ \alpha_0^Q \circ (i_v)^{-1} \implies u = v\alpha_0^Q(v^{-1}).$$

Then we note that $i_u \circ \alpha_0$ is similar to α_0 in $\mathrm{Aut}\,(F)$ since $i_u \circ \alpha_0 = (i_t)^{-1} \circ \alpha_0 \circ i_t$. It follows that $i_u \circ \alpha_0$ and α_i are not similar for $i \geq 1$. By Lemma V.1, $i_u \circ \alpha_0^Q$ and α_i^Q are not similar in $\mathrm{Aut}\,(\mathrm{Stab}\, Q)$. $\square$

VI. Discussion and remarks

In this last section we want to discuss some properties of certain classes of automorphisms of F, with respect to their index. We will not give formal proofs but rather indicate the reason for our claims; details may be worked out elsewhere.

First we recall the definition of the *index* of an outer automorphism of the free group $\hat{\alpha} \in \mathrm{Out}\,(F)$, induced by $\alpha \in \mathrm{Aut}\,(F)$. Let $\mathcal{S}(\hat{\alpha})$ denote the set of similarity classes $[\alpha']$ of automorphisms α' inducing the outer automorphism $\hat{\alpha}' = \hat{\alpha}$. We define

$$\mathrm{ind}\,(\hat{\alpha}) := \sum_{[\alpha'] \in \mathcal{S}(\hat{\alpha})} \max(\mathrm{ind}\,(\alpha'), 0).$$

Our main result, Theorem 1', is equivalent to the inequality

$$\mathrm{ind}\,(\hat{\alpha}) \leq \mathrm{rk}\,(F) - 1 \tag{1}$$

for all $\hat{\alpha} \in \mathrm{Out}\,(F)$.

Next we observe that the techniques for computing a basis of the fixed subgroup as well as representatives for each equivalence class of attracting fixed points, described in [CL 1] for positive automorphisms, can be extended to all automorphisms of F. In [CL 1] the only specific property of positive automorphisms used is the fact that for any $w \in F$ there is an obvious lower bound for the length of $\alpha^k(w)$ for any $k \geq 1$. A similar bound can be deduced for arbitrary $\alpha \in \mathrm{Aut}\,(F)$ if we use a relative train track representative for α as constructed in [BH]. This gives the following, which can be used to verify some of the statements below:

VI.1. *There is a combinatorial algorithm which computes for any* $\alpha \in \operatorname{Aut}(F)$ *the indices* $\operatorname{ind}(\alpha)$ *and* $\operatorname{ind}(\hat{\alpha})$.

There is an obvious question as to when the inequality (1) is an equality. Using Theorem 1' together with standard Nielsen-Thurston theory one can compute the index of any *geometric* automorphism, i.e. an automorphism which is induced by a homeomorphism of a surface M with boundary, where $\pi_1 M \cong F$. For example, if $\hat{\alpha}$ is induced by a pseudo-Anosov automorphism which fixes each separatrix at every singularity of the stable foliation, then $\operatorname{ind}(\hat{\alpha})$ is indeed equal to $\chi(S) = \operatorname{rk}(\pi_1 S) - 1$. These arguments extend to the case where α is a multiple Dehn twist on disjoint closed curves, possibly with pseudo-Anosov components on the complementary subsurfaces. This shows:

VI.2. *Every geometric automorphism* $\hat{\alpha} \in \operatorname{Out}(F)$ *has a power* $\hat{\alpha}^k$ *with*

$$\operatorname{ind}(\hat{\alpha}^k) = \operatorname{rk}(F) - 1.$$

This fact was used in earlier work of the fourth author to describe new classes of non-geometric automorphisms with properties very similar to those of geometric ones. Also, from the above discussion it is easy to construct geometric α's which do not satisfy equality in (1).

Dehn twists on surfaces have been generalized in [CL 2] to Dehn twist automorphisms of free groups. It is shown in [CL 3] that every Dehn twist automorphism $\hat{D} \in \operatorname{Out}(F)$ has maximal index $\operatorname{ind}(\hat{D}) = \operatorname{rk}(F) - 1$. This implies (see [CL 3]):

VI.3. *Every linear growth automorphism* $\hat{\alpha} \in \operatorname{Out}(F)$ *has a power* $\hat{\alpha}^k$ *with*

$$\operatorname{ind}(\hat{\alpha}^k) = \operatorname{rk}(F) - 1.$$

Recall that $\hat{\alpha}$ is said to have *polynomial growth of degree* $\leq k$ if, for every $g \in F$, there exists C such that $\alpha^t(g)$ is conjugate to an element of length $\leq Cn^k$ for every $t \geq 1$ (for some, hence any, α representing $\hat{\alpha}$).

We now want to discuss how or why the index of an outer automorphism can be non-maximal. Our proof of Theorem 1' gives rather explicit information about this, and we can distinguish between two qualitatively different phenomena.

VI.4. Given two automorphisms $\alpha \in \operatorname{Aut}(F')$ and $\beta \in$

Aut (F"), $wecancomposethemcanonicallytoanewautomorphism \alpha * \beta \in \operatorname{Aut}(F' * F'')$. On the level of outer automorphisms, however, there is no such canonical composition. Given $\hat{\alpha}$ and $\hat{\beta}$, one first has to choose representatives α' and β', and then one can define a *freely composed* outer automorphism $\widehat{\alpha' * \beta'}$. If one has both $\operatorname{ind}(\alpha') \geq 0$ and $\operatorname{ind}(\beta') \geq 0$, then one obtains

$$\operatorname{ind}(\widehat{\alpha' * \beta'}) = \operatorname{ind}(\widehat{\alpha'}) + \operatorname{ind}(\widehat{\beta'}) + 1 \,, \tag{2}$$

which gives maximal index for $\widehat{\alpha' * \beta'}$ if both $\hat{\alpha}$ and $\hat{\beta}$ have maximal index. If, however, the index of α' or β' is negative, then the index of $\widehat{\alpha' * \beta'}$ differs from the maximal value (2) by $\frac{1}{2}$ or 1.

Similarly, for $\alpha \in \mathrm{Aut}\,(F)$, the *HNN-composition* $\alpha_+ \in$

$\mathrm{Aut}\,(\mathrm{F}{*}\langle t\rangle)$ given by $\alpha_+(w) = v\alpha(w)v^{-1}$ if $w \in F$, $\alpha_+(t) = tu$ $(v, u \in F)$, which has been considered in case B3 of Parts IV and V, "connects" similarity classes of $\hat{\alpha}$ with varying possibilities for their index (according to the choice of v and u). Again, the index of $\hat{\alpha}_+ \in \mathrm{Out}\,(F * \langle t\rangle)$ can drop by $\frac{1}{2}$ or 1 from the maximal value

$$\mathrm{ind}\,(\widehat{\alpha_+}) = \mathrm{ind}\,(\alpha) + 1\ .$$

We obtain as immediate consequence the existence of outer automorphisms with all positive powers of very small index: for example, for $\alpha : a \to a, b \to ba, c_k \to b^k c_k b^k$ $(k = 1, \ldots, m)$ one has $\mathrm{rk}\,(F) = m + 2$ and $\mathrm{ind}\,(\hat{\alpha}^t) = 1$ for all $t \geq 1$.

VI.5. It follows from [BH] that $\hat{\alpha}$ has exponential growth if for some stratum of some (and hence any) relative train track representative of $\hat{\alpha}$ the transition matrix has Perron-Frobenius eigenvalue strictly bigger than 1. Otherwise, after possibly refining the strata structure, every transition matrix is either 0 or a permutation matrix. In this case $\hat{\alpha}$ has polynomial growth, and it is not hard to see that a suitable power of $\hat{\alpha}$ arises precisely as iterated free composition or HNN-composition as in VI.4, where all **????** starting **????** factors are Dehn twist automorphisms. One can show that at each composition the degree of polynomial growth can be raised by one, but only at the expense of strictly decreasing the index of the composed automorphism with respect to the maximal possible composition value. This shows:

Every polynomially but non-linearily growing (and hence non-geometric) outer automorphism $\hat{\alpha}$ has non-maximal index:

$$\mathrm{ind}\,(\hat{\alpha}) \leq \mathrm{rk}\,(F) - \frac{3}{2}$$

Let us now turn to the second basic phenomenon which produces non-maximal index of an outer automorphism.

If α "commutes" with a homothety H of an $\mathbf{R}$-tree T as in Theorem II.1, then by the analysis of Part V the index of $\hat{\alpha}$ is bounded by the index $\mathrm{ind}(T)$ of the tree T, defined as the maximal value of the sum $\sum_{\ell=1}^{q}(\mathrm{rk}\,\mathrm{Stab}\,Q_\ell + \frac{1}{2}v(Q_\ell) - 1)$ that appears in Theorem 2 (see [GL], which considers $i(T) = 2\,\mathrm{ind}\,(T)$). In [GL] a distinction is drawn between the case when the action of F on T is *geometric*, in which case the index of T is shown to be $\mathrm{rk}\,(F) - 1$ (i.e. maximal), and the case when T is *non-geometric*, where one proves $\frac{1}{2} \leq \mathrm{ind}\,(T) \leq \mathrm{rk}\,(F) - \frac{3}{2}$.

A prototype of geometric actions is given by pseudo-Anosov automorphisms and the $\mathbf{R}$-tree dual to an invariant foliation. However, somehow surprisingly, there are also non-geometric automorphisms for which the invariant $\mathbf{R}$-tree is geometric but not simplicial. These *parageometric* automorphisms can be found even among irreducible automorphisms with irreducible powers (i.e. no α^t with $t \geq 1$ leaves a proper free factor of F invariant), see [BF] or [Le], and those deserve special attention as they seem rather atypical among all irreducible ones. It is shown in [BF] that precisely the non-geometric and non-parageometric irreducible automorphisms admit train track representatives without non-trivial Nielsen paths. Combining their result with ours can be summarized as follows:

VI.6. *Assume that the outer automorphism $\hat{\alpha} \in \mathrm{Out}\,(F)$ is irreducible, let $f\colon \tau \to \tau$ be a train track representative for α with the least number of indivisible Nielsen paths among all such representatives, and let T be the invariant tree obtained from iterating f as in Part II. Then precisely one of the following two cases occurs:*

(i) *The index of $\hat{\alpha}$ satisfies*

$$\frac{1}{2} \leq \mathrm{ind}\,(\hat{\alpha}^t) \leq \mathrm{rk}\,(F) - \frac{3}{2}$$

for all $t \geq 1$. The action of F on T is non-geometric. The train track τ has no non-trivial Nielsen path with respect to f or to any f^t.

(ii) *The index of $\hat{\alpha}$ satisfies*

$$\mathrm{ind}\,(\hat{\alpha}^t) = \mathrm{rk}\,(F) - 1$$

for some $t \geq 1$. The action of F on T is geometric. The train track τ has a non-trivial Nielsen path with respect to f^t.

It is shown in [BF] that in case (ii) there is precisely one f-orbit of non-trivial indivisible Nielsen paths, and that this path is closed precisely if $\hat{\alpha}$ is geometric. It has been shown by the second author that (contrary to the geometric case!) in the parageometric case the similarity class of that representative $\alpha \in \mathrm{Aut}\,(F)$ of $\hat{\alpha}$ which is given by the Nielsen path is not an invariant of $\hat{\alpha}$, but can change when passing over to a different train track representative.

We now turn again to ordinary automorphisms rather than outer ones. For any $\alpha \in \mathrm{Aut}\,(F)$ our Theorem 1' (see (1) above) gives

$$0 \leq \mathrm{rk}\,(\mathrm{Fix}\,(\alpha)) \leq \mathrm{rk}\,(F) \quad \text{and} \quad 0 \leq a(\alpha) \leq 2\mathrm{rk}\,(F).$$

We are interested in the case where one of the two summands $\mathrm{rk}\,(\mathrm{Fix}\,(\alpha))$ or $a(\alpha)$ for the index becomes maximal.

VI.7. Assume $\mathrm{rk}\,(\mathrm{Fix}\,(\alpha)) = \mathrm{rk}\,(F)$. As a consequence we obtain $a(\alpha^t) = 0$ for any $t \geq 1$, i.e. α^t has no attracting fixed point in δF, by Theorem 1. Hence, by Lemma IV.2, there is no α-invariant tree T as in Theorem II.1 with $\lambda > 1$, as it follows from Theorem 2 [GL] that $\mathcal{V}^{H^t}(Q) = \mathcal{V}(Q)$ for some $t \geq 1$. Thus we can decompose α as in the cases B2 or B3, since case B1 yields immediately that $\mathrm{rk}\,(\mathrm{Fix}\,(\alpha)) < \mathrm{rk}\,(F)$. We now consider the restriction of α to the vertex group (case B3) or to the vertex groups (case B2) and observe that they also must have fixed subgroup of maximal rank. Hence (by inverting the HNN-composition process described in VI.4) we can proceed inductively to obtain a "normal form" for such automorphisms, see [CT] and [CL 2].

VI.8. Assume that the action on the invariant tree (satisfying $\lambda > 1$) is free, and that α is replaced by a sufficiently high power so that H_α fixes all orbits of branch points and all orbits of directions at those branch points. Then one obtains $\mathrm{ind}\,(\hat{\alpha}) = \mathrm{ind}\,(T)$. If furthermore T has only one orbit of branch points, then any automorphism α', with $\hat{\alpha}' = \hat{\alpha}$ and with the property that $H_{\alpha'}$ fixes one of the branch points, satisfies

$$\mathrm{ind}\,(\alpha') = \mathrm{ind}\,(T).$$

Thus, if the F-action on T is geometric, one has $a(\alpha) = 2\mathrm{rk}\,(F)$, which is the upper bound. Such examples can be constructed from an irreducible parageometric automorphism through subsequent HNN-compositions chosen so as to connect all similarity classes with positive index (see VI.4). Notice however that the resulting automorphism of a free group of higher rank is of course no longer irreducible. An irreducible automorphism with $a(\alpha) = 2\mathrm{rk}\,(F)$ must be represented by a train track map for α which has one nonclosed Nielsen path connecting the only two fixed points of the map with positive index. Somehow surprisingly such automorphisms exist. An example is given by $a \mapsto aac^{-1}aac^{-1}b^{-1},\ b \mapsto bca^{-1}a^{-1},\ c \mapsto ca^{-1}$.

VI.9. Finally, we compare $\mathrm{ind}\,(\hat{\alpha})$ to $\mathrm{ind}\,(\hat{\alpha}^{-1})$. It is easy to find examples with $\mathrm{ind}\,(\hat{\alpha}) \neq \mathrm{ind}\,(\hat{\alpha}^{-1})$, (and hence $\mathrm{ind}\,(\alpha') \neq \mathrm{ind}\,({\alpha'}^{-1})$ for some α' representing $\hat{\alpha}$). Such examples occur even among irreducible automorphisms with irreducible powers. In particular there are positive automorphisms α satisfying $\mathrm{ind}\,(\alpha) = \mathrm{ind}\,(\hat{\alpha}) = n - \frac{3}{2}$, while $\hat{\alpha}^{-1}$ is parageometric and hence satisfies $\mathrm{ind}\,(\hat{\alpha}^{-t}) = n - 1$ for some $t \geq 1$. An example is the automorphism $\alpha:\ a \mapsto abc,\ b \mapsto bab,\ c \mapsto cabc$, which is the inverse of the one given in VI.8. Notice that this example also satisfies $\lambda(\hat{\alpha}) \neq \lambda(\hat{\alpha}^{-1})$; for further examples satisfying this last inequality see [BH].

VI.10. Notice that the automorphism $\alpha_3 = \alpha$ from VI.9 has 11 fixed points on δF, 5 of them being attractive, and 6 repelling. The analogue is true for $\alpha_4:\ a \mapsto abcd,\ b \mapsto bab,\ c \mapsto cabc,\ d \mapsto dabcd$, which has 7 attractive and 8 repelling fixed points at δF. We believe this pattern repeats for all the analogously defined α_n.

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