Pseudogroups of isometries of $\mathbf{R}$ and Rips' theorem on free actions on $\mathbf{R}$-trees

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Abstract : We give a shorter proof of Rips' theorem saying that a finitely generated group acting freely on an $\mathbf{R}$-tree is a free product of free abelian groups and surface groups. For this, we study the dynamical properties of finite sets of partial isometries of $\mathbf{R}$.

Résumé : Nous donnons une preuve plus courte du théorème de Rips disant qu'un groupe de type fini agissant librement sur un arbre réel est un produit libre de groupes abéliens libres et de groupes de surface. Pour cela, nous étudions les propriétés dynamiques des systèmes finis d'isométries partielles de $\mathbf{R}$.

Key words : group action, $\mathbf{R}$-tree, free group, surface group, partial isometry of $\mathbf{R}$.

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An $\mathbf{R}$-tree is an arcwise connected metric space in which every arc is isometric to an interval of $\mathbf{R}$. See for instance [Sha1, Sha2, Mor1] for historical remarks, references and motivations. This paper contains a proof of the following result, formerly known as the Morgan-Shalen conjecture [MS2] :

Theorem 0.1 *(E. Rips) Let G be a finitely generated group acting freely on an $\mathbf{R}$-tree. Then G is a free product of free abelian groups and surface groups.*

More generally, we shall prove :

Theorem 0.2 *Let G be a finitely generated group acting on an $\mathbf{R}$-tree T. Let G_e be the (normal) subgroup of G generated by all elements acting with a fixed point (elliptic elements). Then G/G_e is a free product of free abelian groups and surface groups.*

There is a restriction on the groups that may occur in theorem 0.1 : the fundamental group of a closed non-orientable surface of Euler characteristic ≥ -1 does not act freely [MS2]. This restriction does not exist in theorem 0.2. For instance, the example in [AY] leads to an action with $G/G_e \simeq \mathbf{Z}/2\mathbf{Z}$, the fundamental group of a projective plane (see [Lev4], example 5).

It is shown in [Lev4] that, given an action of G, there exists a canonical normal subgroup $H_0 \subset G$ such that the quotient space T/H_0, made Hausdorff, is an $\mathbf{R}$-tree $\widehat{T/H_0}$, and the natural action of G/H_0 on $\widehat{T/H_0}$ is free. Theorem 0.2 leads to an effective procedure for finding H_0 : consider the action of G/G_e on $\widehat{T/G_e}$ (an $\mathbf{R}$-tree by [Lev4], theorem 2), divide G/G_e by the subgroup generated by elliptic elements, and iterate. This process stops after finitely many steps (see lemma 2.2).

Fix the finitely generated group G. It is asked in [Lev4] whether G/H_0 is a free product of free abelian groups for a *generic* action of G. More generally, one may ask ("generic Lyndon conjecture") : is it true that, for a generic action of G, the groups H_0 and G_e are equal, and G/G_e is a free product of free abelian groups ?

We now sketch the proof of theorem 0.1 (the proof of theorem 0.2 is similar). We will also mention related results, to be found in [GLP2]. Our proof has been inspired by the one Rips sketched at the Isle of Thorns conference in july 1991 (see [Mor2], [BF]), but it is different in several aspects. In particular, Rips uses a combinatorial complexity introduced by Makanin [Mak] and Razborov [Raz]. We rely instead on ideas developed by Imanishi and Levitt [Ima, Lev1, Lev2] in the context of foliations.

So let G act freely on T (for simplicity, we assume here that G is finitely presented). Let $\{\gamma_1, \ldots, \gamma_p\}$ be a system of generators of G. Let $K \subset T$ be a finite subtree. Each γ_i gives rise to a (partially defined) isometry $g_i : K \cap \gamma_i^{-1}K \to \gamma_i K \cap K$.

If K is large enough, then G may be read off the dynamical system $\mathcal{T} = (K, \{g_i\})$: one gets a presentation of G in terms of the generators γ_i by taking as relations all words $\gamma_{i_1}^{\pm 1} \ldots \gamma_{i_n}^{\pm 1}$ such that there exists $x \in K$ with $g_{i_1}^{\pm 1} \circ \ldots \circ g_{i_n}^{\pm 1}(x) = x$ (staying in K).

This leads to an important idea of Rips' : since theorem 0.1 is about the group G, not about the action, we may forget about T and concentrate on $\mathcal{T}$; the loss of dynamical information involved in this process is not important if one only wants to prove theorem 0.1. In [GLP2], we shall associate to $\mathcal{T}$ a free action of G, and show that this action converges to the original one when K grows to exhaust T.

A simple manipulation allows us to replace $\mathcal{T}$ by our real object of study : a system X consisting of a finite disjoint union D of compact subintervals of $\mathbf{R}$, together with a finite number of partially defined isometries $\varphi_j : A_j \to B_j$, where each *base* A_j, B_j is a compact subinterval of D (for simplicity, we assume here that every component of D and every base has positive length). One may associate a group $G(X)$ to a system X (either by giving generators and relations, as before, or by using a foliated 2-complex $\Sigma(X)$), and thus recover G. Our task then is to determine $G(X)$.

Consider the orbits of X (2 points $x, y \in D$ belong to the same orbit if some word in the generators φ_j and their inverses takes x to y). Furthermore, a given orbit of X has the structure of a metric graph, well defined up to quasi-isometry (just like the Cayley graph of a finitely generated group). It is then possible (see [Gab]) to relate dynamical properties of X to properties of these graphs such as the growth of balls or the number of ends.

The system X splits up canonically into finitely many pieces (theorem 3.1). On each piece, either every orbit is finite or every orbit is dense. This follows from a theorem about singular measured foliations proved by Imanishi [Ima] in 1979 (we provide a direct proof, based on the appendix of [AL]). Imanishi's theorem was rediscovered, in the slightly generalized context of laminations, by Morgan-Shalen [MS1].

This dynamical decomposition of X corresponds to a decomposition of $G(X)$ as a free product (proposition 3.5). Finite orbits are easy to analyze ($G(X)$ is free if every orbit is finite, see corollary 3.7), so that we assume from now on that X is *minimal* : every orbit is dense.

For $t > 0$ small, we define $X_{-t} = (D, \{\varphi_j^{-t}\})$ as follows : if the domain of φ_j is $A_j = [a_j, b_j]$, then φ_j^{-t} is the restriction of φ_j to $[a_j + t, b_j - t]$.

We now have the following dichotomy (cf. [Lev1], lemme III.5) : either every orbit of X_{-t} is finite, or X is *homogenous* (proposition 4.1). By X homogenous, we mean that there exists a finitely generated dense subgroup $P \subset \mathbf{R}$ such that 2 points x, y in the same component of D belong to the same orbit if and only if $x - y \in P$ (there is another type of homogenous system, which we do not mention here). If X is homogenous, it is easy to show that $G(X)$ is isomorphic to P, hence free abelian (proposition 4.2).

Define $\overset{\circ}{G}(X)$ as the direct limit of $G(X_{-t})$ as t goes to 0 (see section 1). It is equal to $G(X)$ for generic X. At this point it is easy to show (using an argument similar to lemma 2.2 below) that $\overset{\circ}{G}(X)$ is a free product of free abelian groups (when all maps φ_j are orientation-preserving, this is essentially théorème 1 of [Lev2]; the proof of [Lev2] extends to the general case [Gus]). Of course we need to compute $G(X)$ for all X, so that we go on.

We say that the generators φ_j are independent if the following holds : if a nontrivial reduced word $\varphi_{i_1}^{\pm 1} \circ \ldots \circ \varphi_{i_n}^{\pm 1}$ has a fixed point, then its domain consists only of that point.

Assuming that X is minimal but not homogenous, we will show (proposition 5.1) how to replace X by an *"equivalent" system Y with independent generators*. Generalizing the equivalence of pseudogroups introduced by Haefliger [Hae1, Hae2], we will take some extra time in [GLP2] to define what it means for systems X, Y to be equivalent. Equivalence implies in particular that the associated groups are isomorphic.

The quest for independent generators was initiated in [Lev2], théorème 5, see also [Rim2]. The optimal result is due to Gaboriau [Gab] : if X is as above, one may replace each φ_j by its restriction to a closed (possibly empty) subinterval of A_j, so as to get independent generators for a system having the same orbits as X. Note that this process does not increase the number of generators.

We assume from now on that X is minimal and that the generators φ_j are independent. Then *the sum of the lengths of the intervals A_j is equal to the total length of D* (proposition 6.1). This is a special case of a much more general result, see [Lev3], cor. II.5 and [Lev5]. Involved here is the *amenability* of the equivalence relation on D whose classes are the orbits of X.

First suppose that every $x \in D$ belongs to at least 2 bases. Then all but finitely many points belong to *exactly* 2 bases, and we say that X is an *interval exchange*. Even if D is connected, this is a generalization of the usual notion [Kea], as in [DN]. For one thing, some of the maps φ_j may reverse orientation. Furthermore, the bases determine 2 partitions of D (up to finitely many points), but for a given j the bases A_j and B_j may belong to the *same* partition. Geometrically, an interval exchange corresponds to a (possibly non-orientable) measured foliation on a (possibly non-orientable) closed surface (of arbitrary Euler characteristic), and $G(X)$ is a surface group (proposition 6.4).

If X is not an interval exchange, let $N \subset D$ be the open set consisting of points belonging to only one base. We define a new system X_1 on $D_1 = D \setminus N$, replacing the generator φ_j by its restriction(s) to $D_1 \setminus \varphi_j^{-1}(N)$ (this may increase the number of generators). It is easy to check that $G(X_1) = G(X)$. Iterate this operation if possible (in Rips' proof, this elementary operation has to be combined with others, so that a certain complexity does not increase).

If this process terminates, some X_n is an interval exchange and $G(X)$ is a surface group. Suppose it does not (what Rips calls the *Levitt case*, cf. [Lev3], [Lev4]). In this case, one shows that the intersection of the family D_n is *nowhere dense* in D (proposition 7.1).

Finally, one proves (section 8) that $G(X_n)$ is *free* for n large enough, completing the proof. This last argument, in our opinion the main novelty in Rips' proof, may be sketched as follows.

Given $X = (D, \{\varphi_j\})$, there is a presentation of $G(X)$ in terms of generators $\overline{\varphi_j}$. There are 2 kinds of relators. First, certain generators ($p-1$ of them if D has p components) have to be set equal to 1. Then one takes as relators words in the $\overline{\varphi_j}^{\pm 1}$ such that the corresponding word in the $\varphi_j{}^{\pm 1}$ has a fixed point. One may restrict relations of the second kind to a fixed, finite set $\mathcal{R}$.

Relations in $\mathcal{R}$ carry over to each X_n, but the labelling of the generators changes : typically, a generator φ_j of X is replaced by many generators of X_n, each defined on a small subinterval of D_n.

Of course a given generator $\overline{\varphi_j}$ of $G(X)$ may appear many times in $\mathcal{R}$. But recall that $\bigcap D_n$ is nowhere dense. It follows that, when the relations are considered for X_n (n large), the total number of occurrences of a given generator and its inverse is at most one, so that $G(X_n)$ is free.

In [GLP2], the systems X (Rips' Unidentified Combinatorial Objects) are interpreted as finite generating systems of closed pseudogroups. We generalize [Lev3], lemme VIII.1, to prove an important technical fact. Suppose $\Phi = \{\varphi_j\}$ and $\Psi = \{\psi_k\}$ generate systems X, Y having the same orbits on a given multi-interval D. Then Φ *admits a finite refinement Φ' such that every element of Φ' may be expressed as a word in the elements of Ψ and their inverses*. We also show that X is *segment closed* (a property introduced by Rimlinger [Rim1]).

This has several consequences. First of all, one may associate to X a free action of $G(X)$ on some $\mathbf{R}$-tree T, provided X satisfies (a property similar to) the following "no reflection condition" : there is no x such that $x+t$ is in the orbit of $x-t$ for $t > 0$ small. Note that there are examples with $G(X) \simeq \mathbf{Z}/2\mathbf{Z}$; this condition rules them out. This condition is satisfied if X is obtained from a free action on a tree as above. In this case, the corresponding free actions are approximations of the original action.

The tree T may be viewed geometrically as follows. There is a compact foliated 2-complex $\Sigma(X)$ canonically associated to X, and $G(X)$ is obtained from $\pi_1\Sigma(X)$ by killing all loops contained in leaves. Let $\Sigma'(X) \to \Sigma(X)$ be the covering with transformation group $G(X)$. The tree T is the space of leaves of the lifted foliation on $\Sigma'(X)$. The absence of reflection implies that every leaf in $\Sigma'(X)$ is closed, and the technical property mentioned above implies that the leaf space is Hausdorff.

Another consequence is the fact that two systems on the same multi-interval D with the same orbits are equivalent in a strong sense. In particular, the group $G(X)$ depends only on the orbits of X, not on a particular system of generators. See also [Rim3].

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1 Systems of isometries

A *multi-interval* D is a union of finitely many disjoint closed intervals of $\mathbf{R}$. Components of D may be *degenerate intervals*, i.e. consist of only one point.

Definition 1.1 *A system of isometries is a pair $X = (D, \{\varphi_j\}_{j=1,\cdots,k})$, where D is a multi-interval and each $\varphi_j : A_j \to B_j$ is an isometry between closed (possibly degenerate) subintervals of D.*

The intervals A_j, B_j are called *bases*. A system of isometries X is said to be *connected* if any two components I, I' of D are equivalent under the equivalence relation generated by $I \sim I'$ if there exists $j \in \{1, \cdots, k\}$ with $A_j \subset I$ and $B_j \subset I'$.

A generator $\varphi_j : A_j \to B_j$ is *a singleton* if A_j is degenerate.

A system of isometries X is said to be *non degenerate* if every component of D and every base A_j, B_j has positive length.

A φ-word is a word in the generators $\varphi_j^{\pm 1}$. It is a partial isometry of D, whose domain (defined in the obvious maximal way) is a closed interval (possibly degenerate or empty). The domain of composition of two partial isometries corresponding to two reduced words may be smaller than the domain of the partial isometry associated to the reduced product word (for instance, consider the product of the words φ_{i_1} and $\varphi_{i_1}^{-1}$).

Two points x, y in D *belong to the same X-orbit* if there exists a φ-word sending one to the other. We denote the orbit of x by $X(x)$. Note that the orbits are countable.

If φ_j is not a singleton, define $\overset{\circ}{\varphi}_j : \overset{\circ}{A}_j \to \overset{\circ}{B}_j$ as the restriction of φ_j to the interior of A_j. A $\overset{\circ}{\varphi}$-word is a word in the generators $\overset{\circ}{\varphi}_j^{\pm 1}$. Its domain is a (possibly empty) open interval. Let $\overset{\circ}{X}(x)$ be the orbit of x under the *pseudogroup* generated by $\overset{\circ}{\varphi}$-words.

An orbit of X or $\overset{\circ}{X}$ is *singular* if it consists of an endpoint of D, or if it meets some ∂A_j or ∂B_j, and is *regular* otherwise. Note that an orbit of $\overset{\circ}{X}$ is contained in an orbit of X with equality except perhaps for a finite number of them, the singular ones.

We can associate a sign $\pm$ to every $\overset{\circ}{\varphi}$-word with nonempty domain : it is precisely the value of the derivative of the associated global isometry of $\mathbf{R}$. We call $\overset{\circ}{X}{}^{+}(x)$ the orbit of x under positive $\overset{\circ}{\varphi}$-words (i.e. restrictions of translations).

A *reflection* is a negative $\overset{\circ}{\varphi}$-word having a fixed point, the *center* of the reflection. For the proof of theorem 0.1, it is enough to consider systems without reflections (see section 2). The orbit of $x \in \overset{\circ}{D}$ by $\overset{\circ}{X}$ is *one-sided* if x is the center of a reflection, and *two-sided* otherwise.

If X is a system of isometries on a multi-interval D, we define a *foliated 2-complex* $(\Sigma(X), \mathcal{F})$ (or simply Σ) associated to X. Start with the disjoint union of D (foliated by points) and strips $A_j \times [0,1]$ (foliated by $\{*\} \times [0,1]$). We get Σ by glueing the $A_j \times [0,1]$ to D, identifying each $(t,0) \in A_j \times \{0\}$ with $t \in A_j \subset D$ and each $(t,1) \in A_j \times \{1\}$ with $\varphi_j(t) \in B_j \subset D$. We will identify D with its image in Σ.

The foliation $\mathcal{F}$ is the decomposition of Σ into the leaves. A leaf is an equivalence class for the equivalence relation $\sim$ generated by $x \sim y$ if there is a $j = 1, \cdots, k$ with x, y corresponding to two points in the same leaf $\{*\} \times [0,1]$ of $A_j \times [0,1]$. Two points of D are in the same leaf of $\mathcal{F}$ if and only if they are in the same X-orbit. For instance, if a point in D belongs to no base, then its leaf consists of itself.

This suspension process is well known for interval exchanges. See J. Morgan's notes [Mor2] for the first appearance under the above generality, and [AL][Lev4] for suspensions as measured foliations with Morse singularities on manifolds.

It is clear that X is connected if and only if Σ is connected. In what follows, we assume that X is connected.

The 2-complex Σ has the homotopy type of a finite graph, so that its fundamental group $\pi_1(\Sigma)$ is a finitely generated free group. We will denote by $\overline{\mathcal{L}}$ the normal subgroup of $\pi_1(\Sigma)$ normally generated by the free homotopy classes of loops contained in leaves of $\mathcal{F}$.

Definition 1.2 *If X is a system of isometries, we define $G(X) = \pi_1(\Sigma)/\overline{\mathcal{L}}$.*

In the case where D is connected, there is an easy presentation of the group $G(X)$ associated to a system of isometries X. The generators are the elements φ_j, and the relations are φ-words having a fixed point. See [Lev2] for instance. See Appendix 1 of [GLP2] for Rips' combinatorial definition when D is not connected.

Our goal will be to prove :

Theorem 1.3 *(E. Rips) Let X be a connected system of isometries. Then $G(X)$ is a free product of free abelian groups and surface groups.*

In the next section, we show that this theorem implies theorem 0.2, which obviously implies theorem 0.1.

There are two other interesting groups $\overset{\circ}{G}(X)$ and $G^+(X)$ associated to a given X. They are studied in [Lev2] and [Gus]. They are related to actions on simply connected non-Hausdorff 1-manifolds.

Assume for simplicity that X is non-degenerate. Let $\operatorname{int}\Sigma$ be the complement in Σ of the open segments $\partial A_j \times (0,1)$, and $\overset{\circ}{\mathcal{F}}$ the foliation induced on $\operatorname{int}\Sigma$. We let $\mathcal{L}$ be the normal subgroup of $\pi_1(\operatorname{int}\Sigma) \simeq \pi_1(\Sigma)$ normally generated by the free homotopy classes of loops contained in leaves of $\overset{\circ}{\mathcal{F}}$, and we define $\overset{\circ}{G}(X) = \pi_1(\Sigma)/\mathcal{L}$.

As mentioned in the introduction, $\overset{\circ}{G}(X)$ is the direct limit of $G(X_{-t})$ as $t > 0$ goes to 0, and $\overset{\circ}{G}(X) = G(X)$ for generic X. Furthermore $\overset{\circ}{G}(X)$ is a free product of free abelian groups; the factors of rank ≥ 2 are isomorphic to the groups of periods of the orientable homogenous components of X (in the sense of section 4). This implies that $G(X)$ is finitely presented, since the set of left classes $\overline{\mathcal{L}}/\mathcal{L}$ is finite. But we won't need this a priori fact, it will follow from our arguments.

Now let $\mathcal{L}^+$ be the normal subgroup of $\pi_1(\Sigma)$ normally generated by the free homotopy classes of loops contained in leaves of $\overset{\circ}{\mathcal{F}}$ and having trivial holonomy, and $G^+(X) = \pi_1(\Sigma)/\mathcal{L}^+$. This group is the fundamental group of Haefliger's classifying space $B\overset{\circ}{X}$. By [Gus], it is a free product whose factors are surface groups, free abelian groups, $\mathbf{Z}/2\mathbf{Z}$, and dihedral groups (twisted product) $\mathbf{Z}^a \times \mathbf{Z}/2\mathbf{Z}$ $(a \geq 2)$; the free abelian factors of rank ≥ 2 are as before, while the dihedral factors are the groups P associated to the nonorientable homogenous components of X (see section 4).

2 From actions on R-trees to systems of isometries

We define an action of a group on a metric space to be a left isometric action. A *branch point* in an $\mathbf{R}$-tree T is a point x such that $T \setminus \{x\}$ has at least 3 components. A finite tree is a compact **R**-tree which is the convex hull of finitely many points, hence has only finitely many branch points.

Let G be a finitely generated group acting on an **R**-tree T, and $\{\gamma_1, \cdots, \gamma_p\}$ a fixed system of generators for G. Let K be a finite subtree of T . We define a system of isometries $X = X(K)$ as follows.

Let $I_1, \cdots, I_n$ be the segments of K that are the closures of the connected components of K minus its branch points. Consider them as embedded disjointly in **R**, and let D be their union. The system X has 2 types of generators.

For every branch point b of K, let $x_{i_1}, \cdots, x_{i_s}$ be the endpoints of the segments $I_{i_1}, \cdots, I_{i_s}$ corresponding to b. Consider the following finite set of singletons on D : $\Phi_0 = \{x^b_{i_1} \mapsto x^b_{i_2}, \cdots, x^b_{i_1} \mapsto x^b_{i_s}$ / b branch point of $K\}$. We could have taken all the possible pairs, but those one suffice.

Now the elements $\gamma_1, \cdots, \gamma_p$ of Γ define partial isometries of K (defined on a maybe empty closed finite subtree of K) $g_i : K \cap \gamma_i^{-1}(K) \to \gamma_i(K) \cap K$ with $g_i(t) = \gamma_i(t)$. The partial isometries g_i of K induce partial isometries of D in the natural way. That is, for every $1 \leq i, j \leq n$ and $1 \leq k \leq p$, g_k defines, by maximal restriction, an isometry φ_{ijk} between a closed interval of I_i and a closed interval of I_j. Set

$$\Phi = \Phi_0 \cup \{\varphi_{ijk}\}_{1 \leq i,j \leq n,\, 1 \leq k \leq p}$$

and $X = (D, \Phi)$. If the action on T is free, then no $\mathring{\Phi}$-word is a reflection.

Lemma 2.1 *The group $G(X(K))$ is generated by $\{\gamma_1, \ldots, \gamma_p\}$, relations being words $\gamma_{i_1}^{\varepsilon_1} \ldots \gamma_{i_n}^{\varepsilon_n}$ ($\varepsilon_i = \pm 1$) such that there exists $x \in K$ with $\gamma_{i_1}^{\varepsilon_1} \ldots \gamma_{i_n}^{\varepsilon_n}(x) = x$ and $\gamma_{i_m}^{\varepsilon_m} \ldots \gamma_{i_n}^{\varepsilon_n}(x) \in K$ for $1 \leq m \leq n$.*

Proof. Let $\Sigma(K)$ be the foliated 2-complex obtained (as in the construction for D) by glueing K (foliated by points) and strips $K_i \times [0,1]$ (foliated by $\{*\} \times [0,1]$), where $K_i = K \cap \gamma_i^{-1} K$, by identifying each $(t, 0) \in K_i \times \{0\}$ with $t \in K_i \subset K$ and each $(t, 1) \in K_i \times \{1\}$ with $g_i(t) \in g_i(K_i) \subset K$. If $\overline{L}(K)$ is the subgroup of $\pi_1\Sigma(K)$ normally generated by closed loops in leaves, then $\pi_1\Sigma(K)/\overline{L}(K)$ obviously has the presentation of the statement of the lemma.

Now first make a foliated homotopy equivalence on $\Sigma(K)$ consisting in pinching to a point every leaf correponding to an element of Φ_0. The new foliated 2-complex is obtained from $\Sigma(X(K))$ by cutting open along leaves $\{*\} \times]0, 1[$ corresponding to the subdivision of g_k into the φ_{ijk} for every k. The new generators introduced in $\pi_1\Sigma(K)$ are killed when we take the quotient by loops in leaves. $\square$

Now consider G as in theorem 0.2. The convex hull of any orbit is an invariant subtree. Since G is countable, we may assume that T is the union of an increasing sequence of finite subtrees K_n. By lemma 2.1, there are natural epimorphisms $\rho_n : G(X(K_n)) \to G(X(K_{n+1}))$, and G/G_e is the direct limit of the sequence ρ_n. Indeed, if F_p is the free group on $\gamma_1, \cdots, \gamma_p$, then the natural maps $F_p \to G(X(K_n))$ defined in lemma 2.1 induce epimorphisms $G \to G(X(K_n))$ that commute with the ρ_n, hence defines an epimorphism from G to the direct limit. The kernel is obviously G_e. Theorem 0.2 then follows from theorem 1.3 and the following fact :

Lemma 2.2 *Suppose $\rho_n : G_n \to G_{n+1}$ is a sequence of epimorphisms, where each G_n is a finitely generated free product of free abelian groups and surface groups. Then ρ_n is an isomorphism for n large enough.*

Proof. First note that only finitely many isomorphism classes of groups may appear in the sequence, since the G_n have a bounded minimal number of generators. Then observe that the groups G_n are residually finite by [Gru], hence hopfian, that is are not isomorphic to proper factors. $\square$

Remark. If G is a finitely presented group acting freely on T, then it follows from the above lemmae that there is a finite subtree K_n of T such that $G \simeq G(X(K_n))$. As Rips has suggested, an explicit K_n may be taken to be the convex hull of the finitely many points $w_i x$ where x is any base point in T, and the w_i's are the generators of G and the words in the γ_i that appear as final subwords (including the empty one) of a finite set of relations for G. Also note that $X(K_n)$ has no reflection.

3 Imanishi's theorem : minimal components

Let X be nondegenerate. Let $E \subset D$ be the union of all finite singular $\overset{\circ}{X}$-orbits. It is finite and contains all endpoints of D, so that $D \setminus E$ is a finite union of open intervals.

Consider the equivalence relation generated on $\pi_0(D \setminus E)$ by saying that two intervals are equivalent if some $\overset{\circ}{X}$-orbit meets them both. Let U be the union of all intervals in an equivalence class. The following theorem states that either U contains only finite $\overset{\circ}{X}$-orbits, or every $\overset{\circ}{X}$-orbit contained in U is dense in U. The important point (compare [Ima], theorem 1) is that every orbit (of X or $\overset{\circ}{X}$) is finite or locally dense : the closure of an orbit cannot be a Cantor set.

Theorem 3.1 *$D \setminus E$ is a disjoint union of open $\overset{\circ}{X}$-invariant sets $U_1, \ldots, U_p$, where each U_i admits one of the following descriptions :*

- *family of finite orbits : U_i consists of intervals of equal length meeting only finite $\overset{\circ}{X}$-orbits ; the family may be untwisted (every $\overset{\circ}{X}$-orbit contained in U_i is two-sided and meets each interval exactly once) or twisted (there is a one-sided orbit meeting each interval once, while all other orbits are two-sided and meet each interval twice).*

- *minimal component : every $\overset{\circ}{X}$-orbit contained in U_i is dense in U_i.*

Remark 3.2
- A minimal component U_i may consist of several intervals.
- Twisted families of finite orbits may occur only if X contains reflections.
- The number p may be bounded in terms of k and the number of components of D.
- We define the *width* e_i of a family of finite orbits as the common length of components of U_i (half this length if the family is twisted).

Proof. (following [AL], appendice) If L is a finite regular $\overset{\circ}{X}$-orbit, orbits close to it are also finite, with the same cardinality (twice this cardinality if L is one-sided). These orbits may be pushed until orbits in E are reached. This shows that intervals of $D \setminus E$ meeting finite $\overset{\circ}{X}$-orbits belong to families of finite orbits.

Claim. No leaf closure is a Cantor set. More precisely, let L be an infinite $\overset{\circ}{X}$-orbit. We claim that *there exists $\delta > 0$ such that every component of $D \setminus \overline{L}$ has length $\geq \delta$.*

To prove this, choose $\delta > 0$ such that :

– any 2 points in E have distance $> \delta$;

– if a is an endpoint of a base A_j, then the distance between a and $\overset{\circ}{A}_j \cap L$ is either 0 or $> \delta$.

Suppose J is a component of $D \setminus \overline{L}$ of length $c < \delta$. At least one endpoint x of J has infinite $\overset{\circ}{X}$-orbit. Indeed, if both endpoints have finite orbits, they cannot be regular, because the infinite L accumulates on them, hence are singular, contradicting the first assumption on δ. Choose $y, z \in \overset{\circ}{X}(x)$ such that $z \in \overset{\circ}{X}^{+}(y)$ and $0 < |y - z| < c$. Interchanging y and z if necessary, fix a word $\overset{\circ}{\varphi}{}_{j_q}^{\varepsilon_q} \circ \ldots \circ \overset{\circ}{\varphi}{}_{j_1}^{\varepsilon_1}$ $(\varepsilon_i = \pm 1)$ taking x to y and sending some subinterval of J between y and z. Define $x' \in J$ by $|x - x'| = |y - z|$.

Since x' does not belong to $\overset{\circ}{X}(z) = L$, there is an i such that $\overset{\circ}{\varphi}{}_{j_i}^{\varepsilon_i} \circ \ldots \circ \overset{\circ}{\varphi}{}_{j_1}^{\varepsilon_1}$ is not defined at x'. Considering the smallest such i shows that $\overset{\circ}{\varphi}{}_{j_{i-1}}^{\varepsilon_{i-1}} \circ \ldots \circ \overset{\circ}{\varphi}{}_{j_1}^{\varepsilon_1}(x')$ (x if $i = 1$), which is not in $\overline{L}$, is an endpoint of the domain of $\overline{\varphi}_{j_i}$, at distance $< \delta$ from $\overset{\circ}{\varphi}{}_{j_{i-1}}^{\varepsilon_{i-1}} \circ \ldots \circ \overset{\circ}{\varphi}{}_{j_1}^{\varepsilon_1}(x) \in \overline{L}$. This contradicts the way δ was chosen, thus proving the claim. □

To complete the proof of the theorem, we consider an infinite $\overset{\circ}{X}$-orbit L_1 meeting a component J_1 of $D \setminus E$, and we show that L_1 is dense in J_1.

Suppose not. Let L be an $\overset{\circ}{X}$-orbit in the frontier of L_1 in J_1. Since L is infinite, we can find δ as above. Choose $y, z \in L$ with $0 < |y - z| < \delta$ and $z \in \overset{\circ}{X}^{+}(y)$. Then by definition of δ, the segment $[y, z]$ is contained in $\overline{L}$, hence in $\overline{L_1}$. Since y and z are in the same $\overset{\circ}{X}^{+}$-orbit, this implies that L belongs to the interior of $\overline{L_1}$, a contradiction. □

Definition 3.3 *A nondegenerate X is minimal if $D \setminus E$ consists of a single minimal component U.*

This implies that every X-orbit is dense. Indeed, more generally, let $a \in E$ be in the closure of a minimal component U of a nondegenerate X. Let J be a closed nondegenerate interval with $a \in \partial J$ and $J \setminus \{a\} \subset U$. Consider images of a by φ-words defined on some nondegenerate interval $[a, b] \subset J$. At least one of them has to be in U : otherwise they would all be in the finite set E, and X-orbits meeting J near a would be finite. In particular, $\overline{X(a)}$ contains U.

Remark. Together with a simple compactness argument, this leads to the following fact (noticed by E. Rips) : given a closed interval H of positive length in a minimal component U, there exists an integer N such that every $x \in \overline{U}$ can be sent into H by a φ-word of length $\leq N$.

Definition 3.4 *A system of isometries X is pure if X is connected, nondegenerate, and E consists only of endpoints of D. Note that either $D \setminus E$ is a family of finite orbits or X is minimal.*

Proposition 3.5 *Let X be connected, possibly degenerate. There exist $X_1, \ldots, X_p$ pure such that $G(X) \simeq G(X_1) * \ldots * G(X_p) * F$, with F a free group.*

Proof. We use 2 simple operations.

– Removing a singleton. Let X' be obtained from X by removing a singleton $\varphi_j : \{a\} \to \{b\}$. It is easy to check that $G(X)$ and $G(X')$ are related as follows.

If X' is connected, then $G(X) \simeq G(X')$ if $b \in X'(a)$, while $G(X) \simeq G(X') * \mathbf{Z}$ if $b \notin X'(a)$. Indeed, in the first case, if one adds a band $A_j \times [0,1]$ to the foliated 2-complex $\Sigma(X')$ such that the points of $X' \subset \Sigma(X')$ joined by a leaf in $A_j \times [0,1]$ were already in the same leaf on $\Sigma(X')$, then $\pi_1 \Sigma(X)$ has a new free generator immediately killed in $G(X)$. In the second case, the new generator is involved in no relation.

If X' has 2 components X_1 and X_2, then $G(X) \simeq G(X_1) * G(X_2)$. Indeed, $\Sigma(X)$ is obtained from $\Sigma(X_1), \Sigma(X_2)$ by joining them with an arc $A_j \times [0,1]$. Hence $\pi_1\Sigma(X) \simeq \pi_1\Sigma(X_1) * \pi_1\Sigma(X_2)$ and only relations in the free factors and their product may appear.

– Splitting an interval. Let x be an interior point of a component I of D. Assume that x is not the domain or range of a singleton φ_j. We are going to split I apart at x.

First we split bases A_j containing x in their interior, replacing φ_j by two generators : the restrictions of φ_j to the closures of the components of $A_j \setminus \{x\}$. Split bases B_j similarly. This does not change $G(X)$.

Then replace I by two disjoint intervals I_1 and I_2, disjoint from the other components of D, isometric to the closures of the components of $I \setminus \{x\}$, so that x is replaced by two points x_1, x_2. We get a new multi-interval D'. Each generator gives rise to a partial isometry of D' in the obvious way, defining X' on D'. To relate $G(X)$ and $G(X')$, simply note $G(X) \simeq G(X'')$, where X'' is obtained from X' by adding a singleton taking x_1 to x_2.

To prove the proposition, first remove singletons and isolated points of D, so as to get a non-degenerate X_{nd}. Then define E as before, and split D apart at each point of $E \cap \overset{\circ}{D}$. We get $X_1, \ldots, X_p$ corresponding to the sets $U_1, \ldots, U_p$ associated to X_{nd} by theorem 3.1. □

Theorem 1.3 will now follow from :

Proposition 3.6 *If X is pure, then $G(X)$ is either a free group, or a free abelian group, or a surface group.*

Recall that either X is minimal or $\overset{\circ}{D}$ is a family of finite orbits. In the first case, the result will follow from propositions 4.2, 6.4 and section 8. We settle the second case right away, as follows.

If the family is untwisted, Σ is the product of an interval by a finite 1-complex σ, with the product foliation $\{*\} \times \sigma$. If it is twisted, Σ is a twisted interval bundle over a finite 1-complex σ. In both cases $G(X)$ is trivial, since every cycle in σ is freely homotopic into a leaf in Σ.

Corollary 3.7 *If X is a system of isometries whose orbits are finite, then $G(X)$ is free.* □

4 Homogeneous components and discrete approximation

Let X be non-degenerate. A minimal component U is *orientable homogenous* if there exists an interval $J \subset U$ of positive length, and a dense subgroup $P \subset \mathbf{R}$, such that $x, y \in J$ belong to the same $\overset{\circ}{X}$-orbit if and only if $x - y \in P$. This property then holds for every interval contained in U, with the same P. The group P should be viewed as a *group of periods*, see below.

If reflections are allowed, there may also be *nonorientable homogenous components*. The definition is the same as above, but now P is a dense subgroup of Isom($\mathbf{R}$) (well defined up to conjugacy), and $x, y \in J$ belong to the same $\overset{\circ}{X}$-orbit if and only if they belong to the same P-orbit.

Note that P has to be finitely generated, since it is a subgroup of the finitely generated subgroup of Isom($\mathbf{R}$) generated by the global isometries of $\mathbf{R}$ whose restrictions are the $\overset{\circ}{\varphi}_j$'s.

Remark. Orientable homogenous components were called weakly complete in [Lev2], complete or equivalent to a group in [Lev3]. Rips calls them axial because they correspond to free actions on $\mathbf{R}$. Of course the word *homogenous* applies to more general situations.

Recall that $X_{-t} = (D, \{\varphi_j^{-t}\})$, where φ_j^{-t} is the restriction of φ_j to $[a_j + t, b_j - t]$ (with $A_j = [a_j, b_j]$). Note that X_{-t} is defined for $t \geq 0$ small enough.

Proposition 4.1 *Assume no minimal component of X is homogenous. For $t > 0$, all orbits of X_{-t} are finite.*

Remark. (cf. [Lev1]) If X has a homogenous component U and $K \subset U$ is compact, then K is contained in a homogenous component of X_{-t} for t small. In the other direction, define X_t by enlarging the domain of each φ_j. Then every minimal component of X is contained in a homogenous component of X_t for $t > 0$.

Proof. (following [Lev1], proof of lemme III.5) We suppose that X_{-t} has a minimal component V, and we prove that X has a homogenous component. Fix an interval $J = [a, a+\ell] \subset V$ of length $\ell \in (0, t)$ with $b \in \overset{\circ}{X}{}^{+}_{-t}(a)$. In the rest of the argument, all points x_0, y_0, x, y will be in J.

First note the following : if x_0, y_0 are in the same $\overset{\circ}{X}{}^{+}_{-t}$-orbit, and $x - y \equiv x_0 - y_0 \bmod \ell$, then $y \in \overset{\circ}{X}{}^{+}(x)$. If X contains no reflection, it follows that only positive elements of $\overset{\circ}{X}$ can send a point of J to a point of J.

Let P_0 be the subgroup of $\mathbf{R}$ generated by $\{x_0 - y_0 / y_0 \in \overset{\circ}{X}{}^{+}_{-t}\}$. It is dense, and $x - y \in P_0$ implies $y \in \overset{\circ}{X}{}^{+}(x)$.

Consider a $\overset{\circ}{\varphi}$-word γ taking some $x_0 \in J$ to some $y_0 \in J$. If γ is positive, we claim that $x - y \equiv x_0 - y_0 \bmod \ell$ implies $y \in \overset{\circ}{X}{}^{+}(x)$. Indeed, choose $p_0 \in P_0$ such that $x + p_0$ belongs to the domain of γ. Then the $\overset{\circ}{X}{}^{+}$-orbit of x contains $\gamma(x + p_0)$, which is congruent to y mod P_0. If γ is negative, a similar argument shows that $x + y \equiv x_0 + y_0 \bmod \ell$ implies $y \in \overset{\circ}{X}(x)$.

It is now clear that the minimal component of X containing V is homogenous. If it is orientable, the group P is generated by $\{x_0 - y_0 \mid y_0 \in \overset{\circ}{X}{}^{+}(x_0)\}$. □

Proposition 4.2 *Suppose X is pure and $\overset{\circ}{D}$ is a homogenous minimal component. If the component is orientable, then $G(X)$ is free abelian (isomorphic to P). If the component is not orientable, then $G(X)$ is trivial.*

Proof. [Conceptually, the system X must be thought of as equivalent to the action of P on $\mathbf{R}$ (cf. [GLP2]). Arguing as in [GLP2], one then gets that $G(X)$ is isomorphic to the quotient of P by the subgroup generated by elements acting with a fixed point, that is P or $\{1\}$. We provide here a direct proof. The argument is similar to the proof of prop. I.2 of [Lev2], which says that $\overset{\circ}{G}(X) \simeq P$ in the orientable case.]

First a remark. Let $[a, a+\tau]$ and $[b, b+\tau]$ be contained in $\overset{\circ}{D}$, with $b \in \overset{\circ}{X}{}^{+}(a)$. Since $\overset{\circ}{D}$ is homogenous, we can choose a path σ_t joining $a + t$ to $b + t$ in a leaf of $\overset{\circ}{\mathcal{F}}$ (defined in section 1) for each $t \in [0, \tau]$. Let λ_t be the loop consisting of $\sigma_0, [b, b+t], \sigma_t, [a, a+t]$. Its class $[\lambda_t]$

in $\overset{\circ}{G}(X)$, hence in $G(X)$, is independent of the choice of σ_0 and σ_t, and it is locally constant in t. It follows that $[\lambda_t] = 1$ for all $t \in [0, \tau]$. A similar result holds for intervals $[a, a+\tau]$ and $[b-\tau, b]$ if some negative $\overset{\circ}{\varphi}$-word takes a to b.

Since X is pure, this remark (and an easy compactness argument) makes it possible to represent any class in $G(X)$ by a piecewise smooth loop γ consisting of segments α_i tangent to $\mathcal{F}$ and segments $\beta_i \subset D$, with γ *quasi-transverse* in the following sense (cf. [FLP]) : for each i, a segment of a leaf of $\mathcal{F}$ close to α_i meets β_{i-1} or β_i, but not both.

If $\overset{\circ}{D}$ is an orientable component, we may change the embedding $D \to \mathbf{R}$ so that all maps φ_j preserve orientation. Given $\gamma \subset \Sigma$ as above (but not necessarily quasi-transverse), define $\rho(\gamma) \in P$ by following γ and adding the variations of the x-coordinate along the segments β_i. [One may think of $\mathcal{F}$ as being defined by a closed differential 1-form ω (equal to dx on D), and $\rho(\gamma)$ is simply $\int_\gamma \omega$. The group P is the group of periods of ω.]

This defines an epimorphism $\rho : \pi_1\Sigma \to P$ which factors through $G(X)$. If γ is quasi-transverse, there is no cancellation in the computation of $\rho(\gamma)$, so that ρ induces an isomorphism $G(X) \to P$.

If the component is not orientable, it is easy to see that $G(X)$ is trivial, using the arguments given above and the fact that centers of reflections are dense in D. □

5 Independent generators

Say that the generators φ_j are *independent* if no nontrivial reduced word in the $\overset{\circ}{\varphi}{}_j^{\pm 1}$ with nonempty domain is a restriction of the identity (in particular, X contains no reflection). Equivalently (when X is connected), the group $\mathcal{L} \subset \pi_1\Sigma$ is trivial. This may be shown to imply that X has no homogenous minimal component.

Proposition 5.1 *Let X be connected nondegenerate without homogenous component. There exists $Y = (J, \{\psi_i\})$ connected nondegenerate such that $G(Y) \simeq G(X)$ and the generators ψ_i are independent.*

Proof. We first assume that X has no reflection, and we argue in 2 steps.

1. First consider any connected nondegenerate X. Fix $t > 0$ smaller than half the length of any base, and assume that all orbits of X_{-t} are finite. For $0 < s \leq t$, we note $\Sigma_{-s} = \Sigma(X_{-s})$, $\mathcal{F}_{-s} = \mathcal{F}(X_{-s})$, $\overline{\mathcal{L}}_{-s} = \overline{\mathcal{L}}(X_{-s})$. We view $\mathcal{F}_{-s}$ as a partial foliation of Σ, defined on the subcomplex Σ_{-s}.

The space of orbits of X_{-t}, equal to the space of leaves of $\mathcal{F}_{-t}$, is a finite metric graph Γ_{-t}. Its vertices are singular orbits, its terminal vertices being the endpoints of D. Its edges are families of finite orbits, their length being the width e_i of the family (see remark 3.2).

The quotient map $p_{-t} : \Sigma_{-t} \to \Gamma_{-t}$ induces an epimorphism $\pi_1\Sigma_{-t} \to \pi_1\Gamma_{-t}$ whose kernel is clearly $\overline{\mathcal{L}}_{-t}$ (for a proof à la Bass-Serre, consider the action of $\pi_1\Sigma_{-t}$ on the space of leaves of the foliation induced by $\mathcal{F}_{-t}$ on the universal covering of Σ_{-t}; also see corollary 3.7).

On the other hand, the natural inclusion $\Sigma_{-t} \to \Sigma$ induces an isomorphism $\pi_1\Sigma_{-t} \to \pi_1\Sigma$ and an injection $\overline{\mathcal{L}}_{-t} \hookrightarrow \overline{\mathcal{L}}$. The group $G(X)$ thus appears as the quotient of $\pi_1\Gamma_{-t} \simeq G(X_{-t})$ by a subgroup $\mathcal{C} \subset \pi_1\Gamma_{-t}$. We want to describe $\mathcal{C}$.

Since we are assuming that X has no reflection, any orbit H of X_{-t} is transversely orientable, in the following sense : we can choose local orientations of D near each point of H in such a way that any φ^{-t}-word whose domain meets H preserves these orientations.

We can then define on Γ_{-t} a *differentiable structure* for which the restriction $p_{-t} : D \to \Gamma_{-t}$ is an immersion. We need only do it near a vertex v of valence ≥ 3. Thanks to the transverse orientation, we partition initial segments of edges near v into exactly 2 classes, and we declare that edges in a given class are tangent to each other and opposite to edges in the other class.

An *immersed cusp of length* s is a map $c : [-s, s] \to \Gamma_{-t}$ such that the restrictions of c to $[-s, 0]$ and $[0, s]$ are locally isometric immersions but c is not an immersion : it folds at 0. For a basic example, consider a segment $[a, a+s] \subset D$, with $s \leq t$, and suppose φ_j^{-t} is defined at a. Then define c by setting $c(u) = p_{-t}(a+u)$ for $u \in [0, s]$ and $c(u) = p_{-t}(\varphi_j(a-u))$ for $u \in [-s, 0]$.

A loop in Γ_{-t} is s*-cuspidal* if it consists of a finite number of immersed cusps of length $\leq s$, joined at their endpoints.

Lemma 5.2 *The group $G(X)$ is isomorphic to the quotient of $\pi_1\Gamma_{-t}$ by the normal subgroup $\mathcal{C}$ generated by free homotopy classes of t-cuspidal loops.*

Proof. (cf. [Lev2], p.743) Choose a strong deformation retraction $r_{-t} : \Sigma \to \Sigma_{-t}$, and consider the epimorphism $\pi_1\Sigma \to \pi_1\Gamma_{-t}$ induced by $p_{-t} \circ r_{-t}$. The image of $\overline{\mathcal{L}}$ is clearly contained in $\mathcal{C}$ (cf. the basic example of immersed cusps).

We complete the proof by showing the following : given a cusp of length $s \leq t$, and points $x, y \in D$ with $p_{-t}(x) = c(-s)$ and $p_{-t}(y) = c(s)$, there exists a path $\delta \subset \Sigma$ from x to y, contained in a leaf of $\mathcal{F}_{s-t}$, such that $p_{-t}(\delta)$ is homotopic to c in Γ_{-t}.

Choose $0 = s_1 < s_2 < \ldots < s_p = s$ such that there exist $h_i^+ : [s_i, s_{i+1}] \to D$ and $h_i^- : [-s_{i+1}, -s_i] \to D$ with $p_{-t} \circ h_i^\pm = c$. Then prove the result for the restriction of c to $[-s_n, s_n]$, by induction on n. $\square$

2. Now let X be as in the proposition. By proposition 4.1, every orbit of X_{-t} is finite for $t > 0$. Since free froups are hopfian, we may choose t so that the natural epimorphism $G(X_{-t}) \to G(X_{-s})$ is an isomorphism for all $s \in (0, t)$.

For every edge e of Γ_{-t}, consider a segment E_e isometric to e. Enlarge it by glueing a segment of length t on every endpoint of e that does not come from an endpoint of D. Let J be the disjoint union of these enlarged segments.

Since any edge of Γ_{-t} containing an endpoint of D has length $\geq t$, there exists an immersion (possibly non unique) $q : J \to \Gamma_{-t}$ whose restriction to each E_e is the natural embedding.

We now associate singletons on J to every vertex v of valence ≥ 3 of Γ_{-t}. Lift a transverse orientation of the X_{-t}-orbit v (see above) to a neighborhood of $q^{-1}(v)$ in J, and partition $q^{-1}(v)$ into two classes by considering the position of E_e, for e edge containing v. Choose x_+ and x_-, one in each class. Then define singletons from x_+ to each point in the opposite class, and from x_- to each point $\neq x_+$ in the opposite class. Let Y_{-t} be the connected degenerate system thus obtained on J.

Define Y by extending each singleton $\{x_1\} \to \{x_2\}$ to $[x_1 - t, x_1 + t]$ (the transverse orientation of $q(x_1)$ dictates which of the 2 possible extensions must be chosen).

Clearly $\overline{\mathcal{L}}(Y_{-t})$ is trivial, and $G(Y_{-t})$ is isomorphic to $\pi_1\Gamma_{-t}$. In fact, the space of orbits Γ'_{-t} of Y_{-t} is obtained from Γ_{-t} by glueing edges of length t at one endpoint (they correspond to the segments that were added), so that $\pi_1\Gamma'_{-t}$ is canonically isomorphic to $\pi_1\Gamma_{-t}$.

Using the retraction from Γ'_{-t} to Γ_{-t} induced by q, we can project s-cuspidal loops from Γ'_{-t} to Γ_{-t} for $s \leq t$. The lemma then implies $G(Y_{-s}) \simeq G(X_{-s})$ for $s \in [0, t]$. In particular

$G(Y) \simeq G(X)$. Furthermore the generators of Y are independent since $\overline{\mathcal{L}}(Y_{-s})$ is trivial for $s > 0$.

This completes the proof of the proposition when X has no reflection. If reflections are allowed, we choose t outside of a countable set, so that all centers of reflections of X belong to regular X_{-t}-orbits. The graph Γ_{-t} may have a new type of terminal vertices, corresponding to one-sided orbits. The definition of an immersion (and of an immersed cusp) must then be changed, so as to allow folding at these vertices. With these changes, the lemma remains valid.

We then define J and Y_{-t} as before, but with a new type of singletons : for each terminal vertex v_i coming from a one-sided orbit, we add the trivial singleton $\{v_i\} \to \{v_i\}$. To define Y, we extend these singletons as reflections. It is still true that $G(Y_{-t}) \simeq \pi_1\Gamma_{-t}$ and $G(Y_{-s}) \simeq G(X_{-s})$ for $s \in [0, t]$.

Of course the generators of Y are not independent, because of the reflections. But the injection $\overline{\mathcal{L}}(Y_{-t}) \to \overline{\mathcal{L}}(Y_{-s})$ is an isomorphism for $s > 0$, and we get independent generators after replacing each reflection defined on $[v_i - t, v_i + t]$ by its restriction to $[v_i, v_i + t]$. $\square$

6 An inequality

Given a system of isometries X, define :

- m as the total length of D;
- ℓ as the sum of the lengths of the domains A_j of the generators;
- e as the sum of the widths e_i of the families of finite orbits (see 3.2).

Proposition 6.1 *(cf. [Lev3], cor. II.5) Let X be nondegenerate.*

1. *The inequality $e + \ell \geq m$ always holds.*
2. *If the generators φ_j are independent, then $e + \ell = m$.*

Remark. We will use (and prove) this proposition only when X has no homogenous component. It is a special case of the following general result [Lev5]. Let μ be a probability measure on a standard Borel space K. Let $\varphi_j : A_j \to B_j$ be measure preserving isomorphisms between Borel subsets of K. They are *independent* if the fixed point set of any nontrivial reduced φ-word has measure 0. Define $\ell = \sum_j \mu(A_j)$, and $e = \inf\{\mu(Z) \mid Z \subset K$ meets every orbit$\}$ (one can show $e = \int_K \frac{1}{n(x)} d\mu(x)$, where $n(x)$ is the cardinality of the orbit of x and $\frac{1}{\infty} = 0$). In this situation, *one has $e + \ell \geq 1$ with equality if and only if the φ_j's are independent and the equivalence relation whose classes are the orbits is amenable*. The proof uses a result of S. Adams [Ada].

Proof. As mentioned above, we assume that X has no homogenous component. The functions $t \mapsto \ell(X_{-t})$ and $t \mapsto e(X_{-t})$ are continuous : ℓ is clearly linear, and e is Lipschitz. Indeed, if k is the number of genrators in X, then going from X_{-t} to X_{-t+s} perturbs at most a subset of D of measure $4ks$, hence e changes by at most $4ks$ (e is even piecewise linear, cf. [Lev3]). Using proposition 4.1, we then reduce to the case when all orbits of X are finite.

Every orbit of X may be viewed as a "Cayley graph" : the vertices are the elements of the orbit, and there is an edge labelled j between x and y whenever $y = \varphi_j(x)$. The generators are independent if and only if all regular orbits are trees.

Consider a family U_i of finite orbits. All orbits in U_i are isometric as graphs (except the one-sided orbit if U_i is twisted). Let v_i be the number of vertices, a_i the number of edges, e_i the width of any two-sided orbit. Note that $1+a_i \geq v_i$, with equality if and only if the graph is a tree. (If U_i is twisted, the graph is not a tree.) The proposition then follows, since

$$\begin{aligned} e &= \sum e_i \\ \ell &= \sum a_i e_i \\ m &= \sum v_i e_i \end{aligned}$$

□

Definition 6.2 *X is an interval exchange if X is connected, nondegenerate, and every $x \in D$ outside of a finite set belongs to exactly 2 bases.*

The number of bases to which a point belongs is the sum of the number of j's such that x is in A_j and of the number of j's such that x is in B_j.

Corollary 6.3 *Let X be connected nondegenerate with independent generators. Suppose X has only finitely many finite orbits (hence $e = 0$), and every $x \in D$ belongs to at least 2 bases. Then X is an interval exchange (otherwise $\ell > m$).* □

By a surface group, we mean the fundamental group of a closed connected (possibly non orientable) surface. We also have :

Proposition 6.4 *If X is a pure, minimal, interval exchange, then $G(X)$ is a surface group.*

Proof. In this case Σ is homeomorphic to a compact surface with boundary, and $\overline{\mathcal{L}}$ is the normal subgroup of $\pi_1\Sigma$ generated by loops contained in $\partial\Sigma$. If one had a closed loop in a leaf of Σ not in the boundary, then it would correspond to a point of $\overset{\circ}{D}$ whose $\overset{\circ}{X}$-orbit consists of itself, contradicting the pureness of X. □

7 Erasing intervals

Let $X_0 = (D_0, \{\varphi_j\})$ be minimal and pure. The set $L(X_0)$ of points belonging to only one base is open in D_0. If it is not empty, define X_1 on the multi-interval $D_1 = D_0 \setminus L(X_0)$ by replacing every φ_j by its restrictions to the finitely many components of the multi-interval $D_1 - \varphi_j^{-1}(L(X_0))$. If some component of $L(X_0)$ coincide with a component of D, then the corresponding is simply removed. Note that by minimality and pureness, the closures of $\varphi_j^{-1}(L(X_0))$ and $L(X_0)$ do not meet.

It is easy to check that X_1 is pure and minimal (every $\overset{\circ}{\varphi}_1$-orbit is still dense in D_1), with $G(X_1) \simeq G(X_0)$. Indeed, $\Sigma(X_1)$ is an obvious strong retract of $\Sigma(X_0)$, where the retraction does not change the loops in leaves contributing to relations, since the leaves of $L(X_n) \times [0,1]$ do not contribute to relations. If the generators of X_0 are independent, so are those of X_1.

If there are points of D_1 belonging to only one base, we can repeat this operation. Iterating leads to an infinite sequence X_n defined on multi-intervals $D_0 \supset D_1 \supset \ldots \supset D_n \supset \ldots$, unless for some n every point of D_n belongs to at least 2 bases.

Proposition 7.1 *Let X_0 be pure minimal with independent generators. If the process (due to Rips) described above does not terminate after a finite number of steps, then $\cap_{n\in\mathbf{N}} D_n$ is nowhere dense in D_0.*

Proof. By absurd, suppose there is a non degenerate closed interval I in $\cap_{n\in\mathbf{N}} D_n$. According to the remark above proposition 3.5, there exists $N in \mathbf{N}$ such that every $y \in D$ can be sent into I by a X_0-word of length $\leq N$.

Every orbit O of X_n is a "Cayley graph" : given a generator $\varphi : A \to B$ of X_n and $x \in O \cap A$, there is an edge labelled φ between x and $\varphi(x)$. The distance between two points of O is the minimum length of a path between them in this graph. By construction, an orbit of X_{n+1} is obtained from an orbit of X_n by removing all the terminating edges.

In particular, since the erasing process has length $\geq N+1$, there is an X_{N+1}-orbit O_{N+1} and a point y in $L(X_0) \cap O_{N+1}$ such that, in the X_0-orbit of y, the distance from the terminating vertex y to O_{N+1} is equal to $N+1$. By minimality of X_0, the X_0-orbit of y meets $\mathring{I} \subset D_{N+1}$. Hence y is, in its X_0-orbit, at distance $\geq N+1$ from any point of I. This contradicts our first assertion. □

8 The group is free

Let X be pure, minimal, not homogenous, not an interval exchange. Replace X by X_0 having independent generators, using proposition 5.1. One checks that X_0 is also pure and minimal. Alternatively, one can check that, if X has independent generators, then so do the X_i's given by proposition 3.5.

Apply the erasing process to X_0. If it stops after n steps, then X_n is an interval exchange by corollary 6.3, and $G(X) \simeq G(X_0) \simeq G(X_n)$ is a surface group.

Assuming the process goes on for ever, consider the foliated 2-complex Σ_0 associated to X_0. Since generators are independent, any loop contained in a leaf and disjoint from each $\delta A_j \times (0,1)$ is nullhomotopic.

This allows us to find a *finite* 1-complex $K \subset \Sigma_0$, whose 0-skeleton is $K^0 = K \cap D_0$, whose edges are segments of leaves, such that $\overline{\mathcal{L}}(X_0) \subset \pi_1 \Sigma_0$ is the normal subgroup generated by loops in K.

First assume that the intersection of K with each strip $A_j \times (0,1)$ is connected. Then we claim that $G(X_0)$ is free. To see this, choose a maximal subtree in each component of K. For each edge not in these trees, remove from Σ_0 the corresponding strip $A_j \times (0,1)$. This does not change the associated group, and brings us to the trivial situation where every component of K is a tree.

To do the general case, consider the finite set $K^0 = K \cap D_0$. Using proposition 7.1, perform the erasing process until all points of K^0 belong to different components of the multi-interval D_n. The 2-complex Σ_n associated to X_n is contained in Σ_0 in a natural way, and $K_n = K \cap \Sigma_n$ generates $\overline{\mathcal{L}}(X_n)$. It follows that $G(X_0) \simeq G(X_n)$ is free. □

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