

A Measurable-Group-Theoretic Solution to von Neumann's Problem

Damien Gaboriau*

Russell Lyons†

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Abstract

We give a positive answer, in the measurable-group-theory context, to von Neumann's problem of knowing whether a non-amenable countable discrete group contains a non-cyclic free subgroup. We also get an embedding result of the free-group von Neumann factor into restricted wreath product factors.

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Amenability of groups is a concept introduced by J. von Neumann in his seminal article [vN29] to explain the so-called Banach-Tarski paradox. He proved that a discrete group containing the free group \mathbf{F}_2 on two generators as a subgroup is non-amenable. Knowing whether this was a characterization of non-amenable groups became known as von Neumann's Problem and was solved in the negative by A. Ol'shanskii [Ol'80]. Still, this characterization could become true after relaxing the notion of "containing a subgroup". K. Whyte gave a very satisfying geometric group-theoretic solution: *A finitely generated group Γ is non-amenable iff it admits a partition with pieces uniformly bilipschitz equivalent to the regular 4-valent tree [Why99].* Geometric group theory admits a measurable counterpart, namely, measurable group theory. The main goal of our note is to provide a solution to von Neumann's problem in this context. We show that any countable non-amenable group admits a measure-preserving free action on some probability space, such that the orbits may be measurably partitioned into pieces given by an \mathbf{F}_2 -action.

To be more precise, we use the following notation. For a finite or countable set M , let μ denote the product $\otimes_M Leb$ on $[0, 1]^M$ of the Lebesgue measures Leb on $[0, 1]$, and for $p \in [0, 1]$, let μ_p denote the product of the discrete measures $(1-p)\delta_{\{0\}} + p\delta_{\{1\}}$ on $\{0, 1\}^M$. Thus, the meaning of μ may vary from use to use as M varies. Usually M will be a countable group Λ or the set E of edges of a Cayley graph.

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Theorem 1 *For any countable discrete non-amenable group Λ , there is a measurable ergodic essentially free action of \mathbf{F}_2 on $([0, 1]^\Lambda, \mu)$ such that almost every Λ -orbit of the Bernoulli shift decomposes into \mathbf{F}_2 -orbits.*

In other words, the orbit equivalence relation of the \mathbf{F}_2 -action is contained in that of the Λ -action. We give two proofs of this theorem, each with its own advantages.

For some purposes, it is useful to get a Bernoulli shift action with a discrete base space. We show:

Theorem 2 *For any finitely generated non-amenable group Γ , there is $n \in \mathbb{N}$ and a non-empty interval (p_1, p_2) of parameters p for which there is an ergodic essentially free action of \mathbf{F}_2 on $\prod_1^n(\{0, 1\}^\Gamma, \mu_p)$ such that almost every Γ -orbit of the diagonal Bernoulli shift decomposes into \mathbf{F}_2 -orbits.*

These results have operator-algebra counterparts:

Corollary 3 *Let Λ be a countable discrete non-amenable group and H be an infinite group. Then the von Neumann factor $L(H \wr \Lambda)$ of the restricted wreath product contains a copy of the von Neumann factor $L(\mathbf{F}_2)$ of the free group.*

Corollary 4 *Let Γ be a finitely generated discrete non-amenable group. Let n, p_1, p_2 be as in Theorem 2 and let $p = \frac{\alpha}{\beta} \in (p_1, p_2)$, with $\alpha, \beta \in \mathbb{N}$. Assume that H contains an abelian subgroup K of order $k = \beta^n$. Then the von Neumann factor $L(H \wr \Lambda)$ of the restricted wreath product contains a copy of the von Neumann factor $L(\mathbf{F}_2)$ of the free group.*

For this paper, we assume a certain familiarity with the results and notation of [Gab05], [Gab00] and [LS99].

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A **(countable standard) equivalence relation** on the standard Borel space (X, ν) is an equivalence relation \mathcal{R} with countable classes that is a Borel subset of $X \times X$ for the product σ -algebra.

A **(measure-preserving oriented) graphing** on (X, ν) is a denumerable family $\Phi = (\varphi_i)_{i \in I}$ of partial measure-preserving isomorphisms $\varphi_i : A_i \rightarrow B_i$ between Borel subsets $A_i, B_i \subset X$.

A graphing Φ **generates** an equivalence relation \mathcal{R}_Φ : the smallest equivalence relation that contains all pairs $(x, \varphi_i(x))$. The **cost** of a graphing $\Phi = (\varphi_i)_{i \in I}$ is the sum of the measures of the domains $\sum_{i \in I} \nu(A_i)$. The **cost**, $\mathcal{C}(\mathcal{R}, \nu)$, of (\mathcal{R}, ν) is the infimum of the costs of the graphings that generate \mathcal{R} . The **graph (structure)** $\Phi[x]$ of a graphing Φ at a point $x \in X$ is the graph whose vertex set is the equivalence class $\mathcal{R}_\Phi[x]$ of x and whose edges are the pairs $(y, z) \in \mathcal{R}_\Phi[x] \times \mathcal{R}_\Phi[x]$ such that for some $i \in I$, either $\varphi_i(y) = z$ or $\varphi_i(z) = y$. For more on cost, see [Gab00] or [KM04].

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Proofs

Since the union of an increasing sequence of amenable groups is still amenable, Λ contains a non-amenable finitely generated subgroup. Let Γ be such a subgroup.

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If S is a finite generating set of Γ (maybe with repetitions), $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ denotes the associated right Cayley graph (with vertex set \mathbf{V}): The set \mathbf{E} of edges is indexed by S and Γ . Given $s \in S$ and $\gamma \in \Gamma$, the corresponding edge is oriented from the vertex γ to γs . Note that Γ acts freely on \mathcal{G} by multiplication on the left. Let $\rho := \text{id}$, the identity of the group Γ , chosen as base vertex for \mathcal{G} .

The set of the subgraphs of \mathcal{G} (with the same set of vertices \mathbf{V}) is naturally identified with $\Omega := \{0, 1\}^{\mathbf{E}}$. The connected components of $\omega \in \Omega$ are called its **clusters**.

Consider a probability-measure-preserving essentially free (left) Γ -action on some standard Borel space (X, ν) together with a Γ -equivariant Borel map $\pi : X \rightarrow \{0, 1\}^{\mathbf{E}}$.

The **full** equivalence relation \mathcal{R}_Γ generated by the Γ -action X is graphed by the graphing $\Phi = (\varphi_s)_{s \in S}$, where φ_s denotes the action by s^{-1} .

We define the following equivalence subrelation on X (see [Gab05, Sect. 1]): the **cluster equivalence subrelation** \mathcal{R}^{cl} , graphed by the graphing $\Phi^{\text{cl}} := (\varphi_s^{\text{cl}})_{s \in S}$ of partial isomorphisms, where φ_s^{cl} is the restriction $\varphi_s^{\text{cl}} := \varphi_s|_{A_s}$ of φ_s to the Borel subset A_s of $x \in X$ for which the edge e labelled s from ρ to ρs lies in $\pi(x)$, i.e., $\pi(x)(e) = 1$. Consequently, *two points $x, y \in X$ are \mathcal{R}^{cl} -equivalent if and only if there is some $\gamma \in \Gamma$ such that $\gamma^{-1}x = y$ and the vertices $\rho, \gamma\rho$ are in the same cluster of $\pi(x)$.*

The graph structure $\Phi^{\text{cl}}[x]$ given by the graphing Φ^{cl} to the \mathcal{R}^{cl} -class of any $x \in X$ is naturally isomorphic with the cluster $\pi(x)_\rho$ of $\pi(x)$ that contains the base vertex. Denote by $U^\infty \subset X$ the Borel set of $x \in X$ whose \mathcal{R}^{cl} -class is infinite and by $\mathcal{R}_{\Gamma|U^\infty}$ (resp. $\mathcal{R}_{|U^\infty}^{\text{cl}}$) the restriction of \mathcal{R}_Γ (resp. \mathcal{R}^{cl}) to U^∞ .

Write $\mathcal{P}(Y)$ for the power set of Y . The map $X \times \mathbf{V} \rightarrow X$ defined by $(x, \gamma\rho) \mapsto \gamma^{-1}.x$ induces a map $\Psi : X \times \mathcal{P}(\mathbf{V}) \rightarrow \mathcal{P}(X)$ that is invariant under the (left) diagonal Γ -action (i.e., $\Psi(\gamma.x, \gamma.C) = \Psi(x, C)$ for all $\gamma \in \Gamma$, $x \in X$, and $C \subset \mathbf{V}$) and such that $\Psi(x, \mathbf{V})$ is the whole \mathcal{R}_Γ -class of x . The restriction of Ψ to the Γ -invariant subset $\mathfrak{C}_\infty^{\text{cl}} := \{(x, C) : x \in X, C \in \mathcal{P}(\mathbf{V}), C \text{ is an infinite cluster of } \pi(x)\}$ sends each (x, C) (and its Γ -orbit) to a whole infinite \mathcal{R}^{cl} -class, namely, the \mathcal{R}^{cl} -class of $\gamma^{-1}.x$ for any γ such that $\gamma\rho \in C$. Moreover, for each $x \in U^\infty$, its $\mathcal{R}_{\Gamma|U^\infty}$ -class decomposes into infinite \mathcal{R}^{cl} -sub-classes that are in one-to-one correspondence with the elements of $\mathfrak{C}_\infty^{\text{cl}}$ that have x as first coordinate. Note that the set $\{(x, y, C) \in X \times X \times \mathcal{P}(\mathbf{V}) : x \in \Psi(y, C)\}$ is Borel, whence for a Borel set $\mathcal{A} \subset \mathfrak{C}_\infty^{\text{cl}}$, the set $\bar{\Psi}(\mathcal{A}) := \bigcup \Psi(\mathcal{A})$ is measurable, being the projection onto the first coordinate of the Borel set $\{(x, y, C) : x \in \Psi(y, C)\} \cap (X \times \mathcal{A})$.

We say that (ν, π) has **indistinguishable infinite clusters** if for every Γ -invariant Borel subset $\mathcal{A} \subset \mathfrak{C}_\infty^{\text{cl}}$, the set of $x \in X$ for which some $(x, C) \in \mathcal{A}$ and some $(x, C) \in \mathfrak{C}_\infty^{\text{cl}} \setminus \mathcal{A}$ has ν -measure 0. In other words, the \mathcal{R}^{cl} -invariant partition $U^\infty = \bar{\Psi}(\mathcal{A}) \cup \complement \bar{\Psi}(\mathcal{A})$ is not allowed to split any $\mathcal{R}_{\Gamma|U^\infty}$ -class (up to a union of measure 0 of such classes). The following proposition, using this refined notion of indistinguishability, corrects [Gab05, Rem. 2.3].

Proposition 5 *Let Γ act ergodically on (X, ν) and $\pi : X \rightarrow \{0, 1\}^{\mathbf{E}}$ be a Γ -equivariant Borel map such that $\nu(U^\infty) \neq 0$. Then $\mathcal{R}_{|\infty}^{\text{cl}}$, the cluster equivalence relation restricted to its infinite locus U^∞ , is ergodic if and only if (ν, π) has indistinguishable infinite clusters.*

Proof. Suppose that $\mathcal{R}_{|\infty}^{\text{cl}}$ is ergodic. Then for every Γ -invariant Borel subset $\mathcal{A} \subset \mathfrak{C}_{|\infty}^{\text{cl}}$, its image $\overline{\Psi}(\mathcal{A})$ is a union of $\mathcal{R}_{|\infty}^{\text{cl}}$ -classes, whence $\overline{\Psi}(\mathcal{A})$ or its complement $\mathfrak{C}\overline{\Psi}(\mathcal{A})$ in U^∞ has measure 0. In particular, the partition $U^\infty = \overline{\Psi}(\mathcal{A}) \cup \mathfrak{C}\overline{\Psi}(\mathcal{A})$ is trivial, whence ν has indistinguishable infinite clusters.

Conversely, suppose that (ν, π) has indistinguishable infinite clusters. An $\mathcal{R}_{|\infty}^{\text{cl}}$ -invariant partition $U^\infty = \mathcal{U} \cup \mathfrak{C}\mathcal{U}$ defines a partition $\mathfrak{C}_{|\infty}^{\text{cl}} = \mathcal{A} \cup \mathfrak{C}\mathcal{A}$ according to whether $\Psi(x, C) \in \mathcal{U}$ or $\mathfrak{C}\mathcal{U}$. Then for ν -almost every $x \in U^\infty$, all $\Psi(x, C)$ are in \mathcal{U} or all are in its complement, i.e., the $\mathcal{R}_{|\infty}^{\text{cl}}$ -subclasses into which the $\mathcal{R}_{|\infty}^{\text{cl}}$ -class of x splits all belong to one side. Since $\mathcal{R}_{|\infty}^{\text{cl}}$ is ν -ergodic, this side has to be the same for almost every x . This means that the other side is a null set. This holds for any partition $\mathcal{U} \cup \mathfrak{C}\mathcal{U}$, whence $\mathcal{R}_{|\infty}^{\text{cl}}$ is ergodic. \blacksquare

If X has the form $X = \Omega \times Y$, then ν is called *insertion tolerant* (see [LS99]) if for each edge $e \in \mathbf{E}$, the map $\Pi_e : X \rightarrow X$ defined by $(\omega, y) \mapsto (\omega \cup \{e\}, y)$ quasi-preserved the measure, i.e., $\nu(A) > 0$ implies $\nu(\Pi_e(A)) > 0$ for every measurable subset $A \subseteq X$. Call a map $\pi : X \rightarrow \Omega$ *increasing* if $\pi(\omega, y) \supseteq \omega$ for all $\omega \in \Omega$. An action of Γ on $\Omega \times Y$ is always assumed to act on the first coordinate in the usual way. A slight extension of [LS99, Th. 3.3, Rem. 3.4], proved in the same way, is the following:

Proposition 6 *Assume that Γ acts on $(\Omega \times Y, \nu)$ preserving the measure and $\pi : \Omega \times Y \rightarrow \Omega$ is an increasing Γ -equivariant Borel map with $\nu(U^\infty) \neq 0$. If ν is insertion tolerant, then (ν, π) has indistinguishable infinite clusters.*

Proposition 7 *If $\Gamma < \Lambda$, then there are Γ -equivariant isomorphisms $([0, 1]^{\mathbf{E}}, \mu) \simeq ([0, 1]^\Gamma, \mu) \simeq ([0, 1]^\Lambda, \mu)$ between the Bernoulli shift actions of Γ . In particular, the orbits of the Bernoulli shift Λ -action on $[0, 1]^\Lambda$ are partitioned into subsets that are identified with the orbits of the standard Bernoulli shift Γ -action on $[0, 1]^\Gamma$.*

Proof. A countable set \mathbf{E} on which Γ acts freely may be decomposed by choosing a representative in each orbit so as to be identified with a disjoint union of Γ -copies, $\mathbf{E} \simeq \coprod_J \Gamma$, and to give Γ -equivariant identifications $[0, 1]^{\mathbf{E}} = [0, 1]^{\coprod_J \Gamma} = ([0, 1]^J)^\Gamma$.

The edge set $\mathbf{E} \simeq \coprod_S \Gamma$ of the Cayley graph of Γ , as well as $\Lambda \simeq \coprod_I \Gamma$ are such countable Γ -sets. Then isomorphisms of standard Borel probability spaces $([0, 1], \text{Leb}) \simeq ([0, 1]^S, \otimes_S \text{Leb}) \simeq ([0, 1]^I, \otimes_I \text{Leb})$ induce Γ -equivariant isomorphisms of the Bernoulli shifts:

$$\begin{array}{ccccc} [0, 1]^\Gamma & \simeq & ([0, 1]^S)^\Gamma & \simeq & ([0, 1]^I)^\Gamma \\ \parallel & & \parallel & & \parallel \\ [0, 1]^\Gamma & \simeq & [0, 1]^{\mathbf{E}} & \simeq & [0, 1]^\Lambda. \end{array}$$

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A subgraph (V', E') of a graph (V, E) is called **spanning** if $V' = V$. A vertex a in a graph is called a **cutvertex** if there are two other vertices in its component with the property that every path joining them passes through a . A **block** of a graph is a maximal connected subgraph that has no cutvertex. Every simple cycle of a graph is contained within one of its blocks.

Lemma 8 *If all vertices of a block have finite degree and for each pair of vertices (a, b) there are only finitely many distinct paths joining a to b , then the block is finite.*

Proof. Suppose for a contradiction that the block is infinite. Then it contains a simple infinite path P of vertices a_1, a_2, \dots . By Menger's theorem, a_1 and a_n belong to a simple cycle C_n for each $n > 1$. But this implies that there are infinitely many distinct paths joining a_1 to a_2 : Fix n and let a_j ($2 \leq j \leq n$) be the vertex on $P \cap C_n$ with minimal index j . We may assume that C_n is oriented so that it visits a_n before it visits a_j . Then simply follow C_n from a_1 until a_j , and then follow P to a_2 . ■

Proposition 9 *(For any Cayley graph) Let $X := \Omega \times [0, 1]^\Gamma$ and $\epsilon > 0$. Let $\nu := \mu_\epsilon \times \mu$. There is a Γ -equivariant Borel map $f : X \rightarrow \Omega$ such that (ν, f) has indistinguishable infinite clusters and for all sufficiently small ϵ , the push-forward measure $f_*(\nu)$ of ν is supported on the set of spanning subgraphs of \mathcal{G} each of whose components is a tree with infinitely many ends.*

Proof. We may equivariantly identify $([0, 1]^\Gamma, \mu)$ with $([0, 1]^{\mathbb{N} \times \Gamma} \times [0, 1]^E, \mu \times \mu)$, so we identify (X, ν) with $(\Omega \times [0, 1]^{\mathbb{N} \times \Gamma} \times [0, 1]^E, \mu_\epsilon \times \mu \times \mu)$. Fix an ordering of $S \amalg S^{-1}$; this determines an ordering of the edges incident to each vertex in \mathcal{G} , where we ignore edge orientations for the rest of this proof. With d denoting the degree of \mathcal{G} , define the function $h(t) := \lceil dt \rceil$ for $t \in [0, 1]$. Given a point $x = (\omega, (r(n, \gamma))_{n \in \mathbb{N}, \gamma \in \Gamma}, u) \in X$, construct the wired spanning forest \mathcal{F}_1 of \mathcal{G} by using the cycle-popping algorithm of D. Wilson [Wil96, Sect. 3] as adapted in [BLPS01, Th. 5.1], also called there “Wilson’s algorithm rooted at infinity”, applied to the stacks where the n th edge in the stack under γ is defined as the $h(r(n, \gamma))$ th edge incident to γ . The measure ν is insertion tolerant and the map $\pi : x \mapsto \omega \cup \mathcal{F}_1$ is increasing, whence by Proposition 6, the pair (ν, π) has indistinguishable infinite clusters. Notice that all clusters are infinite. Now use u to construct the free minimal spanning forest \mathcal{F}_2 in each cluster of $\pi(x)$, that is, for every cycle $\Delta \subset \pi(x)$, delete the edge $e \in \Delta$ with maximum $u(e)$ in that cycle. The map f is $f(x) := \mathcal{F}_2$.

Now the ν -expected number of distinct simple paths in $\pi(x)$ that join any two vertices is finite (equation (13.7) of [BLPS01]) for all sufficiently small ϵ . In particular, the number of such paths is finite ν -a.s. By Lemma 8, this means that all blocks of $\pi(x)$ are finite, so that \mathcal{F}_2 is a spanning tree in each block. Therefore each component of \mathcal{F}_2 spans a component of $\pi(x)$. Thus, \mathcal{F}_2 determines the same cluster relation and so (ν, f) also has indistinguishable (infinite) clusters. Finally, the fact that the clusters of $\pi(x)$, and hence those of \mathcal{F}_2 , have infinitely many ends follows, e.g., from [BLPS01, Th. 13.7]. ■

The cluster relation determined by f of Proposition 9 is treeable and has cost larger than 1 by [Gab00, Cor. IV.24 (2)], has finite cost (since the degree is bounded), and is ergodic by Proposition 5. Since we may equivariantly identify $(\Omega \times [0, 1]^\Gamma, \mu_\epsilon \times \mu)$ with $([0, 1]^\Gamma, \mu)$, we proved:

Proposition 10 *For any Cayley graph of Γ , the Bernoulli action on $([0, 1]^\Gamma, \mu)$ contains a treeable subrelation that is ergodic and has cost in the interval $(1, \infty)$.*

At this point, we already have a reasonable answer to the analogue of von Neumann’s problem, since “treeable relation” is the analogue of “free group” and cost $\mathcal{C} > 1$ is, in the context of treeable relations, equivalent to non-amenability.

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An alternative approach begins with a more explicit f and a more common measure $f_*(\mu)$, namely, the Bernoulli measure μ_p on $\{0, 1\}^\mathbb{E}$ for a certain parameter p , but requires us to choose a particular Cayley graph for Γ . It also requires us to obtain a treeable subrelation in a less explicit way. This is accomplished as follows.

Results of Häggström-Peres [HP99] imply that there are two critical values $0 < p_c \leq p_u \leq 1$ such that

- (finite phase, $p \in [0, p_c)$) μ_p -a.s., the subgraph has only finite clusters;
- (**non-uniqueness phase**, $p \in (p_c, p_u)$) μ_p -a.s., infinitely many of the clusters of the subgraph are infinite, each one with infinitely many ends;
- (the uniqueness phase, $p \in (p_u, 1]$) μ_p -a.s., the subgraph has only one cluster that is infinite.

The situation for the critical values p_c and p_u themselves is far from clear. Benjamini and Schramm [BS96] conjectured that $p_c \neq p_u$ for every Cayley graph of a f.g. non-amenable group. The main result of [PSN00] (Th. 1, p. 498) asserts that given a f.g. non-amenable group Γ , there is a finite set of generators such that the associated Cayley graph admits a non-trivial interval of non-uniqueness. Thus:

Proposition 11 *(For particular Cayley graphs) There exists a Cayley graph of Γ and a non-empty interval (p_c, p_u) such that, for any $p \in (p_c, p_u)$, the Bernoulli measure μ_p on $\{0, 1\}^\mathbb{E}$ is supported on the set of subgraphs admitting infinite components, each one with infinitely many ends.*

Let $\pi : (X, \nu) \rightarrow \{0, 1\}^\mathbb{E}$ denote either

- (i) $f_p : ([0, 1]^\mathbb{E}, \mu) \rightarrow \{0, 1\}^\mathbb{E}$ induced by the characteristic function $\chi_{[0, p]} : [0, 1] \rightarrow \{0, 1\}$ of $[0, p]$, or
- (ii) the identity map $(\{0, 1\}^\mathbb{E}, \mu_p) \rightarrow \{0, 1\}^\mathbb{E}$,

both with the natural Bernoulli Γ -action. Notice that the action is essentially free when $0 < p < 1$.

In case (ii), we have that (μ_p, π) has indistinguishable infinite clusters by [LS99, Th. 3.3]. Case (i) is essentially the same, but first we must identify $([0, 1]^\mathbb{E}, \mu)$ equivariantly as $(\{0, 1\} \times [0, 1]^\mathbb{E} = \{0, 1\}^\mathbb{E} \times [0, 1]^\mathbb{E})$ equipped with the product measure $\mu_p \times \mu$ in such a way that f_p becomes the identity on the first coordinate. Then we have insertion tolerance and so, by [LS99, Rem. 3.4], indistinguishable infinite clusters.

Hence, in both cases, for any p given by Prop. 11, the locus U^∞ of infinite classes of \mathcal{R}^{cl} is non-null and we have ergodicity of the restriction $\mathcal{R}_{|\infty}^{\text{cl}}$ of \mathcal{R}^{cl} to U^∞ by Proposition 5. We claim that its normalized cost (i.e., computed with respect to the normalized probability measure $\nu/\nu(U^\infty)$ on U^∞) satisfies $1 < \mathcal{C}(\mathcal{R}_{|\infty}^{\text{cl}}) < \infty$. The finiteness of the cost is clear since S , the index set for Φ^{cl} , is finite. That it is strictly greater than 1 is a direct application of [Gab00, Cor. IV.24 (2)], since the graph $\Phi^{\text{cl}}[x] \simeq \pi(x)_\rho$ associated with almost every $x \in U^\infty$ has at least 3 ends.

In order to extend $\mathcal{R}_{|\infty}^{\text{cl}}$ to a subrelation of \mathcal{R}_Γ defined on the whole of X , choose an enumeration $\{\gamma_i\}_{i \in \mathbb{N}}$ of Γ . For each $x \in X \setminus U^\infty$, let γ_x be the first element $\gamma_j \in \Gamma$ such that $\gamma_j \cdot x \in U^\infty$. Then the smallest equivalence relation containing $\mathcal{R}_{|\infty}^{\text{cl}}$ and the $(x, \gamma_x \cdot x)$'s is a subrelation of \mathcal{R}_Γ , is ergodic, and has cost in $(1, \infty)$ by the induction formula of [Gab00, Prop. II.6]. We proved:

Proposition 12 *For a Cayley graph and a p given by Proposition 11, the Bernoulli actions on both $([0, 1]^{\mathbb{E}}, \mu)$ and $(\{0, 1\}^{\mathbb{E}}, \mu_p)$ contain a subrelation that is ergodic and has cost in the open interval $(1, \infty)$.*

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Proposition 13 *If an equivalence relation \mathcal{R} is ergodic and has cost in $(1, \infty)$, then it contains a treeable subrelation \mathcal{S}_1 that is ergodic and has cost in $(1, \infty)$.*

Proof. This is ensured by a result proved independently by A. Kechris and B. Miller [KM04, Lem. 28.11; 28.12] and by M. Pichot [Pic05, Cor. 40], through a process of erasing cycles from a graphing of \mathcal{S}_1 with finite cost that contains an ergodic global isomorphism. ■

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Proposition 14 *If a treeable equivalence relation \mathcal{S}_1 is ergodic and has cost > 1 , then it contains a.e. a subrelation \mathcal{S}_2 that is generated by an ergodic free action of the free group \mathbf{F}_2 .*

Proof. If the cost of \mathcal{S}_1 is > 2 , this follows from a result of G. Hjorth [Hjo06] (see also [KM04, Sect. 28]). Otherwise, one first considers the restriction of the treeable \mathcal{S}_1 to a small enough Borel subset V : this increases the normalized cost by the induction formula of [Gab00, Prop. II.6 (2)] to get $\mathcal{C}(\mathcal{S}_1|V) \geq 2$. In fact, it follows from the proof of [KM04, Th. 28.3] that *a treeable probability measure-preserving equivalence relation with cost ≥ 2 contains a.e. an equivalence subrelation that is given by a free action of the free group $\mathbf{F}_2 = \langle a, b \rangle$ in such a way that the generator a acts ergodically.* By considering a subgroup of \mathbf{F}_2 generated by a and some conjugates of a of the form $b^k a b^{-k}$, one gets an ergodic treeable subrelation of $\mathcal{S}_1|V$ with a big enough normalized cost that, when extended to the whole of X (by using partial isomorphisms of \mathcal{S}_1), it gets cost ≥ 2 (by the induction back [Gab00, Prop. II.6 (2)]) and of course remains ergodic. Another application of the above-italicized result gives the desired ergodic action of \mathbf{F}_2 on X . ■

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The proof of Theorem 2 is now complete as a direct consequence of Propositions 12 (for the case $X = \{0, 1\}^{\mathbb{E}}$), 13 and 14. ■

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In case $X = [0, 1]^{\mathbb{E}}$ of Prop. 12, by using Prop. 7, we can see \mathcal{S}_2 (with $\mathcal{S}_2 \subset \mathcal{S}_1 \subset \mathcal{R}_\Gamma$ given by Prop. 14 and 13) as an equivalence subrelation of that given by the Bernoulli shift action of Λ . This finishes the proof of Theorem 1. Alternatively, we may use Prop. 10 and a similar argument to prove Theorem 1. ■

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Proof of Cor. 3. For any diffuse abelian subalgebra A of $L(H)$, the von Neumann factor $L(H \wr \Lambda) = L(\Lambda \rtimes \oplus_\Lambda H)$ contains the von Neumann algebra crossed product $\Lambda \rtimes \otimes_\Lambda A$, which is isomorphic with the group-measure-space factor $\Lambda \rtimes L^\infty([0, 1]^\Lambda, \mu)$ associated with the Bernoulli shift. The corollary then follows from Th. 1. ■

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Proof of Cor. 4. If \hat{K} is the dual group of K , then $L(H \wr \Gamma)$ contains $L(K \wr \Gamma)$, which is isomorphic with the group-measure-space factor $\Gamma \rtimes L^\infty(\hat{K}^\Gamma)$ associated with the Bernoulli shift of Γ on \hat{K}^Γ , where the finite set $\hat{K} \simeq \{1, 2, \dots, k\}$ is equipped with the equiprobability measure ν . The result is then obtained by taking the pull-back of the \mathbf{F}_2 -action on $\prod_1^n \{0, 1\}^\Gamma$, given in Th. 2, by the Γ -equivariant Borel map $\hat{K}^\Gamma \rightarrow (\{0, 1\}^n)^\Gamma \simeq \prod_1^n \{0, 1\}^\Gamma$, sending $\otimes \nu$ to μ_p , that extends a map $\{1, 2, \dots, k\} \rightarrow \{0, 1\}^n$ (whose existence is ensured by the form of $k = \beta^n$). ■

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It is likely that the free minimal spanning forest (FMSF) of a Cayley graph of Γ would serve as the desired ergodic subrelation \mathcal{S}_1 of Prop. 13, but its indistinguishability, conjectured in [LPS06], is not known. Also, it is not known to have $\text{cost} > 1$, but this is equivalent to $p_c < p_u$, which is conjectured to hold and which we know holds for some Cayley graph. See [LPS06] for information on the FMSF and [Tim06] for a weak form of indistinguishability.

A general question remains open:

Question: Does every probability-measure-preserving free ergodic action of a non-amenable countable group contain an ergodic subrelation generated by a free action of a non-cyclic free group? More generally: Does every standard countable probability-measure-preserving non-amenable ergodic equivalence relation contain a treeable non-amenable ergodic equivalence subrelation?

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Damien Gaboriau:

Unité de Mathématiques Pures et Appliquées

Université de Lyon, CNRS, Ens-Lyon

46, allée d'Italie
69364 Lyon cedex 7, France
gaboriau@umpa.ens-lyon.fr

Russell Lyons:
Department of Mathematics
Indiana University Bloomington, IN 47405-5701
USA
rdlyons@indiana.edu