THE RANK OF ACTIONS ON R-TREES

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Abstract.

For $n \geq 2$, let F_n denote the free group of rank n. We define a total branching index i for a minimal small action of F_n on an $\mathbf{R}$-tree. We show $i \leq 2n-2$, with equality if and only if the action is geometric. We thus recover Jiang's bound $2n-2$ for the number of orbits of branch points of free F_n-actions, and we extend it to very small actions (i.e. actions which are limits of free actions).

The $\mathbf{Q}$-rank of a minimal very small action of F_n is bounded by $3n-3$, equality being possible only if the action is free simplicial. There exists a free action of F_3 such that the values of the length function do not lie in any finitely generated subgroup of $\mathbf{R}$.

The boundary of Culler-Vogtmann's outer space Y_n has topological dimension $3n-5$.

INTRODUCTION AND STATEMENT OF RESULTS.

Various problems from geometry and group theory lead to isometric group actions on $\mathbf{R}$-trees. An $\mathbf{R}$-tree is a path-connected metric space in which every arc is isometric to an interval of $\mathbf{R}$. See the surveys [Sh 1], [Sh 2], [Mo] and the papers [AB], [CM] for basic results about $\mathbf{R}$-trees.

These actions on $\mathbf{R}$-trees are most often *small*: no edge stabilizer contains a free non-abelian subgroup. Following work of Rips, it is now known that hyperbolic groups admitting nontrivial small actions on $\mathbf{R}$-trees have nontrivial splittings (see [BF 2] for precise statements and corollaries).

Small actions of a given finitely generated group G determine a closed subspace in the space of all length functions on G. This subspace is often infinite dimensional [CL, Theorem 9.8]. Bestvina-Feighn have proved a finiteness theorem for reduced simplicial small actions [BF 1].

In this paper we consider actions of F_n, the free group of rank n. We obtain finiteness results about *branch points*, *rank*, and Culler-Vogtmann's *outer space*. Our results apply to small actions, and to very small actions.

Recall (Cohen-Lustig [CL]) that a small action of F_n on an $\mathbf{R}$-tree is *very small* if for every nontrivial $g \in F_n$ the fixed subtree $\mathrm{Fix}(g)$ is equal to $\mathrm{Fix}(g^p)$ for $p \geq 2$ (*no obtrusive powers*) and $\mathrm{Fix}(g)$ is isometric to a subset of $\mathbf{R}$ (*no fixed triods*).

Outer space Y_n consists of (projective classes of length functions of) free simplicial actions of F_n, and its closure consists precisely of very small actions [BF 3]. In particular, an action is very small if and only if it is a limit of free actions.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Let T be a small F_n-tree (i.e. an $\mathbf{R}$-tree equipped with a small action of F_n). We always assume that T is *minimal* (there is no proper invariant subtree).

Let $x \in T$ be a *branch point* (i.e. a point such that $T \setminus \{x\}$ has at least 3 components). In Part III we define an *index* $i(x)$ in terms of the isotropy subgroup $\mathrm{Stab}(x)$ and its action on the *set of directions* $\pi_0(T \setminus \{x\})$, by

$$i(x) = 2 \text{ rk } \mathrm{Stab}(x) + v_1(x) - 2,$$

where $v_1(x)$ is the number of $\mathrm{Stab}(x)$-orbits of directions with trivial stabilizer; it turns out that $i(x) \in \mathbf{N}$.

The index $i(x)$ depends only on the orbit $\mathcal{O} = F_n(x)$ and we define the index of T as

$$i(T) = \sum_{\mathcal{O} \in T/F_n} i(\mathcal{O}).$$

Theorem III.2. *Let T be a small minimal F_n-tree. Then $i(T) \leq 2n - 2$.*

If the action is very small, the index of every branch point is positive. We then get:

Corollary III.3. *Let T be a very small minimal F_n-tree. The number b of orbits of branch points satisfies $b \leq 2n - 2$.*

Another corollary is:

Corollary III.4. *Let T be a small F_n-tree. The stabilizer of any $x \in T$ has rank at most n.*

In the case of a free action, $i(x) + 2$ is the number of components of $T \setminus \{x\}$, so that Theorem III.2 specializes to Jiang's theorem [Ji].

It is worth pointing out the analogy with actions of surface groups. Suppose T is an $\mathbf{R}$-tree with a minimal small action of $\pi_1\Sigma$, where Σ is a closed surface. By Skora's theorem [Sk 1], T is dual to a measured foliation $\mathcal{F}$ on Σ. Branch points of T come from singularities of $\mathcal{F}$ and the Euler-Poincaré formula for line fields on surfaces gives the equality $i(T) = -2\chi(\Sigma)$ (see Part III).

In the case of F_n, *equality in Theorem III.2 holds if and only if the action is geometric* (compare [Du]). Roughly speaking, *geometric* means that the action is dual to a measured foliation on a finite 2-complex (see Part II for a discussion). For instance a minimal simplicial F_n-action is geometric if and only if every edge stabilizer is finitely generated.

There is a close connection between branch points and rank. This is best seen on geometric F_n-actions (not necessarily small). Let L be the subgroup of $\mathbf{R}$ generated by the values of the *length function* $\ell(g) = \min_{x \in T} d(x, gx)$.

For a geometric F_n-action, the group L is finitely generated. Its rank r is called the *rank* of the action (or of the length function). Equivalently T may be viewed as the completion of a Λ-tree, with $\Lambda \subset \mathbf{R}$ a subgroup of rank r (see [Sh 2, §1.3.1]).

By studying the 2-group $L/2L$, we show (Corollary IV.3) the inequality

$$r \leq b + n - 1$$

valid for any geometric minimal F_n-action without inversions (the number b of orbits of branch points is always finite). In particular we have $r \leq 3n - 3$ for a geometric very small action.

If the action is not geometric, the group L needs not be finitely generated (this may happen for free actions, see Example II.7). Instead of rank we use **Q**-*rank*: the dimension of the **Q**-vector space generated by L. Actions with low **Q**-rank have been studied extensively ([GS], [GSS]).

Theorem IV.4. *Let T be a very small minimal F_n-tree. The* **Q**-*rank of the action satisfies $r_{\mathbf{Q}} \leq 3n - 3$. Equality may hold only if the action is free simplicial.*

Given a finitely generated group G and an integer k, the space of length functions on G with **Q**-rank $\leq k$ has topological dimension at most k (Proposition V.1). We then get:

Theorem V.2. *The boundary of Culler-Vogtmann's outer space Y_n has dimension $3n - 5$.*

This improves the result $\dim \overline{Y_n} = 3n - 4$ by Bestvina-Feighn [BF 3].

Theorem IV.4 also implies:

Corollary IV.5. *Let T be a very small F_n-tree with length function ℓ. Suppose $\ell \circ \alpha = \lambda \ell$ with $\alpha \in Aut(F_n)$ and $\lambda \in \mathbf{R}^+$. Then λ is algebraic, of degree bounded by $3n - 4$. If T is geometric, then λ is an algebraic unit.*

Such an ℓ represents a fixed point for the action of α on $\overline{Y_n}$ (compare [Lu]). In Example II.7 we use a construction by Bestvina-Handel to get an example with λ not an unit. The corresponding action is free and does not have finite rank.

The theorems mentioned above are proved in Parts III, IV, V. Parts I and II may be viewed as preliminary.

First recall the following construction due to Rips (see [GLP 1]). Let T be a minimal F_n-tree, and $K \subset T$ a finite subtree (i.e. a subtree homeomorphic to a finite simplicial complex). If K is large enough, the action of each generator $g_1, \ldots, g_n$ of F_n defines a partial isometry $\varphi_i \colon g_i^{-1}K \cap K \to K \cap g_i K$ between nonempty closed subtrees of K.

In Part I we show how to associate a canonical geometric F_n-tree $T_{\mathcal{K}}$ to a system $\mathcal{K}$ consisting of a finite metric tree K and n partial isometries $\varphi_i \colon A_i \to B_i$ between closed subtrees of K (Theorem I.1). Similar constructions are known (see e.g. [GLP 2]), but they often require an additional hypothesis to ensure that a certain space is Hausdorff.

In our particular setting this problem does not exist. One consequence, used in the proof of Theorem III.2, is that orbits of branch points of $T_{\mathcal{K}}$ are created only by vertices of the finite trees K and A_i ($i = 1, \ldots, n$). Another consequence, derived in Part V, is a simple proof of the following result announced by Skora [Sk 3]:

Theorem V.4. *Every action of F_n may be approximated by simplicial actions.*

Returning to T as above, we associate an F_n-tree T_K to every finite subtree $K \subset T$. As K grows bigger, these trees approximate T. We define T to be *geometric* if T equals T_K for some K (see Part II for equivalent definitions).

If T is not geometric, it is the strong limit (in the sense of [GS]) of a sequence of geometric actions. This allows us to prove Theorems III.2 and IV.4 first for geometric actions, and then to "pass to the limit".

In Part II we give examples of geometric and non-geometric actions. In particular we take advantage of the non-completeness of certain minimal F_n-trees to construct a lot of non-geometric actions by taking "free products" of actions using a basepoint not in the tree but in its completion (Example II.6).

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I. THE R-TREE ASSOCIATED TO A SYSTEM OF ISOMETRIES.

Let G be a group. A *G-tree* is an $\mathbf{R}$-tree T equipped with a left isometric action of G. Two G-trees are considered equal if they are equivariantly isometric.

A *finite tree* will be an $\mathbf{R}$-tree homeomorphic to a finite simplicial complex. A subtree of an $\mathbf{R}$-tree is a finite tree if and only if it is the convex hull of a finite subset.

A map $j\colon T \to T'$ between $\mathbf{R}$-trees is a *morphism* if every segment in T may be written as a finite union of subsegments, each of which is mapped isometrically into T'. If j is an equivariant morphism between G-trees, with length functions ℓ and ℓ', then $\ell \geq \ell'$ since j does not increase distances.

We let F_n be the free group on n generators $g_1, \dots, g_n$. We write $|g|$ for the length of $g \in F_n$ relative to this generating set.

We consider systems $\mathcal{K}$ consisting of a finite tree K and n isometries $\varphi_i\colon A_i \to B_i$ between closed nonempty subtrees of K. We let S be the (finite) set consisting of all vertices of the trees K, A_i, B_i $(1 \leq i \leq n)$.

For example, take K to be a finite subtree in an F_n-tree T, with $K \cap g_i K \neq \emptyset$ for $i = 1, \dots, n$. Then define φ_i as the restriction of the action of g_i to $A_i = g_i^{-1} K \cap K$.

Theorem I.1. *Let $\mathcal{K}$ be as above. There exists a unique F_n-tree $T_{\mathcal{K}}$ such that:*

1. *$T_{\mathcal{K}}$ contains K (as an isometrically embedded subtree).*
2. *if $x \in A_i$, then $g_i x = \varphi_i(x)$.*
3. *every orbit of the action meets K, indeed every segment of $T_{\mathcal{K}}$ is contained in a finite union of images wK, $w \in F_n$.*
4. *if T' is another F_n-tree satisfying (1) and (2), there exists a unique equivariant morphism $j\colon T_{\mathcal{K}} \to T'$ such that $j(x) = x$ for $x \in K$.*

Remark I.2. If j is as in (4), it is surjective if and only if T' satisfies (3).

Remark I.3. Before proving Theorem I.1, we give a geometric description of $T_{\mathcal{K}}$. Let Γ be the Cayley graph of F_n relative to $g_1, \dots, g_n$. We construct a foliated

2-complex Σ sitting above Γ, as follows. Place a copy $K(g)$ of K above each vertex g of Γ. Above each edge $g - gg_i$, place a strip $A_i \times [0,1]$ foliated by $\{*\} \times [0,1]$. Then glue $A_i \times \{1\}$ to $K(gg_i)$ using the inclusion of A_i into K, and glue $A_i \times \{0\}$ to the subtree of $K(g)$ corresponding to B_i, using φ_i (i.e. identify $(x,0) \in A_i \times [0,1]$ to $\varphi_i(x) \in K(g)$). The tree $T_{\mathcal{K}}$ is the space of leaves of this simply connected foliated 2-complex Σ. The action of F_n on $T_{\mathcal{K}}$ is induced by the natural action of F_n on Γ.

Proof of theorem I.1.

Recall that a *pseudodistance* on a set X is a symmetric map $\delta : X \times X \to \mathbf{R}^+$ satisfying the triangle inequality, with $\delta(x,x) = 0\ \forall x$. The relation "$\delta(x,y) = 0$" is a (possibly nontrivial) equivalence relation $\mathcal{R}$ on X, and δ induces a genuine distance d on the quotient set $Y = X/\mathcal{R}$. We call (Y,d) the *metric space associated to* (X,δ).

Now suppose T' is an F_n-tree satisfying (1) and (2). Write d' and d_K for distance in T' and K respectively. Let δ' be the pseudodistance on $K \times F_n$ defined by $\delta'((x,g),(y,h)) = d'(gx, hy)$.

A simple computation, based on (1) and (2), shows the inequality

$$\begin{aligned}\delta'((x,g),(y,h)) \le \inf\Big\{ & d_K(x,x_p) + d_K(\varphi_{i_p}^{\varepsilon_p}(x_p), x_{p-1}) + \dots \\ & + d_K(\varphi_{i_2}^{\varepsilon_2}(x_2), x_1) + d_K(\varphi_{i_1}^{\varepsilon_1}(x_1), y)\Big\} \qquad (*)\end{aligned}$$

where the infimum is taken over all words $g_{i_1}^{\varepsilon_1} \dots g_{i_p}^{\varepsilon_p}$ representing $h^{-1}g$ (with $\varepsilon_j = \pm 1$) and all points x_j in the domain of $\varphi_{i_j}^{\varepsilon_j}$.

Indeed we write:

$$\begin{aligned}\delta'((x,g),(y,h)) &= d'(gx, hy) \\ &= d'(h^{-1}gx, y) \\ &= d'(g_{i_1}^{\varepsilon_1} \dots g_{i_p}^{\varepsilon_p} x, y) \\ &\le d'(g_{i_1}^{\varepsilon_1} \dots g_{i_p}^{\varepsilon_p} x, g_{i_1}^{\varepsilon_1} \dots g_{i_p}^{\varepsilon_p} x_p) + d'(g_{i_1}^{\varepsilon_1} \dots g_{i_p}^{\varepsilon_p} x_p, g_{i_1}^{\varepsilon_1} \dots g_{i_{p-1}}^{\varepsilon_{p-1}} x_{p-1}) \\ &\qquad + \dots + d'(g_{i_1}^{\varepsilon_1} g_{i_2}^{\varepsilon_2} x_2, g_{i_1}^{\varepsilon_1} x_1) + d'(g_{i_1}^{\varepsilon_1} x_1, y) \\ &\le d_K(x, x_p) + d_K(\varphi_{i_p}^{\varepsilon_p}(x_p), x_{p-1}) + \dots \\ &\qquad + d_K(\varphi_{i_2}^{\varepsilon_2}(x_2), x_1) + d_K(\varphi_{i_1}^{\varepsilon_1}(x_1), y).\end{aligned}$$

With this as a motivation, define $\delta((x,g),(y,h))$ as the infimum in the right hand side of the above inequality $(*)$. This gives a pseudodistance on $K \times F_n$. It induces d_K on each $K \times \{g\}$, and it is invariant under the natural action of F_n given by $h(x,g) = (x, hg)$.

It is important to note that the infimum is always achieved: we need only consider the *reduced* word $g_{i_1}^{\varepsilon_1} \dots g_{i_p}^{\varepsilon_p}$ representing $h^{-1}g$, and then the infimum is taken over a fixed number of points x_j varying in compact sets.

More explicitly, let z_p be the point in the domain of $\varphi_{i_p}^{\varepsilon_p}$ closest to x, let z_{p-1} be the point in the domain of $\varphi_{i_{p-1}}^{\varepsilon_{p-1}}$ closest to $\varphi_{i_p}^{\varepsilon_p}(z_p)$, and so on. Then:

$$\begin{aligned} d_K(x,x_p) &+ d_K(\varphi_{i_p}^{\varepsilon_p}(x_p),x_{p-1}) \\ &= d_K(x,z_p)+d_K(z_p,x_p)+d_K(\varphi_{i_p}^{\varepsilon_p}(x_p),x_{p-1}) \\ &= d_K(x,z_p)+d_K(\varphi_{i_p}^{\varepsilon_p}(z_p),\varphi_{i_p}^{\varepsilon_p}(x_p))+d_K(\varphi_{i_p}^{\varepsilon_p}(x_p),x_{p-1}) \\ &\geq d_K(x,z_p)+d_K(\varphi_{i_p}^{\varepsilon_p}(z_p),x_{p-1}) \end{aligned}$$

and induction on $p=|h^{-1}g|$ yields

$$\delta((x,g),(y,h)) = d_K(x,z_p)+d_K(\varphi_{i_p}^{\varepsilon_p}(z_p),z_{p-1})+\cdots+d_K(\varphi_{i_1}^{\varepsilon_1}(z_1),y). \qquad (**)$$

We claim that the metric space $T_{\mathcal{K}}$ associated to $(K\times F_n,\delta)$ is an $\mathbf{R}$-tree. Since $T_{\mathcal{K}}$ is connected (because $A_i\times\{gg_i\}$ and $B_i\times\{g\}$ have the same image in $T_{\mathcal{K}}$), it suffices by [AB, Theorem 3.17] to show that any 4 points $u_i=(x_i,h_i)$ satisfy the 0-hyperbolicity inequality:

$$\delta(u_1,u_2)+\delta(u_3,u_4)\leq \max\bigl\{\delta(u_1,u_3)+\delta(u_2,u_4),\delta(u_1,u_4)+\delta(u_2,u_3)\bigr\}.$$

This is clear if the elements h_1,h_2,h_3,h_4 are equal, since K is a tree. In general, we consider them as 4 points in a simplicial tree, namely the Cayley graph Γ of F_n relative to $g_1,\dots,g_n$. Let Γ_0 be the finite subtree they span. Assume that some terminal vertex of Γ_0, say h_1, is distinct from the other three elements h_2,h_3,h_4. Then the reduced words representing h_1, $h_2^{-1}h_1$, $h_3^{-1}h_1$, $h_4^{-1}h_1$ all end with the same letter, say g_1. Let z be the point in A_1 closest to x_1. We have $\delta(u_1,u_k)=d_K(x_1,z)+\delta((\varphi_1(z),h_1g_1^{-1}),u_k)$ for $k=2,3,4$, and 0-hyperbolicity follows by induction on the size of Γ_0. We leave to the reader the remaining case, when h_1,h_2,h_3,h_4 are equal in pairs.

The $\mathbf{R}$-tree $T_{\mathcal{K}}$ obviously satisfies (1) and (2), with K embedded in $T_{\mathcal{K}}$ as $K\times\{1\}$. Furthermore K meets every orbit.

Since $K\cap g_iK\neq\emptyset$ for $i=1,\dots,n$, any segment in $T_{\mathcal{K}}$ joining a point of gK to a point of hK, with $hg^{-1}=g_{i_1}^{\varepsilon_1}\dots g_{i_p}^{\varepsilon_p}$, may be covered by a finite union of images wK, namely $gK,(g_{i_p}^{\varepsilon_p}g)K,(g_{i_{p-1}}^{\varepsilon_{p-1}}g_{i_p}^{\varepsilon_p}g)K,\dots,(g_{i_1}^{\varepsilon_1}\dots g_{i_p}^{\varepsilon_p}g)K$. Applied to an arbitrary F_n-tree T' satisfying (1) and (2), this argument shows that the union of all orbits meeting K is a subtree (i.e. it is connected). This means that in proving (4) we may assume that T' also satisfies (3).

Define δ' on $K\times F_n$ as in the beginning of the proof. The map $(x,g)\mapsto gx$ identifies T' with the metric space associated to $(K\times F_n,\delta')$ (while $T_{\mathcal{K}}$ is associated to $(K\times F_n,\delta)$). Since $\delta'\leq\delta$, the identity of $K\times F_n$ induces a continuous equivariant map $j\colon T_{\mathcal{K}}\to T'$. This j induces the identity on K and is a morphism because any segment in $T_{\mathcal{K}}$ is contained in a finite union of images of K. Finally, uniqueness of $T_{\mathcal{K}}$ is easy to check using (4). ∎

Since the infimum defining δ is always achieved, we have the following facts about $T_{\mathcal{K}}$:

Proposition I.4.

(1) *two points* (x,g) *and* (y,h) *in* $K\times F_n$ *define the same point in* $T_{\mathcal K}$ *if and only if one can write* $y=\varphi_{i_1}^{\varepsilon_1}\dots\varphi_{i_p}^{\varepsilon_p}(x)$ *with* $g_{i_1}^{\varepsilon_1}\dots g_{i_p}^{\varepsilon_p}=h^{-1}g$.

(2) *given* $x,y\in K$ *and* $g\in F_n$, *one has* $y=gx$ *if and only if one can write* $y=\varphi_{i_1}^{\varepsilon_1}\dots\varphi_{i_p}^{\varepsilon_p}(x)$ *with* $g=g_{i_1}^{\varepsilon_1}\dots g_{i_p}^{\varepsilon_p}$.

(3) *if* $\gamma\in F_n$ *is represented by a cyclically reduced word* $g_{i_1}^{\varepsilon_1}\dots g_{i_p}^{\varepsilon_p}$, *then*

$$\ell(\gamma)=\min_{x_j\in\operatorname{dom}\varphi_{i_j}^{\varepsilon_j}}\left\{d_K(x_p,\varphi_{i_1}^{\varepsilon_1}(x_1))+d_K(x_1,\varphi_{i_2}^{\varepsilon_2}(x_2))+\dots+d_K(x_{p-1},\varphi_{i_p}^{\varepsilon_p}(x_p))\right\}.$$

$\square$

Remark. In the situation of Assertion 2, note that all points $g_{i_j}^{\varepsilon_j}\dots g_{i_p}^{\varepsilon_p}x$ $(1\le j\le p)$ belong to K.

We now prove a few other properties of $T_{\mathcal K}$.

Proposition I.5. *If* $\gamma\in F_n$ *is represented by a cyclically reduced word, then its fixed point set* $Fix\gamma\subset T_{\mathcal K}$ *is contained in* K.

Proof. Let $a\in T_{\mathcal K}$ be a fixed point of γ. Choose a representative $(x,g)\in K\times F_n$ of a with the length $|g|$ minimal. We shall identify g and the reduced word representing it. We assume $|g|>0$, and we argue towards a contradiction.

Since (x,g) and $(x,\gamma g)$ both represent a, Proposition I.4 (Assertion 1) lets us write $x=\varphi_{i_1}^{\varepsilon_1}\dots\varphi_{i_p}^{\varepsilon_p}(x)$ with $g_{i_1}^{\varepsilon_1}\dots g_{i_p}^{\varepsilon_p}=g^{-1}\gamma g$ (and $g_{i_1}^{\varepsilon_1}\dots g_{i_p}^{\varepsilon_p}$ reduced). Now $(\varphi_{i_1}^{-\varepsilon_1}(x),gg_{i_1}^{\varepsilon_1})$ and $(\varphi_{i_p}^{\varepsilon_p}(x),gg_{i_p}^{-\varepsilon_p})$ also represent a, so that g cannot end with $g_{i_1}^{-\varepsilon_1}$ or $g_{i_p}^{\varepsilon_p}$ (by minimality of $|g|$). It follows that $gg_{i_1}^{\varepsilon_1}\dots g_{i_p}^{\varepsilon_p}g^{-1}$ is the reduced word representing γ. This means that γ cannot be represented by a cyclically reduced word. $\square$

If $\gamma\neq 1$ is not cyclically reduced, then Fixγ is contained in some hK. We then get:

Corollary I.6. *For any* $\gamma\neq 1$ *in* F_n, *the set* $Fix\gamma\subset T_{\mathcal K}$ *is compact. If* $j\colon T_{\mathcal K}\to T'$ *is a morphism as in Theorem I.1, the restriction of* j *to* $Fix\gamma$ *is an isometry.* $\square$

Corollary I.7. *Suppose the action of* F_n *on* $T_{\mathcal K}$ *has no global fixed point. Then its length function is not abelian (i.e. it is not the absolute value of a homomorphism from* F_n *to* $\mathbf{R}$*).*

Proof. Otherwise, commutators would have non-compact fixed point sets (see e.g. [CM, 2.2 and 2.3]). $\square$

Recall that S is the finite set consisting of all vertices of the trees K, A_i, B_i $(1\le i\le n)$.

Proposition I.8. *If $x \in T_K$ is a branch point, its orbit contains a point of S. The action of the isotropy subgroup $Stab(x) \subset F_n$ on the set of directions $\pi_0(T_K \setminus \{x\})$ has only finitely many orbits.*

Proof. We start with a general argument. Suppose $[x, x'] \subset T_K$ is a segment with $[x, x'] \cap K = \{x\}$. Some subsegment $[x, x_1]$ is contained in some wK. We choose $x_1 \neq x$ and w so that $p = |w|$ is minimal. By Proposition I.4 (Assertion 2) we can write $x = \varphi_{i_1}^{\varepsilon_1} \dots \varphi_{i_p}^{\varepsilon_p}(y)$, with $y \in K$ and $w = g_{i_1}^{\varepsilon_1} \dots g_{i_p}^{\varepsilon_p}$. Since $[x, x_1] \cap K = \{x\}$, minimality of p implies that the segment $[y, y_1] = w^{-1}([x, x_1]) \subset K$ meets the domain of $\varphi_{i_p}^{\varepsilon_p}$ only at y. In particular $y \in S$.

This argument shows that the orbit of any branch point $x \in T_K$ meets S: since K meets every orbit we may assume $x \in K$, and if x is not a vertex of K then there is a segment $[x, x']$ as above. The argument also implies the second assertion since the number of possible points $y \in S$, and possible germs of segments $[y, y_1] \subset K$, is finite. $\square$

Corollary I.9. *There are only finitely many orbits of branch points in T_K.* $\square$

Remark. The number of orbits of branch points may be bounded in terms of n and the complexity of K. Our goal for very small actions will be to find a bound involving only n.

Corollary I.9 implies that the action on T_K is a J-action in the sense of [Le 3]. It follows that the closure of any orbit is a discrete union of closed subtrees. If no orbit is discrete, then every orbit is dense.

We shall use the following fact:

Proposition I.10. *Suppose F_n acts on T_K with every orbit dense. If the action is small, then every edge stabilizer is trivial.*

This is well-known (Rips, [BF 3, Remark 1.9]), but we sketch a proof. It is based on a theorem by Imanishi.

Proof.

If the result is false, let E be an edge with stabilizer $\mathbf{Z}$. By shortening E and applying elements of F_n, we may assume that every subarc of E has the same stabilizer, and a generator g of Stab(E) is represented by a cyclically reduced word $g_{i_1}^{\varepsilon_1} \dots g_{i_p}^{\varepsilon_p}$. Note that $E \subset K$ by Proposition I.5.

Choose $x \in \mathring{E}$ such that the orbit $F_n(x)$ contains no point of S. Observe that $F_n(x)$ meets K in an infinite set: otherwise $F_n(x)$ would be discrete. Imanishi's theorem (see [GLP 1, Theorem 3.1]) then implies that $F_n(x) \cap K$ accumulates on x. [Theorem 3.1 of [GLP 1] is stated for systems of isometries on a multi-interval, but it also holds on a finite tree]

Consider $h \in F_n$ such that $hx \neq x$ belongs to $\mathring{E}$. Then hgh^{-1} stabilizes some neighborhood of hx in E, so that hgh^{-1} is a power of g. It follows that h commutes with g. Since g is cyclically reduced, this leads to a contradiction for hx closer to x than any $\varphi_{i_j}^{\varepsilon_j} \dots \varphi_{i_p}^{\varepsilon_p}(x)$, $j = 2, \dots, p$. $\square$

Recall that an F_n-tree with no global fixed point contains a unique minimal invariant subtree, the union of all translation axes (see [CM]). The following fact will be used in Example II.6, but not elsewhere.

Proposition I.11. *Suppose the action of F_n on $T_{\mathcal{K}}$ has no global fixed point. Then the minimal invariant subtree $T_{\min}$ is closed in $T_{\mathcal{K}}$.*

Proof. Assume $T_{\min}$ is not closed. Then there is a segment $[x,y]$ with $[x,y]\cap T_{\min} = (x,y]$. Changing y and applying an element of F_n, we may assume $[x,y]\subset K$. We thus see that the tree $K' = T_{\min}\cap K$ is not closed in K.

It has finitely many limit points $x_1,\dots,x_k$. Let K'' be the tree obtained from K' by removing open segments of equal length $(x_1,y_1),\dots,(x_k,y_k)$ disjoint from S. Since $T_{\min}$ is connected, we have $K'\cap A_i\neq\emptyset$ for each i. The same is then true for K''. This implies that the union of all orbits meeting K'' is a subtree T''. By Proposition I.4 (Assertion 2) the intersection of T'' with K' consists only of K'' since no $\varphi_i^{\varepsilon_i}$ can send a point of some (x_j,y_j) into K'. We thus get an invariant subtree properly contained in $T_{\min}$, a contradiction. □

Remark. The action of F_n on $T_{\min}$ is the action associated to K', $\varphi_1|K',\dots,\varphi_n|K'$.

Corollary I.12. *Suppose the subgroup $F_p\subset F_n$ generated by $g_1,\dots,g_p$ acts with no global fixed point. Then its minimal invariant subtree $T_{\min}(F_p)$ is closed in $T_{\mathcal{K}}$.*

Proof. The union of all F_p-orbits meeting K is a subtree $T(F_p)$, and the action of F_p on $T(F_p)$ is the action associated to $(K,\varphi_1,\dots,\varphi_p)$. The set $T_{\min}(F_p)\cap K$ is closed in K (by Proposition I.11), hence also in $T_{\mathcal{K}}$ (by an argument given above). □

II. GEOMETRIC AND NON-GEOMETRIC ACTIONS.

Let T be a minimal F_n-tree, with length function ℓ. Let $K\subset T$ be a finite subtree such that $K\cap g_iK\neq\emptyset$ $(i=1,\dots,n)$. We consider the system $\mathcal{K} = (K,(\varphi_i)_{i=1,\dots,n})$, with φ_i the restriction of the action of g_i to $g_i^{-1}K\cap K$ (if $T=T_{\mathcal{K}}$, this new $\mathcal{K}$ equals the original $\mathcal{K}$ because $g_i^{-1}K\cap K = A_i$ by Assertion 2 of Proposition I.4: notation is consistent).

Theorem I.1 associates to $\mathcal{K}$ an F_n-tree $T_{\mathcal{K}}$, with a surjective morphism $j_K\colon T_{\mathcal{K}}\to T$. We shall usually write T_K instead of $T_{\mathcal{K}}$, and we denote by ℓ_K the length function of T_K. Recall that $\ell_K\geq\ell$ and ℓ_K is not abelian (Corollary I.7).

If the action on T is free (resp. small, resp. very small), so is the action on T_K: this is clear for free and small actions, and it follows from Corollary I.6 for very small actions.

The tree T_K is not necessarily minimal, but we can find arbitrarily large subtrees K with T_K minimal, as follows. Fix $x_0\in T$. It belongs to some translation axis A_γ (see [CM]). Choose an integer $p\geq|\gamma|$, and define K_p as the convex hull of the set $\{gx_0; |g|\leq p\}$ (note that by minimality T is the increasing union of the subtrees K_p). Since $p\geq|\gamma|$, all images of x_0 by terminal subwords of γ belong to K_p and it follows that the distance between x_0 and γx_0 is the same in T_{K_p} as in

T. The point x_0 thus belongs to the axis of γ in T_{K_p}. Being the convex hull of the orbit of x_0, the F_n-tree T_{K_p} is minimal.

Now consider two finite subtrees K, K' of T, with $K \subset K'$. Theorem I.1 provides an equivariant morphism $j_{K,K'}: T_K \to T_{K'}$, so that the trees T_K form a direct system of F_n-trees.

We now prove the well-known fact that this direct system *converges strongly* towards T in the sense of [GS]. This amounts to showing that, given a segment I in some T_K, there exists $K' \supset K$ such that the set $j_{K,K'}(I) \subset T_{K'}$ is mapped isometrically into T by $j_{K'}$. Choose finitely many elements $h_j \subset F_n$ such that I is covered by the trees $h_j K$. Letting $m = \max |h_j|$, take any K' containing all images of K by words of length $\leq m$.

To be more concrete, T is the strong limit of the sequence of minimal trees T_{K_p} constructed above. The fact that the limit is strong is often used in the following way. Any finite subtree $A \subset T$ may be lifted isometrically to T_{K_p} for p large: there exists a subtree $A^p \subset T_{K_p}$ such that the restriction of $j_{K_p}: T_{K_p} \to T$ is an isometry. Furthermore, given $g \in F_n$ and lifts A^p, A'^p of A and gA respectively, there exists $q \geq p$ such that $A'^q = gA^q$, where A^q and A'^q denote the images of A^p and A'^p in T_{K_q}. In particular $\ell_{K_q}(g) = \ell(g)$ for q large.

Instead of viewing T as the strong limit of a *sequence* T_{K_p}, we can also choose an increasing *continuous* family $K(t)$ ($t \in \mathbf{R}^+$), with $T = \cup K(t)$, and view T as the strong limit of the system $T_{K(t)}$. The following properties then hold.

Fix $g \in F_n$, and consider the function $\sigma_g: t \mapsto \ell_{K(t)}(g)$. It is non-increasing, and it is constant for t larger than some t_0 (depending on g). Furthermore σ_g is continuous: by Proposition I.4 (Assertion 3) we can bound $|\sigma_g(t_1) - \sigma_g(t_2)|$ by $|g|$ times the Hausdorff distance between $K(t_1)$ and $K(t_2)$.

Now we prove:

Proposition II.1. *Let T be a minimal F_n-tree. The following conditions are equivalent:*

(1) *There exists $\mathcal{K} = (K, \varphi_1, \ldots, \varphi_n)$ such that $T = T_{\mathcal{K}}$.*
(2) *There exists a finite subtree $K \subset T$ such that $T = T_K$ (i.e. $j_K: T_K \to T$ is an isometry).*
(2') *There exists a finite subtree $K \subset T$ such that $\ell_{K'} = \ell$ for every $K' \subset T$ containing K.*
(3) *T can only be a strong limit in a trivial way (if T is the strong limit of a sequence of F_n-morphisms $f_p: T_p \to T_{p+1}$ between minimal trees, then f_p is an isometry for p large).*

Proof.

$2 \implies 1$ by definition.

$1 \implies 2$ because $(T_{\mathcal{K}})_K = T_{\mathcal{K}}$ (see the above remark about consistency of notation).

$2 \implies 2'$ because $\ell = \ell_K$ and $\ell_K \geq \ell_{K'} \geq \ell$.

$2' \implies 2$: Take K' containing K such that $T_{K'}$ is minimal. Then $T_{K'}$ and T are equal because they are minimal trees with the same, non-abelian, length function ([AB], [CM]).

$3 \implies 2$ because $T = T_{K_p}$ for p large.

To prove $1 \implies 3$, suppose that $T = T_{\mathcal{K}}$ is the strong limit of a sequence f_p. For p large enough we may lift K isometrically to a subtree K^p of T_p (see above). For $i = 1, \dots, n$, let A_i^p and B_i^p be the subtrees of K^p corresponding to A_i and B_i. Since $g_i A_i = B_i$ we may take p even larger so as to ensure $g_i A_i^p = B_i^p$, and Theorem I.1 yields a morphism $j\colon T \to T_p$. It follows that the morphism from T_p to the limit tree T is an isometry, and the strong limit is trivial. ☐

Definition. *A minimal action of F_n is geometric if it satisfies conditions 1-3 above.* Using condition 3, we see that being geometric or not does not depend on the particular set of generators $g_1, \dots, g_n$.

Example II.2. We have seen that *every minimal action of F_n is the strong limit of a sequence of geometric minimal actions.*

Example II.3. The non-simplicial free F_3-actions constructed in [Le 2] are geometric.

Example II.4. An F_n-action with an abelian length function is not geometric by Corollary I.7 (compare [Le 4]).

Example II.5. It may be shown that *a minimal simplicial F_n-action is geometric if and only if every edge stabilizer has finite rank.* In particular, small simplicial actions are geometric.

Example II.6: *non-geometric free products of actions*

Consider finitely generated free groups G_1, G_2 acting non-trivially on **R**-trees T_1 and T_2. Fix basepoints $p_i \in T_i$. One can combine these two actions ([Sk 2], [CL]), obtaining an **R**-tree T with an action of $G_1 * G_2$. If the actions on T_i are minimal (resp. free), so is the action on T. More generally, the action on T is minimal as soon as no proper G_i-invariant subtree of T_i contains p_i.

Now let T_1 be a minimal G_1-tree. Assume that branch points are dense in T_1 (this happens for instance for the free F_3-actions of [Le 2], or for certain very small F_2-actions). Then segments are nowhere dense closed subsets, and by Baire's theorem T_1 is not complete as a metric space since it is a countable union of segments.

Choose a point p_1 in the completion $\overline{T_1}$ but not in T_1, and let $T_1' \subset \overline{T_1}$ be the smallest G_1-invariant subtree containing p_1. Combine the G_1-tree T_1' with some minimal G_2-tree T_2 (e.g. $G_2 = \mathbf{Z}$, $T_2 = \mathbf{R}$). The resulting $(G_1 * G_2)$-tree T is minimal, but by Corollary I.12 it is not geometric since T_1 is not closed in T.

Example II.7: *a free F_3-action with L infinitely generated*

Bestvina-Handel have shown how iterating an automorphism of F_n may lead to a non-simplicial free F_n-action. An explicit example is worked out in [Sh 2]. It is not geometric because it is a nontrivial strong limit (compare [BF 3]). We give an example where iterating an automorphism of F_3 leads to a free action such that the values of the length function do not lie in any finitely generated subgroup of $\mathbf{R}$.

Let α be the automorphism of F_3 given by $\alpha(a) = ab^{-1}$, $\alpha(b) = bac^{-1}$, $\alpha(c) = ca^{-3}$. Let $\lambda > 1$ be the largest eigenvalue of the associated matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}$$

and let (u, v, w) be a positive eigenvector.

View F_3 as the fundamental group of a wedge of 3 circles of respective lengths u, v, w, and let $\ell\colon F_3 \to \mathbf{R}^+$ be the corresponding length function (associated to the action of F_3 on the universal covering).

Since $\ell(\alpha h) \leq \lambda \ell(h)$ for every $h \in F_n$, each sequence

$$\ell_p(g) = \lambda^{-p} \ell(\alpha^p g)$$

is non-increasing. Taking its limit as $p \to +\infty$, we get a function $\ell_\infty\colon F_3 \to \mathbf{R}^+$ which is the length function of a very small action (provided it is not identically 0).

Our discussion so far holds for any automorphism of F_3 (or even of F_n), as long as the matrix A has a positive eigenvector. We now use the special form of α.

First of all, arguing as in [Sh 2], one shows that each sequence $\ell_p(g)$ is eventually constant, so that $\ell_\infty(g)$ is positive for every nontrivial $g \in F_n$. Thus ℓ_∞ is the length function of a free action.

Now the key feature of our example is that λ is not an algebraic unit, because the determinant of A is 3 (it is always an odd integer because A is invertible mod 2). This implies that $\mathbf{Z}[\lambda, \lambda^{-1}]$ is not a finitely generated subgroup of $\mathbf{R}$. Since ℓ_∞ satisfies the relation

$$\ell_\infty(\alpha^{\pm 1} g) = \lambda^{\pm 1} \ell_\infty(g),$$

the subgroup $L \subset \mathbf{R}$ generated by the values of ℓ_∞ is a $\mathbf{Z}[\lambda, \lambda^{-1}]$-module and therefore is not a finitely generated group.

Remark II.8.

- It is easy to check that α^{-2} is a positive automorphism. On the other hand α could not be positive, since $\det A = \pm 1$ if α is positive.
- One can show that the F_3-action just constructed has only one orbit of branch points. These branch points have index 1 (i.e. $T \setminus \{x\}$ has 3 components).
- The second author has proved that very small F_n-actions with $\mathbf{Q}$-rank $3n-4$ have finite ($\mathbf{Z}$-)rank.

III. COUNTING BRANCH POINTS.

Let T be a minimal small F_n-tree. Given $x \in T$, a *direction* d from x is a component of $T \setminus \{x\}$, or equivalently a germ of edges issuing from x. The isotropy subgroup $\mathrm{Stab}(x) \subset F_n$ acts on the set of directions from x. The stabilizer $\mathrm{Stab}(d)$ of a direction d is either trivial or infinite cyclic.

Let $v_1(x)$ be the (presumably infinite) number of $\mathrm{Stab}(x)$-orbits of directions from x with trivial stabilizer. We define the *index*

$$i(x) = 2 \text{ rk } \mathrm{Stab}(x) + v_1(x) - 2.$$

Theorem III.2 will imply that $i(x)$ is finite. If $\mathrm{Stab}(x)$ is trivial, then $i(x)+2$ is the number of components of $T \setminus \{x\}$.

This definition may be motivated by the analogy with surface groups mentioned in the introduction. Suppose $\mathcal{F}$ is a measured foliation on a closed surface Σ, whose singularities are k_s-prong saddles ($k_s \geq 3$). Then $\sum_s (k_s - 2) = -2\chi(\Sigma)$ by the Euler-Poincaré formula [FLP, p. 75]. A branch point x in the $\pi_1\Sigma$-tree associated to $\mathcal{F}$ corresponds to a set A of saddles linked by saddle connections. Setting $i(x) = \sum_{s \in A}(k_s - 2)$ leads to the formula above, since $v_1(x)$ is the number of infinite separatrices issuing from saddles in A while $\mathrm{Stab}(x)$ is isomorphic to the fundamental group of the 1-complex whose edges are the saddle connections.

Proposition III.1. *The index $i(x)$ is always non-negative. If $i(x) > 0$, then x is a branch point. Conversely, if the action is very small, then every branch point has index ≥ 1.*

Proof.

We fix $x \in T$, and we distinguish three cases according to the rank of $\mathrm{Stab}(x)$.

If $\mathrm{Stab}(x)$ has rank ≥ 2, then $i(x) \geq 2$. Since the action of $\mathrm{Stab}(x)$ on the set of directions has an infinite orbit, x is a branch point.

If $\mathrm{Stab}(x)$ is trivial, then $i(x) = v_1(x) - 2$, with $v_1(x)$ equal to the number of components of $T \setminus \{x\}$. Minimality of the action implies $v_1(x) \geq 2$. We thus have $i(x) \geq 0$, and $i(x) > 0$ if and only if x is a branch point.

If $\mathrm{Stab}(x) \simeq \mathbf{Z}$, then $i(x) = v_1(x)$ is non-negative. If $i(x) > 0$, we deduce that x is a branch point as in the first case. Now we assume that $i(x) = v_1(x)$ is 0 and the action is very small, and we prove x is not a branch point.

Consider a direction from x. The inclusion from its stabilizer into $\mathrm{Stab}(x)$ is an isomorphism because there are no obtrusive powers. This means that every element of $\mathrm{Stab}(x)$ acts on $\pi_0(T \setminus \{x\})$ as the identity. By the no-triod condition, there cannot be 3 distinct directions from x, so that x is not a branch point. $\square$

Remark. The proof shows that a branch point x has index 0 if and only if $\mathrm{Stab}(x) \simeq \mathbf{Z}$ and $v_1(x) = 0$.

Clearly $i(x) = i(x')$ if x and x' belong to the same F_n-orbit $\mathcal{O}$, and we write $i(\mathcal{O}) = i(x)$. We define the *total index* of T as

$$i(T) = \sum_{\mathcal{O} \in T/F_n} i(\mathcal{O}).$$

Theorem III.2. *Let T be a minimal small F_n-tree.*

(1) *If T is geometric, then $i(T) = 2n - 2$.*
(2) *If T is not geometric, then $i(T) < 2n - 2$.*

Corollary III.3. *If T is a minimal very small F_n-tree, the number of orbits of branch points is at most $2n - 2$.* □

Corollary III.4. *If T is a minimal small F_n-tree, the stabilizer of any $x \in T$ has rank at most n.* □

Proof of theorem III.2.

First assume that $T = T_{\mathcal{K}}$ is geometric. Given a finite tree H (such as K or A_i), and $x \in H$, we denote $u_H(x)$ the valence of x in H. Then:

$$\sum_{x \in H} (u_H(x) - 2) = -2. \tag{1}$$

Fix an F_n-orbit $\mathcal{O} \subset T_{\mathcal{K}}$. The interesting case is when $\mathcal{O}$ contains a point of S (since otherwise $i(\mathcal{O}) = 0$ by Propositions I.8 and III.1), but for now $\mathcal{O}$ may be arbitrary. We define a "Cayley graph" $\mathcal{O}_K$ as follows. Vertices of $\mathcal{O}_K$ are the points of $\mathcal{O}$ belonging to K (recall that K meets every orbit). There is an edge labelled g_i from z to $\varphi_i(z)$ whenever $z \in A_i$. Assertion 2 of Proposition I.4 implies that $\mathcal{O}_K$ is connected.

We define the weight $w(e)$ of an edge e labelled g_i as the valence of its origin z in A_i. All but finitely many edges have weight 2.

Next we define a "blown-up" 1-complex $\mathcal{O}'_K$. Vertices of $\mathcal{O}'_K$ will be directions, viewed as germs of edges. If $x \in K$, we shall distinguish between directions from x in K or in $T_{\mathcal{K}}$.

To define $\mathcal{O}'_K$, we start from $\mathcal{O}_K$, replacing each vertex x of $\mathcal{O}_K$ by $u_K(x)$ vertices representing directions d from x in K, and replacing each edge e by $w(e)$ edges in the obvious way (these edges in $\mathcal{O}'_K$ carry the same label and orientation as e). Let π be the natural projection from $\mathcal{O}'_K$ to $\mathcal{O}_K$.

Lemma III.5. *Fix $x \in \mathcal{O} \cap K$.*

(1) *The fundamental group of $\mathcal{O}_K$ is isomorphic to $Stab(x)$.*
(2) *The set of components $\mathcal{O}_1$ of $\mathcal{O}'_K$ is in one-to-one correspondence with the set of orbits under $Stab(x)$ of directions d from x in $T_{\mathcal{K}}$.*
(3) *The fundamental group of a component $\mathcal{O}_1$ is isomorphic to the corresponding isotropy subgroup $Stab(d)$, hence to $\{1\}$ or $\mathbf{Z}$.*

Proof.

There is a natural homomorphism $\rho\colon \pi_1(\mathcal{O}_K, x) \to F_n$, where $\rho(\gamma)$ is the product of labels of edges crossed by a loop γ (taking orientation into account: write g_i if the edge is crossed from z to $\varphi_i(z)$ and g_i^{-1} otherwise). Clearly this homomorphism is injective and takes values in $\mathrm{Stab}(x)$. By Proposition I.4 (Assertion 2), its image is the whole of $\mathrm{Stab}(x)$. This proves the first assertion of the lemma.

Now consider a component $\mathcal{O}_1$ of $\mathcal{O}'_K$. A vertex of $\mathcal{O}_1$ is a direction d_0 in K, at a point $y \in \mathcal{O} \cap K$. Applying any $g \in F_n$ taking y to x, we get a direction d from x in $T_{\mathcal{K}}$. The orbit of d under $\mathrm{Stab}(x)$ is independent of the choice of either d_0 or g, and we obtain a map Φ from components of $\mathcal{O}'_K$ to orbits of directions from x in $T_{\mathcal{K}}$.

The argument used in the proof of Proposition I.8 shows that Φ is onto. To prove injectivity, suppose that d_0, d'_0 are directions in K that correspond to directions d, d' in the same $\mathrm{Stab}(x)$-orbit. Then some $g \in F_n$ maps d_0 to d'_0. Assertion 2 of Proposition I.4 implies that d_0 and d'_0 belong to the same component of $\mathcal{O}'_K$.

Finally, the proof of Assertion (3) is similar to that of Assertion (1). □

Thanks to Lemma III.5 we can now deduce properties of T from combinatorial properties of finite subgraphs of $\mathcal{O}_K$ and $\mathcal{O}'_K$. Let $\mathcal{G}$ be a finite connected subgraph of $\mathcal{O}_K$ containing every vertex belonging to S and every edge of weight $\neq 2$ (if there are any). Let $\mathcal{G}' \subset \mathcal{O}'_K$ be the preimage $\pi^{-1}(\mathcal{G})$.

By Proposition I.8 and Lemma III.5, the 1-complex $\mathcal{O}'_K$ has finitely many components, each with first Betti number 0 or 1. Enlarging $\mathcal{G}$ if necessary, we may assume that $\pi_1\mathcal{G}'$ generates the fundamental group of every component of $\mathcal{O}'_K$.

Note that in general the intersection of $\mathcal{G}'$ with a given component of $\mathcal{O}'_K$ need not be connected. These intersections are connected, however, if $\pi_1\mathcal{G}$ generates $\pi_1\mathcal{O}_K$, since $\mathcal{G}$ then contains any embedded path in $\mathcal{O}_K$ with endpoints in $\mathcal{G}$. It is true that $\pi_1\mathcal{O}_K$ is finitely generated, but we do not know it yet.

In any connected finite 1-complex we have the formula:

$$1 - \mathrm{rk}\ \pi_1 = \sharp\,\mathrm{vertices} - \sharp\,\mathrm{edges}. \tag{2}$$

Applying it to each component $\mathcal{G}'_j$ of $\mathcal{G}'$ and summing up we get:

$$\sum_j (1 - \mathrm{rk}\ \pi_1\mathcal{G}'_j) = \sum_{x \in V(\mathcal{G})} u_K(x) - \sum_{e \in E(\mathcal{G})} w(e),$$

denoting V and E the set of vertices and edges respectively.

Subtracting formula (2) applied to $\mathcal{G}$ and multiplied by 2, we obtain:

$$2\mathrm{rk}\ \pi_1\mathcal{G} - 2 + \sum_j (1 - \mathrm{rk}\ \pi_1\mathcal{G}'_j) = \sum_{x \in V(\mathcal{G})} (u_K(x) - 2) - \sum_{e \in E(\mathcal{G})} (w(e) - 2). \tag{3}$$

The right hand side is independent of $\mathcal{G}$ (only finitely many terms may be nonzero), while every term $(1 - \mathrm{rk}\ \pi_1\mathcal{G}'_j)$ is non-negative. It follows that $\mathrm{rk}\ \pi_1\mathcal{G}$ is bounded, in other words $\pi_1\mathcal{O}_K$ *is finitely generated*.

We may then assume that $\pi_1\mathcal{G}$ generates $\pi_1\mathcal{O}_K$, thereby ensuring that a given component of $\mathcal{O}'_K$ contains only one $\mathcal{G}'_j$ (see above). By Lemma III.5 this implies that the left hand side of (3) equals $i(\mathcal{O})$, so that we have proved:

$$i(\mathcal{O}) = \sum_{x \in V(\mathcal{O}_K)} (u_K(x) - 2) - \sum_{e \in E(\mathcal{O}_K)} (w(e) - 2).$$

If $\mathcal{O}$ does not meet S, the right hand side is 0 and $i(\mathcal{O}) = 0$. For the other orbits we recall that the weight of an edge labelled g_i is the valence of its origin in A_i, and we write:

$$i(\mathcal{O}) = \sum_{x \in \mathcal{O} \cap K} (u_K(x) - 2) - \sum_{i=1}^{n} \sum_{x \in \mathcal{O} \cap A_i} (u_{A_i}(x) - 2). \tag{4}$$

Summing up and using (1) we get the required equation:

$$\sum_{\mathcal{O} \in T/F_n} i(\mathcal{O}) = -2 + 2n.$$

This completes the proof of Theorem III.2 when T is geometric. From now on we assume that T is not geometric (compare [Du, Theorem 5]).

Choose a base point $x_0 \in T$, and let K_p be the convex hull of $\{gx_0; |g| \leq p\}$. Recall from the beginning of Part II that T_{K_p} is a sequence of minimal small F_n-trees converging strongly to T. For convenience we write T_p instead of T_{K_p}, and we denote j_p the morphism $T_p \to T$.

Let x be a branch point of T, and $k \leq i(x)$ an integer (if we knew that $i(x)$ is finite, we would simply take $k = i(x)$). We are going to show that, for p large enough, there exists a lift $x' \in j_p^{-1}(x)$ with $i(x') \geq k$. This will prove $i(T) \leq 2n-2$ (note that lifts x' and y' are in distinct orbits if x, y are in distinct orbits).

Choose $h_1, \ldots, h_q \in \mathrm{Stab}(x)$ belonging to a free basis, and directions $d_1, \ldots, d_r$ from x with trivial stabilizers, in distinct $\mathrm{Stab}(x)$-orbits, with $2q + r - 2 = k$. Because of strong convergence, it is possible for p large to lift x to $x' \in T_p$ in such a way that h_α fixes x' ($\alpha = 1, \ldots, q$). Similarly we may assume that d_β lifts to a direction d'_β from x' in T_p ($\beta = 1, \ldots, r$). Clearly $v_1(x') \geq r$. On the other hand j_p induces an injection from $\mathrm{Stab}(x')$ into $\mathrm{Stab}(x)$ whose image contains $h_1, \ldots, h_q$. Since the subgroup generated by $h_1, \ldots, h_q$ is a free factor of $\mathrm{Stab}(x)$, we get $\mathrm{rk}\,\mathrm{Stab}(x') \geq q$ and $i(x') \geq k$. This shows the existence of x', hence the inequality $i(T) \leq 2n - 2$.

Now we assume $i(T) = 2n - 2$ and we argue towards a contradiction (for T non-geometric). We assume that the basepoint x_0 is a branch point.

Consider $B \subset T$ containing one point from each orbit with positive index. For $x \in B$ choose a basis $h_1, \ldots, h_q$ of $\mathrm{Stab}(x)$ and directions $d_1, \ldots, d_r$ as before with $2q + r - 2 = i(x)$. Choose p so that we can associate $x' \in T_p$ as above to each $x \in B$. Also make sure that x_0 is a branch point of K_p.

Since $i(T_p) = i(T)$, the orbit of every branch point of T_p with positive index contains some x'. Furthermore every direction from x' with trivial stabilizer is $\mathrm{Stab}(x')$-congruent to some d'_β.

The morphism j_p is not an isometry. Thus two distinct germs of edges e_1, e_2 at some $y \in T_p$ are carried by j_p onto the same germ at $j_p(y)$. We show that this leads to a contradiction.

If y is a branch point with positive index, previous remarks imply that e_1 and e_2 both have nontrivial stabilizer. Since e_1 and e_2 get identified in T, the union of their stabilizers generates a cyclic group, so that $e_1 \cup e_2$ is contained in some $\mathrm{Fix}\gamma \subset T_p$. This contradicts Corollary I.6.

If y is a branch point with index 0, then again e_1 and e_2 both have nontrivial stabilizer (see the remark after the proof of Proposition III.1), and the argument is the same.

If y is a regular point, then $e_1 \cup e_2$ is contained in some wK_p: otherwise y would belong to the orbit of a terminal vertex of K_p, hence to the orbit of the branch point x_0 of T_p. We have again reached a contradiction since the restriction of j_p to wK_p is an isometry. □

Remark III.6. The total index of a very small minimal F_n-tree is at least 1. As mentioned in Remark II.8, the free F_3-tree of Example II.7 has index 1. There exist small F_2-actions with total index 0.

IV. BOUNDING THE RANK.

Let G be a finitely generated group. Let T be a G-tree, with length function ℓ.

Let L be the subgroup of $\mathbf{R}$ generated by the values of ℓ. The $\mathbf{Q}$-*rank* $r_{\mathbf{Q}}$ (of the action, or of the length function) is the dimension of the $\mathbf{Q}$-vector space $L \otimes_{\mathbf{Z}} \mathbf{Q}$ generated by L. The $\mathbf{Z}$-*rank* (or simply *rank*) r is the rank of the abelian group L. Both ranks may be infinite. If r is finite, then $r_{\mathbf{Q}} = r$ and $L/2L$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^r$. A minimal action is simplicial if and only if it is topologically conjugate to an action with $r = 1$.

Our main interest will be in very small F_n-actions, but for now we only assume that $T \neq \mathbf{R}$ is a minimal G-tree with non-abelian length function. Define Λ as the subgroup of $\mathbf{R}$ generated by distances between branch points. It is the smallest subgroup $\Lambda \subset \mathbf{R}$ such that T may be viewed as the completion of a Λ-tree (see [Sh 2, §1.3.1], or "base change" in [AB]).

We note the inclusions

$$2\Lambda \subset L \subset \Lambda,$$

which imply that we may use Λ instead of L when computing $r_{\mathbf{Q}}$ and r.

The second inclusion is obvious since we assume $T \neq \mathbf{R}$. The first one comes from [AB, Theorem 7.13 (c)]. Here is a proof based on the fact that, in a minimal G-tree with non-abelian length function, every segment is contained in some translation axis (see Lemma 4.3 of [Pa]). Given two branch points a, b, there exists a translation axis A_α (resp. A_β) containing a (resp. b) and disjoint from the open segment (a, b). The well known formula $\ell(\alpha\beta) = \ell(\alpha) + \ell(\beta) + 2d(a, b)$ (see e.g. [Pa, Proposition 1.6]) then yields $2\Lambda \subset L$.

We shall say that $g \in G$ acts as an *inversion* if it interchanges two distinct points of T. We thank the referee for pointing out that the following result applies only to actions without inversions (unless we count centers of inversions as branch points). Of course a very small F_n-action has no inversion.

Proposition IV.1. *Let $T \neq \mathbf{R}$ be a minimal G-tree with non-abelian length function and no inversion.*

(1) *Let $g_1, \dots, g_n$ be a system of generators for G. The numbers $\ell(g_1), \dots, \ell(g_n)$*

generate $L \bmod 2\Lambda$.

(2) *Let* $(p_j)_{j\in J}$ *be representatives of* G*-orbits of branch points. For* $j_0 \in J$*, the numbers* $d(p_{j_0}, p_j)$ *generate* $\Lambda \bmod L$.

Proof. First we prove the following equalities modulo 2Λ:

$$d(b,b') + d(b',b'') = d(b,b'') \tag{5}$$

$$d(b,gb) = \ell(g), \tag{6}$$

where b, b', b'' are branch points and $g \in G$.

Define c by $[b,b'] \cap [b',b''] = [b',c]$. The point c is a branch point (possibly b, b', or b''), and (5) follows from the formula

$$d(b,b') + d(b',b'') = d(b,b'') + 2d(c,b').$$

For (6), we use the formula

$$d(b,gb) = \ell(g) + 2d(b,C_g)$$

where C_g is the characteristic set of g (its fixed point set or its translation axis), see e.g. [CM, 1.3]. If the point of C_g closest to b is a branch point, we are done. Otherwise g has a unique fixed point, namely the midpoint m of $[b,gb]$. Since m is not a branch point, the segment $g([m,gb]) = [m,g^2b]$ meets $[m,b]$ in a non-degenerate segment and g is an inversion, a contradiction.

If $g_1, g_2 \in G$ we choose an arbitrary branch point b, and using (5) and (6) we write (modulo 2Λ):

$$\ell(g_1g_2) = d(b,g_1g_2b) = d(b,g_1b)+d(g_1b,g_1g_2b) = d(b,g_1b)+d(b,g_2b) = \ell(g_1)+\ell(g_2).$$

This proves Assertion (1) of the proposition since $g \mapsto \ell(g)$ induces a homomorphism from G onto $L/2\Lambda$.

Given two branch points q, r, we write $q = gp_j$ and $r = hp_k$ with $g, h \in G$ and $j, k \in J$. Then (also modulo 2Λ):

$$\begin{aligned} d(q,r) &= d(gp_j, hp_k) \\ &= d(gp_j, gp_{j_0}) + d(gp_{j_0}, gp_k) + d(gp_k, hp_k) \\ &= d(p_j, p_{j_0}) + d(p_{j_0}, p_k) + \ell(g^{-1}h) \\ &= d(p_{j_0}, p_j) + d(p_{j_0}, p_k) \pmod{L}. \end{aligned} \tag{7}$$

This proves the second assertion. ∎

Proposition IV.2.

(1) *Geometric F_n-actions have finite rank.*
(2) *Consider a non-geometric F_n-tree T as the strong limit of a system $T_{K(t)}$, as in Part II. If $\liminf_{t\to+\infty} r(T_{K(t)})$ is finite, then*

$$r_{\mathbf{Q}}(T) \leq \liminf_{t\to+\infty} r(T_{K(t)})$$

and

$$r_{\mathbf{Q}}(T) < \limsup_{t\to+\infty} r(T_{K(t)}).$$

Proof.

Let $T = T_K$. It follows from Proposition I.8 and Equation $(**)$ (from the proof of Theorem I.1) that Λ is contained in the subgroup of $\mathbf{R}$ generated by distances between points in the finite set S, and Assertion (1) holds.

Recall from Part II that for a given $g \in F_n$ the function $t \mapsto \ell_{K(t)}(g)$ is continuous, and constant for t large. Thus every finitely generated subgroup of $L(T)$ is contained in $L(T_{K(t)})$ for t large. This proves the first inequality of Assertion (2).

If the second inequality is false, then $r(T_{K(t)}) = r_{\mathbf{Q}}(T)$ for t large. We choose a finite set of elements $g_j \in G$ such that the numbers $\ell(g_j)$ generate $L(T) \otimes_{\mathbf{Z}} \mathbf{Q}$. Since each function $t \mapsto \ell_{K(t)}(g_j)$ is constant for t large, we see that the $\mathbf{Q}$-vector space generated by $L(T_{K(t)})$ is independent of t for t large.

Since $\ell_{K(t)}$ varies continuously, this means that $\ell_{K(t)}$ is *constant* for t large. As $\ell_{K(t)}$ is not abelian (Corollary I.7), the minimal invariant subtree of $T_{K(t)}$ is independent of t. Therefore T is geometric, a contradiction. ◻

Corollary IV.3. *Let T be a geometric minimal F_n-tree without inversions. Let b be the number of orbits of branch points. Then $r(T) \leq n + b - 1$.*

Proof. We know by Proposition IV.2 that the action has finite rank r. The group $\Lambda/2\Lambda$ is then isomorphic to $(\mathbf{Z}/2\mathbf{Z})^r$. The result follows since $\Lambda/2\Lambda$ is generated by $n+b-1$ elements by Proposition IV.1 (note that b is finite by Corollary I.9). ◻

We now prove:

Theorem IV.4. *Let T be a minimal, very small, F_n-tree. Then $r_{\mathbf{Q}}(T) \leq 3n-3$. Equality may hold only if the action is free and simplicial.*

Proof.

If T is geometric, we have $r(T) \leq 3n-3$ by Corollaries III.3 and IV.3. If T is not geometric, we recall that the geometric trees $T_{K(t)}$ are very small (see Part II), so that $r_{\mathbf{Q}}(T) < 3n-3$ by Proposition IV.2.

From now on we assume that the action is geometric, but not free simplicial. We know that it has finite rank r, with $\Lambda/2\Lambda \simeq (\mathbf{Z}/2\mathbf{Z})^r$, and we show $r < 3n-3$. This will complete the proof.

We consider several cases.

• If the action is simplicial, it is obtained from a graph of groups Γ. Consider the natural epimorphism ρ from F_n to the fundamental group of Γ in the *topological* sense. Since the action is not free, some vertex group is nontrivial and ρ is not injective. Free groups being hopfian, the rank of $\pi_1 \Gamma$ is strictly inferior to n.

On the other hand, every vertex of Γ is the projection of a branch point of T (because there is no inversion). By Corollary III.3, Γ has at most $2n-2$ vertices. It follows that Γ has strictly less than $3n-3$ edges. Since Λ is generated by the lengths of edges, we have $r < 3n-3$.

• Now suppose that every F_n-orbit is dense in T. In the previous case, we had $r < 3n-3$ because $L/2\Lambda$ had 2-rank $< n$. In this case, we prove that Λ/L has 2-rank $< 2n-3$, so that $\Lambda/2\Lambda$ has 2-rank $< 3n-3$.

We write $T = T_K$ as in Part II, making sure that every terminal vertex of K is a branch point of T. If there are less than $2n-2$ distinct orbits of branch points in T, we are done by Proposition IV.1 (Assertion 2). If not, let $p_1, \dots, p_{2n-2}$ be representatives of these orbits, chosen to belong to K. Each p_j has index 1.

By Proposition I.10, every edge stabilizer is trivial. This means that the generators $\varphi_1, \dots, \varphi_n$ are *independent* in the sense of [GLP 1]: a reduced word $\varphi_{i_1}^{\varepsilon_1} \dots \varphi_{i_p}^{\varepsilon_p}$ cannot be equal to the identity on a non-degenerate subinterval of K. Denoting by $|\ \ |$ arclength in K, we then have

$$|K| = \sum_{i=1}^{n} |A_i| \tag{8}$$

([Le 5, Theorem 2], see also [Le 1, corollaire II.5] and [GLP 1, Part 6]).

Equation (8) is an equality between numbers of the form $d(q,r)$, where qr is an edge of K or A_i. We view it as an equation in Λ/L (recall that every vertex of K, hence also of A_i, is a branch point of T).

Using Equation (7) from the proof of Proposition IV.1, we may replace each term $d(q,r)$ by a sum $d(p_1,p_j)+d(p_1,p_k)$. We thus obtain a linear relation between the numbers $d(p_1,p_j)$, $j = 2, \dots, 2n-2$ (whose coefficients are integers mod 2). We have to check that it is not trivial.

The coefficient of $d(p_1,p_j)$ in the expansion of $|K|$ (resp. $|A_i|$) has the same parity as the sum $\sum u_K(x)$ (resp. $\sum u_{A_i}(x)$) taken over vertices of K (resp. A_i) belonging to the orbit of p_j. Since every p_j has index 1, Equation (4) from the proof of Theorem III.2 then yields the nontrivial relation $\sum_{j=2}^{2n-2} d(p_1,p_j) = 0 \bmod L$ between the $2n-3$ generators of Λ/L.

• Finally, we simply assume that the action is not simplicial. We recall [Le 3] that T may be obtained as a *graph of transitive actions*. In particular, there exists a subtree $T_v \subset T$ such that:

- T_v is closed, not equal to a point;
- there exists $\delta > 0$ such that, for $g \in F_n$, either $gT_v = T_v$ (i.e. $g \in \mathrm{Stab}(T_v)$) or the distance between T_v and gT_v is greater than δ;
- $\mathrm{Stab}(T_v)$ acts on T_v with dense orbits.

Let T' be the F_n-tree obtained by collapsing each gT_v to a point. The natural action of F_n on T' is very small. Apply Theorem III.2 to both T and T'. We find that $\mathrm{Stab}(T_v)$ has some finite rank p and

$$i(T) - i(T') = i(T_v) - (2p - 2),$$

where $i(T_v)$ is the index of T_v viewed as a $\mathrm{Stab}(T_v)$-tree. The left hand side is non-negative because T is geometric, while the right hand side is non-positive. This implies $i(T_v) = 2p - 2$: the action of $\mathrm{Stab}(T_v)$ on T_v is geometric.

If there are less than $2p-2$ distinct $\mathrm{Stab}(T_v)$-orbits of branch points in T_v, then there are less than $2n - 2$ distinct F_n-orbits in T, and we are done. Otherwise, the analysis of the previous case yields a nontrivial relation in $\Lambda(T_v)/L(T_v)$, hence also in $\Lambda(T)/L(T)$. ☐

Corollary IV.5. *Let T be a very small F_n-tree with length function ℓ. Suppose $\ell \circ \alpha = \lambda\ell$ with $\alpha \in Aut(F_n)$ and $\lambda \in \mathbf{R}^+$. Then λ is algebraic, of degree bounded by $3n - 4$. If T is geometric, then λ is an algebraic unit.*

Proof. If the action on T is free simplicial, then $\lambda = 1$. If not, multiplication by λ defines an automorphism of $L \otimes \mathbf{Q}$, a $\mathbf{Q}$-vector space of dimension $\leq 3n - 4$. This implies that λ is algebraic of degree $\leq 3n - 4$. If the action is geometric, then λ is a unit because it acts on L, a finitely generated abelian group by Assertion 1 of Proposition IV.2. ☐

V. SPACES OF LENGTH FUNCTIONS.

Let G be a finitely generated group. Let Ω be the set of conjugacy classes in G. Let $LF(G) \subset (\mathbf{R}^+)^\Omega$ be the space of all length functions on G, and $PLF(G)$ the space of projectivized length functions. Recall that $PLF(G)$ is compact [CM]. Also note that the $\mathbf{Q}$-rank of a length function ℓ depends only on its class in $PLF(G)$.

Proposition V.1. *Let G be a finitely generated group. Let $k \geq 1$ be an integer. The space $LF_{\leq k}(G)$ of all length functions with $\mathbf{Q}$-rank $\leq k$ has dimension $\leq k$. The space $PLF_{\leq k}(G)$ of all projectivized length functions with $\mathbf{Q}$-rank $\leq k$ has topological dimension $\leq k - 1$.*

Proof. Fix $k + 1$ rationally independent real numbers $\lambda_0, \ldots, \lambda_k$. For $j = 0, \ldots, k$, let M_j be the space of all $x \in (\mathbf{R}^+)^\Omega$ such that no nonzero coordinate of x is a rational multiple of λ_j. Each M_j has dimension 0: every $x \in M_j$ has arbitrarily small neighborhoods with boundary disjoint from M_j. Next we observe that

every $\ell \in LF_{\leq k}(G)$ belongs to at least one M_j: otherwise $L \otimes \mathbf{Q}$ would contain $\lambda_0, \ldots, \lambda_k$. It follows that $LF_{\leq k}(G)$ has dimension $\leq k$ since it is contained in the union of the 0-dimensional sets M_j, $j = 0, \ldots, k$ (see [HW, p. 29]). A similar argument applies to $PLF_{\leq k}(G)$. ☐

Theorem V.2. *The boundary of Culler-Vogtmann's outer space Y_n has dimension $3n-5$.*

Proof. The boundary of Y_n consists of (projective classes of length functions of) very small actions of F_n which are not free simplicial, so that it has dimension $\leq 3n-5$ by Theorem IV.4 and Proposition V.1. Since it is easy to find in δY_n a $(3n-5)$-simplex consisting of simplicial actions, we have $\dim Y_n = 3n-5$. $\square$

Remark V.3. Let T_1 be a very small F_2-tree with dense orbits (see [CV, §5]). Apply Example II.6, taking T_2 to be the universal covering of the graph Γ pictured below (for $n \geq 3$) and choosing p_2 in the preimage of q. We get a non-geometric very small F_n-tree. Varying the lengths of edges of Γ gives a $(3n-7)$-simplex of non-geometric actions in δY_n (this application of Example II.6 was suggested by M. Bestvina). Since we may choose p_1 arbitrarily in $\overline{T_1} \setminus T_1$, which is one-dimensional (see the proof of Theorem 2.2.2 in [MNO]), the set of non-geometric actions in δY_n has dimension $\geq 3n-6$ for $n \geq 3$.

4.71in by 1.32in (Fig1 scaled 1030)

Finally, we sketch a proof of a theorem announced by Skora [Sk 3].

Theorem V.4. *Length functions of simplicial actions are dense in $LF(F_n)$.*

Proof. We need to approximate any $\ell \in LF(F_n)$ by simplicial length functions. By Example II.2, we may assume that ℓ comes from a geometric F_n-tree $T_{\mathcal{K}}$. The system $\mathcal{K}$ consists of a finite tree K and n isometries $\varphi_i\colon A_i \to B_i$. We may approximate it by a system $\mathcal{K}'$ such that every distance between vertices of K', A_i', B_i' is *rational*. The corresponding length function ℓ' is then simplicial. By Assertion 3 of Proposition I.4, this ℓ' is an approximation of ℓ: for $g \in F_n$ cyclically reduced, $|\ell'(g) - \ell(g)|$ is bounded by $|g|$ times a constant depending only on $\mathcal{K}$ and $\mathcal{K}'$. $\square$

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