AN INTRODUCTION TO II_1 FACTORS

CYRIL HOUDAYER

Contents

2000 *Mathematics Subject Classification.* 46L07; 46L10; 46L54.

Key words and phrases. Finite von Neumann algebras; Equivalence relations; Hilbert bimodules; Amenability; Haagerup property; Relative property (T); Cartan subalgebras.

Research partially supported by ANR grant AGORA.

1. Finite von Neumann algebras

1.1. **Basics on von Neumann algebras.** Let H be a separable complex Hilbert space. We shall denote by $\langle \cdot, \cdot \rangle$ the inner product on H that we assumed to be linear in the first variable (and conjugate linear in the second one). Let $\mathbf{B}(H)$ be the algebra of all bounded linear maps $T : H \to H$. This is a Banach algebra for the *uniform* norm:

$$\|T\|_\infty = \sup_{\|\xi\| \leq 1} \|T\xi\|.$$

We moreover have $\|ST\|_\infty \leq \|S\|_\infty \|T\|_\infty$, $\forall S, T \in \mathbf{B}(H)$. The algebra $\mathbf{B}(H)$ is naturally endowed with $*$-operation called the *adjonction* defined as follows:

$$\langle T^*\xi, \eta \rangle = \langle \xi, T\eta \rangle, \forall \xi, \eta \in H.$$

We have $(T^*)^* = T$, $\|T^*\|_\infty = \|T\|_\infty$ and

$$\|T^*T\|_\infty = \|TT^*\|_\infty = \|T\|_\infty^2.$$

Thus, $\mathbf{B}(H)$ is a C^*-algebra. We can define several *weaker* topologies on $\mathbf{B}(H)$ as well, in the following way. Let (T_i) be a net of operators in $\mathbf{B}(H)$.

Topology	T_i	$\to$	0
norm	$\|T_i\|_\infty$	$\to$	0
ultra-$*$-strong	$\sum_n (\|T_i\xi_i\|^2 + \|T_i^*\xi_n\|^2)$	$\to$	$0, \quad \forall (\xi_n) \in \ell^2 \otimes H$
$*$-strong	$\|T_i\xi\|^2 + \|T_i^*\xi\|^2$	$\to$	$0, \quad \forall \xi \in H$
ultrastrong	$\sum_n \|T_i\xi_n\|^2$	$\to$	$0, \quad \forall (\xi_n) \in \ell^2 \otimes H$
strong	$\|T_i\xi\|$	$\to$	$0, \quad \forall \xi \in H$
ultraweak	$\sum_n \langle T_i\xi_n, \eta_n \rangle$	$\to$	$0, \quad \forall (\xi_n), (\eta_n) \in \ell^2 \otimes H$
weak	$\langle T_i\xi, \eta \rangle$	$\to$	$0, \quad \forall \xi, \eta \in H$

For a non-empty subset $S \subset \mathbf{B}(H)$, define the *commutant* of S in $\mathbf{B}(H)$ by

$$S' := \{T \in \mathbf{B}(H) : xT = Tx, \forall x \in S\}$$

One can then define inductively $S'' = (S')'$, $S^{(3)} = (S'')'$, $S^{(k+1)} = (S^{(k)})'$, for all $k \geq 1$. It is easy to see that

$$\begin{aligned} S &\subset S'' \\ S^{(k)} &= S^{(k+2)}, \forall k \geq 1. \end{aligned}$$

Definition 1.1. Let $M \subset \mathbf{B}(H)$ be a unital $*$-subalgebra. We say that M is a *von Neumann algebra* if $M'' = M$.

Theorem 1.2 (Von Neumann's Bicommutant Theorem)**.** *Let $M \subset \mathbf{B}(H)$ be a unital $*$-subalgebra. The following are equivalent:*

1. *$M'' = M$.*
2. *M is strongly closed.*
3. *M is weakly closed.*

Proof. We only sketch the proof.

(1) $\Longrightarrow$ (2) is clear since commutants are always strongly closed.

(2) $\Longrightarrow$ (1). Let $x \in M''$. Let

$$\mathcal{V}(x, \xi_1, \dots, \xi_n, \varepsilon) := \{y \in \mathbf{B}(H) : \|x\xi_i - y\xi_i\| < \varepsilon, \forall i = 1, \dots, n\}$$

be a strong neighborhood of x in $\mathbf{B}(H)$. Let $K = \ell^2_n \otimes H$ and observe that $\mathbf{B}(K) = \mathbf{M}_n(\mathbf{C}) \otimes \mathbf{B}(H)$. Let $\eta = (\xi_1, \dots, \xi_n) \in K$. Define $V = (1 \otimes M)\eta \subset K$. Since

M is strongly closed, V is a closed subspace of K. Denote by $P_V \in \mathbf{B}(K)$ the corresponding orthogonal projection. Since $(1 \otimes a)P_V = P_V(1 \otimes a)$, $\forall a \in M$, it follows that $1 \otimes x$ commutes with P_V, since $x \in M''$. Thus $(1 \otimes x)\eta \in V$ and we can find $y \in M$ such that $(1 \otimes x)\eta = (1 \otimes y)\eta$, so in particular $y \in N(x, \xi_1, \dots, \xi_n, \varepsilon)$. Then M'' is contained in the strong closure of M and hence $M = M''$.

The fact that (2) and (3) are equivalent follows from Hahn-Banach Separation Theorem (M is convex since it is a vector subspace!). $\square$

1.2. **Finite von Neumann algebras.** A von Neumann algebra M is said to be *finite*, if every isometry $v \in M$ is a unitary, i.e.

$$v^*v = 1 \Longrightarrow vv^* = 1, \forall v \in M.$$

One can show that a von Neumann algebra is finite if and only if it has a *faithful normal tracial state* $\tau : M \to \mathbf{C}$:

- τ is a positive linear functional with $\tau(1) = 1$.
- τ is faithful, i.e. $\forall x \in M$, $\tau(x^*x) = 0 \Longrightarrow x = 0$.
- τ is normal, i.e. τ is weakly continuous on $(M)_1$, the unit ball of M with respect to the uniform norm $\| \cdot \|_\infty$.
- τ is a trace, i.e. $\forall x, y \in M$, $\tau(xy) = \tau(yx)$.

An infinite dimensional finite von Neumann algebra M with trivial center, i.e. $M' \cap M = \mathbf{C}$, is called a II_1 factor.

The most simple examples of finite von Neumann algebras are the following:

(1) Abelian von Neumann algebra. Let (X, μ) be a standard probability space. Represent $L^\infty(X, \mu)$ on the Hilbert space $L^2(X, \mu)$ by multiplication

$$(f\xi)(x) = f(x)\xi(x), \forall f \in L^\infty(x, \mu), \forall \xi \in L^2(X, \mu).$$

The von Neumann algebra $M = L^\infty(X, \mu)$ comes equipped with the trace τ given by integration against the probability measure μ, i.e. $\tau = \int \cdot \mathrm{d}\mu$.

(2) Group von Neumann algebra. Let Γ be a countable discrete group. The *left regular* representation $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ is defined as follows

$$\lambda_s \delta_t = \delta_{st}.$$

The von Neumann algebra of Γ is then defined by

$$L(\Gamma) = \{\lambda_s : s \in \Gamma\}''.$$

The canonical trace τ on $L(\Gamma)$ is $\tau = \langle \cdot \delta_e, \delta_e \rangle$. One checks that $L(\Gamma)$ is a II_1 factor if and only if the group Γ has infinite conjugacy classes (icc), that is, $\forall t \neq e$, the set $\{sts^{-1} : s \in \Gamma\}$ is infinite.

Exercise 1.3. Let $T = [T_{st}]_{s,t \in \Gamma} \in \mathbf{B}(\ell^2(\Gamma))$, with $T_{st} = \langle T\delta_t, \delta_s \rangle$. Show that $T \in L(\Gamma)$ if and only if T is *constant down the diagonals*, i.e. $T_{st} = T_{xy}$ whenever $st^{-1} = xy^{-1}$.

Assume that Γ is abelian. Then the dual $\widehat{\Gamma}$ is a second countable compact abelian group. Write $\mathcal{F} : \ell^2(\Gamma) \to L^2(\widehat{\Gamma})$ for the Fourier transform which is defined by $\mathcal{F}(\delta_s) = \chi \mapsto \langle s, \chi \rangle$. We get a canonical identification

$$L^\infty(\widehat{\Gamma}) = \mathcal{F} L(\Gamma) \mathcal{F}^*.$$

(3) Group measure space construction. Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving (p.m.p.) action. Define an action $\sigma : \Gamma \curvearrowright L^\infty(X)$ by

$(\sigma_s(F))(x) = F(s^{-1}x)$, $\forall F \in L^\infty(X)$. We still denote by

$$\sigma : \Gamma \to \mathcal{U}(L^2(X))$$

the corresponding Koopman representation. We regard $L^\infty(X) = L^\infty(X) \otimes 1 \subset \mathbf{B}(L^2(X) \otimes \ell^2(\Gamma))$. The unitaries $u_s = \sigma_s \otimes \lambda_s \in \mathcal{U}(L^2(X) \otimes \ell^2(\Gamma))$, for $s \in \Gamma$, satisfy the following covariance relation:

$$u_s F u_s^* = \sigma_s(F).$$

Observe that by Fell's absorption principle, the unitary representation $(u_s)_{s\in\Gamma}$ is simply a multiple of the left regular representation. The *crossed product* von Neumann algebra is then defined by

$$L^\infty(X) \rtimes \Gamma = \left\{ \sum_{\text{finite}} a_s u_s : a_s \in L^\infty(X) \right\}'' \subset \mathbf{B}(L^2(X) \otimes \ell^2(\Gamma)).$$

The trace is given by

$$\tau(\sum_{s\in\Gamma} a_s u_s) = \int_X a_e \mathrm{d}\mu.$$

Recall that the action is said to be *free* if

$$\mu(\{x \in X : sx = x\}) = 0, \forall e \neq s \in \Gamma.$$

It is moreover said to be *ergodic* if

$$\Gamma A = A \Longrightarrow \mu(A)(1 - \mu(A)) = 0, \forall A \subset X.$$

One can check that the action is free if and only if $L^\infty(X)$ is maximal abelian in $L^\infty(X) \rtimes \Gamma$. In that case, we say that $L^\infty(X)$ is a *Cartan subalgebra*, i.e. $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ is maximal abelian and regular. Moreover, $L^\infty(X) \rtimes \Gamma$ is a II_1 factor if and only if the action is ergodic.

Observe that when the probability space $X = \{\mathrm{pt}\}$ is a point, then the group von Neumann algebra and the group measure space construction coincide, i.e. $L^\infty(X) \rtimes \Gamma = L(\Gamma)$.

Let M be a finite von Neumann algebra and fix τ a faithful normal trace. We endow M with the following sesquilinear form

$$\langle x, y\rangle_\tau = \tau(y^* x), \forall x, y \in M.$$

Denote by $L^2(M, \tau)$ or simply by $L^2(M)$ the completion of M with respect to $\langle \cdot, \cdot \rangle_\tau$. The corresponding $\|\cdot\|_2$-norm on M is defined by $\|x\|_2 = \sqrt{\tau(x^*x)}$. Write $M \ni x \to x\widehat{1} = \widehat{x} \in L^2(M)$ for the natural embedding. Note that the unit vector $\widehat{1}$ is *cyclic* (i.e. $M\widehat{1}$ is dense in $L^2(M)$) and *separating* (i.e. $x\widehat{1} = 0 \Longrightarrow x = 0$) for M. For every $x, y \in M$,

$$\begin{aligned} \|xy\|_2^2 &= \tau(y^*x^*xy) \\ &\leq \tau(y^*\|x^*x\|_\infty y) \\ &\leq \|x\|_\infty^2 \|y\|_2^2, \end{aligned}$$

so that we can represent M in a *standard way* on $L^2(M)$ by

$$\pi(x)\widehat{y} = \widehat{xy}, \forall x, y \in M.$$

This is the so-called *GNS-representation.* Observe that $\pi : M \to \mathbf{B}(L^2(M))$ is a normal $*$-representation and $\|\pi(x)\|_\infty = \|x\|_\infty$. Abusing notation, we identify $\pi(x)$

with $x \in M$ and regard $M \subset \mathbf{B}(L^2(M))$. Let $J : L^2(M) \ni x\widehat{1} \mapsto x^*\widehat{1} \in L^2(M)$ be the canonical antiunitary.

Theorem 1.4. $JMJ = M'$.

Proof. We first prove $JMJ \subset M'$. Let $x, y, a, b \in M$. We have

$$\begin{aligned}\langle JxJy\widehat{a}, \widehat{b}\rangle &= \langle \widehat{xa^*y^*}, \widehat{b^*}\rangle = \tau(bxa^*y^*) = \tau(y^*bxa^*) \\ &= \langle \widehat{xa^*}, \widehat{b^*y}\rangle = \langle xJ\widehat{a}, J\widehat{y^*b}\rangle = \langle yJxJ\widehat{a}, \widehat{b}\rangle,\end{aligned}$$

so that $JxJy = yJxJ$.

Claim. The faithful normal state $x \mapsto \langle x\widehat{1}, \widehat{1}\rangle$ is a trace on M'.

Let $x \in M'$. We first show that $Jx\widehat{1} = x^*\widehat{1}$. Indeed, for every $a \in M$, we have

$$\begin{aligned}\langle Jx\widehat{1}, a\widehat{1}\rangle &= \langle Ja\widehat{1}, x\widehat{1}\rangle = \langle x^*a^*\widehat{1}, \widehat{1}\rangle \\ &= \langle a^*x^*\widehat{1}, \widehat{1}\rangle = \langle x^*\widehat{1}, a\widehat{1}\rangle.\end{aligned}$$

Let now $x, y \in M'$. We have

$$\begin{aligned}\langle xy\widehat{1}, \widehat{1}\rangle &= \langle y\widehat{1}, x^*\widehat{1}\rangle = \langle y\widehat{1}, Jx\widehat{1}\rangle = \langle x\widehat{1}, Jy\widehat{1}\rangle \\ &= \langle x\widehat{1}, y^*\widehat{1}\rangle = \langle yx\widehat{1}, \widehat{1}\rangle.\end{aligned}$$

Denote the trace $x \mapsto \langle x\widehat{1}, \widehat{1}\rangle$ on M' by τ'. Define the canonical antiunitary K on $L^2(M', \tau') = \overline{M'\widehat{1}} = L^2(M)$ by $Kx\widehat{1} = x^*\widehat{1}$, $\forall x \in M'$. The first part of the proof yields $KM'K \subset M'' = M$. Since K and J coincide on $M'\widehat{1}$, which is dense in $L^2(M)$, it follows that $K = J$. Therefore, we have $JM'J \subset M$ and so $JMJ = M'$. $\square$

This Theorem shows in particular that the commutant of the (left) group von Neumann algebra $L(\Gamma)$ inside $\mathbf{B}(\ell^2(\Gamma))$ is the *right* von Neumann algebra $R(\Gamma)$, that is, the von Neumann algebra generated by the right regular representation of the group Γ.

Exercise 1.5. Show that the strong operator topology on $(M)_1$ is given by the norm $\|\cdot\|_2$. Thus, strong and $*$-strong topologies coincide on $(M)_1$ since $\|x\|_2 = \|x^*\|_2$, $\forall x \in M$.

Let $B \subset M$ be a von Neumann subalgebra. One can show that there exists a unique τ-preserving faithful normal *conditional expectation* $E_B : M \to B$ (see Section 2 for details)[1]. The map $E_B : M \to B$ is unital completely positive and moreover satisfies

$$E_B(b_1xb_2) = b_1E_B(x)b_2, \forall x \in M, \forall b_1, b_2 \in B.$$

We say that E_B is *B-B bimodular.* If we denote by $e_B : L^2(M) \to L^2(B)$ the orthogonal projection, we have $e_B(x\widehat{1}) = E_B(x)\widehat{1}$, for every $x \in M$.

Proposition 1.6 (Fourier coefficients)**.** *Let* $\Gamma \curvearrowright (X, \mu)$ *be a p.m.p. action. let* $A = L^\infty(X)$ *and* $M = L^\infty(X) \rtimes \Gamma$. *Every* $x \in M$ *has a unique* Fourier decomposition

$$x = \sum_{s \in \Gamma} x_s u_s,$$

[1] These notes are nonlinear!

with $x_s = E_A(xu_s^*)$*. The convergence holds for the* $\|\cdot\|_2$*-norm.*[2] *Moreover,* $\|x\|_2^2 = \sum_{s\in\Gamma}\|x_s\|_2^2$.

Proof. Define the unitary $U : L^2(M) \to L^2(X)\otimes \ell^2(\Gamma)$ by the formula

$$U\left(\sum_{\text{finite}} a_s u_s\right) = \sum_{\text{finite}} a_s \otimes \delta_s.$$

Then $U\widehat{1}U^* = \mathbf{1}_X \otimes \delta_e$ is a cyclic separating vector for M represented on the Hilbert space $L^2(X)\otimes\ell^2(\Gamma)$. Abusing notation, we shall identify $L^2(M)$ with $L^2(X)\otimes\ell^2(\Gamma)$. Under this identification e_A is the orthogonal projection $L^2(X)\otimes\ell^2(\Gamma) \to L^2(X)\otimes \mathbf{C}\delta_e$. Moreover $u_s e_A u_s^*$ is the orthogonal projection $L^2(X)\otimes\ell^2(\Gamma) \to L^2(X)\otimes\mathbf{C}\delta_s$ and thus $\sum_{s\in\Gamma} u_s e_A u_s^* = 1$. Let $x \in M$. Regarding $x(\mathbf{1}_X\otimes\delta_e) \in L^2(X)\otimes\ell^2(\Gamma)$, we know that there exists $a_s \in L^2(X)$ such that

$$x(\mathbf{1}_X\otimes\delta_e) = \sum_{s\in\Gamma} a_s\otimes\delta_s \text{ and } \|x\|_2^2 = \sum_{s\in\Gamma}\|a_s\|_2^2.$$

Then we have

$$\begin{aligned} a_s\otimes\delta_s &= u_s e_A u_s^* x(\mathbf{1}_X\otimes\delta_e) \\ &= u_s e_A u_s^* x e_A(\mathbf{1}_X\otimes\delta_e) \\ &= u_s E_A(u_s^* x)(\mathbf{1}_X\otimes\delta_e) \\ &= E_A(xu_s^*)(\mathbf{1}_X\otimes\delta_s). \end{aligned}$$

It follows that $a_s = E_A(xu_s^*)$. Therefore, we have $x = \sum_{s\in\Gamma} E_A(xu_s^*)u_s$ and the convergence holds for the $\|\cdot\|_2$-norm. Moreover, $\|x\|_2^2 = \sum_{s\in\Gamma}\|E_A(xu_s^*)\|_2^2$. $\square$

Exercise 1.7. Let $\Gamma \curvearrowright (X,\mu)$ be a p.m.p. action. Let $A = L^\infty(X)$ and $M = L^\infty(X)\rtimes\Gamma$.

(1) Show that $\Gamma\curvearrowright X$ is free if and only if $A = A'\cap M$ (A is maximal abelian).
(2) Under the assumption that $\Gamma\curvearrowright X$ is free, show that M is a II_1 factor if and only if $\Gamma\curvearrowright X$ is ergodic.
(3) Assume that Γ is icc. Show that M is a II_1 factor if and only if $\Gamma\curvearrowright X$ is ergodic.

Exercise 1.8. A von Neumann algebra M is *diffuse* if it has no minimal projection.

(1) Let $N\subset M$ be an inclusion of von Neumann algebras and let $e\in N$ be a projection. Show that $e(N'\cap M)e = (eNe)'\cap eMe$.
(2) Let M be a finite von Neumann algebra. Show that M is diffuse if and only if there exists a sequence of unitaries $u_n\in\mathcal{U}(M)$ such that $u_n\to 0$ weakly.

For more on C^*-algebras and finite von Neumann algebras, we refer to the excellent book by Brown and Ozawa [2].

1.3. **Orbit equivalence relations.**

[2]The convergence does not hold in the strong topology.

1.3.1. *Basic facts on measured equivalence relations.* In this note, (X, μ) will denote a nonatomic standard Borel probability space. A *countable Borel equivalence relation* $\mathcal{R}$ is an equivalence relation defined on the space $X \times X$ which satisfies:

(1) $\mathcal{R} \subset X \times X$ is a Borel subset.
(2) For any $x \in X$, the *class* or *orbit* of x denoted by $[x]_{\mathcal{R}} := \{y \in X : (x, y) \in \mathcal{R}\}$ is countable.

We shall denote by $[\mathcal{R}]$ the *full group* of the equivalence relation $\mathcal{R}$, i.e. $[\mathcal{R}]$ consists of all Borel isomorphisms $\phi : X \to X$ such that $\mathrm{graph}(\phi) \subset \mathcal{R}$. The set of all *partial* Borel isomorphisms $\phi : \mathrm{dom}(\phi) \to \mathrm{range}(\phi)$ such that $\mathrm{graph}(\phi) \subset \mathcal{R}$ will be denoted by $[[\mathcal{R}]]$. If Γ is a countable group and $(g, x) \to gx$ is a Borel action of Γ on X, then the equivalence relation given by

$$(x, y) \in \mathcal{R}(\Gamma \curvearrowright X) \Longleftrightarrow \exists g \in \Gamma, y = gx$$

is a countable Borel equivalence relation on X. Conversely, we have the following:

Theorem 1.9 (Feldman & Moore, [11])**.** *If $\mathcal{R}$ is a countable Borel equivalence relation on X, then there exist a countable group Γ and a Borel action of Γ on X such that $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$. Moreover, Γ and the action can be chosen such that*

$$(x, y) \in \mathcal{R} \Longleftrightarrow \exists g \in \Gamma, g^2 = 1 \text{ and } y = gx.$$

Given a countable Borel equivalence relation $\mathcal{R}$ on X, we say that μ is $\mathcal{R}$-*invariant* if

$$\phi_* \mu = \mu, \forall \phi \in [\mathcal{R}],$$

where $\phi_* \mu(\mathcal{U}) = \mu(\phi^{-1}(\mathcal{U}))$, for any Borel $\mathcal{U} \subset X$. The following proposition is useful and easy to prove:

Proposition 1.10. *Let $\mathcal{R}$ be a countable Borel equivalence relation defined on X. The following are equivalent:*

(1) *μ is $\mathcal{R}$-invariant.*
(2) *μ is Γ-invariant whenever Γ is a countable group acting in a Borel way on X such that $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$.*
(3) *μ is Γ-invariant for some countable group Γ acting in a Borel way on X such that $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$.*
(4) *$\forall \phi \in [[\mathcal{R}]]$, $\mu(\mathrm{dom}(\phi)) = \mu(\mathrm{range}(\phi))$.*

For any $\mathcal{U} \subset X$, we define the $\mathcal{R}$-*saturation* of $\mathcal{U}$ by

$$\begin{aligned} [\mathcal{U}]_{\mathcal{R}} &= \bigcup_{x \in \mathcal{U}} [x]_{\mathcal{R}} \\ &= \{y \in X : \exists x \in \mathcal{U}, (x, y) \in \mathcal{R}\}. \end{aligned}$$

We have $\mathcal{U} \subset [\mathcal{U}]_{\mathcal{R}}$ and $[\mathcal{U}]_{\mathcal{R}}$ is a Borel subset of X. We say that $\mathcal{U} \subset X$ is $\mathcal{R}$-*invariant* if $[\mathcal{U}]_{\mathcal{R}} = \mathcal{U}$ (up to null sets). The equivalence relation $\mathcal{R}$ is said to be *ergodic* if any $\mathcal{R}$-invariant Borel subset $\mathcal{U} \subset X$ is null or co-null.

Exercise 1.11. Let $\mathcal{R}$ be a measured equivalence relation for which (almost) every orbit is infinite. Show that there exists a sequence (g_n) in $[\mathcal{R}]$ such that $g_0 = \mathrm{Id}_X$ and $\mathcal{R} = \bigsqcup_n \mathrm{graph}(g_n)$. In other words, we can write the equivalence relation $\mathcal{R}$ as a countable disjoint union of graphs of elements in the full group $[\mathcal{R}]$.

Important Convention. In the rest of this paper, when we write an equivalence relation $\mathcal{R}$ defined on (X, μ), we always mean a countable Borel equivalence relation such that the measure μ is $\mathcal{R}$-invariant. When we write $\mathcal{U} \subset X$, we always mean a Borel subset of X. From now on, we will neglect null sets, i.e. whenever a property is true *for every* $x \in X$, we mean *for* μ*-almost every* $x \in X$. From now on, we will always assume that (almost) every orbit of $\mathcal{R}$ is infinite, that is, $\mathcal{R}$ is a type II_1 equivalence relation.

We define now a Borel measure on $\mathcal{R}$. For $\mathcal{W} \subset \mathcal{R}$ a Borel subset, we define $\mathcal{W}_x = \{y \in X : (x, y) \in \mathcal{W}\}$ and $\mathcal{W}^y = \{x \in X : (x, y) \in \mathcal{W}\}$. We define ν on $\mathcal{R}$ by

$$\nu(\mathcal{W}) = \int_X |\mathcal{W}_x| \mathrm{d}\mu(x), \forall \mathcal{W} \subset \mathcal{R}.$$

Lemma 1.12. *Since μ is assumed to be $\mathcal{R}$-invariant, we have:*

$$\int_X |\mathcal{W}_x| \mathrm{d}\mu(x) = \int_X |\mathcal{W}^y| \mathrm{d}\mu(y), \forall \mathcal{W} \subset \mathcal{R}.$$

Proof. From Exercise 1.11, we know that $\mathcal{R} = \bigsqcup_n \mathrm{graph}(g_n)$, for some $g_n \in [\mathcal{R}]$. Let $\mathcal{W} \subset \mathcal{R}$ be a Borel subset. Then $\mathcal{W} = \bigsqcup_n (\mathrm{graph}(g_n) \cap \mathcal{W})$ and $\mathrm{graph}(g_n) \cap \mathcal{W} = \mathrm{graph}(\phi_n)$ for $\phi_n \in [[\mathcal{R}]]$. Thus, we can write $\mathcal{W}$ as a countable disjoint union of graphs of $\phi \in [[\mathcal{R}]]$. Consequently, we just have to prove the equality when $\mathcal{W} = \mathrm{graph}(\phi)$, for $\phi \in [[\mathcal{R}]]$. For any $\phi : \mathrm{dom}(\phi) \to \mathrm{range}(\phi) \in [[\mathcal{R}]]$, we have

$$\begin{aligned} \mu(\mathrm{dom}(\phi)) &= \int_X |\,\mathrm{graph}(\phi)_x| \mathrm{d}\mu(x) \\ \mu(\mathrm{range}(\phi)) &= \int_X |\,\mathrm{graph}(\phi)^y| \mathrm{d}\mu(y). \end{aligned}$$

Since $\mu(\mathrm{dom}(\phi)) = \mu(\mathrm{range}(\phi))$, we are done. ∎

We shall denote by $D := \{(x, x) : x \in X\} \subset \mathcal{R}$ the diagonal. We have $\nu(D) = 1$. For $i = 1, 2$ let $\mathcal{R}_i$ be a countable Borel equivalence relation on (X_i, μ_i), and assumed that μ_i is $\mathcal{R}_i$-invariant. We say that $\mathcal{R}_1$ and $\mathcal{R}_2$ are *isomorphic* and denote $\mathcal{R}_1 \simeq \mathcal{R}_2$ if there exists a Borel isomorphism $\Delta : X_1 \to X_2$, such that $\Delta_* \mu_1 = \mu_2$ and

$$(x, y) \in \mathcal{R}_1 \Longleftrightarrow (\Delta(x), \Delta(y)) \in \mathcal{R}_2.$$

We will be using the following important fact. For $\Delta : X_1 \to X_2$ a Borel isomorphism such that $\Delta_* \mu_1 = \mu_2$, we associate $\pi_\Delta : L^\infty(X_1) \to L^\infty(X_2)$ defined by $(\pi_\Delta F)(x) = F(\Delta^{-1} x)$, for any $F \in L^\infty(X_1)$, and any $x \in X_1$. Moreover the map $\Delta \to \pi_\Delta$ is an onto isomorphism (see [28, Proposition XIII.1.2]).

1.3.2. *Construction of the von Neumann algebra of $\mathcal{R}$.* We define the *left regular representation* of the equivalence relation $\mathcal{R}$ on the Hilbert space $L^2(\mathcal{R}, \nu)$. For $\phi : \mathrm{dom}(\phi) \to \mathrm{range}(\phi) \in [[\mathcal{R}]]$ and $\xi \in L^2(\mathcal{R}, \nu)$, set

$$(u_\phi \xi)(x, y) = \mathbf{1}_{\mathrm{range}(\phi)}(x) \xi(\phi^{-1}(x), y), \forall (x, y) \in \mathcal{R}.$$

In other words, $(u_\phi \xi)(x, y) = \xi(\phi^{-1}(x), y)$ if $x \in \mathrm{range}(\phi)$ and 0 otherwise. The von Neumann algebra $L^\infty(X)$ acts in the following way. If $F \in L^\infty(X)$ and $\xi \in L^2(\mathcal{R}, \nu)$, we have

$$(F\xi)(x, y) = F(x)\xi(x, y), \forall (x, y) \in \mathcal{R}.$$

For $F \in L^\infty(X)$ and $\phi \in [[\mathcal{R}]]$, define $F_\phi \in L^\infty(X)$ by

$$F_\phi(x) = \mathbf{1}_{\mathrm{range}(\phi)}(x) F(\phi^{-1}(x)).$$

In other words, $F_\phi(x) = F(\phi^{-1}(x))$ if $x \in \mathrm{range}(\phi)$ and 0 otherwise.

Note that, we can also define the right regular representation of $\mathcal{R}$ on $L^2(\mathcal{R}, \nu)$ in the following way: for $\phi : \mathrm{dom}(\phi) \to \mathrm{range}(\phi) \in [[\mathcal{R}]]$ and $\xi \in L^2(\mathcal{R}, \nu)$, set

$$(v_\phi \xi)(x, y) = \mathbf{1}_{\mathrm{range}(\phi)}(y) \xi(x, \phi^{-1}(y)), \forall (x, y) \in \mathcal{R}.$$

Exercise 1.13. Show that for any $\phi, \psi \in [[\mathcal{R}]]$, we have

(1) $u_\phi u_\psi = u_{\phi\psi}$.
(2) $u_\phi^* = u_{\phi^{-1}}$.
(3) $u_\phi^* u_\phi = \mathbf{1}_{\mathrm{dom}(\phi)}$ and $u_\phi u_\phi^* = \mathbf{1}_{\mathrm{range}(\phi)}$.
(4) $u_\phi F = F_\phi u_\phi$, for any $F \in L^\infty(X)$.

Definition 1.14 (Feldman & Moore, [12])**.** *The von Neumann algebra $L(\mathcal{R}) \subset \mathbf{B}(L^2(\mathcal{R}, \nu))$ of the equivalence relation $\mathcal{R}$ is then defined as follows:*

$$L(\mathcal{R}) := W^*\{u_\phi : \phi \in [[\mathcal{R}]]\}.$$

Likewise, we define $R(\mathcal{R}) = W^*\{v_\phi : \phi \in [[\mathcal{R}]]\}$. It is trivial to check that $L(\mathcal{R}) \subset R(\mathcal{R})'$. Define the canonical anti-unitary $J : L^2(\mathcal{R}, \nu) \to L^2(\mathcal{R}, \nu)$ by $(J\xi)(x, y) = \overline{\xi(y, x)}$, for any $\xi \in L^2(\mathcal{R}, \nu)$, for any $(x, y) \in \mathcal{R}$.

Proposition 1.15. *Denote by $\xi_0 = \mathbf{1}_D \in L^2(\mathcal{R}, \nu)$ the characteristic function corresponding to the diagonal $D \subset \mathcal{R}$.*

(1) *ξ_0 is a cyclic separating vector for $L(\mathcal{R})$.*
(2) *The vector state $\tau = \langle \cdot \xi_0, \xi_0 \rangle$ is a faithful normal trace on $L(\mathcal{R})$. In particular, $L(\mathcal{R})$ is a finite von Neumann algebra.*
(3) *For any $\phi \in [[\mathcal{R}]]$, $J u_\phi J = v_\phi$. In particular, $L(\mathcal{R}) = R(\mathcal{R})'$.*

Proof. (1) Write $\xi_0 = \mathbf{1}_D$, where $D \subset \mathcal{R}$ is the diagonal. For $\phi \in [[\mathcal{R}]]$, we have $\mathbf{1}_{\mathrm{graph}(\phi)} = u_{\phi^{-1}} \xi_0$. Recall that $\mathcal{R}$ can be written as a countable disjoint union of graphs of Borel isomorphisms $\mathcal{R} = \bigsqcup_n \mathrm{graph}(g_n)$. Take any $\mathcal{W} \subset \mathcal{R}$. Then $\mathcal{W} = \bigsqcup_n \mathcal{W}_n$, with $\mathcal{W}_n = \mathcal{W} \cap \mathrm{graph}(g_n)$. Since $\mathcal{W}_n$ is the graph of a partial Borel isomorphism, it follows that $\mathbf{1}_{\mathcal{W}} \in \overline{L(\mathcal{R})\xi_0}$. Consequently, ξ_0 is cyclic for $L(\mathcal{R})$. Exactly in the same way, ξ_0 is cyclic for $R(\mathcal{R})$, hence separating for $L(\mathcal{R})$.

(2) It suffices to prove that $\tau(u_\phi u_\psi) = \tau(u_\psi u_\phi)$, for every $\phi, \psi \in [[\mathcal{R}]]$. Let $\phi, \psi \in [[\mathcal{R}]]$. We have

$$\begin{aligned}
\tau(u_\phi u_\psi) &= \int_X \mathbf{1}_{\{x = (\phi\psi)^{-1}(x)\}}(x) \mathrm{d}\mu(x) \\
&= \int_X \mathbf{1}_{\{x = \psi^{-1}\phi^{-1}(x)\}}(x) \mathrm{d}\mu(x) \\
&= \int_X \mathbf{1}_{\{x' = \phi^{-1}\psi^{-1}(x')\}}(x') \mathrm{d}\mu(x') \ (x' = \phi^{-1}(x)) \\
&= \int_X \mathbf{1}_{\{x = (\psi\phi)^{-1}(x)\}}(x) \mathrm{d}\mu(x) \\
&= \tau(u_\psi u_\phi).
\end{aligned}$$

(3) A straightforward computation shows that $J u_\phi J = v_\phi$, for any $\phi \in [[\mathcal{R}]]$. It follows that $J L(\mathcal{R}) J = R(\mathcal{R})$. By the general theory of finite von Neumann algebras (see Proposition 1.4), we get $L(\mathcal{R}) = R(\mathcal{R})'$. $\square$

Denote $A = L^\infty(X)$ and $M = L(\mathcal{R})$. Observe that $A \subset M$. We know that there exists a unique τ-preserving faithful normal conditional expectation $E_A : M \to A$. In order to know E_A, it is sufficient to compute $E_A(u_\phi)$ for any $\phi \in [[\mathcal{R}]]$. Denote by $e_D : L^2(\mathcal{R}) \to L^2(D, \nu_0)$ the orthogonal projection.

Proposition 1.16. *We have*

(1) $E_A(u_g) = \mathbf{1}_{\{x = g^{-1}x\}}$, $\forall g \in [\mathcal{R}]$.
(2) $e_D(x\xi_0) = E_A(x)\xi_0$, $\forall x \in L(\mathcal{R})$.

Proof. In order to prove (1) and (2), it suffices to show that $E_A(u_\phi) = \mathbf{1}_{\{x = \phi^{-1}(x)\}}$, for every $\phi \in [[\mathcal{R}]]$. Let $\phi, \psi \in [[\mathcal{R}]]$. Write $f = \mathbf{1}_{\{x=\phi^{-1}(x)\}} \in L^\infty(X)$. Then, we have

$$\begin{aligned}
\tau(E_A(u_\phi)u_\psi) &= \int_X E_A(u_\phi)(x)\mathbf{1}_{\{x=\psi^{-1}(x)\}}(x)\mathrm{d}\mu(x) \\
&= \int_X E_A(u_\phi \mathbf{1}_{\{x=\psi^{-1}(x)\}})(x)\mathrm{d}\mu(x) \\
&= \tau(E_A(u_\phi \mathbf{1}_{\{x=\psi^{-1}(x)\}})) = \tau(u_\phi \mathbf{1}_{\{x=\psi^{-1}(x)\}}) \\
&= \int_X \mathbf{1}_{\{x=\psi^{-1}(x)\}}(x) f(x)\mathrm{d}\mu(x) \\
&= \int_X f(x)\mathbf{1}_{\{x=\psi^{-1}(x)\}}(x)\mathrm{d}\mu(x) = \tau(f u_\psi).
\end{aligned}$$

Thus, $\tau((E_A(u_\phi) - f)x) = 0$, for any $x \in M$. Consequently, $E_A(u_\phi) = f = \mathbf{1}_{\{x=\phi^{-1}(x)\}}$. ☐

Proposition 1.17. *Let (g_n) be a sequence in $[\mathcal{R}]$ such that $g_0 = \mathrm{Id}_X$ and $\mathcal{R} = \bigsqcup_n \mathrm{graph}(g_n^{-1})$. Then any $x \in L(\mathcal{R})$ can be uniquely written*

$$x = \sum_n a_n u_{g_n},$$

where $a_n \in A$.

Proof. Since $\mathcal{R} = \bigsqcup_n \mathrm{graph}(g_n^{-1})$, we have

$$L^2(\mathcal{R}, \nu) = \bigoplus_n L^2(\mathrm{graph}(g_n^{-1}), \nu_n),$$

where ν_n is the restriction of ν to $\mathrm{graph}(g_n^{-1})$. Let $x \in L(\mathcal{R})$. Define $a_n = E_A(xu_{g_n}^*)$. Recall that $\mathbf{1}_{\mathrm{graph}(g_n^{-1})} = u_{g_n}\xi_0$. Denote by $e_D : L^2(\mathcal{R}, \nu) \to L^2(D, \nu_0)$ the orthogonal projection. It is easy to check that $u_{g_n} e_D u_{g_n}^*$ is the orthogonal projection $L^2(\mathcal{R}, \nu) \to L^2(\mathrm{graph}(g_n^{-1}), \nu_n)$. We have

$$\begin{aligned}
u_{g_n} e_D u_{g_n}^*(x\xi_0) &= u_{g_n}(e_D u_{g_n}^* x\xi_0) \\
&= u_{g_n} E_A(u_{g_n}^* x)\xi_0 \\
&= u_{g_n} E_A(u_{g_n}^* x) u_{g_n}^* u_{g_n}\xi_0 \\
&= E_A(xu_{g_n}^*)u_{g_n}\xi_0.
\end{aligned}$$

Therefore $x\xi_0 = \sum_n E_A(xu_{g_n}^*)u_{g_n}\xi_0$ in $L^2(\mathcal{R}, \nu)$ and so $x = \sum_n E_A(xu_{g_n}^*)u_{g_n}$ where the convergence holds for the $\|\cdot\|_2$-norm. ☐

The above proposition yields in particular $L(\mathcal{R}) = (L^\infty(X) \cup \{u_g : g \in [\mathcal{R}]\})''$.

Proposition 1.18. *Denote $M = L(\mathcal{R})$ and $A = L^\infty(X)$. Then*

(1) *$A = A' \cap M$, i.e. $A \subset M$ is a maximal abelian $*$-subalgebra.*
(2) *$\mathcal{N}_M(A)'' = M$, i.e. $A \subset M$ is regular.*

Proof. (1) Let $u \in \mathcal{U}(A' \cap M)$. As before we may write $u = \sum_n a_n u_{g_n}$ for some $g_n \in [\mathcal{R}]$. Fix $F \in L^\infty(X)$. Since $uF = Fu$, we get $a_n(F_{g_n} - F) = 0$, for any n. Thus, for any $x \in \mathrm{supp}(a_n)$, $F(g_n^{-1}(x)) = F(x)$. Using a previous remark, we get $g_n^{-1}(x) = x$, for any $x \in \mathrm{supp}(a_n)$. Thus, $u = \sum_n a_n \mathbf{1}_{\mathrm{supp}(a_n)} \in A$.

(2) It is trivial once we noticed that $M = (A \cup \{u_g : g \in [\mathcal{R}]\})''$. $\square$

From this proposition, it follows that $\mathcal{Z}(M) = M' \cap M \subset A' \cap M = A$. Moreover, for any $\mathcal{U} \subset X$, we have the following

$$\mathcal{U} = [\mathcal{U}]_{\mathcal{R}} \iff g\mathcal{U} = \mathcal{U}, \forall g \in [\mathcal{R}].$$

Indeed, assume that $\mathcal{U} = [\mathcal{U}]_{\mathcal{R}}$ and fix $g \in [\mathcal{R}]$. For any $x \in \mathcal{U}$, since $(x, gx) \in \mathcal{R}$, then $gx \in \mathcal{U}$. Conversely, assume that $g\mathcal{U} = \mathcal{U}$, for any $g \in [\mathcal{R}]$. Recall that there exists a countable group and a p.m.p. action of Γ on X such that $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$. If $(x, y) \in \mathcal{R}$, with $x \in \mathcal{U}$, there exists $g \in \Gamma$ such that $y = gx$. But then $y \in \mathcal{U}$. Thus we obtain:

Proposition 1.19. *$L(\mathcal{R})$ is a factor if and only if $\mathcal{R}$ is ergodic.*

Then we summarize what we did so far in the following theorem:

Theorem 1.20. *Let (X, μ) be a nonatomic probability space. Let $\mathcal{R}$ be an ergodic countable Borel equivalence relation on X such that μ is $\mathcal{R}$-invariant. Then $L(\mathcal{R})$ is a II_1 factor and $L^\infty(X) \subset L(\mathcal{R})$ is a Cartan subalgebra.*

1.3.3. *The full group of $\mathcal{R}$ and consequences.* Denote $A = L^\infty(X)$ and $M = L(\mathcal{R})$. We prove the following:

Theorem 1.21. *We have*

$$[\mathcal{R}] = \mathcal{N}_M(A)/\mathcal{U}(A).$$

Proof. Let $u \in \mathcal{N}_M(A)$. As before, we may write $u = \sum_n a_n u_{g_n}$, for some $a_n \in A$, $g_n \in [\mathcal{R}]$. We know that there exists a Borel isomorphism $\Delta : X \to X$ such that $\Delta_* \mu = \mu$ and $uFu^* = F_\Delta$, for any $F \in A$. Thus, $uF = F_\Delta u$ and so $a_n(F_{g_n} - F_\Delta) = 0$, for any n and any $F \in A$. Hence, for any $x \in \mathrm{supp}(a_n)$, for any $F \in A$

$$F(g_n^{-1}(x)) = F(\Delta^{-1}(x)).$$

Denote $Y = \bigcup_n \mathrm{supp}(a_n)$. The Borel subset Y is co-null. Indeed, for all $n \in \mathbf{N}$, we have

$$\mathbf{1}_{X \setminus Y} a_n u_{g_n} = 0$$

and so $\mathbf{1}_{X\setminus Y} u = 0$. Thus, $\mu(X \setminus Y) = 0$. This finally proves that $\Delta \in [\mathcal{R}]$.

We have constructed a group morphism

$$\Phi : \begin{array}{ccc} \mathcal{N}_M(A) & \to & [\mathcal{R}] \\ u & \mapsto & \Delta \end{array}$$

which is onto since $\Delta = u_\Delta$. Moreover, $u \in \ker(\Phi)$ if and only if $u \in A' \cap M$. Thus, $\ker(\Phi) = \mathcal{U}(A)$. This completes the proof. $\square$

Corollary 1.22. *We have $\mathcal{N}_M(A) = \mathcal{U}(A) \rtimes [\mathcal{R}]$.*

Proof. We already know that the sequence

$$1 \longrightarrow \mathcal{U}(A) \longrightarrow \mathcal{N}_M(A) \longrightarrow [\mathcal{R}] \longrightarrow 1$$

is exact. It moreover splits with the following section

$$s: \begin{array}{ccc} [\mathcal{R}] & \to & \mathcal{N}_M(A) \\ g & \mapsto & u_g \end{array}.$$

∎

Theorem 1.23. *For $i = 1, 2$, let $\mathcal{R}_i$ be a measured equivalence relation on (X_i, μ_i). Denote $A_i = L^\infty(X_i)$ and $M_i = L(\mathcal{R}_i)$. Then*

$$\mathcal{R}_1 \simeq \mathcal{R}_2 \Longleftrightarrow (A_1 \subset M_1) \simeq (A_2 \subset M_2).$$

Proof. $\Longrightarrow$ First assume that $\mathcal{R}_1 \simeq \mathcal{R}_2$. Then there exists a Borel isomorphism $\Delta : X_1 \to X_2$ such that $\Delta_*\mu_1 = \mu_2$ and for any $(x, y) \in X_1 \times X_1$, $(x, y) \in \mathcal{R}_1$ iff $(\Delta(x), \Delta(y)) \in \mathcal{R}_2$. We define a unitary as follows:

$$U: \begin{array}{ccc} L^2(\mathcal{R}_1, \nu_1) & \to & L^2(\mathcal{R}_2, \nu_2) \\ \xi & \mapsto & \left((x, y) \mapsto \xi(\Delta^{-1}(x), \Delta^{-1}(y))\right). \end{array}$$

For any $g \in [\mathcal{R}_1]$, $F \in L^\infty(X_1)$ and any $\xi \in L^2(\mathcal{R}_2, \nu_2)$, we have

$$\begin{array}{rcl} (Uu_gU^*\xi)(x, y) & = & \xi(\Delta g^{-1}\Delta^{-1}(x), y) \\ (UFU^*\xi)(x, y) & = & F(\Delta^{-1}(x))\xi(x, y). \end{array}$$

Thus, $Uu_gU^* = u_{\Delta g \Delta^{-1}}$ and $UFU^* = F_\Delta$. Consequently, $\theta = \mathrm{Ad}(U) : M_1 \to M_2$ is an onto $*$-isomorphism such that $\theta(A_1) = A_2$.

$\Longleftarrow$ Assume now that there exists an onto $*$-isomorphism $\theta : M_1 \to M_2$ such that $\theta(A_1) = A_2$. We know that there exists a Borel isomorphism $\Delta : X_1 \to X_2$ such that $\Delta_*\mu_1 = \mu_2$ and $\theta(F) = F_\Delta$, for any $F \in A_1$. Fix $g \in [\mathcal{R}_1]$. Since u_g normalizes A_1 inside M_1, it follows that $\theta(u_g)$ normalizes A_2 inside M_2. Thus there exist $h \in [\mathcal{R}_2]$ and $v \in \mathcal{U}(A_2)$ such that $\theta(u_g) = u_h v$.

Fix $F \in A_1$. From the one hand, we know that $u_g F u_g^* = F_g$. Thus we obtain $\theta(u_g F u_g^*) = \theta(F_g) = (F_g)_\Delta$. On the other hand, we have

$$\begin{array}{rcl} \theta(u_g F u_g^*) & = & \theta(u_g)\theta(F)\theta(u_g)^* \\ & = & u_h v F_\Delta v^* u_h^* \\ & = & u_h F_\Delta u_h^* \\ & = & (F_\Delta)_h. \end{array}$$

Consequently, $(F_g)_\Delta = (F_\Delta)_h$, and so $g^{-1}\Delta^{-1} = \Delta^{-1}h^{-1}$ on X_2. Equivalently, $\Delta g = h\Delta$ on X_1. For any $x \in X_1$, $(\Delta(x), \Delta g(x)) = (\Delta(x), h\Delta(x)) \in \mathcal{R}_2$, since $h \in [\mathcal{R}_2]$.

Let now $(x, y) \in \mathcal{R}_1$. We know that there exists $g \in [\mathcal{R}_1]$ such that $y = gx$. Thus, $(\Delta(x), \Delta(y)) = (\Delta(x), \Delta g(x)) \in \mathcal{R}_2$. Reasoning exactly the same way with Δ^{-1}, we obtain that Δ is an isomorphism of equivalence relations. ∎

1.3.4. *Group actions and their orbit equivalence relations.* Given a p.m.p. action $\Gamma \curvearrowright (X, \mu)$, one can associate the *orbit equivalence relation* $\mathcal{R}(\Gamma \curvearrowright X)$ defined by

$$(x, y) \in \mathcal{R}(\Gamma \curvearrowright X) \Longleftrightarrow \exists s \in \Gamma, y = sx.$$

When the action $\Gamma \curvearrowright X$ is free, the map

$$(\Gamma \times X, \text{counting} \otimes \mu) \ni (s, x) \mapsto (x, sx) \in (\mathcal{R}(\Gamma \curvearrowright X), \nu)$$

is a p.m.p. Borel isomorphism.

Exercise 1.24. Let $\Gamma \curvearrowright (X,\mu)$ be a free p.m.p. action. Show that the von Neumann algebra of the orbit equivalence relation $L(\mathcal{R}(\Gamma \curvearrowright X))$ and the group measure space construction $L^\infty(X) \rtimes \Gamma$ are $*$-isomorphic.

Definition 1.25. Let $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ be two free p.m.p. actions. We shall say that

(1) $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are *conjugate* if there exist a p.m.p. Borel isomorphism $\Delta : (X,\mu) \simeq (Y,\nu)$ and a group isomorphism $\delta : \Gamma \simeq \Lambda$ such that $\Delta(sx) = \delta(s)\Delta(x)$, $\forall s \in \Gamma, \forall x \in X$.
(2) $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are *orbit equivalent* (abbreviated OE) if there exist a p.m.p. Borel isomorphism $\Delta : (X,\mu) \simeq (Y,\nu)$ such that $\Delta(\Gamma x) = \Lambda\Delta(x)$, $\forall x \in X$.
(3) $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are *W^*-equivalent* (abbreviated W*E) if $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$.

Let $A = L^\infty(X) \subset L^\infty(X) \rtimes \Gamma = M$ and $B = L^\infty(Y) \subset L^\infty(Y) \rtimes \Lambda = N$. Observe that Theorem 1.23 yields

$$\begin{aligned}\Gamma \curvearrowright (X,\mu) \sim_{\mathrm{OE}} \Lambda \curvearrowright (Y,\nu) &\iff \mathcal{R}(\Gamma \curvearrowright X) \simeq \mathcal{R}(\Lambda \curvearrowright Y)\\ &\iff (A \subset M) \simeq (B \subset N)\end{aligned}$$

Therefore the following implications are true

$$\text{conjugacy} \implies \text{orbit equivalence} \implies \text{W}^*\text{-equivalence}.$$

2. Hilbert bimodules. Completely positive maps

2.1. Generalities.

2.1.1. *Hilbert bimodules.* The discovery of the appropriate notion of representations for von Neumann algebras, as so-called *correspondences* or *bimodules*, is due to Connes [5].

Definition 2.1. Let M, N be finite von Neumann algebras. A Hilbert space $\mathcal{H}$ is said to be an *M-N-bimodule* if it comes equipped with two commuting normal $*$-representations $\pi : M \to \mathbf{B}(\mathcal{H})$ and $\rho : N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$. We shall intuitively write $x \cdot \xi \cdot y = \pi(x)\rho(y^{\mathrm{op}})\xi$, $\forall \xi \in \mathcal{H}, \forall x \in M, \forall y \in N$.

We shall see that an M-M bimodule $\mathcal{H}$ is the analog of a unitary group representation $\pi : \Gamma \to \mathcal{U}(H_\pi)$.

Example 2.2. The following are important examples of bimodules:

(1) The identity bimodule $L^2(M)$ with $x \cdot \xi \cdot y = x\xi y$.
(2) The coarse bimodule $L^2(M) \otimes L^2(N)$ with $x \cdot (\xi \otimes \eta) \cdot y = (x\xi) \otimes (\eta y)$. It can be checked that as M-N-bimodules,
$$L^2(M) \otimes L^2(N) \simeq \mathcal{HS}(L^2(M), L^2(N))$$
where $\mathcal{HS}(L^2(M), L^2(N))$ denotes the M-N-bimodule of Hilbert-Schmidt operators on from $L^2(M)$ to $L^2(N)$.
(3) For any τ-preserving automorphism $\theta \in \mathrm{Aut}(M)$, we regard $L^2(M)$ with the following structure: $x \cdot \xi \cdot y = x\xi\theta(y)$.

(4) Let $B \subset M$ be a von Neumann subalgebra and denote by $e_B : L^2(M) \to L^2(B)$ the orthogonal projection. Consider the *basic construction* $\langle M, e_B\rangle$ which is the von Neumann subalgebra of $\mathbf{B}(L^2(M))$ generated by M and e_B. We endow $\langle M, e_B\rangle$ with the following semifinite trace: $\mathrm{Tr}(xe_By) = \tau(xy)$, for all $x, y \in M$ (see Subsection 2.3). Then $L^2(\langle M, e_B\rangle, \mathrm{Tr})$ is naturally endowed with a structure of M-M-bimodule. Moreover, as M-M-bimodules,

$$L^2(\langle M, e_B\rangle, \mathrm{Tr}) \simeq L^2(M) \otimes_B L^2(M)$$

where $\otimes_B$ denotes Connes' fusion tensor product (see [5, Appendix V.B]).

2.1.2. *Unital completely positive maps.* Let $(M, \tau_M), (N, \tau_N)$ be finite von Neumann algebras endowed with a fixed faithful normal trace. A linear map $\phi : M \to N$ is said to be *completely positive* if the maps

$$\phi_n = \mathrm{Id}_n \otimes \phi : \mathbf{M}_n(\mathbf{C}) \otimes M \to \mathbf{M}_n(\mathbf{C}) \otimes N$$

are positive for every $n \geq 1$. We shall say that ϕ is unital if $\phi(1) = 1$, and *trace-preserving* if moreover $\tau_N(\phi(x)) = \tau_M(x)$, for every $x \in M$.

Theorem 2.3 (Stinespring dilation)**.** *Let $\phi : M \to N$ be a (normal) u.c.p. map. Then there exist a Hilbert space $\mathcal{H}$, an isometry $V : L^2(N) \to \mathcal{H}$ and a (normal) $*$-representation $\pi : M \to \mathbf{B}(\mathcal{H})$ such that*

$$\phi(x) = V^*\pi(x)V, \forall x \in M.$$

Proof. Equip $\mathcal{H}_0 = M \otimes_{\mathrm{alg}} L^2(N)$ with the following sesquilinear form

$$\langle \sum_i a_i \otimes \eta_i, \sum_j b_j \otimes \zeta_j\rangle = \sum_{i,j} \langle \phi(b_j^* a_i)\eta_i, \zeta_j\rangle_{L^2(N)}$$

and promote it to a Hilbert space $\mathcal{H}$ by separation and completion. Denote by $(\sum_j b_j \otimes \zeta_j)^\bullet$ the vector in $\mathcal{H}$ which it represents. Define now $V : L^2(N) \to \mathcal{H}$ by $V\zeta = (1 \otimes \zeta)^\bullet$. It is clear that V is an isometry, i.e. $V^*V = 1_{L^2(N)}$. For every $x \in M$, we define a bounded linear operator $\pi(x)$ on $\mathcal{H}$ by

$$\pi(x)(\sum_j b_j \otimes \zeta_j)^\bullet = (\sum_j xb_j \otimes \zeta_j)^\bullet.$$

As expected, $\pi : M \to \mathbf{B}(H)$ is a (normal) $*$-representation such that $\phi(x) = V^*\pi(x)V$, $\forall x \in M$. $\square$

It follows that a u.c.p. map $\phi : M \to N$ satisfies for every $x \in M$,

$$\begin{aligned} \phi(x^*x) &= V^*\pi(x^*x)V \\ &= V^*\pi(x^*)VV^*\pi(x)V + V^*\pi(x^*)(1 - VV^*)\pi(x)V \\ &\geq \phi(x)^*\phi(x). \end{aligned}$$

If ϕ is moreover assumed to be trace-preserving, the operator $T_\phi : L^2(M) \to L^2(N)$ defined by

$$T_\phi(\widehat{x}) = \widehat{\phi(x)}, \forall x \in M.$$

is bounded and $\|T_\phi\|_\infty = 1$.

Example 2.4. The following are important examples of τ-preserving u.c.p. maps:

(1) The trace $\tau : M \to M$, the identity map $\mathrm{Id} : M \to M$ and more generally all $*$-automorphisms $\theta : M \to M$ which preserve the trace.

(2) Let $B \subset M$ be a von Neumann subalgebra. Denote by $e_B : L^2(M) \to L^2(B)$ the orthogonal projection. Denote by $E_B : M \to B$ the unique τ-preserving *conditional expectation* from M onto B which satisfies
$$\widehat{E_B(x)} = e_B(\widehat{x}), \forall x \in M.$$
It is easy to see that E_B is indeed a u.c.p. map.

(3) Let $M = L(\Gamma)$ be the von Neumann algebra of a countable group Γ. Let $\varphi : \Gamma \to \mathbf{C}$ be a normalized *positive definite* function, i.e. for any finite set $F \subset \Gamma$ the matrix $(\varphi(st^{-1}))_{s,t\in F}$ is positive. Define the corresponding τ-preserving u.c.p. map $\phi : L(\Gamma) \to L(\Gamma)$ by
$$\phi(\sum_{s\in\Gamma} a_s u_s) = \sum_{s\in\Gamma} \varphi(s) a_s u_s.$$

Exercise 2.5. Let $\phi : L(\Gamma) \to L(\Gamma)$ be a τ-preverving u.c.p. map. Show that $\varphi : \Gamma \to \mathbf{C}$ defined by $\varphi(s) = \tau(\phi(u_s)u_s^*)$ is a positive definite function.

2.2. **Dictionary between Hilbert bimodules and u.c.p. maps.**

2.2.1. *From u.c.p. maps to Hilbert bimodules.* Let $\phi : M \to N$ be a trace-preserving u.c.p. map. Equip $\mathcal{H}_0 = M \odot L^2(N)$ with the following sesquilinear form
$$\langle \sum_i a_i \otimes \eta_i, \sum_j b_j \otimes \zeta_j \rangle = \sum_{i,j} \langle \phi(b_j^* a_i)\eta_i, \zeta_j \rangle_{L^2(N)}$$
and promote it to a Hilbert space $\mathcal{H}_\phi$ by separation and completion. Observe that $\mathcal{H}_\phi$ is a Stinespring Dilation of ϕ. Abusing notation, denote by $b \otimes \zeta$ the vector in $\mathcal{H}_\phi$ which it represents. The action is given by
$$a(b \otimes \zeta)x = (ab) \otimes (\zeta x),$$
for $a \in M$ and $x \in N$. With the unit vector $\xi = 1 \otimes \widehat{1} \in \mathcal{H}_\phi$, we have
$$\langle a\xi x, \xi y \rangle_{\mathcal{H}_\phi} = \langle \phi(a)\widehat{x}, \widehat{y} \rangle_{L^2(N)}$$
for every $a \in M$, $x, y \in N$. Since the u.c.p. map ϕ is assumed to be trace-preserving, we get
$$\langle \cdot \xi, \xi \rangle = \tau_M \text{ and } \langle \xi \cdot, \xi \rangle = \tau_N.$$

2.2.2. *From Hilbert bimodules to u.c.p. maps.* Let $\mathcal{H}$ be an M-N bimodule, with a *tracial* unit vector ξ, i.e. $\langle \cdot\xi, \xi \rangle = \tau_M$ and $\langle \xi\cdot, \xi \rangle = \tau_N$. Then the linear operator $L_\xi : L^2(N) \to \mathcal{H}$ defined by $L_\xi(\widehat{x}) = \xi x$ is bounded and $\|L_\xi\|_\infty = 1$. For any $x, y \in N$, we have
$$\begin{aligned}
\langle \widehat{x}, L_\xi^*(\xi y) \rangle_{L^2(N)} &= \langle \xi x, \xi y \rangle_{\mathcal{H}} \\
&= \tau(xy^*) \\
&= \langle \widehat{x}, \widehat{y} \rangle_{L^2(N)},
\end{aligned}$$
so that $L_\xi^*(\xi y) = \widehat{y}$. Therefore L_ξ is an isometry, i.e. $L_\xi^* L_\xi = 1$. Denote by $J = J_N$ the canonical antiunitary. Moreover, for any $a \in M$, $L_\xi^* a L_\xi \in N$. Indeed for any $y, z_1, z_2 \in N$, we have
$$\begin{aligned}
\langle (L_\xi^* a L_\xi)(Jy^*J)\widehat{z_1}, \widehat{z_2} \rangle_{L^2(N)} &= \langle aL_\xi Jy^*J\widehat{z_1}, L_\xi \widehat{z_2} \rangle_{\mathcal{H}} \\
&= \langle a\xi z_1 y, \xi z_2 \rangle_{\mathcal{H}} \\
\langle (Jy^*J)(L_\xi^* a L_\xi)\widehat{z_1}, \widehat{z_2} \rangle_{L^2(N)} &= \langle xL_\xi \widehat{z_1}, L_\xi JyJ\widehat{z_2} \rangle_{\mathcal{H}} \\
&= \langle a\xi z_1, \xi z_2 y^* \rangle_{\mathcal{H}}.
\end{aligned}$$

Define the u.c.p. map $\phi : M \to N$ by $\phi(a) = L_\xi^* a L_\xi$. Since

$$\tau_N(\phi(a)) = \langle L_\xi^* a L_\xi \widehat{1}, \widehat{1}\rangle = \langle a\xi, \xi\rangle = \tau_M(x),$$

it follows that ϕ is trace-preserving. We moreover have

$$\langle \phi(a)\widehat{x}, \widehat{y}\rangle_{L^2(N)} = \langle a\xi x, \xi y\rangle_{\mathcal{H}},$$

We can now prove the uniqueness and existence of the trace-preserving faithful normal conditional expectation $E_B : M \to B$.

Proposition 2.6. *Let (M, τ) be a finite von Neumann algebra with a fixed trace and let $B \subset M$ be a von Neumann subalgebra such that $\tau_B = \tau_{|B}$. Then there exists a unique normal faithful trace-preserving conditional expectation $E_B : M \to B$.*

Proof. Consider the M-B-bimodule $\mathcal{H} = {}_M L^2(M)_B$. The vector $\widehat{1} \in \mathcal{H}$ is obviously a unit tracial vector. Denote by E_B the corresponding normal trace-preserving u.c.p. map $E_B : M \to B$. Recall

$$\langle x\widehat{1}a, \widehat{1}b\rangle_{\mathcal{H}} = \langle E_B(x)\widehat{a}, \widehat{b}\rangle_{L^2(B)}, \forall x \in M, \forall a, b \in B.$$

Let $x \in M$, $a, b, c, d \in B$. We have

$$\begin{aligned}
\langle E_B(axb)\widehat{c}, \widehat{d}\rangle_{L^2(B)} &= \langle axb\widehat{1}c, \widehat{1}d\rangle_{\mathcal{H}} \\
&= \langle x\widehat{1}bc, \widehat{1}a^*d\rangle_{\mathcal{H}} \\
&= \langle E_B(x)\widehat{bc}, \widehat{a^*d}\rangle_{L^2(B)} \\
&= \langle aE_B(x)b\widehat{c}, \widehat{d}\rangle_{L^2(B)},
\end{aligned}$$

hence $E_B(axb) = aE_B(x)b$. Assume now that $E_B(x^*x) = 0$. Then

$$0 = \langle E_B(x^*x)\widehat{1}, \widehat{1}\rangle_{L^2(B)} = \langle x^*x\widehat{1}, \widehat{1}\rangle_{\mathcal{H}} = \|x\|_2^2,$$

and so $x = 0$. Let $E : M \to B$ be another trace-preserving conditional expectation. Then

$$\begin{aligned}
\langle E_B(x)\widehat{c}, \widehat{d}\rangle_{L^2(B)} &= \langle x\widehat{1}c, \widehat{1}d\rangle_{\mathcal{H}} \\
&= \tau(d^*xc) \\
&= \tau(E(d^*xc)) \\
&= \tau(d^*E(x)c) \\
&= \langle E(x)\widehat{c}, \widehat{d}\rangle_{L^2(B)},
\end{aligned}$$

hence $E_B(x) = E(x)$. Therefore $E_B : M \to B$ is the unique normal faithful trace-preserving conditional expectation. □

2.2.3. *From unitary group representations to Hilbert bimodules.* Let $\pi : \Gamma \to \mathcal{U}(H_\pi)$ be a unitary representation of a countable discrete group Γ. Let $M = L(\Gamma)$ be the group von Neumann algebra and denote by $(u_s)_{s\in\Gamma}$ the canonical unitaries. Define on $\mathcal{K}_\pi = H_\pi \otimes \ell^2(\Gamma)$ the following left and right commuting multiplications: for every $\xi \in H_\pi$ and every $s, t \in \Gamma$,

$$\begin{aligned}
u_s \cdot (\xi \otimes \delta_t) &= (\pi_s \otimes \lambda_s)(\xi \otimes \delta_t) = \pi_s\xi \otimes \delta_{st} \\
(\xi \otimes \delta_t) \cdot u_s &= (1_{H_\pi} \otimes \rho_{s^{-1}})(\xi \otimes \delta_t) = \xi \otimes \delta_{ts}.
\end{aligned}$$

It is clear that the right multiplication extends to the whole von Neumann algebra M. Observe now that the unitary representations $\pi \otimes \lambda$ and $1_{H_\pi} \otimes \lambda$ are unitarily conjugate. Indeed, define $U : H_\pi \otimes \ell^2(\Gamma) \to H_\pi \otimes \ell^2(\Gamma)$ by

$$U(\xi \otimes \delta_t) = \pi_t \xi \otimes \delta_t.$$

It is routine to check that U is a unitary and $U(1_{H_\pi} \otimes \lambda_s)U^* = \pi_s \otimes \lambda_s$, for every $s \in \Gamma$. Therefore, the left multiplication extends to M. We have proven:

Proposition 2.7. *The formulae above endow the Hilbert space $\mathcal{K}_\pi$ with a structure of $L(\Gamma)$-$L(\Gamma)$-bimodule.*

Observe that in the case $\pi = \lambda$ is the left regular representation of Γ, the M-M-bimodule $\mathcal{K}_\lambda$ is nothing but the coarse bimodule $L^2(M) \otimes L^2(M)$.

Exercise 2.8. Let $\varphi : \Gamma \to \mathbf{C}$ be a normalized positive definite function. Let (π, H_π, ξ), with $\xi \in H_\pi$ unit vector, be the GNS-representation of φ, i.e., $\varphi(s) = \langle \pi_s \xi, \xi \rangle$, for all $s \in \Gamma$. Show that the u.c.p. map ϕ associated to the bimodule $\mathcal{K}_\pi$ satisfies

$$\phi(\sum_{s \in \Gamma} a_s u_s) = \sum_{s \in \Gamma} \varphi(s) a_s u_s.$$

Exercise 2.9. Prove the following dictionary between u.c.p. maps and Hilbert bimodules:

u.c.p. maps	**Hilbert bimodules**
$\mathrm{Id} : M \to M$	Identity bimodule $L^2(M)$
$\tau : M \to M$	Coarse bimodule $L^2(M) \otimes L^2(M)$
Automorphism $\theta : M \to M$	$L^2(M)$ with $x \cdot \xi \cdot y = x\xi\theta(y)$
$E_B : M \to M$	$L^2\langle M, e_B \rangle$

2.3. **Popa's intertwining techniques.**

2.3.1. *The basic construction.* Throughout this section, we will denote by M a finite von Neumann algebra with a distinguished faithful normal trace τ. Let $B \subset M$ be a unital von Neumann subalgebra. Let $e_B : L^2(M) \to L^2(B)$ be the orthogonal projection. We will denote by $E_B : M \to B$ the unique faithful normal τ-preserving *conditional expectation.* It satisfies the following:

$$\begin{aligned} \widehat{E_B(x)} &= e_B(\widehat{x}), \\ E_B(axb) &= aE_B(x)b, \forall x \in M, \forall a, b \in B. \end{aligned}$$

The *basic construction* $\langle M, e_B \rangle$ is the von Neumann subalgebra of $\mathbf{B}(L^2(M))$ generated by M and the projection e_B. Observe that $Je_B = e_B J$ and $e_B x e_B = E_B(x) e_B$, $\forall x \in M$.

Proposition 2.10. *The following are true.*

(1) $\langle M, e_B \rangle = JB'J \cap \mathbf{B}(L^2(M))$.
(2) *The central support of* e_B *in* $\langle M, e_B \rangle$ *equals* 1*. In particular, the* $*$*-subalgebra* $\mathrm{span}(Me_BM)$ *is a* $*$*-strongly dense in* $\langle M, e_B \rangle$*. The formula* $e_B x e_B = E_B(x)e_B$ *extends the conditional expectation* $E_B : \langle M, e_B \rangle \to B$*.*
(3) $\langle M, e_B \rangle$ *is endowed with a semifinite faithful normal trace defined by*

$$\mathrm{Tr}(xe_By) = \tau(xy), \forall x, y \in M$$

.

Proof. (1) For $x \in B$, we clearly have $xL^2(B) \subset L^2(B)$ and $xL^2(B)^\perp \subset L^2(B)^\perp$, hence $xe_B = e_B x$. If $x \in M \cap \{e_B\}'$, then

$$E_B(x)\widehat{1} = e_B(x\widehat{1}) = xe_B(\widehat{1}) = x\widehat{1}.$$

Therefore $x = E_B(x) \in B$. It follows that $B = M \cap \{e_B\}'$. Thus,

$$JB'J = \langle JM'J, Je_BJ\rangle = \langle M, e_B\rangle.$$

(2) The map $B \ni x \mapsto xe_B \in Be_B$ is a $*$-isomorphism. Indeed, if $xe_B = 0$, then $x\eta = 0$, for every $\eta \in L^2(B)$. Since $x \in B$, it follows that $x = 0$. Denote by $z(e_B)$ the central support of e_B in B'. Then $z(e_B) \in B$ and $z(e_B)e_B = e_B$. Hence $z(e_B) = 1$. Thus the central support of $e_B = Je_BJ$ in $JB'J$ is equal to 1. It is clear that $\mathcal{I} = \operatorname{span}(Me_BM)$ is a $*$-subalgebra of $\langle M, e_B\rangle$ and a 2-sided ideal of the $*$-algebra generated by M and e_B. Thus $\overline{\mathcal{I}}$ is a closed 2-sided ideal of $\langle M, e_B\rangle$. Moreover

$$\mathcal{I}L^2(M) = Me_BL^2(M) = ML^2(B) \supset M\widehat{1}.$$

Since $\mathcal{I}$ is nondegenerate, we get $\overline{\mathcal{I}} = \langle M, e_B\rangle$.

(3) Since e_B has central support 1 in $\langle M, e_B\rangle$, one can find partial isometries (v_i) in $\langle M, e_B\rangle$ such that $v_i^*v_i \leq e_B$ and $\sum_i v_iv_i^* = 1$. It follows that

$$\bigoplus_i v_iL^2(B) = L^2(M).$$

Define the following normal weight Tr on $\langle M, e_B\rangle$ by

$$\operatorname{Tr}(x) = \sum_i \langle xv_i\widehat{1}, v_i\widehat{1}\rangle, \forall x \in \langle M, e_B\rangle_+.$$

Assume that $\operatorname{Tr}(x^*x) = 0$. Then $xv_i\widehat{1} = 0$, for every i. For every $b \in B$, we have

$$xv_i\widehat{b1} = xv_iJb^*J\widehat{1} = Jb^*Jxv_i\widehat{1} = 0.$$

Therefore $x = 0$ and Tr is faithful. For every $x, y \in M$, we have

$$\begin{aligned}
\operatorname{Tr}(xe_By) &= \sum_i \langle xe_Byv_i\widehat{1}, v_i\widehat{1}\rangle = \sum_i \langle e_Byv_ie_B\widehat{1}, e_Bx^*v_ie_B\widehat{1}\rangle \\
&= \sum_i \langle E_B(yv_i)e_B\widehat{1}, E_B(x^*v_i)e_B\widehat{1}\rangle = \sum_i \tau(E_B(x^*v_i)^*E_B(yv_i)) \\
&= \sum_i \tau(E_B(v_i^*y^*)^*E_B(v_i^*x)) = \sum_i \langle E_B(v_i^*x)e_B\widehat{1}, E_B(v_i^*y^*)e_B\widehat{1}\rangle \\
&= \sum_i \langle e_Bv_i^*xe_B\widehat{1}, e_Bv_i^*y^*e_B\widehat{1}\rangle = \sum_i \langle v_iv_i^*x\widehat{1}, y^*\widehat{1}\rangle \\
&= \langle \sum_i v_iv_i^*x\widehat{1}, y^*\widehat{1}\rangle = \langle x\widehat{1}, y^*\widehat{1}\rangle = \tau(xy).
\end{aligned}$$

We get that Tr is semifinite since $\operatorname{span}(Me_BM)$ is a $*$-strongly dense $*$-subalgebra in $\langle M, e_B\rangle$. For every $x, y, z, t \in \langle M, e_B\rangle$, we have

$$\begin{aligned}
\operatorname{Tr}(xe_Byze_Bt) &= \operatorname{Tr}(xE_B(yz)e_Bt) = \tau(xE_B(yz)t) \\
&= \tau(E_B(yz)E_B(tx)) = \tau(zE_B(tx)y) \\
&= \operatorname{Tr}(zE_B(tx)e_By) = \operatorname{Tr}(ze_Btxe_By).
\end{aligned}$$

Thus Tr is a trace. This completes the proof. $\square$

It follows from the previous proposition that

$$\langle M, e_B\rangle = \{T \in \mathbf{B}(L^2(M)) : T(\xi b) = T(\xi)b, \forall \xi \in L^2(M), \forall b \in B\}.$$

Let $\mathcal{H}_B$ be a right B-submodule of $L^2(M)_B$. Write $P_{\mathcal{H}} : L^2(M) \to \mathcal{H}$ for the orthogonal projection. It is clear that $P_{\mathcal{H}} \in \langle M, e_B\rangle$. We define the *von Neumann dimension* of $\mathcal{H}_B$ by $\dim(\mathcal{H}_B) := \mathrm{Tr}(P_{\mathcal{H}})$.

Exercise 2.11 ([2])**.** Let (N, Tr) be a semifinite von Neumann algebra. Let $\Omega = \{x \in N : \|x\|_{2,\mathrm{Tr}} \leq 1\}$. Prove that the formal inclusion $\Omega \hookrightarrow L^2(N, \mathrm{Tr})$ is ultraweak-weak continuous.

2.3.2. *Intertwining subalgebras.* In [23, 19], Popa introduced a very powerful tool to prove the unitary conjugacy of two von Neumann subalgebras of a tracial von Neumann algebra (M, τ). If $A \subset (M, \tau)$ is a (possibly non-unital) von Neumann subalgebras, denote by 1_A the unit of A.

Theorem 2.12 (Popa, [23, 19])**.** *Let (M, τ) be a finite von Neumann algebra. Let $A \subset M$ be a possibly non-unital von Neumann subalgebra and $B \subset M$ be a unital von Neumann subalgebra. The following are equivalent:*

(1) *There exist $n \geq 1$, a possibly non-unital $*$-homomorphism $\psi : A \to \mathbf{M}_n(\mathbf{C}) \otimes B$ and a non-zero partial isometry $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes 1_A M$ such that*
$$xv = v\psi(x), \forall x \in A.$$
(2) *There exists a nonzero A-B-subbimodule $\mathcal{H}$ of ${}_A L^2(1_A M)_B$ such that*
$$\dim(\mathcal{H}_B) < \infty.$$
(3) *There exists a nonzero element $d \in A' \cap 1_A \langle M, e_B\rangle_+ 1_A$ such that*
$$\mathrm{Tr}(d) < \infty.$$
(4) *There is no sequence of unitaries (u_k) in A such that*
$$\lim_{k \to \infty} \|E_B(a^* u_k b)\|_2 = 0, \forall a, b \in 1_A M.$$

If one of the previous equivalent conditions is satisfied, we shall say that A *embeds into B inside M* and denote $A \preceq_M B$. For simplicity, we shall write $M^n := \mathbf{M}_n(\mathbf{C}) \otimes M$.

Proof. We first prove that $(1), (2), (3)$ are equivalent. Then we show $(1) \Longrightarrow (4)$ and $(4) \Longrightarrow (3)$.

$(1) \Longrightarrow (2)$. Take a nonzero component of $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes 1_A M$ that we may assume to be v_1. Set $w = v_1$. We have $Aw \subset \sum_{j=1}^n v_j B$. Define $\mathcal{H} = \overline{AwB}$. Therefore $\mathcal{H} \subset v(\ell^2_n \otimes L^2(B))$ and $\dim(\mathcal{H}_B) \leq n$.

$(2) \Longrightarrow (3)$. Write $d = P_{\mathcal{H}}$. Then $d \in \langle M, e_B\rangle_+$ and $\mathrm{Tr}(d) = \dim(\mathcal{H}_B) < \infty$. Since $\mathcal{H}$ is moreover a left A-module, we have $ad = da$, for every $a \in A$, hence $d \in A' \cap 1_A \langle M, e_B\rangle_+ 1_A$ such that $0 < \mathrm{Tr}(d) < \infty$.

$(3) \Longrightarrow (1)$. Write $q = \mathbf{1}_{[\|d\|_\infty/2, \|d\|_\infty]}(d)$ for the nonzero spectral projection of d. We get that $\mathcal{K} = qL^2(M)$ is a nonzero A-B-subbimodule of $L^2(1_A M)$ such that $\dim(\mathcal{K}_B) = \mathrm{Tr}(q) < \infty$. Thus, cutting down by a central projection of B (see [29, Lemma C.1]), we get a nonzero A-B-subbimodule $\mathcal{H} \subset L^2(1_A M)$ which is finitely generated over B. Hence, we can take $n \geq 1$, a projection $p \in B^n$ and a right B-module isomorphism
$$\psi : pL^2(B)^{\oplus n} \to \mathcal{H}.$$

Since $\mathcal{H}$ is a left A-module, we get a (unital) $*$-homomorphism $\theta : A \to pB^n p$ satisfying $x\psi(\eta) = \psi(\theta(x)\eta)$ for all $x \in A$, and $\eta \in pL^2(B)^{\oplus n}$. Define now $e_j \in L^2(B)^{\oplus n}$ as $e_j = (0, \dots, \widehat{1}, \dots, 0)$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes \mathcal{H}$, with $\xi_j = \psi(pe_j)$. Let $j \in \{1, \dots, n\}$. For any $x \in A$, write $\theta(x) = (\theta_{kl}(x))_{kl} \in pB^n p$. We have

$$\begin{aligned}
x\xi_j &= x\psi(pe_j) = \psi(\theta(x)pe_j) = \psi(p\theta(x)e_j) = \psi(p\sum_{i=1}^n \theta_{ij}(x)e_i) \\
&= \sum_{i=1}^n \psi(p(0, \dots, \theta_{ij}(x), \dots, 0)) \\
&= \sum_{i=1}^n \psi((pe_i)\theta_{ij}(x)) \\
&= \sum_{i=1}^n \psi(pe_i)\theta_{ij}(x) \; (\psi \text{ is a right } B\text{-module isomorphism}) \\
&= \sum_{i=1}^n \xi_i \theta_{ij}(x).
\end{aligned}$$

Consequently, for every $x \in A$, $x\xi = \xi\theta(x)$. In the von Neumann algebra $M^{n+1} \subset \mathbf{B}(L^2(M) \oplus L^2(M)^{\oplus n})$, define

$$X_x = \begin{pmatrix} x & 0 \\ 0 & \theta(x) \end{pmatrix}, \forall x \in A.$$

In the space $L^2(M^{n+1})$, define

$$\Xi = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}.$$

Note that $X_x \in 1_{A^{n+1}} M^{n+1} 1_{A^{n+1}}$, $\forall x \in A$, and $\Xi \in 1_{A^{n+1}} L^2(M^{n+1}) 1_{A^{n+1}}$. We obtain $X_x \Xi = \Xi X_x$, for all $x \in A$. Denote by T_Ξ the corresponding unbounded operator affiliated with M^{n+1}. and write $T_\Xi = V|T_\Xi|$ for its polar decomposition (see Appendix A.1). We get $X_x V = V X_x$, for every $x \in A$, and $VV^* \leq 1_{A^{n+1}}$. Write

$$V = \begin{pmatrix} u & v \\ v' & w \end{pmatrix}.$$

It is straightforward to check that $v \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes 1_A M$ is a partial isometry from $\ker w$ onto $\ker u^*$ such that $xv = v\theta(x)$, for every $x \in A$.

(1) $\Longrightarrow$ (4). By contradiction, assume that there exists a sequence of unitaries (u_k) be a sequence of unitaries in A such that $\lim_k \|E_B(a^* u_k b)\|_2 = 0$ for all $a, b \in 1_A M$. Then $\|(\mathrm{Id}_n \otimes E_B)(v^* u_k v)\|_2 \to 0$. But for every $k \in \mathbf{N}$, $v^* u_k v = \theta(u_k) v^* v$. Moreover, $\theta(u_k) \in \mathcal{U}(pB^n p)$ and $v^* v \leq p$. Thus,

$$\begin{aligned}
\|(\mathrm{Id}_n \otimes E_B)(v^* v)\|_2 &= \|\theta(u_k)(\mathrm{Id}_n \otimes E_B)(v^* v)\|_2 \\
&= \|(\mathrm{Id}_n \otimes E_B)(\theta(u_k) v^* v)\|_2 \\
&= \|(\mathrm{Id}_n \otimes E_B)(v^* u_k v)\|_2 \to 0.
\end{aligned}$$

We conclude that $(\mathrm{Id}_n \otimes E_B)(v^* v) = 0$ and so $v = 0$. Contradiction.

(4) $\Longrightarrow$ (3). We can take $\varepsilon > 0$ and $K \subset 1_A M$ finite subset such that

$$\max_{a,b \in K} \|E_B(a^* ub)\|_2 \geq \varepsilon, \forall u \in \mathcal{U}(A).$$

Note that

$$\begin{aligned} \|E_B(a^*ub)\|_2^2 &= \tau(E_B(a^*ub)^*E_B(a^*ub)) \\ &= \mathrm{Tr}(e_B(a^*ub)^*e_B(a^*ub)e_B). \end{aligned}$$

Define now the element $c = \sum_{a\in K} ae_Ba^*$ in $1_A\langle M, e_B\rangle_+1_A$. Note that $\mathrm{Tr}(c) = \sum_{a\in K}\tau(aa^*) < \infty$. Denote by $\mathcal{C}$ the ultraweak closure of the convex hull of $\{u^*cu : u \in \mathcal{U}(A)\}$. Observe that $\mathcal{C} \subset \langle M, e_B\rangle_+ \cap L^2(\langle M, e_B\rangle)$ is bounded for both $\|\cdot\|_\infty$ and $\|\cdot\|_{2,\mathrm{Tr}}$ and is closed in $L^2(\langle M, e_B\rangle)$. Let $d \in \mathcal{C}$ be the unique element of minimal $\|\cdot\|_{2,\mathrm{Tr}}$-norm. Since $\|udu^*\|_{2,\mathrm{Tr}} = \|d\|_{2,\mathrm{Tr}}$ for all $u \in \mathcal{U}(A)$, we get $u^*du = d$, and so $d \in A' \cap 1_A\langle M, e_B\rangle_+1_A$. We show now that $d \neq 0$. For all $u \in \mathcal{U}(A)$, we have

$$\begin{aligned} \sum_{b\in K}\mathrm{Tr}(e_Bb^*(u^*cu)be_B) &= \sum_{a,b\in K}\mathrm{Tr}(e_B(a^*ub)^*e_B(a^*ub)e_B) \\ &= \sum_{a,b\in K}\tau(E_B(a^*ub)^*E_B(a^*ub)) \\ &= \sum_{a,b\in K}\|E_B(a^*ub)\|_2^2 \geq \varepsilon^2. \end{aligned}$$

Consequently, using the facts that $\mathrm{Tr}(e_B\cdot e_B)$ is a normal state on the basic construction $\langle M, e_B\rangle$ and $d \in \mathcal{C}$, we get

$$\sum_{b\in K}\mathrm{Tr}(e_Bb^*dbe_B) \geq \varepsilon^2.$$

It follows that $d \neq 0$. The proof is complete. $\square$

Assume that $M = B \rtimes \Lambda$ where $\Lambda \curvearrowright B$ is a trace-preserving action of a countable group Λ on a finite von Neumann algebra B. Denote by $(v_s)_{s\in\Lambda}$ the canonical unitaries in M which implement the action. It is straightforward to see that $A \npreceq_M B$ if and only if there a sequence of unitaries $u_n \in \mathcal{U}(A)$ such that

$$\lim_n \|E_B(u_nv_s^*)\|_2 = 0, \forall s \in \Lambda.$$

In the case when A and B are maximal abelian in M, one can get a more precise result (see [19, Theorem A.1]).

Proposition 2.13 (Popa, [19])**.** *Let (M,τ) be a finite von Neumann algebra. Let $A, B \subset M$ be a maximal abelian von Neumann subalgebras. The following are equivalent:*

1. *$A \preceq_M B$.*
2. *There exists a nonzero partial isometry $v \in M$ such that $vv^* \in A$, $v^*v \in B$ and $v^*Av = Bv^*v$.*

Proof. We only need to prove (1) $\Longrightarrow$ (2). The proof follows the one of [19, Theorem A.1]. We will use exactly the same reasoning as in the proof of [29, Theorem C.3].

Since $A \preceq_M B$, we can find $n \geq 1$, a nonzero projection $q \in \mathbf{M}_n(\mathbf{C}) \otimes B$, a nonzero partial isometry $w \in \mathbf{M}_{1,n}(\mathbf{C}) \otimes pM$ and a unital $*$-homomorphism $\psi : A \to q(\mathbf{M}_n(\mathbf{C}) \otimes B)q$ such that $xw = w\psi(x)$, $\forall x \in A$. Since we can replace q by an equivalent projection in $\mathbf{M}_n(\mathbf{C}) \otimes B$, we may assume $q = \mathrm{Diag}_n(q_1, \ldots, q_n)$ (see for instance second item in [29, Lemma C.2]). Observe now that $\mathrm{Diag}_n(q_1B, \ldots, q_nB)$ is maximal abelian in $q(\mathbf{M}_n(\mathbf{C}) \otimes B)q$. Since B is abelian, $q(\mathbf{M}_n(\mathbf{C}) \otimes B)q$ is of finite type I. Since A is abelian, up to unitary conjugacy by a unitary in $q(\mathbf{M}_n(\mathbf{C}) \otimes B)q$, we may assume that $\psi(A) \subset \mathrm{Diag}_n(q_1B, \ldots, q_nB)$ (see [29, Lemma C.2]). We can now cut down ψ and w by one of projections $(0, \ldots, q_i, \ldots, 0)$ and assume $n = 1$ from the beginning.

Write $e = ww^* \in A$ (since $A' \cap pMp = A$) and $f = w^*w \in \psi(A)' \cap qMq$. By spatiality, we have

$$f(\psi(A)' \cap qMq)f = (\psi(A)f)' \cap fMf = (w^*Aw)' \cap fMf = w^*Aw,$$

which is abelian. Let $Q := \psi(A)' \cap qMq$, which is a finite von Neumann algebra. Since $Bq \subset Q$ is maximal abelian and $f \in Q$ is an abelian projection, [29, Lemma C.2] yields a partial isometry $u \in Q$ such that $uu^* = f$ and $u^*Qu \subset Bq$. Define now $v = wu$. We get

$$v^*Av = u^*w^*Awu = u^*f(\psi(A)' \cap qMq)fu \subset Bq.$$

Moreover $vv^* = wuu^*w^* = wfw^* = e \in A$. Since v^*Av and Bv^*v are both maximal abelian, we get $v^*Av = Bv^*v$. $\square$

We can even go further if we moreover assume that $A, B \subset M$ are both Cartan subalgebras and M is a II_1 factor (see [19, Theorem A.1]).

Theorem 2.14 (Popa, [19])**.** *Let M be a II_1 factor. Let $A, B \subset M$ be Cartan subalgebras. The following are equivalent:*

1. *$A \preceq_M B$.*
2. *There exists $u \in \mathcal{U}(M)$ such that $uAu^* = B$.*

Proof. We only need to prove (1) $\Longrightarrow$ (2). By Proposition 2.13, there exists a nonzero partial isometry $v \in M$ such that $vv^* \in A$, $v^*v \in B$ and $v^*Av = Bv^*v$. Since A is diffuse, we may shrink $vv^* \in A$ so that $\tau(vv^*) = 1/n$, for some $n \in \mathbf{N}$. Write $p_1 = vv^*$, $q_1 = v^*v$ and take projections $p_2, \ldots, p_n \in A$, $q_2, \ldots, q_n \in B$ such that $\tau(p_i) = \tau(q_j) = 1/n$. Since $\mathcal{N}_M(A)'' = \mathcal{N}_M(B)'' = M$ and M is a II_1 factor, a classical exhaustion argument gives partial isometries $u_i, w_j \in M$ such that $p_1 = u_i^*u_i$, $p_i = u_iu_i^*$, $u_i^*Au_i = Au_i^*u_i$, $u_iAu_i^* = Au_iu_i^*$ and likewise $q_1 = w_jw_j^*$, $q_j = w_j^*w_j$, $w_j^*Bw_j = Bw_j^*w_j$, $w_jBw_j^* = Bw_jw_j^*$. Define

$$u = \sum_{i=1}^{n} u_i v^* w_i.$$

It is now routine to check that $u \in \mathcal{U}(M)$ and $uAu^* = B$. $\square$

3. Approximation properties

3.1. **Amenability.**

3.1.1. *Noncommutative L_p spaces.* We refer to [27, Chapter IX] for the details of the following facts on noncommutative L_p spaces. Let $(\mathcal{N}, \mathrm{Tr})$ be a semifinite von Neumann algebra endowed with a faithful, normal, semifinite trace Tr. For $1 \leq p < \infty$, we define the L_p-norm on $\mathcal{N}$ by $\|x\|_p = \mathrm{Tr}(|x|^p)^{1/p}$. By completing $\{x \in \mathcal{N} : \|x\|_p < \infty\}$ with respect to the L_p-norm, we obtain a Banach space $L_p(\mathcal{N})$. We only deal with $L_1(\mathcal{N})$, $L_2(\mathcal{N})$, and $L_\infty(\mathcal{N}) = \mathcal{N}$. The trace Tr extends to a contractive linear functional on $L_1(\mathcal{N})$. We occasionally write $\widehat{x}$ for $x \in \mathcal{N}$ when regarded as an element in $L_2(\mathcal{N})$. For any $1 \leq p, q, r \leq \infty$, with $1/p + 1/q = 1/r$, there is a natural product map

$$L_p(\mathcal{N}) \times L_q(\mathcal{N}) \ni (x, y) \mapsto xy \in L_r(\mathcal{N})$$

which satisfies $\|xy\|_r \leq \|x\|_p \|y\|_q$, $\forall x, y$. The Banach space $L_1(\mathcal{N})$ is identified with the predual of $\mathcal{N}$ under the duality

$$L_1(\mathcal{N}) \times \mathcal{N} \ni (\zeta, x) \mapsto \mathrm{Tr}(\zeta x) \in \mathbf{C}.$$

The Banach space $L_2(\mathcal{N})$ si identified with the GNS-Hilbert space $L^2(\mathcal{N}, \mathrm{Tr})$. Elements in $L_p(\mathcal{N})$ can be regarded as closed operators on $L^2(\mathcal{N})$ which are affiliated with $\mathcal{N}$ and hence in addition to the above-mentioned product, there are well-defined notions of positivity, square root, etc... We shall use the generalized Powers-Størmer Inequality (see [27, Theorem IX.1.2]):

$$\||\eta| - |\zeta|\|_2^2 \leq \|\eta^2 - \zeta^2\|_1 \leq \|\eta + \zeta\|_2 \|\eta - \zeta\|_2, \forall \eta, \zeta \in L_2(\mathcal{N}).$$

The Hilbert space $L_2(\mathcal{N})$ is an $\mathcal{N}$-$\mathcal{N}$ bimodule such that $\langle x\xi y, \eta\rangle = \mathrm{Tr}(x\xi y\eta^*)$, $\forall x, y \in \mathcal{N}$, $\forall \xi, \eta \in L_2(\mathcal{N})$. We also recall the following formulae. Let f_a be the characteristic function of the interval $(\sqrt{a}, +\infty)$. For any $\xi, \eta \in L_2(\mathcal{N})_+$, we have (see [7, Proposition 1.1] and [27, Theorem IX.2.14])

$$\begin{aligned} \int_0^\infty \|f_a(\xi)\|_2^2 \,\mathrm{d}a &= \|\xi\|_2^2, \\ \int_0^\infty \|f_a(\xi) - f_a(\eta)\|_2^2 \,\mathrm{d}a &\leq \|\xi - \eta\|_2 \|\xi + \eta\|_2. \end{aligned}$$

Let H be a complex separable Hilbert space and let $(\mathcal{N}, \mathrm{Tr}) = (\mathbf{B}(H), \mathrm{Tr})$, where Tr is the canonical trace on $\mathbf{B}(H)$. Then, $L_1(\mathbf{B}(H))$ can be identified with the space of *trace-class* operators on H, denoted by $\mathbf{S}_1(H)$ in the sequel. In the same way, $L_2(\mathbf{B}(H))$ can be identified with the space of *Hilbert-Schmidt* operators on H, denoted by $\mathbf{S}_2(H)$ in the sequel.

Let (M, τ) be a finite von Neumann algebra, denote by $H = L^2(M, \tau)$ its L^2-space with respect to the finite trace τ. The Hilbert space H is endowed with a canonical anti-unitary J defined by $J\widehat{x} = \widehat{x^*}$, $\forall x \in M$. In the sequel, we shall simply denote $H \ni \eta \mapsto \eta^* \in H$. We regard $M \subset \mathbf{B}(H)$ through the GNS-construction. The following linear map

$$U : \begin{array}{ccc} H \otimes H & \to & \mathbf{S}_2(H) \\ \sum_k \xi_k \otimes \eta_k & \mapsto & \sum_k \langle \cdot, \eta_k^* \rangle \xi_k \end{array}$$

defines a unitary. We shall identify $\mathbf{S}_2(H)$ and $H \otimes H$ through this unitary U. Moreover, for any $\xi, \eta \in H$, for any $x \in M$,

$$\begin{aligned} U(x\xi \otimes \eta) &= \langle \cdot, \eta^* \rangle x\xi = x\langle \cdot, \eta^* \rangle \xi = xU(\xi \otimes \eta) \\ U(\xi \otimes \eta x) &= \langle \cdot, (\eta x)^* \rangle \xi = \langle \cdot, x^*\eta^* \rangle \xi = \langle x\cdot, \eta^* \rangle \xi = U(\xi \otimes \eta)x. \end{aligned}$$

Thus, U preserves the M-M-bimodule structure: $\mathbf{S}_2(L^2(M))$ with its bimodule structure, as a two-sided ideal of $\mathbf{B}(L^2(M))$, is identified with the so-called *coarse correspondence* $L^2(M) \otimes L^2(M)$.

Finally, note that the symbol "Lim" will be used for a state on $\ell^\infty(\mathbf{N})$ or more generally on $\ell^\infty(I)$ with I directed set.

3.1.2. *Amenable finite von Neumann algebras.* Recall that a countable discrete group Γ is *amenable* if one the following equivalent conditions holds:

- There exists a Γ-invariant state $\varphi : \ell^\infty(\Gamma) \to \mathbf{C}$, i.e. $\varphi(\lambda_s f) = \varphi(f)$, for all $s \in \Gamma$, $f \in \ell^\infty(\Gamma)$.

- There exists a sequence of almost invariant unit vectors $\xi_n \in \ell^2(\Gamma)$, i.e. $\lim_n \|\lambda_s \xi_n - \xi_n\|_2 = 0$, for all $s \in \Gamma$.

Let (M, τ) be a finite von Neumann algebra with separable predual. Denote $H = L^2(M, \tau)$. We regard $M \subset \mathbf{B}(H)$ through the GNS-construction. A state φ on $\mathbf{B}(H)$ is said to be *M-central* if $\varphi \circ \mathrm{Ad}(u) = \varphi$, $\forall u \in \mathcal{U}(M)$.

Theorem 3.1 (Connes, [7]). *Let M be a finite von Neumann algebra. The following are equivalent:*

1. *There exists a conditional expectation $E : \mathbf{B}(H) \to M$.*
2. *There exists an M-central state φ on $\mathbf{B}(H)$ such that $\varphi_{|M} = \tau$.*
3. *There exists a net of unit vectors (ξ_n) in $\mathbf{S}_2(H)$ such that $\langle x\xi_n, \xi_n\rangle \to \tau(x)$, $\forall x \in M$, and $\|[\xi_n, u]\|_2 \to 0$, $\forall u \in \mathcal{U}(M)$.*

Proof. (1) $\Longrightarrow$ (2). Let E be a conditional expectation from $\mathbf{B}(H)$ onto M. Denote $\varphi = \tau \circ E$. Then $\forall x \in \mathbf{B}(H)$, $\forall u \in M$, one has

$$\varphi(uxu^*) = \tau(E(uxu^*)) = \tau(uE(x)u^*) = \tau(E(x)) = \varphi(x).$$

Thus, the state φ is M-central and $\varphi_{|M} = \tau$.

(2) $\Longrightarrow$ (3). Let φ be an M-central state on $\mathbf{B}(H)$ such that $\varphi_{|M} = \tau$. Take a net (ζ_n) of positive norm-one elements in $\mathbf{S}_1(H)$ such that $\mathrm{Tr}(\zeta_n \cdot)$ converges to φ pointwise. Then $\forall x \in \mathbf{B}(H)$, $\forall u \in \mathcal{U}(M)$, one has

$$\begin{aligned} \lim_n \mathrm{Tr}((\zeta_n - \mathrm{Ad}(u)\zeta_n)x) &= \lim_n \mathrm{Tr}(\zeta_n x) - \lim_n \mathrm{Tr}(u\zeta_n u^* x) \\ &= \lim_n \mathrm{Tr}(\zeta_n x) - \lim_n \mathrm{Tr}(\zeta_n u^* x u) \\ &= \varphi(x) - \varphi(\mathrm{Ad}(u^*)(x)) = 0 \end{aligned}$$

by assumption. It follows that the net $(\zeta_n - \mathrm{Ad}(u)\zeta_n)$ in $\mathbf{S}_1(H)$ converges to 0 in the weak topology. By the Hahn-Banach Separation Theorem, one may assume, by passing to finite convex combinations, that the net $(\zeta_n - \mathrm{Ad}(u)\zeta_n)$ in $\mathbf{S}_1(H)$ converges to 0 in norm. Thus, $\|[u, \zeta_n]\|_1 \to 0$, $\forall u \in \mathcal{U}(M)$. Define the unit vectors $\xi_n = \zeta_n^{1/2} \in \mathbf{S}_2(H)$. Using the Powers-Størmer Inequality, $\forall u \in \mathcal{U}(M)$,

$$\begin{aligned} \|[u, \xi_n]\|_2^2 &= \|u\xi_n u^* - \xi_n\|_2^2 \\ &\leq \|u\zeta_n u^* - \zeta_n\|_1 \\ &= \|[u, \zeta_n]\|_1 . \end{aligned}$$

This implies that $\|[u, \xi_n]\|_2 \to 0$, $\forall u \in \mathcal{U}(M)$. Moreover, $\forall x \in M$,

$$\begin{aligned} \lim_n \langle x\xi_n, \xi_n\rangle &= \lim_n \mathrm{Tr}(x\xi_n\xi_n^*) = \lim_n \mathrm{Tr}(x\xi_n^2) \\ &= \lim_n \mathrm{Tr}(\zeta_n x) = \varphi(x) = \tau(x). \end{aligned}$$

This proves (3).

(3) $\Longrightarrow$ (1). Assume that there exists a net of unit vectors (ξ_n) in $\mathbf{S}_2(H)$ such that $\langle x\xi_n, \xi_n\rangle \to \tau(x)$, $\forall x \in M$, and $\|[\xi_n, u]\|_2 \to 0$, $\forall u \in \mathcal{U}(M)$. Note that we also have $\|[\xi_n^*, u]\|_2 \to 0$, $\forall u \in \mathcal{U}(M)$. Write $\xi_n^* = w_n \eta_n$ for the polar decomposition, with $\eta_n = (\xi_n \xi_n^*)^{1/2} \geq 0$, $\|\eta_n\|_2 = 1$, and w_n partial isometry in $\mathbf{B}(H)$. Thus, $\forall n$, $\forall x \in \mathbf{B}(H)$,

$$\langle x\xi_n, \xi_n\rangle = \mathrm{Tr}(x\xi_n\xi_n^*) = \mathrm{Tr}(x\eta_n^2) = \langle x\eta_n, \eta_n\rangle.$$

Consequently, $\lim_n \langle x\eta_n, \eta_n\rangle = \tau(x)$, $\forall x \in M$. Moreover, using the Powers-Størmer Inequality, $\forall n$, $\forall u \in \mathcal{U}(M)$,

$$\begin{aligned}
\|[u, \eta_n]\|_2^2 &= \|u\eta_n u^* - \eta_n\|_2^2 \\
&\leq \|u\eta_n^2 u^* - \eta_n^2\|_1 \\
&= \|u(\xi_n\xi_n^*) - (\xi_n\xi_n^*)u\|_1 \\
&\leq \|(u\xi_n - \xi_n u)\xi_n^*\|_1 + \|\xi_n(u\xi_n^* - \xi_n^* u)\|_1 \\
&\leq \|u\xi_n - \xi_n u\|_2 + \|u\xi_n^* - \xi_n^* u\|_2.
\end{aligned}$$

Therefore, $\lim_n \|[u, \eta_n]\|_2 = 0$. So, we may moreover assume that $\xi_n \geq 0$, $\forall n$. Thus, $\forall c \in M$,

$$\begin{aligned}
\lim_n \|\xi_n c\|_2^2 &= \lim_n \langle \xi_n c, \xi_n c\rangle = \lim_n \mathrm{Tr}(\xi_n c c^* \xi_n^*) \\
&= \lim_n \mathrm{Tr}(\xi_n^2 c c^*) = \tau(cc^*) = \|c\|_{2,\tau}^2.
\end{aligned}$$

Define the following state

$$\varphi(x) = \operatorname*{Lim}_n \langle x\xi_n, \xi_n\rangle, \forall x \in \mathbf{B}(H).$$

Note that $\varphi_{|M} = \tau$. Moreover, $\forall b, c \in M$, $\forall y, z \in \mathbf{B}(H)$, since $\|[\xi_n, bc]\|_2 \to 0$, we get

$$\begin{aligned}
|\varphi(bcyz)| &= |\operatorname*{Lim}_n \mathrm{Tr}(\xi_n^2 bcyz)| \\
&= |\operatorname*{Lim}_n \mathrm{Tr}(\xi_n bc\xi_n yz)| \\
&= |\operatorname*{Lim}_n \mathrm{Tr}((z\xi_n b)(c\xi_n y))| \\
&\leq \limsup_n \|z\xi_n b\|_2 \|c\xi_n y\|_2 \\
&\leq \limsup_n \|\xi_n b\|_2 \|c\xi_n\|_2 \|y\|_\infty \|z\|_\infty \\
&= \|b\|_{2,\tau} \|c\|_{2,\tau} \|y\|_\infty \|z\|_\infty.
\end{aligned}$$

Take now $x \in \mathbf{B}(H)$, $a \in M$ and write $a = v|a|$ its polar decomposition in M. Thus, we have $a = (v|a|^{1/2})|a|^{1/2}$ and

$$\begin{aligned}
|\varphi(ax)| &\leq \|v|a|^{1/2}\|_{2,\tau} \||a|^{1/2}\|_{2,\tau} \|x\|_\infty \\
&\leq \||a|^{1/2}\|_{2,\tau} \||a|^{1/2}\|_{2,\tau} \|x\|_\infty \\
&= \|a\|_{1,\tau} \|x\|_\infty.
\end{aligned}$$

It follows that $|\varphi(ax)| \leq \|a\|_{1,\tau}\|x\|_\infty$, $\forall a \in M$, $\forall x \in \mathbf{B}(H)$. By the duality $M = L_1(M)^*$, we know that there exists $\Phi(x) \in M$, such that $\tau(a\Phi(x)) = \varphi(ax)$, $\forall a \in M$, $\forall x \in \mathbf{B}(H)$. It is straightforward to check that $\Phi : \mathbf{B}(H) \to M$ is a conditional expectation. $\square$

The state φ in (2) of Theorem 3.1 is called a *hypertrace*. Condition (3) says that the identity bimodule $L^2(M)$ is "weakly contained" in the coarse bimodule $L^2(M) \otimes L^2(M)$, that we usually denote by $L^2(M) \prec_{\text{weak}} L^2(M) \otimes L^2(M)$. This is the analog of the notion of amenability in the case of groups since the identity bimodule plays the rôle of the trivial representation and the identity bimodule plays the one of the left regular representation. For this reason, a finite von Neumann algebra M which satisfies one of the equivalent conditions of Theorem 3.1 is said to be *amenable*. Note that more generally for any M-M-bimodule $\mathcal{H}$, $L^2(M) \prec_{\text{weak}} \mathcal{H}$

if there exists a net (ξ_n) of unit vectors in $\mathcal{H}$ such that $\lim_n \langle \cdot \xi_n, \xi_n \rangle = \tau$ pointwise and $\lim_n \|[u, \xi_n]\| = 0$.

Proposition 3.2. *Let Γ be a countable discrete group. Then, Γ is amenable if and only if the group von Neumann algebra $L(\Gamma)$ is amenable.*

Proof. Assume that Γ is amenable. Denote by λ_Γ the left regular representation of Γ on $\ell^2(\Gamma)$. Then $1_\Gamma \prec \lambda_\Gamma$, i.e. there exists a net of unit vectors (η_n) in $\ell^2(\Gamma)$ such that $\lim_n \|\lambda_g \eta_n - \eta_n\|_2 = 0$, $\forall g \in G$. We shall identify $\ell^2(\Gamma)$ with the L^2-space of $L(\Gamma)$. Denote the unit vector $\xi_n = \eta_n \otimes \eta_n^* \in \ell^2(\Gamma) \otimes \ell^2(\Gamma) = \mathbf{S}_2(\ell^2(\Gamma))$. Moreover, we have for all $g \in \Gamma$,

$$\begin{aligned}
\|u_g \xi_n - \xi_n u_g\|_2 &= \|u_g \eta_n \otimes \eta_n^* - \eta_n \otimes \eta_n^* u_g\|_2 \\
&\leq \|u_g \eta_n \otimes \eta_n^* - \eta_n \otimes \eta_n^*\|_2 + \|\eta_n \otimes \eta_n^* - \eta_n \otimes \eta_n^* u_g\|_2 \\
&\leq \|\lambda_g \eta_n - \eta_n\|_2 + \|\lambda_{g^{-1}} \eta_n - \eta_n\|_2.
\end{aligned}$$

Therefore, $\lim_n \|u_g \xi_n - \xi_n u_g\|_2 = 0$, $\forall g \in \Gamma$. Thus Condition (3) in Theorem 3.1 is satisfied, and $L(\Gamma)$ is amenable.

Assume that $L(\Gamma)$ is amenable. Let $H = \ell^2(\Gamma)$. By Condition (3) in Theorem 3.1, we know that there exists a sequence of unit vectors (ξ_n) in $H \otimes H$ such that for any $g \in \Gamma$, $\lim_n \|u_g \xi_n - \xi_n u_g\|_2 = 0$. Define the unitary representation $\pi : \Gamma \to \mathcal{U}(H \otimes H)$ by

$$\pi(g)\xi = u_g \xi u_g^*.$$

It is straightforward to see that π is a mutiple of λ_Γ and that $1_\Gamma \prec \pi$. Consequently, $1_\Gamma \prec \lambda_\Gamma$ and Γ is amenable. $\square$

Exercise 3.3. Let M be a diffuse amenable finite von Neumann algebra. Show that a hypertrace φ given by Theorem 3.1 can never be normal on $\mathbf{B}(H)$.

Recall that a finite von Neumann algebra M is said to be *approximately finite dimensional* (AFD) if there exists an increasing sequence of finite dimensional unital $*$-subalgebras $Q_n \subset M$ such that $\bigcup_n Q_n$ is ultraweakly dense in M. Murray and von Neumann showed in their seminal work the uniqueness of the AFD II_1 factor. The following is easy to prove.

Proposition 3.4. *Let M be a finite AFD von Neumann algebra. Then M is amenable.*

Proof. Let $Q_n \subset M$ be a sequence of finite dimensional unital $*$-subalgebras such that $\bigcup_n Q_n$ is ultraweakly dense in M. Denote by μ_n the (probability) Haar measure on the compact group $\mathcal{U}(Q_n)$. Fix $\omega \in \beta(\mathbf{N}) \backslash \mathbf{N}$ a free ultrafilter. For every $x \in \mathbf{B}(L^2(M))$ define

$$E'(x) = \lim_{n \to \omega} \int_{\mathcal{U}(Q_n)} u x u^* \mathrm{d}\mu_n(u).$$

It is clear that $E' : \mathbf{B}(L^2(M)) \to M'$ is a conditional expectation. Denote by J the canonical antiunitary. Thus $E : \mathbf{B}(L^2(M)) \to M$ defined by $E(x) = JE'(JxJ)J$ is a conditional expectation. $\square$

The converse is Connes' fundamental result.

Theorem 3.5 (Connes, [7])**.** *Let M be a finite von Neumann algebra. Then M is AFD if and only if M is amenable. There exists a unique amenable II_1 factor. In particular, all icc amenable countable discrete groups give the same II_1 factor.*

Exercise 3.6. Let $\Gamma \curvearrowright P$ be a trace-preserving action of an amenable group Γ on an amenable finite von Neumann algebra P. Show that $M = P \rtimes \Gamma$ is amenable.

3.2. **Property Gamma.**

Definition 3.7. A II_1 factor M is said to have the *property Gamma* if for every $\varepsilon > 0$ and every $x_1, \ldots, x_n \in (M)_1$, there exists $u \in \mathcal{U}(M)$ such that $\tau(u) = 0$ and $\|[u, x_i]\|_2 < \varepsilon$, for all $1 \leq i \leq n$.

Theorem 3.8 (Connes, [8])**.** *Let M be a II_1 factor. The following are equivalent:*

1. *M does not have property Gamma.*
2. *There exists a non-trivial central sequence in M, i.e. $M' \cap M^\omega \neq \mathbf{C}$.*
3. *$M' \cap M^\omega$ is diffuse.*

Proof. For a free ultafilter ω, we will denote by $\pi_\omega : \ell^\infty(\mathbf{N}, M) \to M^\omega$ the quotient map. For (1) $\Longrightarrow$ (2), fix a free ultrafilter ω. Since M does not have property Gamma, there exists a sequence of unitaries $u_n \in \mathcal{U}(M)$ such that $\tau(u_n) = 0$ and $\lim_n \|[u_n, x]\|_2 = 0$, for all $x \in M$. Define $u = \pi_\omega((u_n)) \in \mathcal{U}(M' \cap M^\omega)$. We have that $\tau_\omega(u) = 0$ and so $M' \cap M^\omega \neq \mathbf{C}$. For (2) $\Longrightarrow$ (3), let $e \in M' \cap M^\omega$ be a nonzero projection such that $e \neq 1$. Write $\lambda = \tau_\omega(e) \in (0, 1)$. We may find a sequence of projections e_n which represents e such $\pi_\omega((e_n)) = e$ and $\tau(e_n) = \lambda$, for all $n \in \mathbf{N}$.

Observe that (e_n) is a central sequence and since M is a II_1 factor, we have that $e_n \to \lambda$ weakly (by weak compactness of the unit ball $(M)_1$). Thus we construct a subsequence e_{k_n} which satisfies for every $n \in \mathbf{N}$,

$$\begin{aligned}
\|[e_j, e_{k_n}]\|_2 &< \frac{1}{n} \\
|\tau(e_j e_{k_n}) - \lambda^2| &< \frac{1}{n}, \forall 1 \leq j \leq n.
\end{aligned}$$

Define $f := \pi_\omega((e_n e_{k_n})) \in M' \cap M^\omega$. The previous inequalities show that $f \in M' \cap M^\omega$ is indeed a nonzero projection such that $f \leq e$ and $f \neq e$, since $\tau_\omega(f) = \lambda^2 < \lambda = \tau_\omega(e)$. Therefore, $M' \cap M^\omega$ cannot have any minimal projections and hence is diffuse.

(3) $\Longrightarrow$ (1). Assume that $M' \cap M^\omega$ is diffuse. Let $e \in M' \cap M^\omega$ be a projection such that $\tau_\omega(e) = 1/2$. We can then represent $e = \pi_\omega((e_n))$ with projections $e_n \in \mathcal{U}(M)$ such that $\tau(e_n) = 1/2$, for all $n \in \mathbf{N}$. Therefore, $u = 2e - 1 \in M' \cap M^\omega$, with $u = \pi_\omega((u_n))$ and $\tau(u_n) = 0$, for all $n \in \mathbf{N}$. Hence M does not have property Gamma. $\square$

The next Theorem, due to Connes, gives a *spectral gap* characterization of property Gamma.

Theorem 3.9 (Connes, [7])**.** *Let M be a II_1 factor. The following are equivalent:*

1. *M has property Gamma.*
2. *There exists a sequence of unit vectors $\xi_n \in L^2(M) \ominus \mathbf{C}\widehat{1}$ such that*
$$\lim_n \|x\xi_n - \xi_n x\|_2 = 0, \forall x \in M.$$
3. $\mathbf{K}(L^2(M)) \cap C^*(M, M') = \{0\}$.

Proof. (1) $\Longrightarrow$ (2) is clear. For (2) $\Longrightarrow$ (1), write $\xi = (\xi_n) \in L^2(M)^\omega$. There are two cases to consider.

Case (1): ξ defines an element in $L^2(M^\omega)$. We have $\langle \xi, \widehat{1} \rangle = 0$ and $x\xi = \xi x$, for all $x \in M$. Write $\xi = v|\xi|$ for the polar decomposition of ξ. We have that

$v \in M' \cap M^\omega$ is a partial isometry such that $v \neq 0, 1$. Thus M has property Gamma.

Case (2): ξ does not define an element in $L^2(M^\omega)$. We start by proving the following claim.

Claim. For every finite subset $F \subset \mathcal{U}(M)$, for every $\varepsilon > 0$, there exists a projection $e \in M$ such that $\tau(e) < \varepsilon$ and $\|[u, e]\|_2 < \varepsilon \|e\|_2$, for all $u \in F$.

By Proposition C.2, we know that

$$\exists c > 0, \forall a > 0, \lim_{n \to \omega} \|f_a(|\xi_n|)|\xi_n|\|_2 > c.$$

We may then choose a subsequence (k_n) such that $\|f_n(|\xi_{k_n}|)|\xi_{k_n}|\|_2 \geq c$, for all $n \in \mathbf{N}$. Then with $\eta_n = \frac{1}{\|f_n(|\xi_{k_n}|)|\xi_{k_n}|\|_2} f_n(|\xi_{k_n}|)|\xi_{k_n}|$, we still have that $\lim_{n \to \omega} \|x\eta_n - \eta_n x\|_2 = 0$, for all $x \in M$. Observe that for all $a > 0$,

$$\eta_n \geq f_a(\eta_n)\eta_n \geq \sqrt{a} f_a(\eta_n)$$

and so $\tau(f_a(\eta_n)) \leq 1/\sqrt{a}$.

Let $F \subset \mathcal{U}(M)$ be a finite subset and $\varepsilon > 0$. Let $\delta = \varepsilon^4/(4|F|)$. Choose $n \in \mathbf{N}$ large enough such that $1/\sqrt{n} < \varepsilon$ and $\eta = \eta_n$ satisfies

$$\sum_{u \in F} \|\eta - u\eta u^*\|_2^2 < \delta.$$

It is clear that $f_a(u\eta u^*) = u f_a(\eta) u^*$. Hence, we get

$$\int_0^\infty \|f_a(\eta)\|_2^2 \mathrm{d}a = \|\eta\|_2^2 = 1$$

and

$$\begin{aligned}
\sum_{u \in F} \int_0^\infty \|f_a(\eta) - u f_a(\eta) u^*\|_2^2 \mathrm{d}a &\leq \sum_{u \in F} \|\eta - u\eta u^*\|_2 \|\eta + u\eta u^*\|_2 \\
&\leq 2(|F| \sum_{u \in F} \|\eta - u\eta u^*\|_2^2)^{1/2} \\
&< 2\sqrt{|F|\delta} \\
&= 2\sqrt{|F|\delta} \int_0^\infty \|f_a(\eta)\|_2^2 \mathrm{d}a.
\end{aligned}$$

Therefore there exists $a \geq n$ such that

$$\sum_{u \in F} \|f_a(\eta) - u f_a(\eta) u^*\|_2^2 < 2\sqrt{|F|\delta} \|f_a(\eta)\|_2^2.$$

Letting $e = f_a(\eta)$, we have $\tau(e) \leq \tau(f_a(\eta)) < \varepsilon$ and $\|ue - eu\|_2 < \varepsilon \|e\|_2$, for all $u \in F$. The claim is proven.

The next claim uses a maximality argument.

Claim. For every finite subset $F \subset \mathcal{U}(M)$, for every $\varepsilon > 0$, there exists a projection $e \in M$ such that $\tau(e) = 1/2$ and $\|[u, e]\|_2 < \varepsilon$, for all $u \in F$.

Once the claim is proven, we are done. Indeed, we can construct a sequence of projections $e_n \in M$ such that $\tau(e_n) = 1/2$ and $\lim_n \|[x, e_n]\|_2 = 0$, for all $x \in M$. We the get a sequence of unitaries $u_n = 2e_n - 1$, with $\tau(u_n) = 0$ and such that $\lim_n \|[x, u_n]\|_2 = 0$, for all $x \in M$. Hence, M has property Gamma. It only remains to prove the claim.

Let $u_1, \dots, u_k \in \mathcal{U}(M)$ and $\varepsilon > 0$. Let $\mathcal{I}$ be the set of families $i = (E, U_1, \dots, U_k)$ such that:

- $E \in M$ is a projection such that $\tau(E) \leq 1/2$.
- Each $U_j \in \mathcal{U}(M)$ is a unitary commuting with E.
- $\|U_j - u_j\|_1 \leq \varepsilon\tau(E)$, for all $1 \leq j \leq k$.

We define a partial ordering on $\mathcal{I}$ in the following way: we write $i \leq i'$ if

$$E \leq E' \text{ and } \|U_j - U_j'\|_1 \leq \varepsilon\tau(E' - E), \forall 1 \leq j \leq k.$$

It is easy to see that if $i \leq i'$ and $i' \leq i''$, then $i \leq i''$. The set $(\mathcal{I}, \leq)$ is moreover inductive. By Zorn's Lemma, there exists a maximal element $i = (E, U_1, \dots, U_k) \in \mathcal{I}$. Assume that $\tau(E) < 1/2$. We will deduce a contradiction. Let $\delta > 0$ such that $\tau(E) + \delta < 1/2$ and $4\delta \leq \varepsilon$. Let $v_j = (1-E)U_j = U_j(1-E) \in \mathcal{U}(N)$ where $N = (1-E)M(1-E)$. By the previous claim (with N), we know that there exists a projection $e \in N$ such that $\tau_N(e) < \delta$ and $\|[v_j, e]\|_{2,\tau_N} \leq \delta\|e\|_{2,\tau_N}$, for all $1 \leq j \leq k$. By Proposition D.1, there exists $w_j \in \mathcal{U}(N)$ such that $w_j v_j e v_j^* w_j^* = e$ and

$$\|w_j - (1-E)\|_{1,\tau_N} \leq \sqrt{2}\|v_j e v_j^* - e\|_{1,\tau_N} \leq 4\|v_j e v_j^* - e\|_{2,\tau_N}\|e\|_{2,\tau_N} \leq 4\delta\tau_N(e).$$

Let $E' = E + e \in M$. It is a projection stricly larger than E and such that $\tau(E') \leq 1/2$. Let $U_j' = U_j E + w_j v_j$. It is easy to see that $U_j' \in \mathcal{U}(M)$ and U_j' is commuting with E'. Moreover, $\|w_j v_j - v_j\|_1 \leq 4\delta\tau(e)$, $\|U_j' - U_j\|_1 \leq 4\delta\tau(e)$. Since $4\delta \leq \varepsilon$ and $\|U_j - u_j\|_1 \leq \varepsilon\tau(E)$ by assumption, we get

$$\|U_j' - u_j\|_1 \leq \varepsilon\tau(E + e) = \varepsilon\tau(E'), \forall 1 \leq j \leq k.$$

The element $i' = (E', U_1', \dots, U_k')$ satisfies $i \leq i'$ and $i \neq i'$, which contradicts the maximality of $i \in \mathcal{I}$. Therefore, we have that $\tau(E) = 1/2$. We have thus shown for each $\varepsilon > 0$, the existence of a projection $E \in M$ such that $\tau(E) = 1/2$, $[E, U_j] = 0$ and $\|U_j - u_j\|_1 \leq \varepsilon$. We finally get

$$\|[u_j, E]\|_2 = \|[u_j - U_j, e]\|_2 \leq 2\|u_j - U_j\|_2 \leq 2(2\varepsilon)^{1/2}.$$

As $\varepsilon > 0$ is arbitrary, we are done.

(1) $\Longrightarrow$ (3). Assume that $\mathbf{K}(L^2(M)) \cap C^*(M, M') \neq \{0\}$. Since $C^*(M, M')$ is a simple C^*-algebra, we get $\mathbf{K}(L^2(M)) \subset C^*(M, M')$. Let (x_n) be a (uniformly bounded) central sequence in M. We get that $[y, x_n] \to 0$ $*$-strongly for all $y \in C^*(M, M')$. Denote by $P_{\mathbf{C}} : L^2(M) \to \mathbf{C}$ the orthogonal projection. Since $\mathbf{K}(L^2(M)) \subset C^*(M, M')$, we get $[P_{\mathbf{C}}, x_n] \to 0$ $*$-strongly. We have that

$$\lim_n \|x_n - \tau(x_n)1\|_2 = \lim_n \|(x_n P_{\mathbf{C}} - P_{\mathbf{C}} x_n)\widehat{1}\| = 0,$$

and so (x_n) is trivial.

(3) $\Longrightarrow$ (2). Assume that it is impossible to find a sequence of unit vectors ξ_n like in (2). Then then exists $\delta > 0$, a finite subset $F \subset \mathcal{U}(M)$ such that

$$\max_{u \in F} \|\xi - u\xi u^*\|_2 \geq \delta\|\xi\|_2, \forall \xi \in L^2(M).$$

We may assume that $F = F^*$. Define the self-adjoint operator

$$T = \frac{1}{|F|}\sum_{u \in F} uJuJ \in C^*(M, M').$$

We have $\|T\|_\infty \leq 1$, $T\widehat{1} = \widehat{1}$, so that $\widehat{1}$ is an eigenvector for the eigenvalue 1. We show that $T - 1$ is invertible on $L^2(M) \ominus \mathbf{C}\widehat{1}$, so that there is a spectral gap at 1 for the operator T. Otherwise, since T is selfadjoint, there would be a sequence of unit vectors $\xi_k \in L^2(M) \ominus \mathbf{C}\hat{1}$ such that $\lim_k \|(T-1)\xi_k\|_2 = 0$. We have

$$\|(1-T)\xi_k\|_2 = \|\frac{1}{|F|}\sum_{u\in F}(1-uJuJ)\xi_k\|_2.$$

Recall that a Hilbert space is *uniformly convex*. For all $k \in \mathbf{N}$, we have that

$$\sum_{u\in F}\frac{1}{|F|}\|\xi_k - uJuJ\xi_k\|^2 = 2 - 2\Re\langle\xi_k, T\xi_k\rangle = 2\Re\langle\xi_k, \xi_k T\xi_k\rangle \leq 2\|\xi_k - T\xi_k\|_2.$$

We have that $\lim_k \|\xi_k - uJuJ\xi_k\|_2 = 0$, for all $u \in F$ and so $\lim_k \|\xi_k\|_2 = 0$, which is absurd. Thus $T-1$ is invertible on $L^2(M) \ominus \mathbf{C}\hat{1}$ and there exists $\varepsilon > 0$ such that $\mathrm{Sp}(T) \subset [-1, 1-\varepsilon] \cup \{1\}$. By continuous functional calculus, we get that $P_{\mathbf{C}} \in C^*(M, M')$ and so $\mathbf{K}(L^2(M)) \subset C^*(M, M')$. $\square$

A closely related concept for groups is the one of inner amenability. A countable discrete group G is said to be *inner amenable* if the adjoint representation $\mathrm{Ad} : G \to \mathcal{U}(\ell^2(G) \ominus \mathbf{C}\delta_e)$ defined by $\mathrm{Ad}_g \delta_h = \delta_{ghg^{-1}}$ contains a sequence of almost invariant unit vectors. Examples of inner amenable groups include amenable groups, direct product groups $G \times H$, where H is infinite amenable, Baumslag-Solitar groups. Examples of groups which are not inner amenable include free groups $\mathbf{F}_n$, $n \geq 2$, and property (T) groups. The following is easy to prove.

Proposition 3.10 (Effros, [10]). *Then G be an icc countable discrete group. If $L(G)$ has property Gamma, then G inner amenable.*

Proof. Assume $L(G)$ has property Gamma. There exists a sequence of unitaries $v_n \in \mathcal{U}(M)$ such that $\tau(v_n) = 0$ and $\lim_n \|[v_n, x]\|_2 = 0$, for all $x \in M$. Define $\xi_n := u_n\delta_e \in \ell^2(G) \ominus \mathbf{C}\delta_e$. Since

$$\|\mathrm{Ad}_g \xi_n - \xi_n\| = \|u_g v_n - v_n u_g\|_2,$$

we get that (ξ_n) is a sequence of almost invariant unit vectors for the adjoint representation Ad. $\square$

The converse is false though, as it was recently discovered by Vaes [30]. We can illustrate the subtle difference between the property Gamma of $L(G)$ and the inner amenability of G.

- The group G is inner amenable if and only if there exists a sequence of unit vectors $\xi_n \in \ell^2(G) \ominus \mathbf{C}\delta_e$ such that
$$\lim_n \|x\xi_n - \xi_n x\|_2 = 0, \forall x \in C^*_\lambda(G).$$
- The von Neumann algebra $L(G)$ has property Gamma if and only if there exists a sequence of unit vectors $\xi_n \in \ell^2(G) \ominus \mathbf{C}\delta_e$ such that
$$\lim_n \|x\xi_n - \xi_n x\|_2 = 0, \forall x \in L(G).$$

3.3. **Haagerup property.**

Definition 3.11 (Haagerup, [14]). A countable discrete group Γ is said to have the *property* (H) if there exists a sequence of positive definite functions $\varphi_n : \Gamma \to \mathbf{C}$ such that $\lim_n \varphi_n = 1$ pointwise and $\varphi_n \in c_0(\Gamma)$, for all $n \in \mathbf{N}$.

We will often write Haagerup property for property (H).

Example 3.12. The following groups have the Haagerup property: amenable groups, free groups and more generally all groups which act properly on a tree. The Haagerup property is moreover stable under taking subgroups, amenable extentions, free products, wreath products.

We refer to the book by P.A. Cherix, M. Cowling, P. Jolissaint, P. Julg and A. Valette [3] for a comprehensive of groups with the Haagerup property

Definition 3.13 (Choda, [4])**.** Let (N, τ) be a finite von Neumann algebra endowed with a fixed faithful normal trace. We say that N has the Haagerup property if there exists a sequence $\theta_n : N \to N$ of τ-preserving ucp maps which satisfies:

- $\lim_n \|\theta_n(x) - x\|_2 = 0$, for all $x \in N$.
- Whenever $w_k \in (N)_1$ is a sequence such that $w_k \to 0$ weakly, then we have $\lim_k \|\theta_n(w_k)\|_2 = 0$, for all $n \in \mathbf{N}$.

Proposition 3.14 (Choda, [4])**.** *Let Γ be a countable discrete group. Then Γ has the Haagerup property if and only if $L(\Gamma)$ has the Haagerup property.*

Proof. Assume that Γ has the Haagerup property. Then there exists a sequence $\varphi_n : \Gamma \to \mathbf{C}$ such that $\lim_n \varphi_n = 1$ pointwise and $\varphi_n \in c_0(\Gamma)$, for all $n \in \mathbf{N}$. We may assume that $\varphi_n(e) = 1$, for all $n \in \mathbf{N}$. Define $\theta_n : L(\Gamma) \to L(\Gamma)$ by

$$\theta_n(\sum_{s \in \Gamma} a_s u_s) = \varphi_n(s) a_s u_s.$$

It is straightforward to check that θ_n is a sequence of τ-preserving ucp maps which satisfies conditions of Definition 3.13.

Conversely, assume that $L(\Gamma)$ has the Haagerup property. Let θ_n be a sequence of τ-preverving ucp maps given by Definition 3.13. Define $\varphi_n(s) = \tau(\theta_n(u_s)u_s^*)$, for all $s \in \Gamma$. Then φ_n is a sequence of positive definite functions that does the job. ☐

Theorem 3.15 (Haagerup, [14])**.** *The free groups $\mathbf{F}_n$ have the Haagerup property.*

Proof. Denote by $\mathbf{F}_n \ni g \mapsto |g| \in \mathbf{R}_+$ the natural length function. We will show that for all $0 < \rho < 1$, the function φ_ρ defined by

$$\mathbf{F}_n \ni g \mapsto \rho^{|g|}$$

is a positive definite function on $\mathbf{F}_n$. Since $\varphi_\rho \in c_0(\mathbf{F}_n)$ and $\lim_{\rho \to 1} \varphi_\rho = 1$, we get that $\mathbf{F}_n$ has the Haagerup property.

We give a proof using Popa's *free malleable deformation* [21, 24]. We may assume that $n < \infty$. Let $M = L(\mathbf{F}_n)$ and $\widetilde{M} = L(\mathbf{F}_n * \widetilde{\mathbf{F}}_n)$, where $\widetilde{\mathbf{F}}_n$ is a copy of $\mathbf{F}_n$ which is free from $\mathbf{F}_n$. Denote by $a_1, \dots, a_n$ the canonical generators of $L(\mathbf{F}_n)$ (resp. $b_1, \dots, b_n$ the ones of $L(\widetilde{\mathbf{F}}_n)$). For $1 \leq k \leq n$, let

$$h_k = \frac{1}{\sqrt{-1}\pi} \log(b_k),$$

where log denotes the principal branch of the logarithm. We get that h_k is selfadjoint and $b_k = \exp(\sqrt{-1}\pi h_k)$. For $t \in \mathbf{R}$, define

$$b_k^t := \exp(\sqrt{-1}\pi t h_k) \in \mathcal{U}(L(\widetilde{\mathbf{F}}_n)).$$

It is straightforward to check that

$$\gamma(t) := \tau(b_k^t) = \int_{-1}^{1} \exp(\sqrt{-1}\pi t x)\mathrm{d}x = \frac{\sin(\pi t)}{\pi t}$$

Define the following $*$-automorphism $\alpha_t : \widetilde{M} \to \widetilde{M}$ by

$$\alpha_t(a_k) = a_k b_k^t \text{ and } \alpha_t(b_k) = b_k.$$

Since the unitaries $\{a_1, \ldots, a_n, a_1 b_1^t, \ldots, a_k b_k^t\}$ generate $\widetilde{M}$ and are $*$-free from each other, we check easily that (α_t) is a one-paramater family of trace-preserving $*$-automorphisms. Write $N = L(\mathbf{F}_n)$ and $\widetilde{N} = \alpha_1(N)$. We see that $N \vee \widetilde{N} = \widetilde{M}$ and $\widetilde{N}$ is $*$-free from N. Consequently, we get $\widetilde{M} = N * \widetilde{N}$ and (α_t) satisfies $\alpha_1(x * 1) = 1 * x$, for all $x \in N$. Moreover, we have

$$(E_N \circ \alpha_t)(u_g) = \gamma(t)^{2|g|} u_g, \forall g \in \mathbf{F}_n,$$

Since $E_N \circ \alpha_t$ is u.c.p., it follows that $\psi_t : \mathbf{F}_n \to \mathbf{C}$ defined by $\psi_t(g) = \tau((E_N \circ \alpha_t)(u_g)u_g^*) = \gamma(t)^{2|g|}$ is a positive definite function. We are done. $\square$

4. Rigidity of II_1 factors

4.1. **Rigid inclusions of von Neumann algebras.** The notion of property (T) for a II_1 factor was introduced by Connes and Jones in their seminal work [9]. Its *relative* version for a inclusion of finite von Neumann algebras is due to Popa [19].

Definition 4.1 (Popa, [19])**.** Let (M, τ) be a finite von Neumann algebra with a fixed trace and $B \subset M$ be a subalgebra. The inclusion is said to be *rigid* if for every $\varepsilon > 0$, there exist $\delta > 0$ and a finite subset $F \subset M$ such that for every τ-preserving u.c.p. map $\phi : M \to M$, we have

$$\sup_{x \in F} \|\phi(x) - x\|_2 \leq \delta \Longrightarrow \sup_{x \in (B)_1} \|\phi(x) - x\|_2 \leq \varepsilon.$$

The von Neumann algebra M has property (T) if the identity inclusion $M \subset M$ is rigid. Note that we can relax the assumptions in Definition 4.1. Indeed, let $\phi : M \to M$ be a completely positive map such that $\phi(1) \leq 1$ and $\tau \circ \phi \leq \tau$. Then, $\widetilde{\phi} : M \to M$ defined by

$$\widetilde{\phi}(x) = \phi(x) + \frac{(\tau - \tau \circ \phi)(x)}{(\tau - \tau \circ \phi)(1)}(1 - \phi(1))$$

is a τ-preserving u.c.p. map.

Theorem 4.2 (Popa, [19])**.** *Let $B \subset M$ be an inclusion of finite von Neumann algebras and let τ be a fixed trace on M. The following are equivalent:*

(1) *The inclusion $B \subset M$ is rigid;*
(2) *For every $\varepsilon > 0$, there exist $\delta > 0$ and a finite subset $F \subset M$ such that for any M-M bimodule $\mathcal{H}$ and any tracial unit vector $\xi \in \mathcal{H}$ for which $\|x\xi - \xi x\| \leq \delta$, $\forall x \in F$, there exists a B-central vector $\eta \in \mathcal{H}$ such that $\|\eta - \xi\| \leq \varepsilon$.*

Proof. We prove both directions.

(1) $\Longrightarrow$ (2). Let $0 < \varepsilon < 1$. Let $F \subset M$ be a finite subset and $\delta > 0$ given by Condition (1). Let $\mathcal{H}$ be an M-M-bimodule and a tracial unit vector $\xi \in \mathcal{H}$ such that $\|x\xi - \xi x\| \leq \delta$, $\forall x \in F$. Let $\phi : M \to M$ be the τ-preserving u.c.p. map

associated with $(\mathcal{H}, \xi)$. Recall that $\langle a\xi x, \xi y\rangle_{\mathcal{H}} = \langle \phi(a)\widehat{x}, \widehat{y}\rangle_{L^2(N)}$, $\forall a, x, y \in M$. Then we have

$$\begin{aligned} \|\phi(x) - x\|_2^2 &= \|\phi(x)\|_2^2 + \|x\|_2^2 - 2\Re\langle x\xi, \xi x\rangle \\ &\leq 2\|x\|_2^2 - 2\Re\langle x\xi, \xi x\rangle \\ &= \|x\xi - \xi x\|^2 \leq \delta^2 \end{aligned}$$

for every $x \in F$. It follows that $\|\phi(x) - x\|_2 \leq \varepsilon$, for every $x \in (B)_1$. We get for every $u \in \mathcal{U}(B)$,

$$\begin{aligned} \|\xi - u\xi u^*\|^2 &= 2 - 2\Re\tau(\phi(u)u^*) \\ &= 2\Re\tau(1 - \phi(u)u^*) \\ &\leq 2\|1 - \phi(u)u^*\|_2 \leq 2\varepsilon. \end{aligned}$$

Denote by η the *circumcenter* of the bounded set $\mathcal{C} = \{u\xi u^* : u \in \mathcal{U}(B)\}$. Since $u\mathcal{C}u^* = \mathcal{C}$, $\forall u \in \mathcal{U}(B)$, by uniqueness it follows that $u\eta u^* = \eta$, $\forall u \in \mathcal{U}(B)$. Thus η follows B-central and moreover $\|\xi - \eta\| \leq (2\varepsilon)^{1/2}$.

(1) $\Longleftarrow$ (2). Let $0 < \varepsilon < 1$. Let $F \subset M$ be a finite subset and $0 < \delta < 1$ given by Condition (2). Let $\phi : M \to M$ be a τ-preserving u.c.p. such that $\|\phi(x) - x\|_2 \leq (2\|x\|_2)^{-1}\delta^2$, $\forall x \in F$. Let $\mathcal{H}$ be the M-M bimodule and ξ be the tracial vector associated with ϕ. We have $\langle a\xi x, \xi y\rangle_{\mathcal{H}} = \langle \phi(a)\widehat{x}, \widehat{y}\rangle_{L^2(N)}$, $\forall a, x, y \in M$. For every $x \in F$, we get

$$\begin{aligned} \|x\xi - \xi x\|_2^2 &= \|x\xi\|_2^2 + \|\xi x\|_2^2 - 2\Re\langle x\xi, \xi x\rangle \\ &= 2\|x\|_2^2 - 2\Re\tau(\phi(x)x^*) \\ &= 2\Re\tau((x - \phi(x))x^*) \\ &\leq 2\|x - \phi(x)\|_2\|x\|_2 \leq \delta^2. \end{aligned}$$

Therefore, there exists a B-central vector $\eta \in \mathcal{H}$ such that $\|\eta - \xi\| \leq \varepsilon$. For every $x \in (B)_1$, we get

$$\begin{aligned} \|x - \phi(x)\|_2^2 &= \|x\|_2^2 + \|\phi(x)\|_2^2 - 2\Re\tau(\phi(x)x^*) \\ &\leq 2\|x\xi\|^2 - 2\Re\langle x\xi, \xi x\rangle \\ &\leq 2\|x\xi\|\|x\xi - \xi x\| \\ &= 2\|x\xi\|\|x(\xi - \eta) - (\xi - \eta)x\| \\ &\leq 4\|x\|_\infty\|x\xi\|\|\xi - \eta\| \leq 4\varepsilon. \end{aligned}$$

□

Exercise 4.3. For $i = 1, 2$ let $B_i \subset M_i$ be an embedding of finite von Neumann algebras. Show that the following are equivalent:

(1) For $i = 1, 2$, the inclusion $B_i \subset M_i$ is rigid.
(2) The inclusion $B_1 \overline{\otimes} B_2 \subset M_1 \overline{\otimes} M_2$ is rigid.

Recall that a pair (Γ, Λ) consisting of a countable Γ with subgroup Λ is said to have the *relative property (T)* of Kazhdan-Margulis [16, 17] if every unitary representation of Γ which admits a sequence of almost invariant unit vectors, admits a non-zero Λ-invariant vector. Equivalently, for every $\varepsilon > 0$, there exist $\delta > 0$ and a finite subset $F \subset \Gamma$ such that for any unitary representation $\pi : \Gamma \to \mathcal{U}(H_\pi)$ which has a $(\pi(F), \delta)$-invariant unit vector ξ, there exists a non-zero $\pi(\Lambda)$-invariant vector such that $\|\eta - \xi\| \leq \varepsilon$ (see [15, Theorem 1.2(a3)]). A group Γ is said to have *property (T)* if the pair (Γ, Γ) has the relative property (T).

Example 4.4. Here are a few classical examples.

(1) The example by excellence of a pair with relative property (T) is $(\mathbf{Z}^2 \rtimes \mathrm{SL}(2, \mathbf{Z}), \mathbf{Z}^2)$.
(2) For any $n \geq 3$, $\mathrm{SL}(n, \mathbf{Z})$ has property (T).
(3) For any property (T) group Γ and any group H, the pair $(H \times \Gamma, \Gamma)$ has relative property (T).

We refer to the book by Bekka, de la Harpe and Valette [1] for a comprehensive list of groups with (relative) property (T).

Theorem 4.5 (Popa, [19])**.** *Let $\Lambda \subset \Gamma$ be an inclusion of countable groups. The following are equivalent:*

(1) *The pair (Γ, Λ) has the relative property (T);*
(2) *The inclusion $L(\Lambda) \subset L(\Gamma)$ is rigid.*

Proof. Write $B = L(\Lambda) \subset L(\Gamma) = M$.

(1) $\Longrightarrow$ (2). Assume that the pair (Γ, Λ) has the relative property (T). Fix $\varepsilon > 0$. We know that there exist $\delta > 0$ and a finite subset $F \subset \Gamma$ such that for any unitary representation $\pi : \Gamma \to \mathcal{U}(H_\pi)$ which has a $(\pi(F), \delta)$-invariant unit vector ξ, there exists a non-zero $\pi(\Lambda)$-invariant vector such that $\|\eta - \xi\| \leq \varepsilon$. Let now $\mathcal{H}$ be a M-M-bimodule and a unit tracial vector $\xi \in \mathcal{H}$ such that $\|u_s\xi - \xi u_s\| \leq \delta$, $\forall s \in \Gamma$. Define the unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ by $\pi_s(\eta) = u_s \eta u_s^*$. The vector ξ is then $(\pi(F), \delta)$-invariant. Thus there exists a $\pi(\Lambda)$-invariant vector $\eta \in \mathcal{H}$ such that $\|\eta - \xi\| \leq \varepsilon$. It is then clear that η is B-central.

(1) $\Longleftarrow$ (2). Assume that the inclusion $B \subset M$ is rigid. Let $\varepsilon = 1/2$. We know that there exist $\delta > 0$ and a finite subset $F \subset M$ such that for any M-M-bimodule $\mathcal{H}$ and any tracial vector $\xi \in \mathcal{H}$ for which $\|x\xi - \xi x\| \leq \delta$, $\forall x \in F$, there exists a B-central vector $\eta \in \mathcal{H}$ such that $\|\eta - \xi\| \leq \varepsilon$. Let $\pi : \Gamma \to \mathcal{U}(H_\pi)$ be a unitary representation. Consider the M-M-bimodule $\mathcal{K}_\pi = H_\pi \otimes \ell^2(\Gamma)$ associated with π. Take a sequence $(\zeta_n) \in H_\pi$ of almost invariant unit vectors and set $\xi_n = \zeta_n \otimes \delta_e$. It is then clear that $(\xi_n) \in \mathcal{K}_\pi$ is a sequence of tracial vectors for which $\lim_n \|x\xi_n - \xi_n x\| = 0$, $\forall x \in M$. For $n \in \mathbf{N}$ large enough, we have $\|x\xi_n - \xi_n x\| \leq \delta$, $\forall x \in F$. Write $\xi = \xi_n$. Therefore there exists a nonzero B-central vector $\eta \in \mathcal{K}_\pi$ such that $\|\eta - \xi\| \leq 1/2$. Regard $\eta \in \ell^2(\Gamma, H_\pi)$ and write $\eta = \sum_{s \in \Gamma} \eta_s \otimes \delta_s$, where $\eta_s \in H_\pi$. We have

$$\|\eta_e - \zeta\|^2 + \sum_{s \in \Gamma - \{e\}} \|\eta_s\|^2 = \|\eta - \xi\|^2 \leq 1/4.$$

Since $\zeta \in H_\pi$ is a unit vector, we have that $\eta_e \neq 0$. Since η_e is moreover $\pi(\Lambda)$-invariant, the proof is complete. ☐

The previous theorem shows in particular that the inclusion $L^\infty(\mathbf{T}^2) \subset L^\infty(\mathbf{T}^2) \rtimes \mathrm{SL}(2, \mathbf{Z})$ is rigid.

4.2. **Applications to rigidity of II_1 factors.** We use now the tools we introduced in the previous sections to get structural properties for property (T) II_1 factors. Most of the proofs are based on a "separability vs property (T)" argument that goes back to Connes [6].

4.2.1. *Symmetry groups of property (T) factors are countable.* For a II_1 factor M, we endow the group $\mathrm{Aut}(M)$ with the topology of pointwise $\|\cdot\|_2$-convergence: for a sequence (θ_n) in $\mathrm{Aut}(M)$, we have

$$\theta_n \to \mathrm{Id} \Longleftrightarrow \|\theta_n(x) - x\|_2 \to 0, \forall x \in M.$$

Note that $\mathrm{Aut}(M)$ is a polish group. We shall denote by $\mathrm{Inn}(M) \subset \mathrm{Aut}(M)$ the subgroup of *inner* automorphisms. For any $u \in \mathcal{U}(M)$, we shall write $\mathrm{Ad}_u(x) = uxu^*$, $\forall x \in M$. We denote by $\mathrm{Out}(M)$ the quotient group $\mathrm{Aut}(M)/\mathrm{Inn}(M)$.

For the hyperfinite II_1 factor R, the outer automorphisms group $\mathrm{Out}(R)$ is "huge". Indeed, one can embed any second countable locally compact group G in $\mathrm{Out}(R)$. For property (T) II_1 factors, the situation is dramatically different.

Theorem 4.6 (Connes, [6])**.** *Let M be a property (T) II_1 factor. Then $\mathrm{Out}(M)$ is countable.*

Proof. We show that $\mathrm{Inn}(M) \subset \mathrm{Aut}(M)$ is an open subgroup. Thus, it follows that $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$ is a Hausdorff discrete group, and by separability, $\mathrm{Out}(M)$ is necessarily countable. Fix $\varepsilon = 1/2$.

By property (T) of M, we know that there exists $\delta > 0$ and $F \subset M$ finite subset such that for every u.c.p. τ-preserving map $\phi : M \to M$, we have

$$\sup_{x \in F} \|\phi(x) - x\|_2 < \delta \Longrightarrow \sup_{x \in (M)_1} \|\phi(x) - x\|_2 < 1/2.$$

Let $\mathcal{V}_{\delta,F} = \{\theta \in \mathrm{Aut}(M) : \|\theta(x) - x\|_2 < \delta, \forall x \in F\}$ be an open neighborhood of Id in $\mathrm{Aut}(M)$. Let $\theta \in \mathcal{V}_{\delta,F}$. Since $\|\theta(x) - x\|_2 < \delta$, $\forall x \in F$, we know that $\|\theta(x) - x\|_2 \leq 1/2$, for every $x \in (M)_1$. Observe that $\|\theta(u)u^* - 1\|_2 \leq 1/2$, for every $u \in \mathcal{U}(M)$. Consider

$$\mathcal{C} = \overline{\mathrm{co}}^w\{\theta(u)u^* : u \in \mathcal{U}(M)\}$$

the weak closure of the convex hull of all the $\theta(u)u^*$'s, for $u \in \mathcal{U}(M)$. Observe that $\mathcal{C} \subset (M)_1$ is closed in $L^2(M)$.

Denote by $a \in \mathcal{C}$ the unique element of minimum $\|\cdot\|_2$-norm. It follows that $\|a - 1\|_2 \leq 1/2$. We get $a \neq 0$. Observe that for every $u \in \mathcal{U}(M)$, $\theta(u)au^* \in \mathcal{C}$ and $\|\theta(u)au^*\|_2 = \|a\|_2$. By uniqueness, we get $\theta(u)au^* = a$, for every $u \in \mathcal{U}(M)$. So we have $a^*au = ua^*a$, for every $u \in \mathcal{U}(M)$. It follows that $a^*a = \lambda \in \mathbf{R}^*_+$ since M is a factor. Therefore $v = a/\sqrt{\lambda} \in \mathcal{U}(M)$ and $\theta = \mathrm{Ad}_v$. $\square$

Observe that a property (T) factor cannot have property Gamma. Recall that for a II_1 factor M, the *fundamental group* of M is defined as follows:

$$\mathcal{F}(M) = \{\tau(p)/\tau(q) : pMp \simeq qMq\}.$$

Murray & von Neumann showed that the unique AFD II_1 factor R has full fundamental group, i.e. $\mathcal{F}(M) = \mathbf{R}^*_+$. There is an alternative way of defining the fundamental group of M. Denote by $M^\infty = M \overline{\otimes} \mathbf{B}(\ell^2)$ the corresponding II_∞ factor with semifinite trace Tr given by $\mathrm{Tr} = \tau \otimes \mathrm{Tr}_{\mathbf{B}(\ell^2)}$. For any $\theta \in \mathrm{Aut}(M^\infty)$, there exists a unique $\lambda > 0$ such that $\mathrm{Tr} \circ \theta = \lambda$. We shall denote this λ by $\mathrm{mod}(\theta)$. Moreover, the map $\mathrm{mod} : \mathrm{Aut}(M^\infty) \to \mathbf{R}^*_+$ is a group homomorphism. It is then easy to check that

$$\mathcal{F}(M) = \{\mathrm{mod}(\theta) : \theta \in \mathrm{Aut}(M^\infty)\}.$$

Using property (T), Connes [6] gave the first example of II_1 factor with countable fundamental group.

Theorem 4.7 (Connes, [6]). *Let M be a property (T) II_1 factor. Then $\mathcal{F}(M)$ is countable.*

Proof. We construct a one-to-one map $\alpha : \mathcal{F}(M) \to \mathrm{Out}(M\overline{\otimes}M)$. Since M has property (T), $M\overline{\otimes}M$ has property (T) as well (cf Exercise 4.3) and then $\mathrm{Out}(M\overline{\otimes}M)$ is countable by Theorem 4.6. Therefore, $\mathcal{F}(M)$ follows countable.

Claim. Let N be a II_1 factor. Let $\theta \in \mathrm{Aut}(N^\infty)$ such that $\mathrm{mod}(\theta) = 1$. Then there exist $u \in \mathcal{U}(N^\infty)$ and $\rho \in \mathrm{Aut}(N)$ such that

$$\theta = \mathrm{Ad}_u \circ (\rho \otimes \mathrm{Id}_{\mathbf{B}(\ell^2)}).$$

Therefore the group homomorphism

$$\{\beta \in \mathrm{Aut}(N^\infty) : \mathrm{mod}(\beta) = 1\} \ni \theta \mapsto [\rho] \in \mathrm{Out}(N)$$

is well-defined.

Denote by $(e_{ij}) \in \mathbf{B}(\ell^2)$ the canonical matrix unit such that $\mathrm{Tr}(e_{ii}) = 1$, $\forall i \in \mathbf{N}$. Let $\theta \in \mathrm{Aut}(N^\infty)$ such that $\mathrm{mod}(\theta) = 1$. Write $f_{ij} = \theta(1 \otimes e_{ij})$. Since $\mathrm{Tr}(f_{00}) = \mathrm{Tr}(1 \otimes e_{00}) = 1$, f_{00} and $1 \otimes e_{00}$ are equivalent projections in the factor N^∞, so that there exists a partial isometry $v \in N^\infty$ such that $vv^* = f_{00}$ and $v^*v = 1 \otimes e_{00}$. Define $u = \sum f_{j0} v(1 \otimes e_{0j})$. It is routine to check that $u \in \mathcal{U}(N^\infty)$ and $u(1 \otimes e_{ij})u^* = f_{ij}$, which finishes the proof of the claim.

Let $t \in \mathcal{F}(M)$ and choose $\theta_t \in \mathrm{Aut}(M^\infty)$ such that $\mathrm{mod}(\theta_t) = t$. Since $\theta_t \otimes \theta_{t^{-1}} \in \mathrm{Aut}((M\overline{\otimes}M)^\infty)$ has modulus 1, the claim yields a unique $\alpha_t = [\rho_t] \in \mathrm{Out}(M\overline{\otimes}M)$, so that the map

$$\mathcal{F}(M) \ni t \mapsto \alpha_t \in \mathrm{Out}(M\overline{\otimes}M) \tag{1}$$

is well-defined. If $s \neq t$, $\theta_s^{-1}\theta_t$ is outer, and so is $(\theta_s \otimes \theta_{s^{-1}})^{-1}(\theta_t \otimes \theta_{t^{-1}})$. Thus $\rho_s \neq \rho_t$. Since the map (1) is one-to-one, we are done. $\square$

4.2.2. *Connes' rigidity conjecture.* Connes in the late '70s conjectured the following: any countable icc property (T) groups Γ, Λ,

$$\Gamma \simeq \Lambda \Longleftrightarrow L(\Lambda) \simeq L(\Lambda).$$

Popa suggested the following strenghthening of Connes' rigidity conjecture:

Conjecture 4.8 (Popa, [22]). If Γ is an icc property (T) group and Λ is a group, then any $*$-isomorphism $\theta : L(\Gamma) \simeq L(\Lambda)^t$ forces $t = 1$ and there exist a group isomorphism $\rho : \Gamma \simeq \Lambda$ and a character $\chi \in \mathrm{Hom}(\Gamma, \mathbf{T})$ such that

$$\theta(\sum_{s\in\Gamma} a_s u_s) = \sum_{s\in\Gamma} \chi(s) a_s u_{\rho(s)}.$$

In particular,

$$\mathcal{F}(L(\Gamma)) = \{1\} \text{ and } \mathrm{Out}(L(\Gamma)) = \mathrm{Out}(\Gamma) \times \mathrm{Hom}(\Gamma, \mathbf{T}).$$

Popa observed in [20] that Ozawa's original result [18] could be used to prove Connes' rigidity conjecture up to "countable classes".

Theorem 4.9 (Ozawa, [18]). *For icc countable property (T) groups, the map $\Gamma \to L(\Gamma)$ is countable-to-one.*

Proof. The proof is an analog of the one of Theorem 2 in [18]. We prove the result by contradiction and assume that there are uncountably many pairwise non-isomorphic icc property (T) groups $(\Gamma_i)_{i\in I}$ which give the same II_1 factor M. By Shalom's result [25], we know that each property (T) group is the quotient of a finitely presented property (T) group. Since there are only countably many finitely presented groups, we may assume that all the Γ_i's are quotients of the same property (T) group Γ.

We regard $M \subset \mathbf{B}(L^2(M))$ represented in its standard form and $M = L(\Gamma_i)$ for every $i \in I$, so that $\Gamma_i \subset \mathcal{U}(M)$. Denote by $\pi_i : \Gamma \to \mathcal{U}(M)$ a group homomorphism such that $\pi_i(\Gamma) = \Gamma_i$. Fix $\varepsilon = 1/4$. Since Γ is a property (T) group, there exist $0 < \delta < 1$ and a finite subset $E \subset \Gamma$ such that for any unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ and any $(\pi(E), \delta)$-invariant unit vector $\xi \in \mathcal{H}$, there exists a $\pi(\Gamma)$-invariant vector $\eta \in \mathcal{H}$ such that $\|\eta - \xi\| \leq 1/4$.

Since the finite von Neumann $\ell^\infty(E, M)$ is $\|\cdot\|_2$-separable, there exist $i \neq j \in I$ such that $\max_{s\in E} \|\pi_i(s) - \pi_j(s)\|_2 \leq \delta$. Let J be the canonical antiunitary defined on $L^2(M)$ and define the unitary representation $\pi : \Gamma \to \mathcal{U}(M)$ by $\pi(s) = \pi_i(s)J\pi_j(s)J$. We have

$$\|\pi(s)\widehat{1} - \widehat{1}\|_{L^2(M)} = \|\pi_i(s)\pi_j(s)^* - 1\|_2 \leq \delta.$$

Since the vector $\widehat{1}$ is $(\pi(E), \delta)$-invariant, it follows that there exists a $\pi(\Gamma)$-invariant vector $\eta \in L^2(M)$ such that $\|\eta - \widehat{1}\|_{L^2(M)} \leq 1/4$. Thus, for every $s \in \Gamma$ we have

$$\begin{aligned}\|\pi_i(s) - \pi_j(s)\|_2 &= \|\pi(s)\widehat{1} - \widehat{1}\|_{L^2(M)} \\ &= \|\pi(s)(\widehat{1} - \eta) - (\widehat{1} - \eta)\|_{L^2(M)} \leq 1/2.\end{aligned}$$

Exactly as in the proof of Theorem 4.6, denote by a the unique element of minimum $\|\cdot\|_2$-norm in the weakly closed convex set $\mathcal{C} = \overline{\mathrm{co}}^w\{\pi_i(s)\pi_j(s)^* : s \in \Gamma\}$. Since $\|a - 1\|_2 \leq \delta < 1$, it follows that $a \neq 0$. Observe that $\pi_i(s)a\pi_j(s)^* \in \mathcal{C}$ and $\|\pi_i(s)a\pi_j(s)^*\|_2 = \|a\|_2$. By uniqueness, we have $\pi_i(s)a\pi_j(s)^* = a$. Moreover, $a^*a\pi_j(s) = \pi_j(s)a^*a$, for every $s \in \Gamma$. Since M is a factor and $\pi_j(\Gamma)'' = M$, we have $a^*a = \lambda \in \mathbf{R}^*_+$. Thus $v = a/\sqrt{\lambda} \in \mathcal{U}(M)$ and $\pi_i(s) = v\pi_j(s)v^*$, for every $s \in \Gamma$. It follows in particular that Γ_i and Γ_j are isomorphic, contradiction. □

4.3. **Uniqueness of Cartan subalgebras.**

Theorem 4.10 (Popa, [19]). *Let $\Gamma \curvearrowright B$ be a trace-preserving action of a countable group Γ with the Haagerup property on a finite von Neumann algebra B. Denote by $M = B \rtimes \Gamma$ the crossed product. Let $A \subset M$ be a rigid von Neumann subalgebra. Then $A \preceq_M B$.*

Proof. Since Γ has the Haagerup property, let $\varphi_n : \Gamma \to \mathbf{C}$ be a sequence of positive definite functions such that $\lim_n \varphi_n = 1$ pointwise and $\varphi_n \in C_0(\Gamma)$, for all $n \in \mathbf{N}$. Define $\theta_n : M \to M$ the sequence of τ-preserving ucp maps as follows:

$$\theta_n(\sum_{s\in\Gamma} a_s u_s) = \sum_{s\in\Gamma} \varphi_n(s) a_s u_s.$$

It is straightforward to check that every θ_n satisfies the following relative compactness property: if (w_k) is a sequence in $(M)_1$ which satisfies $\lim_k \|E_B(aw_kb)\|_2 = 0$, for all $a, b \in M$, then $\lim_k \|\theta_n(w_k)\|_2 = 0$. Indeed, write $w_k = \sum_{s\in\Gamma} (w_k)^s u_s$, where

$(w_k)^s = E_B(w_k u_s^*)$. Then we have $\theta_n(w_k) = \sum_{s\in\Gamma} \varphi_n(s)(w_k)^s u_s$ and thus

$$\|\theta_n(w_k)\|_2^2 = \sum_{s\in\Gamma} |\varphi_n(s)|^2 \|(w_k)^s\|_2^2.$$

Assume that $\lim_k \|(w_k)^s\|_2 = 0$, for all $s \in \Gamma$. Fix $\varepsilon > 0$. Since $\varphi_n \in c_0(\Gamma)$, $\mathcal{V} := \{s \in \Gamma : |\varphi_n(s)| \geq \varepsilon^2/2\}$ is a finite set. Then we have

$$\begin{aligned}\|\theta_n(w_k)\|_2^2 &= \sum_{s\in\Gamma-\mathcal{V}} |\varphi_n(s)|^2 \|(w_k)^s\|_2^2 + \sum_{s\in\mathcal{V}} |\varphi_n(s)|^2 \|(w_k)^s\|_2^2 \\ &\leq \varepsilon^2/2 \sum_{s\in\Gamma-\mathcal{V}} \|(w_k)^s\|_2^2 + \sum_{s\in\mathcal{V}} |\varphi_n(s)|^2 \|(w_k)^s\|_2^2 \\ &\leq \varepsilon^2/2 + \sum_{s\in\mathcal{V}} |\varphi_n(s)|^2 \|(w_k)^s\|_2^2.\end{aligned}$$

We can choose $k_0 \in \mathbf{N}$ large enough so that $\sum_{s\in\mathcal{V}} |\varphi_n(s)|^2 \|(w_k)^s\|_2^2 \leq \varepsilon^2/2$. Therefore, we have $\|\theta_n(w_k)\|_2 \leq \varepsilon$, for all $k \geq k_0$.

Since $A \subset M$ is rigid, there exists $n \in \mathbf{N}$ such that $\|\theta_n(w) - w\|_2 \leq 1/4$, for all $w \in \mathcal{U}(A)$. By contradiction, assume that $A \npreceq_M B$. Then there exists a sequence $w_k \in \mathcal{U}(A)$ such that $\lim_k \|E_B(a w_k b)\|_2 = 0$, for all $a, b \in M$. For $k \in \mathbf{N}$ large enough, we get

$$1 = \|w_k\|_2 \leq \|\theta_n(w_k) - w_k\|_2 + \|\theta_n(w_k)\|_2 \leq 1/4 + 1/4 = 1/2,$$

which is absurd. □

Corollary 4.11 (Popa, [19]). *Consider the linear action* $\mathrm{SL}_2(\mathbf{Z}) \curvearrowright \mathbf{T}^2$. *Then, up to unitary conjugacy,* $L^\infty(\mathbf{T}^2)$ *is the unique rigid Cartan subalgebra in* $L^\infty(\mathbf{T}^2) \rtimes \mathrm{SL}_2(\mathbf{Z})$.

Proof. Write $M = L^\infty(\mathbf{T}^2) \rtimes \mathrm{SL}_2(\mathbf{Z})$. Let $A \subset M$ be rigid Cartan subalgebra. We get $A \preceq_M L^\infty(\mathbf{T}^2)$ by the previous Theorem. Since $A, L^\infty(\mathbf{T}^2) \subset M$ are both Cartan subalgebras of the II_1 factor M, Theorem 2.14 yields $u \in \mathcal{U}(M)$ such that $uAu^* = L^\infty(\mathbf{T}^2)$. □

We can now apply this last result to compute explicitely the fundamental group of $L(\mathbf{Z}^2 \rtimes \mathrm{SL}_2(\mathbf{Z}))$. Gaboriau [13] showed that the group $\mathrm{SL}_2(\mathbf{Z})$ has *fixed price* and its *cost* equals 13/12. This means that for every free ergodic p.m.p. $\mathrm{SL}_2(\mathbf{Z}) \curvearrowright X$, the equivalence relation $\mathcal{R}(\mathrm{SL}_2(\mathbf{Z}) \curvearrowright X)$ has cost 13/12. It follows from the induction formula [13, Proposition II.6] that $\mathcal{R}(\mathrm{SL}_2(\mathbf{Z}) \curvearrowright X)$ has trivial fundamental group.

Corollary 4.12 (Popa, [19]). *We have* $\mathcal{F}(L(\mathbf{Z}^2 \rtimes \mathrm{SL}_2(\mathbf{Z}))) = \{1\}$.

Proof. Let $M = L(\mathbf{Z}^2 \rtimes \mathrm{SL}_2(\mathbf{Z})) = A \rtimes \mathrm{SL}_2(\mathbf{Z})$. Let $\mathcal{R}$ be the equivalence relation induced by the action $\mathrm{SL}_2(\mathbf{Z}) \curvearrowright \mathbf{T}^2$. Let $t \leq 1$ such that $M \simeq M^t$. We can assume that $p \in A$ is a projection of trace t so that $(pAp \subset pMp) \simeq (A^t \subset M^t)$. It follows that $A^t \subset M$ is a rigid Cartan subalgebra and thus there exists $u \in \mathcal{U}(M)$ such that $uA^t u^* = A$, by Corollary 4.11. This shows that $\mathcal{R} \simeq \mathcal{R}^t$ (see Theorem 1.23) and thus $t = 1$. □

Popa's result was the first explicit computation of a fundamental group of a II_1 factor that was different from $\mathbf{R}_+$, solving then a long-standing open problem of Kadison.

Appendix A. Polar decomposition of a vector

Let (N, τ) be a finite von Neumann algebra. Since τ is fixed, we simply denote $L^2(N, \tau)$ by $L^2(N)$. We regard $N \subset \mathbf{B}(L^2(N))$. Let $\xi \in L^2(N)$ such that $\xi \neq 0$. Let $T_\xi^0 : \widehat{N} \to L^2(N)$ be the linear operator defined by $T_\xi^0(\widehat{x}) = \xi x$, for all $x \in N$.

Proposition A.1. *The densily defined operator T_ξ^0 is closable. Denote by T_ξ its closure. The operator T_ξ is affiliated with N. Write $T_\xi = v|T_\xi|$ for its polar decomposition. Then $v \in N$ and $|T_\xi|$ is affiliated with N.*

Let $B \subset N$ be a von Neumann subalgebra such that $x\xi = \xi x$, for all $x \in B$. Then we have $xv = vx$, for all $x \in B$.

Proof. First, we prove that the operator T_ξ^0 is closable. It suffices to show that $(T_\xi^0)^*$ is densily defined. Let $y \in N$ and $z \in N$. Then,

$$\begin{aligned} \langle T_\xi^0(\widehat{y}), \widehat{z} \rangle &= \langle \xi y, \widehat{z} \rangle = \langle J y^* \xi, \widehat{z} \rangle \\ &= \langle z^* J y^* J \xi, \widehat{1} \rangle = \langle J y^* J z^* \xi, \widehat{1} \rangle \\ &= \langle \widehat{y}, J z^* \xi \rangle = \langle \widehat{y}, (J\xi) z \rangle = \langle \widehat{y}, T_{J\xi}^0(\widehat{z}) \rangle. \end{aligned}$$

Then $T_{J\xi}^0 \subset (T_\xi^0)^*$ and so $(T_\xi^0)^*$ is densily defined. Thus T_ξ^0 is closable and we denote by T_ξ its closure. We prove now that T_ξ is affiliated with N. Let $a, x \in N$. On the one hand,

$$T_\xi^0 J a^* J(\widehat{x}) = T_\xi^0(\widehat{xa}) = \xi x a.$$

On the other hand,

$$J a^* J T_\xi^0(\widehat{x}) = \xi x a.$$

Consequently, we have $Ja^*JT_\xi^0 \subset T_\xi^0 J a^* J$, for all $a \in N$. Since $JNJ = N'$, it follows that T_ξ is affiliated with N. Write $T_\xi = v|T_\xi|$ for the polar decomposition of T_ξ. Since v is bounded and affiliated with N, we have that $v \in N$. Moreover, $|T_\xi|$ is affiliated with N.

At last, let $B \subset N$ be a von Neumann subalgebra such that for any $x \in B$, $x\xi = \xi x$. Fix $x \in B$. It is straightforward to check that $xT_\xi \subset T_\xi x$. We also have $x(T_\xi)^* \subset (T_\xi)^* x$, and so $x(T_\xi)^* T_\xi \subset (T_\xi)^* T_\xi x$. By functional calculus, it follows that $x|T_\xi| \subset |T_\xi| x$. Moreover, since N is a finite von Neumann algebra, since $x \in N$ and $|T_\xi|$ is affiliated with N, it follows that $x|T_\xi|$ and $|T_\xi|x$ are closed, affiliated with N and consequently the equality $x|T_\xi| = |T_\xi|x$ holds. Thus,

$$xv|T_\xi| = xT_\xi \subset T_\xi x \subset v|T_\xi|x \subset vx|T_\xi|.$$

It follows that xv and vx coincide on the range of $|T_\xi|$, and so $xv = vx$. Thus, $xv = vx$, for every $x \in B$. $\square$

Appendix B. Von Neumann's dimension theory

Let (N, τ) be a finite von Neumann algebra with a distinguished faithful normal trace. Let $\mathcal{H}$ be a right Hilbert N-module, i.e. $\mathcal{H}$ is a complex (separable) Hilbert space together with a normal $*$-representation $\pi : N^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$. For any $b \in N$, and $\xi \in \mathcal{H}$, we shall simply write $\pi(b^{\mathrm{op}})\xi = \xi b$.

Proposition B.1. *Let $\mathcal{H}$ be a right N-module. Then there exists an isometry $v : \mathcal{H} \to \ell^2 \otimes L^2(N)$ such that $v(\xi b) = v(\xi) b$, for all $\xi \in \mathcal{H}$, $b \in N$.*

Proof. Let $\rho : N^{\text{op}} \to \mathbf{B}(\mathcal{H} \oplus (\ell^2 \otimes L^2(N)))$ be the $*$-representation defined by

$$\rho(y^{\text{op}}) = \begin{pmatrix} \pi(y^{\text{op}}) & 0 \\ 0 & 1_{\ell^2} \otimes y^{\text{op}} \end{pmatrix}.$$

Let $Q = \rho(N^{\text{op}})$. It is clear that Q is a finite von Neumann algebra and the projections

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix}$$

belong to $Q' \cap \mathbf{B}(\mathcal{H} \oplus (\ell^2 \otimes L^2(N)))$ which is a semifinite von Neumann algebra. Since q is infinite and p is finite, [26, Theorem V.1.8] yields a nonzero central projection $z \in \mathcal{Z}(Q')$ such that $zp \precsim zq$. There exists an isometry $w \in Q'$ such that $w^*w = zp$ and $ww^* \leq zq$. If we write

$$z = \begin{pmatrix} z_1 & 0 \\ 0 & 1 \otimes z_2 \end{pmatrix} \text{ and } w = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get

$$\begin{aligned} a^*a + c^*c &= z_1 \\ b^*b + d^*d &= 0 \\ aa^* + bb^* &= 0 \\ cc^* + dd^* &\leq 1 \otimes z_2. \end{aligned}$$

Thus $a = 0$, $b = 0$, $d = 0$, $c^*c = z_1$ and $v = c : \mathcal{H}z_1 \to \ell^2 \otimes L^2(N)$ is an isometry (since $u^*u = z_1$) which moreover satifies $v\pi(x) = (1 \otimes x)v$ (since $w \in Q'$). Then a simple maximality argument finishes the proof. □

Since $p = vv^*$ commutes with the right N-action on $\ell^2 \otimes L^2(N)$, it follows that $p \in \mathbf{B}(\ell^2)\overline{\otimes}N$. Thus, as right N-modules, we have

$$\mathcal{H}_N \simeq p(\ell^2 \otimes L^2(N))_N.$$

On $\mathbf{B}(\ell^2)\overline{\otimes}N$, we define the following faithful normal semifinite trace Tr (which depends on τ): for any $x = [x_{ij}]_{i,j} \in (\mathbf{B}(\ell^2)\overline{\otimes}N)_+$,

$$\text{Tr}\left([x_{ij}]_{i,j}\right) = \sum_i \tau(x_{ii}).$$

We set $\dim(\mathcal{H}_N) = \text{Tr}(vv^*)$. Note that the dimension of $\mathcal{H}$ depends on τ but does not depend on the isometry v. Indeed take another isometry $w : \mathcal{H} \to \ell^2 \otimes L^2(N)$, satisfying $w(\xi b) = w(\xi)b$, for any $\xi \in \mathcal{H}$, $b \in N$. Note that $vw^* \in \mathbf{B}(\ell^2)\overline{\otimes}N$ and $w^*w = v^*v = 1$. Thus, we have

$$\text{Tr}(vv^*) = \text{Tr}(vw^*wv^*) = \text{Tr}(wv^*vw^*) = \text{Tr}(ww^*).$$

We define $\dim(\mathcal{H}_N) := \text{Tr}(vv^*)$, for any isometry v as in Proposition B.1.

Appendix C. Ultraproducts

Let ω be a free ultrafilter on $\mathbf{N}$ and (N, τ) a finite von Neumann algebra. Let $\mathcal{I}_\omega$ be the norm closed ideal of $\ell^\infty(\mathbf{N}, N)$ defined by

$$\mathcal{I}_\omega = \left\{(x_n) \in \ell^\infty(\mathbf{N}, N) : \lim_{n \to \omega} \|x_n\|_2 = 0\right\}.$$

Denote by $\pi_\omega : \ell^\infty(\mathbf{N}, N) \to \ell^\infty(\mathbf{N}, N)/\mathcal{I}_\omega$ the quotient map. The tracial ultraproduct of N is defined as $N^\omega := \pi_\omega(\ell^\infty(\mathbf{N}, N))$ with tracial faithful state

$\tau_\omega(\pi_\omega((x_n))) = \lim_{n\to\omega} \tau(x_n)$. It is easy to check that N^ω is indeed a von Neumann algebra and τ_ω is normal on N^ω.

Exercise C.1. Let ω be a free ultrafilter and (N, τ) a finite von Neumann algebra.

- Prove that every projection $e \in N^\omega$ lifts to a projection $(e_n) \in \ell^\infty(\mathbf{N}, N)$ such that $\lim_{n\to\omega} \tau(e_n) = \tau_\omega(e)$.
- Show that if N is a II_1 factor, so is N^ω.

Regard $L^2(N^\omega)$ as a Hilbert subspace of $L^2(N)^\omega$. For $a \geq 0$, denote by f_a the characteristic function of the interval $(\sqrt{a}, +\infty)$.

Proposition C.2 ([7])**.** *Let $\xi = (\xi_n) \in L^2(N)^\omega$. Then $\xi \in L^2(N^\omega)$ if and only if ξ satisfies the following* equi-integrability *condition:*

$$\forall \varepsilon > 0, \exists a > 0, \lim_{n\to\omega} \|f_a(|\xi_n|)|\xi_n|\|_2 < \varepsilon. \tag{2}$$

Proof. Assume that $\xi \in L^2(N)^\omega$ satisfies (2). Simply write $\xi = (\xi_n)$. Fix $\varepsilon > 0$. Then for some $a > 0$, one has $\lim_{n\to\omega} \|f_a(|\xi_n|)\xi_n\|_2 < \varepsilon$ so that the vector $\eta = (\eta_n)$ with $\eta_n = \xi_n(1-f_a)(|\xi_n|)$ satisfies $\|\eta-\xi\|_2 = \lim_{n\to\omega} \|\eta_n-\xi_n\|_2 \leq \varepsilon$ and $\|\eta_n\|_\infty \leq a$. Thus, $\xi \in L^2(N^\omega)$.

Conversely assume that $\xi \in L^2(N^\omega)$. We may assume that $\|\xi_n\|_2 \leq 1$, for all $n \in \mathbf{N}$. Let $\varepsilon \in (0, 1)$. We can then find $a > 0$ and $x_n \in N$ such that $\|\xi_n - x_n\|_2 \leq \varepsilon$ and $\|x_n\|_\infty \leq a$, for all $n \in \mathbf{N}$. Up to extracting, Powers-Størmer Inequality yields in particular

$$\||\xi_n| - |x_n|\|_2^2 \leq \||\xi_n|^2 - |x_n|^2\|_1 \leq \|\xi_n - x_n\|_2\|\xi_n + x_n\|_2 \leq 3\varepsilon, \forall n \in \mathbf{N}.$$

Using the same trick as in the proof of Theorem 3.9, we get

$$\int_0^\infty \|f_b(|\xi_n|) - f_b(|x_n|)\|_2^2 \leq \||\xi_n|^2 - |x_n|^2\|_1 \leq 3\varepsilon.$$

Since $E_b(|x_n|) = 0$ for all $b > a$, we get $\|f_{2a}(|\xi_n|)\|_2 \leq (3\varepsilon)^{1/2}/a$, for all $n \in \mathbf{N}$. Then, for all $n \in \mathbf{N}$,

$$\begin{aligned}
\|f_{2a}(|\xi_n|)|\xi_n|\|_2 &\leq \|f_{2a}(|\xi_n|)(|\xi_n| - |x_n|)\|_2 + \|x_n\|_\infty\|f_{2a}(|\xi_n|)\|_2 \\
&\leq \||\xi_n| - |x_n|\|_2 + \|x_n\|_\infty\|f_{2a}(|\xi_n|)\|_2 \leq 2(3\varepsilon)^{1/2}.
\end{aligned}$$

This finishes the proof. $\square$

Appendix D. On the geometry of two projections

Proposition D.1 ([7])**.** *Let M be a finite von Neumann algebra. If e and f are equivalent projections in M, then there exists $u \in \mathcal{U}(M)$ such that*

$$\begin{aligned}
ueu^* &= f \\
u|e-f| &= |e-f|u \\
|u-1| &\leq \sqrt{2}|e-f|.
\end{aligned}$$

Proof. Recall the analysis of two projections in [26, Chapter V]. Let $e^\perp = 1 - e$ and $f^\perp = 1 - f$. Set

$$\begin{aligned}
e_0 &= e - e \wedge f - e \wedge f^\perp \\
f_0 &= f - f \wedge e - f \wedge e^\perp.
\end{aligned}$$

We know that $e_0 \wedge f_0 = 0$ and $e_0 - e_0 \wedge f_0 = e_0 \sim f_0$. We then represent

$$\begin{aligned} e &= e \wedge f \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus e \wedge f^{\perp} \oplus 0 \oplus 0, \\ f &= e \wedge f \oplus \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus 0 \oplus e^{\perp} \wedge f \oplus 0, \end{aligned}$$

with $c, s \in (e_0 \vee f_0)M(e_0 \vee f_0)$ positive elements such that $c^2 + s^2 = e_0 \vee f_0$. Since $e \sim f$ and $e_0 \sim f_0$, the finiteness of M yields $e \wedge f^{\perp} \sim e^{\perp} \wedge f$. Consequently, $e \wedge f^{\perp}$ and $e^{\perp} \wedge f$ are represented by matrices:

$$\begin{aligned} e \wedge f^{\perp} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ e^{\perp} \wedge f &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore we come to the following situation:

$$\begin{aligned} e &= e \wedge f \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0, \\ f &= e \wedge f \oplus \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus 0. \end{aligned}$$

and

$$|e - f| = 0 \oplus \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus 0.$$

We set

$$u = e \wedge f \oplus \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus (e \vee f)^{\perp}.$$

It is straightforward to check that u is a unitary, $ueu^* = f$, $u|e - f| = |e - f|u$ and that

$$(u - 1)^*(u - 1) \leq 2(e - f)^2.$$

$\square$

References

[1] B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's property (T).* New Mathematical Monographs, **11**. Cambridge University Press, Cambridge, 2008. xiv+472 pp.

[2] N. Brown, N. Ozawa, *C*-algebras and finite-dimensional approximations.* Grad. Stud. Math., **88**. Amer. Math. Soc., Providence, RI, (2008).

[3] P.A. Cherix, M. Cowling, P. Jolissaint, P. Julg, A. Valette, *Groups with the Haagerup Property.* Progress in Mathematics **197**. Birkhäuser Verlag, Basel, Boston, Berlin, 2001.

[4] M. Choda, *Group factors of the Haagerup type.* Proc. Japan Acad. **59** (1983), 174–177.

[5] A. Connes, *Noncommutative geometry.* Academic Press. San Diego, California, 1994.

[6] A. Connes, *A factor of type* II_1 *with countable fundamental group* J. Operator Theory **4** (1980), no. 1, 151–153.

[7] A. Connes, *Classification of injective factors.* Ann. of Math. **104** (1976), 73–115.

[8] A. Connes, *Almost periodic states and factors of type* III_1. J. Funct. Anal. **16** (1974), 415–445.

[9] A. Connes, V.F.R. Jones, *Property (T) for von Neumann algebras.* Bull. London Math. Soc. **17** (1985), 57–62.

[10] E.G. Effros, *Property* Γ *and inner amenability.* Proc. Amer. Math. Soc. **47** (1975), 483–486.

[11] J. Feldman, C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras.* I. Trans. Amer. Math. Soc. **234** (1977), 289–324.

[12] J. Feldman, C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras.* II. Trans. Amer. Math. Soc. **234** (1977), 325–359.

[13] D. Gaboriau, *Coût des relations d'équivalence et des groupes.* Invent. Math. **139** (2000), 41–98.

[14] U. Haagerup, *An example of non-nuclear C^*-algebra which has the metric approximation property.* Invent. Math. **50** (1979), 279–293.

[15] P. Jolissaint, *On property (T) for pairs of topological groups.* Ens. Math. (2) **51** (2005), 31–45.

[16] D. Kazhdan, *Connection of the dual space of a group with the structure of its subgroups.* Funct. Anal. Appl. **1** (1967), 63–65.

[17] G. Margulis, *Finitely-additive invariant measures on Euclidean spaces.* Ergodic Th. and Dynam. Sys. **2** (1982), 383–396.

[18] N. Ozawa, *There is no separable universal* II_1*-factor.* Proc. Amer. Math. Soc., **132** (2004), 487–490.

[19] S. Popa, *On a class of type* II_1 *factors with Betti numbers invariants.* Ann. of Math. **163** (2006), 809–899.

[20] S. Popa, *Deformation and rigidity for group actions and von Neumann algebras.* International Congress of Mathematicians. Vol. I, 445–477, Eur. Math. Soc., Zurich, 2007.

[21] S. Popa, *On Ozawa's property for free group factors.* Int. Math. Res. Notices. Vol. 2007 : article ID rnm036, 10 pages.

[22] S. Popa, *Some results and problems in W^*-rigidity.* Available at http://www.math.ucla.edu/~popa/tamu0809rev.pdf.

[23] S. Popa, *Strong rigidity of* II_1 *factors arising from malleable actions of w-rigid groups* I. Invent. Math. **165** (2006), 369-408.

[24] S. Popa, *Some rigidity results for non-commutative Bernoulli Shifts.* J. Funct. Anal. **230** (2006), 273–328.

[25] Y. Shalom, *Rigidity of commensurators and irreducible lattices.* Invent. Math. **141** (2000), 1–54.

[26] M. Takesaki, *Theory of Operator Algebras* I. EMS **124**. Springer-Verlag, Berlin, Heidelberg, New-York, 2000.

[27] M. Takesaki, *Theory of Operator Algebras* II. EMS **125**. Springer-Verlag, Berlin, Heidelberg, New-York, 2000.

[28] M. Takesaki, *Theory of Operator Algebras* III. EMS **127**. Springer-Verlag, Berlin, Heidelberg, New-York, 2003.

[29] S. Vaes, *Rigidity results for Bernoulli actions and their von Neumann algebras (after S. Popa).* Séminaire Bourbaki, exposé 961. Astérisque **311** (2007), 237-294.

[30] S. Vaes, *An inner amenable group whose von Neumann algebra does not have property Gamma.* Acta Math., to appear. `arXiv:0909.1485`

CNRS-ENS Lyon, UMPA UMR 5669, 69364 Lyon cedex 7, France
E-mail address: `cyril.houdayer@umpa.ens-lyon.fr`