Shalom: Harmonic functions on groups and Gromov's polynomial growth theorem

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Disclaimer

These notes are full of typos and mistakes as they are made available as soon as possible and depend on the author's comprehension of the talk... We still hope they can be of some interest for the people who follow the lectures.

Growth

Let G be a finitely generated group $G = \langle S \rangle$ where S is finite. We can then define the growth:

$$a_{G,S}(n) = |g \in G : \|g\|_S \leqslant n|$$

It is easy to see that the growth type of this function does not depend on S. So the property $\exists K$ such that $a_{G,S}(n) \leqslant Kn^d$ does not depend on S and then we say that G has **polynomial growth**.

Example. : $G = \mathbb{Z}^d$, then $a_{G,S}(n) \leqslant Kn^d$. More generally, nilpotent groups have polynomial growth and there is an exact formula for d in terms on the structure of G. For instance, G is the Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

we have $d(G) = 4$

Nilpotent: consider sequences $G^{i+1} = [G, G^i]$ and $G^0 = G$; nilpotent means that $G^i = \{e\}$ for some i. Other inductive definition: if $Z < Z(G)$ and G/Z is nilpotent then G is nilpotent.

Fact. Passing to finite index subgroup does not change the growth type. So if G is almost (virtually) nilpotent, then G has polynomial growth.

Theorem 1 (Gromov). *The converse is also true.*

Gromov's proof depended on the Montgomery-Zippin work on Hilbert's fifth problem. In 2007, Kleiner replaced all of the Montgomery-Zippin argument by results on harmonic functions on groups with polynomial growth. In 2009, Shalom-Tao proved a quantitative finitery version of Gromov's theorem.

Theorem 2 (Shalom-Tao)**.** *There exists an absolute (explicit) constant C such that for every finitely generated group G, $d > 0$, the following holds: if there exists some $R_0 > \exp(\exp(Cd^C))$, such that for some S finite generating G, $a_{(G,S)}(R_0) < R_0^d$, then G is almost nilpotent; , and even almost polycyclic with an explicit bound to the index of the finite index polycyclic subgroup.*

The proof uses simplified steps of Montgomery-Zippin, so we are going to see an elementary proof of Gromov's theorem. today, we do the first reduction:

Theorem 3. *If G infinite has polynomial growth, then there is a finite index subgroup $G_0 < G$ and $\phi : G_0 \twoheadrightarrow \mathbb{Z}$ and $\operatorname{Ker}\phi$ is finitely generated.*

Proposition 4. *This theorem implies Gromov's theorem.*

Proof. We will prove it by induction on $\lfloor d \rfloor$ that $a_{(G,S)}(n) \leqslant K \cdot n^d$ implies that G has a finite index subgroup which is nilpotent and torsion free.

Base: $\lfloor d \rfloor = 0$, then G is finite (exercise; $d < 1$!).

Induction: assume $a_{(G,S)}(n) \leqslant Kn^d$. By theorem 3 there exists $G_0 < G$ of finite index and $\rho : G_0 \twoheadrightarrow \mathbb{Z}$. Let H be the kernel. We have $1 \to H \to G_0 \to \mathbb{Z} \to 1$. Note that H has polynomial growth of degree $\leqslant d - 1$. This follows from

Lemma 1. Suppose we have $1 \to B \to G \to A \to 1$ where B generate by F and A by S_A, the we can lift S'_A such taht G is generated by $F \cup S'_A$, and then

$$a_{(B,F)}(n) \cdot a_{(A,S_A)}(n) \leqslant a_{(G,F \cup S_A)}(2n)$$

So by induction we get that there is $H_0 < H$ of finite index nilpotent torsion free. We can replace it by H_1 normal in H of finite index. Let $K = [H : H_1]$. Then for all $h \in H$, $h^k \in H_1$ by Lagrange's theorem. Let

$$L = \bigcup_{h \in H} \langle h^k \rangle < H.$$

Clearly $L < H_1$ is a characteristic subgroup. Note that it has finite index in H_1 because H_1/L is a finitely generated nilpotent group which has torsion so is finite.

Let $g_0 \in G_0$, such taht $\phi(g_0) = 1 \in \mathbb{Z}$, ie $G_0 = \langle g_0, H \rangle$ so $\langle G_0, L \rangle$ has finite index in G_0. We will show this is almost nilpotent.

L is a torsion free finitely generated nilpotent group, and all G^i are finitely generated as well (in the nilpotent sequence). So $\{0\} \triangleleft L^m \cdots \triangleleft L^1 \triangleleft L$, and L^m is finitely generated abelian torsion free so it is a $\mathbb{Z}^r$. Now it is normalised by g_0 so the conjugation action of g_0 is an automorphism of the abelian group $\mathbb{Z}^r$, so represented by $A \in Gl_r(\mathbb{Z})$.

Claim. Every eigenvalue λ of a power of A has modulus one. Indeed otherwise for some k $B = A^k$ has an eigenvalue $|\lambda| > 2$. [This will imply exponential growth] λ is also an eigenvalue of B^t, and let $v \in \mathbb{C}^r$ such that $B^t v = \lambda v$ where $|\lambda| > 2$. Take some $u \in \mathbb{Z}^r$ such that $\langle u, v \rangle \neq 0$. Now look at all the vectors of the form $\sum_{i=1}^M \alpha_i B^i u$, $\alpha_i = 0, 1$. Notice that all are different, because if not $\beta_i = \pm 1, 0$, and $\langle \sum_{i=0}^M \beta_i B^i u, v \rangle = \sum \beta_i \lambda^i \langle u, v \rangle \neq 0$. This translates back to something inside $\mathbb{Z}^r = L^m$, which gives exponential growth, a contradiction.

Claim. Every eigenvalue λ of A^k is a root of unity. Indeed, all A^k are integer matrices of r dimension; their characteristic polynomials are

$$\chi_{A^k}(t) = (t - \lambda_1) \cdots (t - \lambda_r)$$

So $\chi_{A^k}(t) = \sum_{i=0}^{r} a_i t^i$ where $|a_i| < 2^r$, so there are finitely many λ^i, so some $i \neq j$ such that $\lambda^i = \lambda^j$ and it is thus a root of one.

Thus some high power of g_0 has eigenvalue one for its conjugation action on L, and we can find a rational eigenvector, so an integer one: we have $g_0 w g_0^{-1} = w$ for some $w \in L^m$, it has infinite order and is in the center; we can divide by it and get something again of polynomial growth lesser than $d-1$, but this time w it is in the center of infinite order so we get finite index nilpotent... $\square$