INTEGRAL p-ADIC ÉTALE COHOMOLOGY OF DRINFELD SYMMETRIC SPACES

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Abstract. We compute the integral p-adic étale cohomology of Drinfeld symmetric spaces of any dimension. This refines the computation of the rational p-adic étale cohomology from [10]. The main tools are: the computation of the integral de Rham cohomology from [10] and the integral p-adic comparison theorems of Bhatt-Morrow-Scholze and Česnavičius-Koshikawa which replace the quasi-integral comparison theorem of Tsuji used in [10]. Along the way we compute A_{inf}-cohomology of Drinfeld symmetric spaces.

Contents

1. Introduction

Let p be a prime number, K a finite extension of $\mathbf{Q}_p$, and C the p-adic completion of an algebraic closure $\overline{K}$ of K. Drinfeld's symmetric space

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of dimension d over K is the rigid analytic variety

$$\mathbb{H}^d_K := \mathbb{P}^d_K \setminus \cup_{H\in\mathscr{H}} H,$$

where $\mathscr{H}$ is the space of K-rational hyperplanes in K^{d+1}. It is equipped with an action of $G = \mathrm{GL}_{d+1}(K)$. One of the main results of [10] is the description of the $G\times\mathscr{G}_K$-modules $H^i_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Q}_p(i))$, where $\mathbb{H}^d_C := \mathbb{H}^d_K \otimes_K C$ and $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$. The analogous result for ℓ-adic étale cohomology, $\ell \neq p$, is a classical result of Schneider and Stuhler [21]. It relies on the fact that ℓ-adic étale cohomology satisfies a homotopy property with respect to the open ball (a fact that is false for p-adic étale cohomology).

The goal of this paper is to refine our result, by describing the integral p-adic étale cohomology groups $H^i_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Z}_p(i))$. Recall that, for $i \geq 0$, there is a natural generalized Steinberg representation $\mathrm{Sp}_i(\mathbf{Z}_p)$ of G (see Section 4.1 for the precise definition). We endow it with the trivial action of $\mathscr{G}_K$ and we write $\mathrm{Sp}_i(\mathbf{Z}_p)^*$ for its $\mathbf{Z}_p$-dual.

The main result of this paper is the following:

Theorem 1.1. *For $i \geq 0$, there are compatible topological isomorphisms of $G\times\mathscr{G}_K$-modules*

$$H^i_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Z}_p(i)) \simeq \mathrm{Sp}_i(\mathbf{Z}_p)^*, \quad H^i_{\text{ét}}(\mathbb{H}^d_C, \mathbf{F}_p(i)) \simeq \mathrm{Sp}_i(\mathbf{F}_p)^*,$$

compatible with the isomorphism $H^i_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Q}_p(i)) \simeq \mathrm{Sp}_i(\mathbf{Z}_p)^* \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ from [10]. *In particular, for $i > d$ these cohomology groups are trivial.*

Remark 1.2. (i) If $d = 1$ and $i = 1$, this result is due to Drinfeld [11] (with a shaky proof corrected in [13]; see also [9, Th. 1.7]).

(ii) In [8], the étale cohomology with compact support of many p-adic period domains is computed. The methods are very different from the ones used in the present paper: they avoid p-adic Hodge theory, follow Orlik's ℓ-adic computations [20], for $\ell \neq p$, and use computations of Ext groups between mod p representations of groups like G. This includes the case of $\mathbb{H}^d_C$ for which the result reads as follows:

$$H^i_{\text{ét},c}(\mathbb{H}^d_C, \mathbf{Z}/p^n) \simeq \mathrm{Sp}_{2d-i}(\mathbf{Z}/p^n)(d-i), \quad p \geq 5,$$

as $G\times\mathscr{G}_K$-modules.

(iii) Hence étale cohomology and étale cohomology with compact support of $\mathbb{H}^d_C$ are in abstract duality

$$H^i_{\text{ét},c}(\mathbb{H}^d_C, \mathbf{Z}/p^n) \simeq H^{2d-i}_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Z}/p^n)^*(-d)$$

as $G\times\mathscr{G}_K$-modules. However, there is no known Poincaré duality for spaces like $\mathbb{H}^d_C$ that would explain this abstract duality and would allow to recover the results of this paper from the computation of the cohomology with compact support. In fact, computations with some basic analytic spaces like the unit ball show that there is no naive Poincaré duality for general p-adic rigid analytic spaces.

Étale cohomology and A_{inf}-cohomology. We will describe now the key ideas and difficulties occurring in the proof of Theorem 1.1. As in [10, Sec. 5.1, Sec. 6.2], a key input is the pro-ordinarity of the standard semistable formal model $\mathfrak{X}_{\mathscr{O}_K}$ of $\mathbb{H}^d_K$, a result due to Grosse-Klönne [14]. More precisely, he proved that

$$H^i(\mathfrak{X}_{\mathscr{O}_K}, \Omega^j_{\mathfrak{X}_{\mathscr{O}_K}}) = 0, \quad i \geq 1, j \geq 0, \tag{1.3}$$

where $\Omega^\bullet_{\mathfrak{X}_{\mathscr{O}_K}}$ is the logarithmic de Rham complex of $\mathfrak{X}_{\mathscr{O}_K}$ over $\mathscr{O}_K$ (for the canonical log-structures of $\mathfrak{X}_{\mathscr{O}_K}$ and $\mathscr{O}_K$). One easily infers from this that $\mathfrak{X}_{\mathscr{O}_K}$ is ordinary in the usual sense [10, Sec. 6.2]. The strongest (and easiest) integral p-adic comparison theorems are available for ordinary varieties, making it natural to try to adapt them to $\mathfrak{X}_{\mathscr{O}_K}$. Nevertheless, the fact that $\mathfrak{X}_{\mathscr{O}_K}$ is not quasi-compact seems to be a serious obstacle in implementing the usual strategy [3, Ch. 7] to our setup. The syntomic method, suitably adapted [10], works well only up to some absolute constants, and reduces the computation of $H^i_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Q}_p(i))$ to that of the (integral) Hyodo-Kato cohomology of the special fiber of $\mathfrak{X}_{\mathscr{O}_K}$, which was done in [10]. The latter computation can be done integrally and also shows that the de Rham cohomology of $\mathfrak{X}_{\mathscr{O}_K}$ is p-torsion-free.

The results of Bhatt-Morrow-Scholze [5] (adapted to the semistable reduction setting by Česnavičius-Koshikawa [7]) show that, for proper rigid analytic varieties with semistable reduction, if the de Rham cohomology of the semistable integral model is p-torsion free (equivalently, if the integral Hyodo-Kato cohomology of the special fiber is p-torsion free) so is the p-adic étale cohomology of the generic fiber. Combined with [10] and with the rigidity of G-invariant lattices in $\mathrm{Sp}_i(\mathbf{Q}_p)$ (a result due to Grosse-Klönne [17]), this would yield our main result. The problem is that the proofs in [5] and [7] rely on the properness of the varieties and it is not clear how to adapt them to our context. However, the key actor in *loc. cit.* makes perfect sense: the A_{inf}-cohomology. One then needs a way to read the p-adic étale cohomology in terms of the A_{inf}-cohomology, which can be done even for non quasi-compact varieties thanks to a remarkable (especially due to its simplicity!) formula in [6] (the way p-adic étale cohomology and A_{inf}-cohomology are related in [5] is rather different and does not seem to be very useful in our case).

This reduces the proof of our main theorem to the computation of the A_{inf}-cohomology.

More precisely, let $A_{\mathrm{inf}} = W(\mathscr{O}_C^\flat)$ be Fontaine's ring associated to C. The choice of a compatible system of primitive p-power roots of unity $(\zeta_{p^n})_n$ gives rise to an element $\mu = [\varepsilon] - 1 \in A_{\mathrm{inf}}$ (where ε corresponds to $(\zeta_{p^n})_n$ under the identification $\mathscr{O}_C^\flat = \varprojlim_{x\mapsto x^p} \mathscr{O}_C$). This, in turn, induces a modified Tate twist $M \to M\{i\} := M \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}\{i\}$, $i \geq 0$, on the category of A_{inf}-modules, where $A_{\mathrm{inf}}\{1\} := \frac{1}{\mu} A_{\mathrm{inf}}(1) \subset W(C^\flat)(1)$, $A_{\mathrm{inf}}\{i\} := A_{\mathrm{inf}}\{1\}^{\otimes i}$. Let $X = \mathbb{H}^d_C$ and $\mathfrak{X} = \mathfrak{X}_{\mathscr{O}_K} \widehat{\otimes}_{\mathscr{O}_K} \mathscr{O}_C$. Using the projection from the pro-étale site of X to the étale site of $\mathfrak{X}$ and a relative version of Fontaine's construction of the ring A_{inf}, one constructs in [5], [7] a complex of sheaves of A_{inf}-modules $A\Omega_{\mathfrak{X}}$ on the étale site of $\mathfrak{X}$, which allows one to interpolate between étale, crystalline, and de Rham cohomology of X and $\mathfrak{X}$.

The technical result we prove is then:

Theorem 1.4. *For $i \geq 0$, there is a topological φ^{-1}-equivariant isomorphism of $G \times \mathscr{G}_K$-modules*

$$H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}) \simeq A_{\mathrm{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \mathrm{Sp}_i(\mathbf{Z}_p)^*.$$

Theorem 1.1 is now obtained from this and the description of p-adic nearby cycles in [6] in terms of $A\Omega_{\mathfrak{X}}$ (a twisted version of the Artin-Schreier exact sequence): an exact sequence

(1.5)

$$0 \to \frac{H^{i-1}_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})}{(1-\varphi^{-1})} \to H^i_{\mathrm{\acute{e}t}}(X, \mathbf{Z}_p(i)) \to H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1} \to 0.$$

Proof of Theorem 1.4. We end the introduction by briefly explaining the key steps in the proof of Theorem 1.4. Fix $i \geq 0$ and write for simplicity $M = H^i(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$. This is an A_{inf}-module, which is derived $\tilde{\xi}$-complete, for $\tilde{\xi} = \varphi(\mu)/\mu$.

In the first step, we interpret (following Schneider-Stuhler [21] and Iovita-Spiess [18]) $\mathrm{Sp}_i(\mathbf{Z}_p)^*$ as a suitable quotient of the space of $\mathbf{Z}_p$-valued measures on $\mathscr{H}^{i+1}$ (recall that $\mathscr{H}$ is the space of K-rational hyperplanes in K^{d+1}). This allows us to construct an étale regulator (an "integration of étale symbols") map

$$r_{\mathrm{\acute{e}t}} : \mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^i_{\mathrm{\acute{e}t}}(X, \mathbf{Z}_p(i))$$

which induces a regulator map

$$r_{\mathrm{inf}} : A_{\mathrm{inf}} \widehat{\otimes}_{\mathbf{Z}_p} \mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}). \tag{1.6}$$

To prove that r_{inf} is an isomorphism we use the derived Nakayama Lemma: since both sides of (1.6) are derived $\tilde{\xi}$-complete it suffices to show that $r_{\rm inf}$ is a quasi-isomorphism when reduced modulo $\tilde{\xi}$ (in the derived sense). That is, that the morphism

$$\begin{aligned} & r_{\rm inf} \otimes^{\rm L} \mathrm{Id}_{A_{\rm inf}/\tilde{\xi}} : \\ & \qquad (A_{\rm inf} \widehat{\otimes}_{\mathbf{Z}_p} \mathrm{Sp}_i(\mathbf{Z}_p)^*) \otimes^{\rm L}_{A_{\rm inf}} (A_{\rm inf}/\tilde{\xi}) \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}) \otimes^{\rm L}_{A_{\rm inf}} (A_{\rm inf}/\tilde{\xi}) \end{aligned}$$

is a quasi-isomorphism. To compute the naive reduction $\overline{r}_{\rm inf}$ modulo $\tilde{\xi}$ of (1.6) we use the Hodge–Tate specialization of $A\Omega_{\mathfrak{X}}$, which identifies $H^i(A\Omega_{\mathfrak{X}}/\tilde{\xi})$ with the (twisted) sheaf of i'th logarithmic differential forms on $\mathfrak{X}$. And, globally, those are well controlled by the acyclicity result (1.3). Combined with a compatibility between the étale and the Hodge–Tate Chern class maps and the Hodge–Tate specialization this implies that $\overline{r}_{\rm inf}$ is isomorphic to the Hodge–Tate regulator

$$r_{\rm HT} : \mathscr{O}_C \widehat{\otimes}_{\mathbf{Z}_p} \mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^0_{\text{ét}}(\mathfrak{X}, \Omega^i_{\mathfrak{X}}).$$

And this we have shown to be an isomorphism in [10].

Along the way we also compute that the target $H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$ of $r_{\rm inf}$ is $\tilde{\xi}$-torsion free. Since the domain $A_{\rm inf} \widehat{\otimes}_{\mathbf{Z}_p} \mathrm{Sp}_i(\mathbf{Z}_p)^*$ of $r_{\rm inf}$ is also $\tilde{\xi}$-torsion free this shows that $r_{\rm inf} \otimes^{\rm L} \mathrm{Id}_{A_{\rm inf}/\tilde{\xi}} \simeq \overline{r}_{\rm inf}$ and hence, by the above, it is a quasi-isomorphism, as wanted.

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Notation and conventions. Throughout the paper p is a fixed prime. The field K is a finite extension of $\mathbf{Q}_p$ with the ring of integers $\mathscr{O}_K$ and the residue field k; the field C is the p-adic completion of an algebraic closure $\overline{K}$ of K.

All formal schemes are p-adic and locally of finite type (over specified bases). A formal scheme over $\mathscr{O}_K$ is called *semistable* if, locally for the Zariski topology, it admits étale maps to the formal spectrum $\mathrm{Spf}(\mathscr{O}_K\{X_1, \ldots, X_n\}/(X_1 \cdots X_r - \varpi))$, $1 \leq r \leq n$, where ϖ is a uniformizer of K. We equip it with the log-structure coming from the special fiber.

If A is a ring and $f \in A$ is a non zero-divisor and $T \in D(A)$, we will often write T/f for $T \otimes^{\rm L}_A A/f$ if there is no confusion.

2. Preliminaries

2.1. **Derived completions and the décalage functor.**

2.1.1. *Derived completions.* We will need the following derived version of completeness

Definition 2.1. ([24, 091S]) Let I be a finitely generated ideal of a ring A. We say that $M \in D(A)$ is *derived I-complete* if, for all $f \in I$, we have

$$\mathrm{holim}(\cdots \to M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} M) = 0.$$

Let A be a ring and let $I \subset A$ be a finitely generated ideal. We list the following basic properties of derived completeness [24, 091N]:

(1) Let M be an A-module. If M is classically I-complete, i.e., the map $M \to \varprojlim_n M/I^n M$ is an isomorphism, then M is also derived I-complete [24, 091R]; the converse is true if M is I-adically separated [24, 091T]
(2) (a) The collection of all derived I-complete A-complexes forms a full triangulated subcategory $D_{\mathrm{comp}}(A, I)$ of $D(A)$ [24, 091U].
    (b) The inclusion functor $D_{\mathrm{comp}}(A, I) \to D(A)$ has a left adjoint, i.e., given an object M of $D(A)$ there exists a map $M \mapsto \widehat{M}$ of M into a derived complete object of $D(A)$ such that the map $\mathrm{Hom}_{D(A)}(\widehat{M}, E) \to \mathrm{Hom}_{D(A)}(M, E)$ is bijective whenever E is a derived complete object of $D(A)$ [24, 091V]. The object $\widehat{M}$ is called the *derived I-completion*[1] *of M*.
(3) $M \in D(A)$ is derived I-complete if and only if so are its cohomology groups $H^i(M)$, $i \in \mathbf{Z}$ [24, 091P].
(4) (*Derived Nakayama Lemma*) A derived I-complete complex $M \in D(A)$ is 0 if and only if $M \otimes^{\mathrm{L}}_A A/I \simeq 0$ [24, 0G1U].
(5) If I is generated by $x_1, ..., x_n \in A$, then $M \in D(A)$ is derived I-complete if and only if M is derived (x_i)-complete for $1 \leq i \leq n$ [24, 091V].
(6) If f is a morphism of ringed topoi, then the functor $\mathrm{R}f_*$ commutes with derived completions [24, 0A0G].

2.1.2. *The Berthelot-Deligne-Ogus décalage functor.* For any ring A and any non zero-divisor $f \in A$ there is a functor $\mathrm{L}\eta_f : D(A) \to D(A)$ (which

[1] The terminology here is misleading. In general, the derived I-completion is not given by $M \mapsto \mathrm{holim}_n (M \otimes^{\mathrm{L}}_A A/I^n)$, as one would naturally guess.

in general is **not** exact) with the key property [5, Lemma 6.4] that there is a functorial isomorphism[2]

$$H^i(\mathrm{L}\eta_f(T)) \simeq H^i(T)/(H^i(T)[f]),$$

where $M[f] := \{x \in M |\, fx = 0\}$. Concretely, choose a representative $T^\bullet$ of $T \in D(A)$ such that $T^i[f] = 0$ for all i, and consider the sub-complex $\eta_f(T^\bullet) \subset T^\bullet[1/f]$ defined by

$$\eta_f(T^\bullet)^i = \{x \in f^iT^i |\, dx \in f^{i+1}T^{i+1}\}.$$

Up to a canonical isomorphism, its image $\mathrm{L}\eta_f(T)$ in $D(A)$ depends only on T.

We list the following properties of the above construction (sometimes extended to ringed topoi; in this case one needs to be careful when dealing with derived completions and assume that the involved topoi are replete):

(1) $\mathrm{L}\eta_f$ commutes with truncations [5, Cor. 6.5] and with restriction of scalars[3] [5, Lemma 6.14]. Moreover,

$$\mathrm{L}\eta_f(\mathrm{L}\eta_g(T)) \simeq \mathrm{L}\eta_{fg}(T)$$

for $f, g \in A$ non zero-divisors and $T \in D(A)$ [5, Lemma 6.11].

(2) For all $T \in D(A)$, we have $\mathrm{L}\eta_f(T)[1/f] \simeq T[1/f]$ and there is a canonical quasi-isomorphism

$$\mathrm{L}\eta_f(T)/f = \mathrm{L}\eta_f(T) \otimes^{\mathrm{L}}_A A/f \simeq (H^*(T/f), \beta_f),$$

where $(H^*(T/f), \beta_f)$ is the Bockstein complex equal to $H^i(T \otimes^{\mathrm{L}}_A (f^iA/f^{i+1}A))$ in degree i, the differential being the boundary map associated to the triangle

$$T \otimes^{\mathrm{L}}_A (f^{i+1}A/f^{i+2}A) \to T \otimes^{\mathrm{L}}_A (f^iA/f^{i+2}A) \to T \otimes^{\mathrm{L}}_A (f^iA/f^{i+1}A).$$

This is discussed in [5, Chapter 6] and [4, Lemma 5.9].

(3) (a) If $T \to L \to M$ is a distinguished triangle in $D(A)$, then $\mathrm{L}\eta_f(T) \to \mathrm{L}\eta_f(L) \to \mathrm{L}\eta_f(M)$ is also a distinguished triangle **if** the boundary map $H^i(M/f) \to H^{i+1}(T/f)$ is the 0 map for all i [4, 5.14].

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[2] Depending on f, not only on the ideal fA. If we want to avoid this, the "correct" isomorphism is

$$H^i(\mathrm{L}\eta_f(T)) \simeq (H^i(T)/H^i(T)[f]) \otimes_A (f^i),$$

where $(f^i) \subset A[1/f]$ is the fractional A-ideal generated by f^i.

[3] The latter means that $\alpha_*(\mathrm{L}\eta_{\alpha(f)}(M)) \simeq \mathrm{L}\eta_f(\alpha_* M)$ for $M \in D(B)$ and $\alpha : A \to B$ a map of rings such that $\alpha(f) \in B$ is a non zero-divisor.

(b) For a non zero-divisor $g \in A$ and a $T \in D(A)$, the natural map $\mathrm{L}\eta_f(T)/g \to \mathrm{L}\eta_f(T/g)$ is a quasi-isomorphism **if** $H^*(T/f)$ has no g-torsion [4, 5.16].

(4) If $I \subset A$ is a finitely generated ideal in a ring A and if $T \in D(A)$ is derived I-complete, then so is $\mathrm{L}\eta_f(T)$ [4, Lemma 5.19]. Let T be a replete topos and let $\mathscr{I} \subset \mathscr{O}_T$ be an invertible ideal sheaf. If $K \in D(\mathscr{O}_T)$ is derived $\mathscr{I}$-complete, then so is $\mathrm{L}\eta_{\mathscr{I}}(K)$ [5, Lemma 6.19].

(5) If $T \in D^{[0,d]}(A)$ and $H^0(T)$ is f-torsion-free then there are natural maps $\mathrm{L}\eta_f(T) \to T$ and $T \to \mathrm{L}\eta_f(T)$ whose compositions are f^d. More precisely, if $T^\bullet$ is a representative concentrated in degrees $0, \dots, d$ and with f-torsion-free terms, then the first map is induced by $\eta_f(T^\bullet) \subset T^\bullet$. Multiplication by f^d on each of the two complexes factors over this inclusion map. When $T \in D^{\geq 0}(A)$, we will refer to the map $\mathrm{L}\eta_f(T) \to T$ as the *canonical map.*

2.2. The complexes $A\Omega_{\mathfrak{X}}$ and $\widetilde{\Omega}_{\mathfrak{X}}$.

2.2.1. *Fontaine rings.* Let

$$\mathscr{O}_C^\flat := \varprojlim_{x \mapsto x^p} \mathscr{O}_C \simeq \varprojlim_{x \mapsto x^p} \mathscr{O}_C/p$$

be the tilt of $\mathscr{O}_C$ (so that $C^\flat = \mathrm{Frac}(\mathscr{O}_C^\flat)$ is an algebraically closed field of characteristic p). Let $A_{\mathrm{inf}} = W(\mathscr{O}_C^\flat)$ and choose once and for all a compatible sequence $(1, \zeta_p, \zeta_{p^2}, ...)$ of primitive p-power roots of 1, giving rise to $\varepsilon = (1, \zeta_p, \zeta_{p^2}, ...) \in \mathscr{O}_C^\flat$. Letting φ be the natural Frobenius automorphism of A_{inf}, define

$$\mu := [\varepsilon] - 1, \quad \xi := \frac{\mu}{\varphi^{-1}(\mu)} = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} \in A_{\mathrm{inf}}.$$

The natural surjective map $\mathscr{O}_C^\flat \to \mathscr{O}_C/p$ lifts to a map $\theta : A_{\mathrm{inf}} \to \mathscr{O}_C$ with kernel generated by ξ; the map θ, in turn, lifts to a map $\theta_\infty : A_{\mathrm{inf}} \to W(\mathscr{O}_C)$ with kernel generated by μ (however, contrary to θ, θ_∞ is not always surjective, see [5, Lemma 3.23]). The kernel of the twisted map $\widetilde{\theta} := \theta\varphi^{-1} : A_{\mathrm{inf}} \to \mathscr{O}_C$ is generated by

$$\tilde{\xi} := \varphi(\xi) = \frac{\varphi(\mu)}{\mu} = \frac{[\varepsilon^p]-1}{[\varepsilon]-1}.$$

We have $\tilde{\theta}(\mu) = \zeta_p - 1$.

We list the following properties [4, 2.25].

(1) $\tilde{\xi}$ modulo μ is equal to p.

(2) Since $A_{\rm inf}$ and its reduction mod p are integral domains and since $\xi, \tilde{\xi}, \mu$ are not 0 modulo p, $(p,\xi),(p,\tilde{\xi}),(p,\mu)$ are regular sequences, and so is the sequence $(\tilde{\xi},\mu)$.

(3) The ideals (p,ξ), $(p,\tilde{\xi})$, and $(\tilde{\xi},\mu)$ define the same topology on $A_{\rm inf}$.

The above constructions naturally generalize to the case when $\mathscr{O}_C$ is replaced by a perfectoid ring.

2.2.2. *Modified Tate twists.* The compatible sequence of roots of unity $(\zeta_{p^n})_n$ gives a trivialization $\mathbf{Z}_p(1)\simeq\mathbf{Z}_p$, and we will write $\zeta=(\zeta_{p^n})_n$ for the corresponding basis of $\mathbf{Z}_p(1)$. By Fontaine's theorem [12], the $\mathscr{O}_C$-module

$$\mathscr{O}_C\{1\} := T_p(\Omega^1_{\mathscr{O}_C/\mathbf{Z}_p})$$

is free of rank 1 and the natural map dlog : $\mu_{p^\infty}\to\Omega^1_{\mathscr{O}_C/\mathbf{Z}_p}$ induces an $\mathscr{O}_C$-linear injection

$$\mathrm{dlog}:\mathscr{O}_C(1)\to\mathscr{O}_C\{1\},\quad \mathrm{dlog}(\zeta)=(\mathrm{dlog}(\zeta_{p^n}))_{n\geq 1}.$$

The $\mathscr{O}_C$-module $\mathscr{O}_C\{1\}$ is generated by

$$\omega := \frac{1}{\zeta_p-1}\mathrm{dlog}(\zeta),$$

thus the annihilator of coker(dlog) is (ζ_p-1). For any $\mathscr{O}_C$-module M, let $M\{1\}:=M\otimes_{\mathscr{O}_C}\mathscr{O}_C\{1\}$, and we will often write $m\{1\}$ for the element of $M\{1\}$ corresponding to $m\in M$ (in particular, $a\{1\}=a\cdot\omega$ in $\mathscr{O}_C\{1\}$).

Finally, define

$$A_{\rm inf}\{1\} := \frac{1}{\mu}A_{\rm inf}(1)\subset W(C^\flat)(1),$$

and let $a\{1\}=\frac{1}{\mu}a(1)\in A_{\rm inf}(1)$, if $a\in A_{\rm inf}$. The Frobenius φ on $W(C^\flat)(1)$ induces an isomorphism

$$\varphi : A_{\rm inf}\{1\}\xrightarrow{\sim}(1/\tilde{\xi})A_{\rm inf}\{1\}.$$

Its inverse defines a map

$$\varphi^{-1}:A_{\rm inf}\{1\}\to A_{\rm inf}\{1\}.$$

There is a natural map

$$\tilde{\theta}:=\theta\circ\varphi^{-1}:A_{\rm inf}\{1\}\to\mathscr{O}_C\{1\}$$

sending $a\{1\}$, for $a\in A_{\rm inf}$, to $\theta(\varphi^{-1}(a))\,\omega$.

If M is an $A_{\rm inf}$-module, let $M\{i\}:=M\otimes_{A_{\rm inf}}A_{\rm inf}\{1\}^{\otimes i}$, $i\in\mathbf{Z}$. The map $\tilde{\theta}:A_{\rm inf}\{1\}\to\mathscr{O}_C\{1\}$ induces a map $\tilde{\theta}:M\{1\}\to(M/\tilde{\xi})\{1\}$ of $A_{\rm inf}$-modules (via the map $A_{\rm inf}\to A_{\rm inf}/\tilde{\xi}$).

2.2.3. *The complexes $A\Omega_{\mathfrak{X}}$ and $\widetilde{\Omega}_{\mathfrak{X}}$.* Let $\mathfrak{X}$ be a flat formal scheme[4] over $\mathcal{O}_C$, with smooth generic fibre X, seen as an adic space over C. There is a natural morphism of sites

$$\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{ét}},$$

as well as a sheaf[5] $\mathbb{A}_{\text{inf}} := \mathbb{A}_{\text{inf,X}} := \widehat{W(\varprojlim_\varphi \mathcal{O}_X^+/p)}$ of A_{inf}-modules on $X_{\text{proét}}$, where the hat denotes the derived p-adic completion (see [5, Rem. 5.5] for an explanation why the hat might be necessary). Even though $\mathbb{A}_{\text{inf}}$ is a sheaf of complexes, for all practical purposes, it behaves as if it were defined naively by $W(\varprojlim_\varphi \mathcal{O}_X^+/p)$: for an affinoid perfectoid $U = \text{Spa}(R, R^+)$ we have $H^0(U, \mathbb{A}_{\text{inf}}) = A_{\text{inf}}(R^+)$ and $[\mathfrak{m}^\flat]H^i(U, \mathbb{A}_{\text{inf}}) = 0$, for $i > 0$ (cf. [5, Lemma 5.6]). The sheaf of complexes $\mathbb{A}_{\text{inf}}$ is endowed with a Frobenius φ, which is a quasi-isomorphism, as well as with a map

$$\theta : \mathbb{A}_{\text{inf}} \to \widehat{\mathcal{O}}_X^+ := \varprojlim \mathcal{O}_X^+/p^n,$$

which is compatible with the map $\theta : A_{\text{inf}} \to \mathcal{O}_C$ and with Frobenius.

Define

$$A\Omega_{\mathfrak{X}} := \mathrm{L}\eta_\mu(\mathrm{R}\nu_*\mathbb{A}_{\text{inf,X}}) \in D^{\geq 0}(\mathfrak{X}_{\text{ét}}, A_{\text{inf}}),$$

and

$$\widetilde{\Omega}_{\mathfrak{X}} := \mathrm{L}\eta_{\zeta_p - 1}(\mathrm{R}\nu_*\widehat{\mathcal{O}}_X^+) \in D^{\geq 0}(\mathfrak{X}_{\text{ét}}).$$

Since the functors $\mathrm{L}\eta_\mu$ and $\mathrm{R}\nu_*$ are lax symmetric monoidal (see [5, Prop. 6.7] for the first functor), $A\Omega_{\mathfrak{X}}$ is naturally a commutative ring in $D(\mathfrak{X}_{\text{ét}})$, and an algebra over the constant sheaf A_{inf}. Similarly, $\widetilde{\Omega}_{\mathfrak{X}}$ is a commutative $\mathcal{O}_{\mathfrak{X}}$-algebra object in $D(\mathfrak{X}_{\text{ét}})$ (see also the discussions after Definition 8.1 and 9.1 in [5]).

2.3. The Hodge–Tate and de Rham specializations.

2.3.1. *The smooth case.* Suppose first that $\mathfrak{X}$ is smooth over $\mathcal{O}_C$. The following result is proved in [5] (for the Zariski site, but the proof is identical in our case).

Theorem 2.2. (Bhatt-Morrow-Scholze, [5, Th. 8.3]) *There is a natural isomorphism of $\mathcal{O}_{\mathfrak{X}}$-modules on $\mathfrak{X}_{\text{ét}}$*

$$H^i(\widetilde{\Omega}_{\mathfrak{X}}) \simeq \Omega^i_{\mathfrak{X}/\mathcal{O}_C}\{-i\}.$$

We will recall the key relevant points since we will need some information about the construction of this isomorphism.

Let R be a formally smooth $\mathcal{O}_C$-algebra, such that $\text{Spf}(R)$ is connected, together with an étale map $A := \mathcal{O}_C\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\} \to R$. We

[4] Recall that all our formal schemes are p-adic and locally of finite type.

[5] Actually, a sheaf of complexes.

will simply say that R is a *small algebra* and call the map $A \to R$ a *framing*. Let $\widehat{\overline{R}}$ be the (perfectoid) completion of the normalization $\overline{R}$ of R in the maximal pro-finite étale extension of $R[1/p]$, and let $\Delta := \mathrm{Gal}(\overline{R}[1/p]/R[1/p])$. Define

$$A_\infty := \mathscr{O}_C\{T_1^{\pm 1/p^\infty}, \ldots, T_d^{\pm 1/p^\infty}\}, \quad R_\infty = R\widehat{\otimes}_A A_\infty.$$

We have $\Gamma := \mathrm{Gal}(R_\infty/R) \simeq \mathbf{Z}_p(1)^d = \oplus_{i=1}^d \mathbf{Z}_p\gamma_i$, where γ_i sends T_i^{1/p^n} to $\zeta_{p^n}T_i^{1/p^n}$ and fixes T_j^{1/p^n} for $j \neq i$. By the almost purity theorem of Faltings, the natural map (group cohomology is always continuous below) $\mathrm{R}\Gamma(\Gamma, R_\infty) \to \mathrm{R}\Gamma(\Delta, \widehat{\overline{R}})$ is an almost quasi-isomorphism. We have the following more precise results:

Theorem 2.3. (Bhatt-Morrow-Scholze, [5, Cor. 8.13, proof of prop. 8.14]) *Let R be a small algebra together with a framing, as above, and let $X = \mathrm{Sp}(R[1/p])$ and $\mathfrak{X} = \mathrm{Spf}(R)$.*

a) *The natural maps*

$$\mathrm{L}\eta_{\zeta_p-1}\mathrm{R}\Gamma(\Gamma, R_\infty) \to \mathrm{L}\eta_{\zeta_p-1}\mathrm{R}\Gamma(X_{\text{proét}}, \widehat{\mathscr{O}}_X^+) \to \mathrm{R}\Gamma(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}})$$

are quasi-isomorphisms.

b) *Writing $\widetilde{\Omega}_R$ for any of these objects, the map $\widetilde{\Omega}_R \otimes_R \mathscr{O}_{\mathfrak{X}} \to \widetilde{\Omega}_{\mathfrak{X}}$ is a quasi-isomorphism in $D(\mathfrak{X}_{\text{ét}})$.*

c) *If $R \to S$ is a formally étale map of small algebras, the natural map $\widetilde{\Omega}_R \otimes_R^L S \to \widetilde{\Omega}_S$ is a quasi-isomorphism.*

Note that

$$H^i(\widetilde{\Omega}_R) \simeq H^i(\mathrm{L}\eta_{\zeta_p-1}\mathrm{R}\Gamma(\Gamma, R_\infty)) \simeq \frac{H^i(\Gamma, R_\infty)}{H^i(\Gamma, R_\infty)[\zeta_p - 1]} \simeq H^i(\Gamma, R),$$

the last isomorphism[6] being a standard decompletion result ([5, Prop. 8.9]).

The key result (not obvious since one needs to define the isomorphisms canonically, independent of coordinates!) is then:

Theorem 2.4. (Bhatt-Morrow-Scholze, [5, Chapter 8]) *Let R be a small algebra.*

a) *There is a natural R-linear isomorphism $H^1(\widetilde{\Omega}_R) \simeq \Omega^1_{R/\mathscr{O}_C}\{-1\}$.*

b) *The cup-products maps induce R-linear isomorphisms $\wedge^i H^1(\widetilde{\Omega}_R) \simeq H^i(\widetilde{\Omega}_R)$ and hence isomorphisms $H^i(\widetilde{\Omega}_R) \simeq \Omega^i_{R/\mathscr{O}_C}\{-i\}$.*

The isomorphism in a) is constructed in [5, Prop. 8.15] using completed cotangent complexes. We will make it explicit, as follows: consider

[6] Induced by the natural map $H^i(\Gamma, R) \to H^i(\Gamma, R_\infty)$.

a framing $A \to R$ (recall that $A = \mathscr{O}_C\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\}$). By compatibility with base change from A to R of all objects involved, it suffices to construct the isomorphism for $R = A$. Moreover we may reduce to describing the isomorphism for $A = \mathscr{O}_C\{T^{\pm 1}\}$, i.e., for $d = 1$. Then the twisted map

$$\begin{aligned}\alpha : \Omega^1_{R/\mathscr{O}_C} &\simeq H^1(\widetilde{\Omega}_R)\{1\} \\ &\simeq \frac{H^1(\Gamma, R_\infty)}{H^1(\Gamma, R_\infty)[\zeta_p - 1]}\{1\} \xrightarrow{x \mapsto (\zeta_p - 1)x} (\zeta_p - 1)H^1(\Gamma, R_\infty)\{1\}\end{aligned}$$

is an isomorphism, described explicitly by

$$\alpha\left(\frac{dT}{T}\right) = (\gamma \mapsto 1 \otimes \operatorname{dlog}(\zeta_\gamma)) = (\gamma \mapsto (\zeta_p - 1) \otimes \frac{1}{\zeta_p - 1}\operatorname{dlog}(\zeta_\gamma)),$$

where $\zeta_\gamma = (\zeta_{\gamma,n})_n$, for $\gamma \in \Gamma$, is defined by the formula $\zeta_{\gamma,n} := \gamma(T^{1/p^n})/T^{1/p^n}$.

2.3.2. *The semistable case.* Suppose now that $\mathfrak{X}$ is semistable. This means that, locally on $\mathfrak{X}$ for the étale topology, $\mathfrak{X} = \operatorname{Spf}(R)$, where R admits an étale morphism of $\mathscr{O}_C$-algebras

$$A := \mathscr{O}_C\{T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}\}/(T_0 \cdots T_r - p^q) \to R$$

for some $d \geq 0$, $r \in \{0, 1, \dots, d\}$ and some rational number $q > 0$ (we fix once and for all an embedding $p^{\mathbf{Q}} \subset C$). Equip $\mathscr{O}_C$ with the log-structure $\mathscr{O}_C \setminus \{0\} \to \mathscr{O}_C$ and $\mathfrak{X}$ with the canonical log-structure, i.e. given by the sheafification of the subpresheaf $\mathscr{O}_{\mathfrak{X},\text{ét}} \cap (\mathscr{O}_{\mathfrak{X},\text{ét}}[1/p])^*$ of $\mathscr{O}_{\mathfrak{X},\text{ét}}$. Let $\Omega_{\mathfrak{X}/\mathscr{O}_C}$ be the finite locally free $\mathscr{O}_{\mathfrak{X}}$-module of logarithmic differentials on $\mathfrak{X}$ over $\mathscr{O}_C$. We have the following result:

Theorem 2.5. (Česnavičius-Koshikawa, [7, Th. 4.2, Cor. 4.6, Prop. 4.8, Th. 4.11])

a) *There is a unique $\mathscr{O}_{\mathfrak{X},\text{ét}}$-module isomorphism $H^1(\widetilde{\Omega}_{\mathfrak{X}}) \simeq \Omega^1_{\mathfrak{X}/\mathscr{O}_C}\{-1\}$ whose restriction to the smooth locus $\mathfrak{X}^{\text{sm}}$ is the one given by Theorem 2.2.*

b) *The cup-product map $\wedge^i(H^1(\widetilde{\Omega}_{\mathfrak{X}})) \to H^i(\widetilde{\Omega}_{\mathfrak{X}})$ is an isomorphism and so there is a natural $\mathscr{O}_{\mathfrak{X},\text{ét}}$-module isomorphism*

$$H^i(\widetilde{\Omega}_{\mathfrak{X}}) \simeq \Omega^i_{\mathfrak{X}/\mathscr{O}_C}\{-i\}.$$

Remark 2.6. 1) The construction of the map in part a) goes as follows. The same arguments as in [5] (using completed cotangent complexes) give a map $\Omega^{1,\text{cl}}_{\mathfrak{X}/\mathscr{O}_C}\{-1\} \to \mathrm{R}^1\nu_*(\widehat{\mathscr{O}}^+_X)$, where we denoted by the superscript $(-)^{\text{cl}}$ the classical, non logarithmic, differential forms. The results in [5] ensure that the resulting map

$$\Omega^{1,\mathrm{cl}}_{\mathfrak{X}/\mathscr{O}_C}\{-1\} \to \mathrm{R}^1\nu_*(\widehat{\mathscr{O}}^+_X) \to \frac{\mathrm{R}^1\nu_*(\widehat{\mathscr{O}}^+_X)}{\mathrm{R}^1\nu_*(\widehat{\mathscr{O}}^+_X)[\zeta_p-1]} \simeq H^1(\widetilde{\Omega}_{\mathfrak{X}}) \tag{2.7}$$

restricts to an isomorphism $\Omega^1_{\mathfrak{X}^{\mathrm{sm}}/\mathscr{O}_C}\{-1\} \simeq (\zeta_p-1)H^1(\widetilde{\Omega}_{\mathfrak{X}})|_{\mathfrak{X}^{\mathrm{sm}}}$. Moreover, one shows that $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ is a vector bundle. Hence one can divide the map (2.7) by ζ_p-1 to obtain a map

$$\Omega^1_{\mathfrak{X}/\mathscr{O}_C}\{-1\} \to H^1(\widetilde{\Omega}_{\mathfrak{X}})$$

which is an isomorphism over $\mathfrak{X}^{\mathrm{sm}}$. One shows that this extends to the isomorphism in a).

2) The cup-product maps in b) are constructed as follows. Setting $T = \mathrm{R}\nu_*(\widehat{\mathscr{O}}^+_X)$ and using the identifications

$$H^1(\widetilde{\Omega}_{\mathfrak{X}}) \simeq \frac{H^1(T)}{H^1(T)[\zeta_p-1]},\ H^i(\widetilde{\Omega}_{\mathfrak{X}}) \simeq \frac{H^i(T)}{H^i(T)[\zeta_p-1]},$$

they are induced by the product maps $H^1(T)^{\otimes i} \to H^i(T)$, which, in turn, are induced by the product maps

$$H^j(T)\otimes_{\mathscr{O}_{\mathfrak{X},\text{ét}}} H^k(T) \to H^{j+k}(T\otimes^{\mathrm{L}}_{\mathscr{O}_{\mathfrak{X},\text{ét}}} T) \to H^{j+k}(T).$$

We continue assuming that $\mathfrak{X}$ is semistable. Recall that the map $\tilde{\theta} = \theta\circ\varphi^{-1} : \mathbb{A}_{\mathrm{inf},X} \to \widehat{\mathscr{O}}^+_X$ is surjective with kernel generated by the non zero-devisor $\tilde{\xi} = \varphi(\xi)$. It thus gives a quasi-isomorphism

$$\mathrm{R}\nu_*\mathbb{A}_{\mathrm{inf},X}\otimes^{\mathrm{L}}_{A_{\mathrm{inf}},\tilde{\theta}}\mathscr{O}_C \xrightarrow{\sim} \mathrm{R}\nu_*\widehat{\mathscr{O}}^+_X$$

Since $\widetilde{\theta}$ sends μ to ζ_p-1, it induces a morphism

$$A\Omega_{\mathfrak{X}}/\tilde{\xi} := A\Omega_{\mathfrak{X}}\otimes^{\mathrm{L}}_{A_{\mathrm{inf}},\tilde{\theta}}\mathscr{O}_C \to \widetilde{\Omega}_{\mathfrak{X}}.$$

Theorem 2.8. (Česnavičius-Koshikawa, [7, Th. 4.2, Th. 4.17, Cor. 4.6 and its proof])

(1) *The above morphism* $A\Omega_{\mathfrak{X}}/\tilde{\xi} \to \widetilde{\Omega}_{\mathfrak{X}}$ *is a quasi-isomorphism.*
(2) *There is a natural quasi-isomorphism* $A\Omega_{\mathfrak{X}}/\xi \xrightarrow{\sim} \Omega^\bullet_{\mathfrak{X}/\mathscr{O}_C}$, *where* $A\Omega_{\mathfrak{X}}/\xi := A\Omega_{\mathfrak{X}}\otimes^{\mathrm{L}}_{A_{\mathrm{inf}},\theta}\mathscr{O}_C$.
(3) *The complex* $A\Omega_{\mathfrak{X}}$ *is derived* $\tilde{\xi}$*-complete. Hence so is* $\mathrm{R}\Gamma_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}})$ *(and its cohomology groups).*

For $i\geq 0$, (using the above theorems) we define:

- the *Hodge–Tate specialization* map
  (1) (on sheaves) as the composition

$$\tilde{\iota}_{\mathrm{HT}} : A\Omega_{\mathfrak{X}} \to A\Omega_{\mathfrak{X}}/\tilde{\xi} \to \widetilde{\Omega}_{\mathfrak{X}};$$

(2) (on cohomology)

$$\iota_{\rm HT} : H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}) \to H^0_{\text{ét}}(\mathfrak{X}, \Omega^i_{\mathfrak{X}/\mathscr{O}_C}\{-i\})$$

as the composition[7]

$$\begin{array}{ccc} \iota_{\rm HT} : H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}) \xrightarrow{\tilde{\iota}_{\rm HT}} H^i_{\text{ét}}(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}}) & & \\ \downarrow & & \\ H^0_{\text{ét}}(\mathfrak{X}, H^i(\widetilde{\Omega}_{\mathfrak{X}})) & \xrightarrow{\sim} & H^0_{\text{ét}}(\mathfrak{X}, \Omega^i_{\mathfrak{X}/\mathscr{O}_C}\{-i\}) \end{array}$$

where the second map is the edge morphism in the spectral sequence

$$E_2^{i,j} = H^i_{\text{ét}}(\mathfrak{X}, H^j(\widetilde{\Omega}_{\mathfrak{X}})) \Rightarrow H^{i+j}_{\text{ét}}(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}});$$

- the *de Rham specialization* map as the composition

$$\tilde{\iota}_{\rm dR} : A\Omega_{\mathfrak{X}} \to A\Omega_{\mathfrak{X}}/\xi \xrightarrow{\sim} \Omega^\bullet_{\mathfrak{X}/\mathscr{O}_C},$$

which on cohomology yields a map

$$\iota_{\rm dR} : H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}) \xrightarrow{\tilde{\iota}_{\rm dR}} H^i_{\rm dR}(\mathfrak{X}).$$

2.4. **p-adic nearby cycles and $A_{\rm inf}$-cohomology.** We review here a result from [6], which describes integral p-adic étale cohomology in terms of the complex $A\Omega_{\mathfrak{X}}$. Let X be a smooth adic space over C and let $\mathfrak{X}$ be a flat formal model of X (not necessarily semistable). Fix an integer $i \geq 0$. Recall that there is an endomorphism[8]

$$\xi^i\varphi^{-1} : \tau_{\leq i}A\Omega_{\mathfrak{X}} \to \tau_{\leq i}A\Omega_{\mathfrak{X}}$$

defined as the composition[9]

$$\begin{array}{ccc} \tau_{\leq i}A\Omega_{\mathfrak{X}} \simeq {\rm L}\eta_\mu\tau_{\leq i}{\rm R}\nu_*\mathbb{A}_{\rm inf} \xrightarrow{\varphi^{-1}} & {\rm L}\eta_{\varphi^{-1}(\mu)}\tau_{\leq i}{\rm R}\nu_*\mathbb{A}_{\rm inf} & \\ & \downarrow \xi^i & \\ & {\rm L}\eta_\xi{\rm L}\eta_{\varphi^{-1}(\mu)}\tau_{\leq i}{\rm R}\nu_*\mathbb{A}_{\rm inf} = \tau_{\leq i}A\Omega_{\mathfrak{X}} & \end{array}$$

---

[7] We abuse notation and write $\tilde{\iota}_{\rm HT}$ instead of $H^i_{\text{ét}}(\mathfrak{X}, \tilde{\iota}_{\rm HT})$.

[8] As an object of $D(\mathfrak{X}_{\text{ét}})$ but **not** as an object of $D(\mathfrak{X}_{\text{ét}}, A_{\rm inf})$, i.e. the endomorphism is not $A_{\rm inf}$-linear.

[9] The first isomorphism follows from the fact that ${\rm L}\eta_\mu$ commutes with truncations, see [5, Lemma 6.5], while the definition of the map ξ^i implicitly uses [5, Lemma 6.9]

The following commutative diagram defines an operator $1-\varphi^{-1}$ on $\tau_{\leq i}A\Omega_{\mathfrak{X}}\{i\}$:

$$\begin{tikzcd}
\tau_{\leq i}A\Omega_{\mathfrak{X}} \arrow[r, "1-\xi^i\varphi^{-1}"] \arrow[d, "\mu^{-i}", "\wr"'] & \tau_{\leq i}A\Omega_{\mathfrak{X}} \arrow[d, "\mu^{-i}", "\wr"'] \\
\tau_{\leq i}A\Omega_{\mathfrak{X}}\{i\} \arrow[r, "1-\varphi^{-1}"] & \tau_{\leq i}A\Omega_{\mathfrak{X}}\{i\}
\end{tikzcd}$$

The following result is proved in [6] in the good reduction case. As we show below the proof goes through in a more general setting. We define the sheaf $\widehat{\mathbf{Z}}_p$ on $X_{\text{proét}}$ by $\widehat{\mathbf{Z}}_p = \varprojlim_n \mathbf{Z}/p^n\mathbf{Z}$ and we recall that $\mathrm{R}^i \varprojlim \mathbf{Z}/p^n\mathbf{Z} = 0$ for $i > 0$ (see [22, Prop. 8.2]; this is not tautological since $X_{\text{proét}}$ is not a replete topos).

Theorem 2.9. (Bhatt-Morrow-Scholze, [6, Chapter 10]) *Let X be a smooth adic space over C with a flat formal model $\mathfrak{X}$. Let $i \geq 0$. There is a natural quasi-isomorphism*

$$\gamma : \tau_{\leq i}\mathrm{R}\nu_*\widehat{\mathbf{Z}}_p(i) \xrightarrow{\sim} \tau_{\leq i}[\tau_{\leq i}A\Omega_{\mathfrak{X}}\{i\} \xrightarrow{1-\varphi^{-1}} \tau_{\leq i}A\Omega_{\mathfrak{X}}\{i\}],$$

where $[\cdot]$ denotes the homotopy fiber. In particular, there is a natural exact sequence

$$0 \to \frac{H^{i-1}_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})}{(1-\varphi^{-1})} \to H^i_{\text{proét}}(X, \widehat{\mathbf{Z}}_p(i)) \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1} \to 0.$$

Everything is Galois equivariant if $\mathfrak{X}$ is defined over $\mathscr{O}_K$.

Proof. We follow [6] faithfully, but work directly on the p-adic level. Using the commutative diagram

$$\begin{tikzcd}
\tau_{\leq i}A\Omega_{\mathfrak{X}} \arrow[r, "1-\xi^i\varphi^{-1}"] \arrow[d, "\mu^{-i}", "\wr"'] & \tau_{\leq i}A\Omega_{\mathfrak{X}} \arrow[d, "\mu^{-i}", "\wr"'] \\
\tau_{\leq i}A\Omega_{\mathfrak{X}}\{i\} \arrow[r, "1-\varphi^{-1}"] & \tau_{\leq i}A\Omega_{\mathfrak{X}}\{i\}
\end{tikzcd},$$

it suffices to construct a quasi-isomorphism

$$\mu^i : \tau_{\leq i}\mathrm{R}\nu_*\widehat{\mathbf{Z}}_p \xrightarrow{\sim} \tau_{\leq i}[\tau_{\leq i}A\Omega_{\mathfrak{X}} \xrightarrow{1-\xi^i\varphi^{-1}} \tau_{\leq i}A\Omega_{\mathfrak{X}}].$$

Let $\psi_i = \xi^i\varphi^{-1}$, seen as an endomorphism of $\tau_{\leq i}A\Omega_{\mathfrak{X}}$ (as explained above) or of $T := \mathrm{R}\nu_*\mathbb{A}_{\inf}$ (defined in the obvious way). These two endomorphisms are compatible with the canonical map can : $A\Omega_{\mathfrak{X}} \to T$.

We start with the following simple fact:

Lemma 2.10. a) *For $i \geq j$, the map $1-\psi_i : \mathbb{A}_{\inf}/\mu^j \to \mathbb{A}_{\inf}/\mu^j$ is a quasi-isomorphism.*

b) *There is a quasi-isomorphism of complexes of sheaves on* $X_{\text{proét}}$

$$\widehat{\mathbf{Z}}_p \xrightarrow[\sim]{\mu^i} [\mathbb{A}_{\text{inf}} \xrightarrow{1-\psi_i} \mathbb{A}_{\text{inf}}].$$

Proof. a) This follows from the proof of [19, Lemma 3.5 (iii)].

b) Consider the following commutative diagram:

$$\begin{tikzcd}
 & {[\mathbb{A}_{\text{inf}}} \arrow[r, "1-\varphi^{-1}"] & {\mathbb{A}_{\text{inf}}]} \arrow[d, "(\mu^i,\mu^i)", "\wr"'] \\
\widehat{\mathbf{Z}}_p \arrow[ur, "\text{can}"] \arrow[r, "\mu^i"] & {[\mathbb{A}_{\text{inf}}} \arrow[r, "1-\xi^i\varphi^{-1}"] & {\mathbb{A}_{\text{inf}}]} \arrow[d, "\text{can}"] \\
 & {[\mathbb{A}_{\text{inf}}/\mu^j} \arrow[r, "1-\xi^i\varphi^{-1}"] & {\mathbb{A}_{\text{inf}}/\mu^j]}
\end{tikzcd}$$

The vertical map is a quasi-isomorphism by (a). It suffices thus to show that we have a quasi-isomorphism

$$\widehat{\mathbf{Z}}_p \xrightarrow{\sim} [\mathbb{A}_{\text{inf}} \xrightarrow{1-\varphi^{-1}} \mathbb{A}_{\text{inf}}].$$

But this is just the derived p-adically complete version of the Artin-Schreier exact sequence [19, Lemma 3.5 (ii)]. This finishes the proof of the lemma. $\square$

Write $U^{\psi_i=1}$ for the homotopy fiber of $1-\psi_i : U \to U$ for $U \in \{A\Omega_{\mathfrak{X}}, T\}$. The above lemma gives rise to a distinguished triangle

$$\mathrm{R}\nu_*\widehat{\mathbf{Z}}_p \xrightarrow{\mu^i} T \xrightarrow{1-\psi_i} T,$$

inducing a quasi-isomorphism

$$\mu^i : \tau_{\leq i}\mathrm{R}\nu_*\widehat{\mathbf{Z}}_p \xrightarrow{\sim} \tau_{\leq i}T^{\psi_i=1}.$$

To finish the proof of the theorem, it remains to show (and this is the hard part) that the map (induced by the natural maps can : $A\Omega_{\mathfrak{X}} \to T$ and $\tau_{\leq i}A\Omega_{\mathfrak{X}} \to A\Omega_{\mathfrak{X}}$)

$$\tau_{\leq i}(\tau_{\leq i}A\Omega_{\mathfrak{X}})^{\psi_i=1} \to \tau_{\leq i}T^{\psi_i=1}$$

is a quasi-isomorphism.

By homological algebra, this happens if $1-\psi_i$ acts bijectively on the kernel and cokernel of $\text{can}_j : H^j(A\Omega_{\mathfrak{X}}) \to H^j(T)$ for $j < i$, bijectively on the kernel for $j = i$, and injectively on the cokernel for $j = i$. We first treat the case $j = 0$, showing that the map can_0 is bijective. It suffices to check that $H^0(A\Omega_{\mathfrak{X}}) = H^0(T)$. This follows from the isomorphism $H^0(A\Omega_{\mathfrak{X}}) \simeq H^0(T)/H^0(T)[\mu]$ and the vanishing of $H^0(T)[\mu]$, which is a consequence of the fact that $\mathbb{A}_{\rm inf}$ is μ-torsion-free, which in turn follows from the description of $\mathbb{A}_{\rm inf}(U)$ for affinoid perfectoid objects U of $X_{\rm pro\acute{e}t}$, see [5, Lemma 5.6].

Assume now that $j>0$ and set $M_i=H^i(T)$. Recall [5, Lemma 6.4] that the map $\mu^j: M_j/M_j[\mu]\to H^j(A\Omega_{\mathfrak{X}})$ is an isomorphism. It follows that, for $0<j\leq i$, the map ${\rm can}_j$ fits into an exact sequence

$$0\to M_j[\mu]\to M_j[\mu^j]\to H^j(A\Omega_{\mathfrak{X}})\xrightarrow{{\rm can}_j} M_j\to M_j/\mu^j\to 0.$$

This sequence is compatible with the operators $1-\psi_{i-j}$, $1-\psi_{i-j}$, $1-\psi_i$, $1-\psi_i$, $1-\psi_i$, respectively. Thus it suffices to show that $1-\psi_{i-j}$ is bijective on $M_j[\mu^j]/M_j[\mu]$, that $1-\psi_i$ is bijective on M_j/μ^j for $j<i$, and is injective for $j=i$. This follows from the following lemma (modulo a change of the roles of i and j). ☐

Lemma 2.11. ([6, Lemma 10.5]) *Let $j\geq 1$, $i\geq 0$.*

a) $1-\psi_{l+j}$ *is bijective on* M_i/μ^j *for* $l>0$ *and is injective for* $l=0$.

b) $1-\psi_l$ *is bijective on* $M_i[\mu^j]$ *for* $l>0$*, surjective for* $l=0$.

c) $1-\psi_l$ *is bijective on* $M_i[\mu^j]/M_i[\mu]$*, for* $l\geq 0$.

Proof. We first prove that $1-\psi_l$ is injective on $M_i[\mu^j]$ for $l>0$. If $\psi_l(x)=x$ and $\mu^j x=0$, then $\psi_{l+1}(\mu x)=\mu x$ and $\mu x\in M_i[\mu^{j-1}]$, thus, arguing by induction on j, we may assume that $j=1$. Suppose that $\mu x=0$ and $\psi_l(x)=x$, i.e., $x-\xi^l\varphi^{-1}(x)=0$. Since $\xi\equiv p \pmod{\varphi^{-1}(\mu)}$ in $A_{\rm inf}$ and $\varphi^{-1}(\mu)$ kills $\varphi^{-1}(x)$, we deduce that $(1-p\xi^{l-1}\varphi^{-1})(x)=0$. This forces $x=0$, since $1-p\xi^{l-1}\varphi^{-1}$ is an automorphism of the derived p-complete module M_i ($\mathbb{A}_{\rm inf}$ is derived p-adically complete, hence so are $T={\rm R}\nu_*\mathbb{A}_{\rm inf}$ and $M_i=H^i(T)$). This proves the first step.

Next, the commutative diagram of distinguished triangles

$$\begin{tikzcd} T \arrow[r, "\mu^j"] \arrow[d, "1-\psi_l"] & T \arrow[r] \arrow[d, "1-\psi_{l+j}"] & T/\mu^j \arrow[d, "1-\psi_{l+j}"] \\ T \arrow[r, "\mu^j"] & T \arrow[r] & T/\mu^j \end{tikzcd}$$

gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M_i/\mu^j \arrow[r] \arrow[d, "1-\psi_{l+j}"] & H^i(T/\mu^j) \arrow[r] \arrow[d, "1-\psi_{l+j}"] & M_{i+1}[\mu^j] \arrow[r] \arrow[d, "1-\psi_l"] & 0 \\ 0 \arrow[r] & M_i/\mu^j \arrow[r] & H^i(T/\mu^j) \arrow[r] & M_{i+1}[\mu^j] \arrow[r] & 0 \end{tikzcd}$$

Since $1-\psi_{l+j}$ is bijective on $H^i(T/\mu^j)$ (Lemma 2.10 implies that the map $1-\psi_{l+j}: T/\mu^j\to T/\mu^j$ is a quasi-isomorphism), we deduce that $1-\psi_{l+j}$ is injective on M_i/μ^j, the map $1-\psi_l$ is surjective on $M_{i+1}[\mu^j]$ and the cokernel of $1-\psi_{l+j}$ on M_i/μ^j is identified with the kernel of

$1-\psi_l$ on $M_{i+1}[\mu^j]$. This last kernel is 0 for $l>0$ (by the first step), thus $1-\psi_l$ is bijective on $M_{i+1}[\mu^j]$ (this also holds trivially on $M_0[\mu^j]=0$) and $1-\psi_{l+j}$ is bijective on M_i/μ^j for $l>0$.

Finally, we need to show that $1-\psi_l$ is bijective on $M_i[\mu^j]/M_i[\mu]$. We may assume that $j>1$. Surjectivity follows from that of $1-\psi_l$ on $M_i[\mu^j]$. For injectivity, note that if $\mu\psi_l(x)=\mu x$, then $\psi_{l+1}(\mu x)=\mu x$ and, since $1-\psi_{l+1}$ is injective on $M_i[\mu^{j-1}]$, we obtain $x\in M_i[\mu]$, as needed.

□

3. $A_{\rm inf}$-SYMBOL MAPS

Let X be a smooth adic space over C and let $\mathfrak{X}$ be a flat p-adic formal model of X over $\mathscr{O}_C$. Let $\nu : X_{\text{proét}}\to\mathfrak{X}_{\text{ét}}$ be the map discussed in the previous section.

3.1. **The construction of symbol maps.**

We will define compatible continuous pro-étale and $A_{\rm inf}$-symbol maps[10]

$$\begin{aligned} r_{\text{proét}} &: \mathscr{O}(X)^{*,\otimes i}\to H^i_{\text{proét}}(X,\widehat{\mathbf{Z}}_p(i)),\\ r_{\rm inf} &: \mathscr{O}(X)^{*,\otimes i}\to H^i_{\text{ét}}(\mathfrak{X},A\Omega_{\mathfrak{X}}\{i\}),\quad i\geq 1. \end{aligned}\tag{3.1}$$

For $i=1$, we will construct below compatible maps of sheaves

$$\begin{aligned} c_1^{\text{proét}} &: \tau_{\leq 1}(\mathrm{R}\nu_*\mathbb{G}_m[-1])\to\tau_{\leq 1}(\mathrm{R}\nu_*\widehat{\mathbf{Z}}_p(1)),\\ c_1^{\rm inf} &: \tau_{\leq 1}(\mathrm{R}\nu_*\mathbb{G}_m[-1])\to\tau_{\leq 1}A\Omega_{\mathfrak{X}}\{1\}. \end{aligned}\tag{3.2}$$

Applying $H^1_{\text{ét}}(\mathfrak{X},-)$ and observing that

$$\begin{array}{ccc} H^1_{\text{ét}}(\mathfrak{X},\tau_{\leq 1}(\mathrm{R}\nu_*\mathbb{G}_m[-1])) & \xrightarrow{\sim} & H^1_{\text{ét}}(\mathfrak{X},\mathrm{R}\nu_*\mathbb{G}_m[-1])\\ & & \|\\ & & H^0_{\text{ét}}(\mathfrak{X},\mathrm{R}\nu_*\mathbb{G}_m)\simeq\mathscr{O}(X)^*, \end{array}$$

we get that the maps $c_1^{\text{proét}}, c_1^{\rm inf}$ induce global symbol maps

$$r_{\text{proét}} : \mathscr{O}(X)^*\to H^1_{\text{proét}}(X,\widehat{\mathbf{Z}}_p(1)),\quad r_{\rm inf} : \mathscr{O}(X)^*\to H^1_{\text{ét}}(\mathfrak{X},A\Omega_{\mathfrak{X}}\{1\}).$$

For $i\geq 1$, we define the symbol maps (3.1) using cup product:

$$x_1\otimes\cdots\otimes x_i\mapsto r_*(x_1)\cup\cdots\cup r_*(x_i).$$

[10]We refer the reader to [10, Sec. 2.2] for a discussion of topology on cohomologies of rigid analytic varieties and formal schemes. Integrally, we work in the category of pro-discrete modules, rationally – in the category of locally convex topological vector spaces over $\mathbf{Q}_p$. But, in this paper, we work with the naive topology on cohomology groups, i.e., the quotient topology, as opposed to the refined cohomology groups (denoted $\widetilde{H}$ in [10]) taken in the derived category of pro-discrete modules.

The construction of the first map in (3.2) uses the Kummer exact sequence on $X_{\text{proét}}$

$$0 \to \widehat{\mathbf{Z}}_p(1) \to \varprojlim_{x \mapsto x^p} \mathbb{G}_m \to \mathbb{G}_m \to 0,$$

obtained by passing to the limit in the usual Kummer exact sequences

$$0 \to \mu_{p^n} \to \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \to 0$$

and using the vanishing of $\mathrm{R}^1 \varprojlim \mathbf{Z}/p^n\mathbf{Z}$ (see [22, Prop. 8.2]). The above exact sequence induces, by projection to $\mathfrak{X}_{\text{ét}}$, the Chern class map

$$c_1^{\text{proét}} : \mathrm{R}\nu_*\mathbb{G}_m[-1] \to \mathrm{R}\nu_*(\widehat{\mathbf{Z}}_p(1)).$$

The construction of the second map in (3.2) uses the above Kummer exact sequence and the *twisted* Artin-Schreier quasi-isomorphism on $X_{\text{proét}}$ (cf. lemma 2.10)

$$\widehat{\mathbf{Z}}_p(1) \underset{\sim}{\xrightarrow{\gamma}} [\mathbb{A}_{\text{inf}}\{1\} \xrightarrow{1-\varphi^{-1}} \mathbb{A}_{\text{inf}}\{1\}],$$

where the map γ is defined by $x(1) \mapsto \mu x\{1\}, x \in \widehat{\mathbf{Z}}_p$. By pushing down to $\mathfrak{X}_{\text{ét}}$ we obtain a map

$$\beta : \tau_{\leq 1}\mathrm{R}\nu_*\widehat{\mathbf{Z}}_p(1) \to \tau_{\leq 1}\mathrm{R}\nu_*\mathbb{A}_{\text{inf}}\{1\}.$$

On the other hand, Theorem 2.9 gives us a natural map

$$\gamma : \tau_{\leq 1}\mathrm{R}\nu_*\widehat{\mathbf{Z}}_p(1) \to \tau_{\leq 1}A\Omega_{\mathfrak{X}}\{1\}.$$

The above two maps are compatible via the map

$$\tau_{\leq 1}A\Omega_{\mathfrak{X}}\{1\} \to \tau_{\leq 1}\mathrm{R}\nu_*\mathbb{A}_{\text{inf}}\{1\}$$

and the composition

$$\tau_{\leq 1}\mathrm{R}\nu_*\widehat{\mathbf{Z}}_p(1) \xrightarrow{\gamma} \tau_{\leq 1}A\Omega_{\mathfrak{X}}\{1\} \xrightarrow{1-\varphi^{-1}} \tau_{\leq 1}A\Omega_{\mathfrak{X}}\{1\}$$

is the 0 map.

The symbol map is simply the composition

$$c_1^{\text{inf}} : \tau_{\leq 1}(\mathrm{R}\nu_*\mathbb{G}_m[-1]) \xrightarrow{c_1^{\text{proét}}} \tau_{\leq 1}\mathrm{R}\nu_*\widehat{\mathbf{Z}}_p(1) \xrightarrow{\gamma} \tau_{\leq 1}A\Omega_{\mathfrak{X}}\{1\}.$$

3.2. **Compatibility with the Hodge–Tate symbol map.** Let $\mathfrak{X}_{\mathscr{O}_K}$ be a semistable formal scheme over $\mathscr{O}_K$. Let M be the sheaf of monoids on $\mathfrak{X}_{\mathscr{O}_K}$ defining the log-structure, M^{gp} its group completion. This log-structure is canonical, in the terminology of Berkovich [2, 2.3], i.e., $M(U) = \{x \in \mathscr{O}_{\mathfrak{X}_{\mathscr{O}_K}}(U) | x_K \in \mathscr{O}^*_{\mathfrak{X}_K}(U_K)\}$ when U is an affine open of $\mathfrak{X}_{\mathscr{O}_K}$. This is shown in [2, Th. 2.3.1], [1, Th. 5.3] and applies also to semistable formal schemes with self-intersections. It follows that $M^{\text{gp}}(U) = \mathscr{O}^*_{\mathfrak{X}_K}(U_K)$. Set $X_K := \mathfrak{X}_{\mathscr{O}_K,K}, \mathfrak{X} := \mathfrak{X}_{\mathscr{O}_C}, X := X_{K,C}$.

For $i \geq 1$, the Hodge–Tate symbol maps

$$r_{\mathrm{HT}}: \quad \mathscr{O}(X_K)^{*,\otimes i} \to H^0_{\mathrm{\acute{e}t}}(\mathfrak{X}, \Omega^i_{\mathfrak{X}})$$

are defined by taking cup products of the Chern class maps

$$c_1^{\mathrm{HT}}: \tau_{\leq 1}(\mathrm{R}\nu_*\mathbb{G}_m[-1]) \to \Omega_{\mathfrak{X}}[-1], \quad x \mapsto \mathrm{dlog}(x) := \frac{dx}{x}.$$

The purpose of this section is to prove the following fact:

Proposition 3.3. *Let $i \geq 1$. The symbol maps $r_{\inf}$ and r_{HT} are compatible under the Hodge–Tate specialization map ι_{HT}, i.e.,*

$$\iota_{\mathrm{HT}} \circ (r_{\inf} | \mathscr{O}(X_K)^{*,\otimes i}) = r_{\mathrm{HT}}.$$

Proof. The case $i = 1$. Consider the composition

$$\begin{array}{ccc} \mathscr{O}(X_K)^* \to \mathscr{O}(X)^* \simeq H^1_{\mathrm{\acute{e}t}}(\mathfrak{X}, \tau_{\leq 1}(\mathrm{R}\nu_*\mathbb{G}_m[-1])) & & \\ \downarrow{\scriptstyle c_1^{\inf}} & & \\ H^1_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{1\}) & \xrightarrow{\iota_{\mathrm{HT}}\{1\}} & H^0_{\mathrm{\acute{e}t}}(\mathfrak{X}, \Omega^1_{\mathfrak{X}}). \end{array}$$

We need to show that

Lemma 3.4. *The above composition is equal to the map*

$$c_1^{\mathrm{HT}} = \mathrm{dlog}: \mathscr{O}(X_K)^* \to H^0_{\mathrm{\acute{e}t}}(\mathfrak{X}, \Omega^1_{\mathfrak{X}}).$$

Proof. Let $\varepsilon : X_{\mathrm{\acute{e}t}} \to \mathfrak{X}_{\mathrm{\acute{e}t}}$ and $\nu_X : X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ be the canonical projections, so that $\nu = \varepsilon\nu_X$. The natural map $\iota : \Omega^1_{\mathfrak{X}} \to \varepsilon_*\Omega^1_X$ is injective (since $\mathfrak{X}$ is flat over $\mathscr{O}_C$) and induces an isomorphism $C \otimes_{\mathscr{O}_C} \Omega^1_{\mathfrak{X}} \simeq \varepsilon_*\Omega^1_X$.

We start by constructing an isomorphism

$$\alpha_2 : \mathrm{R}^1\nu_*\widehat{\mathscr{O}}(1) \to \varepsilon_*\Omega^1_X$$

as well as the commutative diagram (3.5) below, where:

- the isomorphism $\alpha_1 : H^1(\widetilde{\Omega}_{\mathfrak{X}}\{1\}) \simeq \Omega^1_{\mathfrak{X}}$ is defined by Theorem 2.5;
- the map $\mathrm{R}^1\nu_*\widehat{\mathscr{O}}^+\{1\} \to \mathrm{R}^1\nu_*\widehat{\mathscr{O}}(1)$ is induced by the inclusion $\widehat{\mathscr{O}}^+ \subset \widehat{\mathscr{O}}$ and by the map $\mathscr{O}_C\{1\} \to C(1)$, which is the composite of the inclusion $\mathscr{O}_C\{1\} \subset C\{1\}$ and of the isomorphism $C\{1\} \simeq C(1)$ induced by dlog (see Section 2.2.2). Informally but intuitively the map $\mathscr{O}_C\{1\} \to C(1)$ is $x\{1\} \to \frac{x}{\zeta_p - 1}(1)$ (and this can be made rigorous by defining $x\{1\}$ as $x\omega$, where $\omega = \frac{\mathrm{dlog}(\zeta)}{\zeta_p - 1}$, see Section 2.2.2).

$$\begin{array}{ccccc} H^1(\widetilde{\Omega}_{\mathfrak{X}}\{1\}) & \xrightarrow[\alpha_1]{\sim} & \Omega^1_{\mathfrak{X}} & & \\ \downarrow{\scriptstyle \mathrm{can}} & & & \searrow^{\iota} & \\ \mathrm{R}^1\nu_*\widehat{\mathscr{O}}^+\{1\} & \longrightarrow & \mathrm{R}^1\nu_*\widehat{\mathscr{O}}(1) & \xrightarrow[\alpha_2]{\sim} & \varepsilon_*\Omega^1_X \end{array} \tag{3.5}$$

In order to define the map α_2 we start by considering Scholze's isomorphism ([23, Lemma 3.24]),

$$\alpha_2 : \mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}(1) \xrightarrow{\sim} \Omega^1_X, \tag{3.6}$$

which is uniquely characterized by the property that its inverse is the unique $\mathscr{O}_X$-linear map $\alpha_2^{-1} : \Omega^1_X \to \mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}(1)$ making the following diagram commute

$$\begin{tikzcd} \mathscr{O}_X^* \arrow[r, "c_1^{\text{proét}}"] \arrow[d, "\mathrm{dlog}"'] & \mathrm{R}^1\nu_{X,*}\widehat{\mathbf{Z}}_p(1) \arrow[d] \\ \Omega^1_X \arrow[r, "\alpha_2^{-1}"] & \mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}(1) \end{tikzcd} \tag{3.7}$$

The isomorphism α_2 extends to isomorphisms [23, Prop. 3.23]:

$$\mathrm{R}^i\nu_{X,*}\widehat{\mathscr{O}}(i) \simeq \Omega^i_X, \quad i \geq 0.$$

The spectral sequence $\mathrm{E}_2^{i,j} : \mathrm{R}^i\varepsilon_*(\mathrm{R}^j\nu_{X,*}\widehat{\mathscr{O}}) \Rightarrow \mathrm{R}^{i+j}\nu_*\widehat{\mathscr{O}}$ degenerates thanks to the vanishing of $\mathrm{R}^i\varepsilon_*\Omega^j_X$ when $i > 0$ (since coherent sheaves have vanishing higher cohomology on affinoids, by Tate's acyclicity theorem). It follows that we have isomorphisms

$$\mathrm{R}^1\nu_*\widehat{\mathscr{O}}(1) \xrightarrow{\sim} \varepsilon_*\mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}(1) \simeq \varepsilon_*\Omega^1_X, \tag{3.8}$$

and we let (abusively) $\alpha_2 : \mathrm{R}^1\nu_*\widehat{\mathscr{O}}(1) \to \varepsilon_*\Omega^1_X$ be their composition.

Let us prove the commutativity of the diagram (3.5), i.e., the compatibility of the maps α_1 and α_2. Call ρ the composition

$$\rho: \quad \Omega^1_{\mathfrak{X}} \xrightarrow{\alpha_1^{-1}} H^1(\widetilde{\Omega}_{\mathfrak{X}}\{1\}) \xrightarrow{\text{can}} \mathrm{R}^1\nu_*\widehat{\mathscr{O}}^+\{1\} \to \mathrm{R}^1\nu_*\widehat{\mathscr{O}}(1) \xrightarrow{\alpha_2} \varepsilon_*\Omega^1_X. \tag{3.9}$$

We want to show that $\rho = \iota$. It suffices to check this on the smooth locus of $\mathfrak{X}$, which reduces us to the case when $\mathfrak{X}$ is smooth. We claim that the maps ρ, ι are $\mathscr{O}_{\mathfrak{X}}$-linear. This is clear for ι; for ρ we look at the individual maps in the composition (3.9) that defines it: the second and the third map are clearly $\mathscr{O}_{\mathfrak{X}}$-linear, for the first map we use Theorem 2.5, and for the last map linearity is clear by the $\mathscr{O}_X$-linearity of Scholze's isomorphism $\alpha_2 : \mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}(1) \xrightarrow{\sim} \Omega^1_X$. Now, the claim that $\rho = \iota$ is local, so we way assume that $\mathfrak{X}$ is associated to a small algebra R with a framing $A = \mathscr{O}_C\{T_i^{\pm 1}\} \to R$. By functoriality, we may reduce to the case when $R = A$ and $A = \mathscr{O}_C\{T^{\pm 1}\}$. Now, the desired compatibility follows from the very construction of the isomorphism α_1. More precisely, since $\text{can} \circ \gamma_2$ is the multiplication by $\zeta_p - 1$, we have

$$(\zeta_p - 1)\gamma_2^{-1}(\alpha_1^{-1}(dT/T)) = \text{can}(\alpha_1^{-1}(dT/T)).$$

As we have already seen (cf. the discussion after Theorem 2.4) this corresponds to $(\gamma \mapsto (\zeta_p - 1) \otimes \frac{1}{\zeta_p - 1}\,\mathrm{dlog}(\zeta_\gamma))$ in $(\zeta_p - 1)H^1(\Gamma, A_\infty)\{1\}$. Now the compatibility of the map α_2 with the Kummer map (see the diagram (3.7)) shows that $\rho(dT/T) = dT/T$, as wanted.

Next, we claim that the composite

$$\begin{tikzcd} \mathscr{O}_X^* \arrow[r] & \mathrm{R}^1\nu_{X,*}\widehat{\mathbf{Z}}_p(1) \arrow[r] & \mathrm{R}^1\nu_{X,*}\mathbb{A}_{\mathrm{inf}}\{1\} \arrow[d, "\tilde{\theta}"] & & \\ & & \mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}^+\{1\} \arrow[r] & \mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}(1) \arrow[r] & \Omega_X^1 \end{tikzcd}$$

is the dlog map. Using the characterization of Scholze's isomorphism (3.6), this comes down to checking that the map

$$\mathrm{R}^1\nu_{X,*}\widehat{\mathbf{Z}}_p(1) \to \mathrm{R}^1\nu_{X,*}\mathbb{A}_{\mathrm{inf}}\{1\} \to \mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}^+\{1\} \to \mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}(1)$$

is the obvious one. But, by construction, this map is induced by the map

$$\widehat{\mathbf{Z}}_p(1) \to \mathbb{A}_{\mathrm{inf}}\{1\} \to \widehat{\mathscr{O}}^+\{1\} \to \widehat{\mathscr{O}}(1)$$

sending $x(1)$ to $\mu x\{1\}$, then to $\tilde{\theta}(\mu x\{1\}) = (\zeta_p - 1)x\{1\}$, then to $x(1)$, as desired.

The commutative diagram (3.5) extends to a commutative diagram

$$\begin{tikzcd} & \frac{\mathrm{R}^1\nu_*\mathbb{A}_{\mathrm{inf}}}{(\mathrm{R}^1\nu_*\mathbb{A}_{\mathrm{inf}})[\mu]}\{1\} \arrow[r, "\tilde{\theta}"] \arrow[dd, "\gamma_1"', "\wr"] & \frac{\mathrm{R}^1\nu_*\widehat{\mathscr{O}}^+}{(\mathrm{R}^1\nu_*\widehat{\mathscr{O}}^+)[\zeta_p - 1]}\{1\} \arrow[dd, "\gamma_2"', "\wr"] & \\ \varepsilon_*\mathscr{O}_X^* \arrow[d, "c_1^{\mathrm{pro\acute{e}t}}"] & & & \\ \mathrm{R}^1\nu_*\widehat{\mathbf{Z}}_p(1) \arrow[r, "\gamma"] \arrow[dr, "\beta"'] & H^1(A\Omega_{\mathfrak{X}}\{1\}) \arrow[r, "\tilde{\theta}"] \arrow[d, "\mathrm{can}"] & H^1(\widetilde{\Omega}_{\mathfrak{X}}\{1\}) \arrow[r, "\sim", "\alpha_1"'] \arrow[d, "\mathrm{can}"] & \Omega_{\mathfrak{X}}^1 \arrow[dd, "\iota"] \\ & \mathrm{R}^1\nu_*\mathbb{A}_{\mathrm{inf}}\{1\} \arrow[r, "\tilde{\theta}"] & \mathrm{R}^1\nu_*\widehat{\mathscr{O}}^+\{1\} \arrow[d] & \\ & & \mathrm{R}^1\nu_*\widehat{\mathscr{O}}(1) \arrow[r, "\sim", "\alpha_2"'] & \varepsilon_*\Omega_X^1 \end{tikzcd}$$

The only nonobvious commutativity is that of the right-bottom trapezoid, i.e. of diagram (3.5), which has already been checked. Using the diagram, the injectivity of ι and the fact that

$$\begin{tikzcd} \mathscr{O}_X^* \arrow[r] & \mathrm{R}^1\nu_{X,*}\widehat{\mathbf{Z}}_p(1) \arrow[r] & \mathrm{R}^1\nu_{X,*}\mathbb{A}_{\mathrm{inf}}\{1\} \arrow[d, "\tilde{\theta}"] & & \\ & & \mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}^+\{1\} \arrow[r] & \mathrm{R}^1\nu_{X,*}\widehat{\mathscr{O}}(1) \arrow[r] & \Omega_X^1 \end{tikzcd}$$

is the dlog map, we deduce that the composition

$$\varepsilon_*\mathscr{O}_X^* \xrightarrow{c_1^{\mathrm{pro\acute{e}t}}} \mathrm{R}^1\nu_*\widehat{\mathbf{Z}}_p(1) \xrightarrow{\gamma} H^1(A\Omega_{\mathfrak{X}}\{1\}) \xrightarrow{\tilde{\theta}} H^1(\widetilde{\Omega}_{\mathfrak{X}}\{1\}) \simeq \Omega_{\mathfrak{X}}^1$$

is the map dlog. Passing to global sections, it follows that the map

$$\begin{tikzcd}
\mathscr{O}(X)^* \arrow[r, "c_1^{\text{proét}}"] & H^0_{\text{ét}}(\mathfrak{X}, \mathrm{R}^1\nu_*\widehat{\mathbf{Z}}_p(1)) \arrow[r, "\gamma"] & H^0_{\text{ét}}(\mathfrak{X}, H^1(A\Omega_{\mathfrak{X}}\{1\})) \arrow[d, "\tilde{\theta}"] \\
& & H^0_{\text{ét}}(\mathfrak{X}, H^1(\widetilde{\Omega}_{\mathfrak{X}}\{1\})) \simeq H^0_{\text{ét}}(\mathfrak{X}, \Omega^1_{\mathfrak{X}})
\end{tikzcd}$$

is the dlog map.

Finally, coming back to the definitions of c_1^{inf} and $\iota_{\text{HT}}\{1\}$ we see that the composition

$$\begin{tikzcd}
\mathscr{O}(X_K)^* \arrow[r] & \mathscr{O}(X)^* \simeq H^1_{\text{ét}}(\mathfrak{X}, \tau_{\leq 1}(\mathrm{R}\nu_*\mathbb{G}_m[-1])) \arrow[d, "c_1^{\text{inf}}"] & \\
& H^1_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{1\}) \arrow[r, "\iota_{\text{HT}}\{1\}"] & H^0_{\text{ét}}(\mathfrak{X}, \Omega^1_{\mathfrak{X}})
\end{tikzcd}$$

is

$$\begin{tikzcd}
\mathscr{O}(X_K)^* \to \mathscr{O}(X)^* \simeq H^1_{\text{ét}}(\mathfrak{X}, \tau_{\leq 1}(\mathrm{R}\nu_*\mathbb{G}_m[-1])) \arrow[r, "c_1^{\text{proét}}"] & H^1_{\text{ét}}(\mathfrak{X}, \tau_{\leq 1}R\nu_*\widehat{\mathbf{Z}}_p(1)) \arrow[d, "\wr"] \\
H^1_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{1\}) \arrow[dr, "\tilde{\theta}"] & H^1_{\text{ét}}(\mathfrak{X}, R\nu_*\widehat{\mathbf{Z}}_p(1)) \arrow[l, "\gamma"'] \\
H^0_{\text{ét}}(\mathfrak{X}, \Omega^1_{\mathfrak{X}}) \simeq H^0_{\text{ét}}(\mathfrak{X}, H^1(\widetilde{\Omega}_{\mathfrak{X}}\{1\})) & H^1_{\text{ét}}(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}}\{1\}), \arrow[l, "e"']
\end{tikzcd}$$

where e is the (twisted) edge map in the local-global spectral sequence

$$E_2^{i,j} = H^i_{\text{ét}}(\mathfrak{X}, H^j(\widetilde{\Omega}_{\mathfrak{X}})) \Rightarrow H^{i+j}_{\text{ét}}(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}});$$

We conclude using the following commutative diagram, in which the vertical maps are edge maps in spectral sequences similar to the one above:

$$\begin{tikzcd}
\mathscr{O}(X)^* \arrow[r, "c_1^{\text{proét}}"] & H^1_{\text{ét}}(\mathfrak{X}, \mathrm{R}\nu_*\widehat{\mathbf{Z}}_p(1)) \arrow[r, "e"] \arrow[d, "\gamma"] & H^0_{\text{ét}}(\mathfrak{X}, \mathrm{R}^1\nu_*\widehat{\mathbf{Z}}_p(1)) \arrow[d, "\tilde{\theta}"] \\
& H^1_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{1\}) \arrow[r, "e"] \arrow[d, "\tilde{\theta}"] & H^0_{\text{ét}}(\mathfrak{X}, H^1(A\Omega_{\mathfrak{X}}\{1\})) \arrow[d] \\
& H^1_{\text{ét}}(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}}\{1\}) \arrow[r, "e"] & H^0_{\text{ét}}(\mathfrak{X}, H^1(\widetilde{\Omega}_{\mathfrak{X}}\{1\})) \simeq H^0_{\text{ét}}(\mathfrak{X}, \Omega^1_{\mathfrak{X}})
\end{tikzcd}$$

□

The case $i \geq 1$. Take now the symbol maps

$$r_{\text{inf}} : \mathscr{O}(X)^{*, \otimes i} \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$$

and consider the composition $\iota_{\rm HT} r_{\rm inf}$:

$$\begin{tikzcd}
\mathscr{O}(X)^{*,\otimes i} \arrow[r] & H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}) \arrow[r] & H^i_{\text{ét}}(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}}\{i\}) \arrow[d, "e"] \\
& & H^0_{\text{ét}}(\mathfrak{X}, H^i(\widetilde{\Omega}_{\mathfrak{X}}\{i\})) \simeq H^0_{\text{ét}}(\mathfrak{X}, \Omega^i_{\mathfrak{X}}).
\end{tikzcd}$$

To finish the proof of our proposition, in view of Lemma 3.4, it suffices to check that this composition is compatible with products. But, the edge map e is clearly compatible with products (it is induced by the restrictions $H^i_{\text{ét}}(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}}) \to H^i_{\text{ét}}(\mathfrak{U}, \widetilde{\Omega}_{\mathfrak{U}})$, for étale maps $\mathfrak{U} \to \mathfrak{X}$) and the first map is compatible with products by definition. The second map is induced by the map $\tilde{\theta} : A\Omega_{\mathfrak{X}} \to \widetilde{\Omega}_{\mathfrak{X}}$ which is compatible with products as can be easily seen from its definition (see the paragraph just before Theorem 2.8) using the fact that the functor L_η is lax (symmetric) monoidal [5, Prop. 6.7]. Finally, the last map is the isomorphism given by Theorem 2.5 hence is compatible with products by its very definition. □

4. The $A_{\rm inf}$-cohomology of Drinfeld symmetric spaces

Let $\mathscr{H} = \mathbb{P}((K^{d+1})^*) \simeq \mathbb{P}^d(K)$ be the space of K-rational hyperplanes in K^{d+1}. Let

$$\mathbb{H}^d_K := \mathbb{P}^d_K \setminus \cup_{H \in \mathscr{H}} H$$

be the Drinfeld symmetric space of dimension d. It is a rigid analytic space. Let $\mathfrak{X}_{\mathscr{O}_K}$ be the standard semistable formal model over $\mathscr{O}_K$ of $\mathbb{H}^d_K$ (see [15, Section 6.1]). Let $\mathfrak{X} := \mathfrak{X}_{\mathscr{O}_K} \widehat{\otimes}_{\mathscr{O}_K} \mathscr{O}_C$, let $X := \mathbb{H}^d_K \hat{\otimes}_K C$ be the rigid analytic generic fiber of $\mathfrak{X}$, and let $X_K = \mathbb{H}^d_K$. The group $G = \mathrm{GL}_{d+1}(K)$ acts naturally on all these objects.

The main goal of this section is to prove the following (here and elsewhere in the paper, the completed tensor product is taken in the category of pro-discrete modules):

Theorem 4.1. *Let $i \geq 0$. There is a $G \times \mathscr{G}_K$-equivariant isomorphism of topological $A_{\rm inf}$-modules*

$$A_{\rm inf} \widehat{\otimes}_{\mathbf{Z}_p} \mathrm{Sp}_i(\mathbf{Z}_p)^* \simeq H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}), \tag{4.2}$$

where $\mathrm{Sp}_i(\mathbf{Z}_p)^*$ is the $\mathbf{Z}_p$-dual of a generalized Steinberg representation (see Section 4.1 for a definition). This isomorphism is compatible with the operator φ^{-1}.

4.1. **Generalized Steinberg representations and their duals.**

4.1.1. *Generalized Steinberg representations.* Let B be the upper triangular Borel subgroup of G and $\Delta = \{1, 2, \ldots, d\}$, identified with the set of simple roots associated to B. For each subset J of Δ we let P_J be the corresponding standard parabolic subgroup of G and set $X_J = G/P_J$, a compact topological space.

If A is an abelian group and $J \subset \Delta$, let

$$\mathrm{Sp}_J(A) = \frac{\mathrm{LC}(X_J, A)}{\sum_{i \in \Delta \setminus J} \mathrm{LC}(X_{J \cup \{i\}}, A)},$$

where LC means locally constant (automatically with compact support). This is a smooth G-module over A and we have a canonical isomorphism $\mathrm{Sp}_J(A) \simeq \mathrm{Sp}_J(\mathbf{Z}) \otimes A$. For $J = \varnothing$ we obtain the usual Steinberg representation with coefficients in A, while for $J = \Delta$ we have $\mathrm{Sp}_J(A) = A$. For $r \in \{0, 1, \ldots, d\}$, we use the simpler notation

$$\mathrm{Sp}_r := \mathrm{Sp}_{\{1,2,\ldots,d-r\}}$$

and we set $\mathrm{Sp}_r = 0$, for $r > d$.

We will need the following result:

Theorem 4.3. (Grosse-Klönne, [17, Cor. 4.3]) *If A is a field of characteristic p then* $\mathrm{Sp}_J(A)$ *(for varying J) are the irreducible constituents of* $\mathrm{LC}(G/B, A)$*, each occurring with multiplicity* 1.

4.1.2. *Duals of generalized Steinberg representations.* If Λ is a topological ring, then $\mathrm{Sp}_J(\Lambda)$ has a natural topology: the space X_J being profinite, we can write $X_J = \varprojlim_n X_{n,J}$ for finite sets $X_{n,J}$ and then $\mathrm{LC}(X_J, \Lambda) = \varinjlim_n \mathrm{LC}(X_{n,J}, \Lambda)$, each $\mathrm{LC}(X_{n,J}, \Lambda)$ being a finite free Λ-module endowed with the natural topology; $\mathrm{Sp}_J(\Lambda)$ has the induced quotient topology.

Let $M^* := \mathrm{Hom}_{\mathrm{cont}}(M, \Lambda)$ for any topological Λ-module M, and equip M^* with the weak topology. Then $\mathrm{LC}(X_J, \Lambda)^*$ is naturally isomorphic to $\varprojlim_n \mathrm{LC}(X_{n,J}, \Lambda)^*$, i.e., it is a countable inverse limit of finite free Λ-modules. In particular, suppose that L is a finite extension of $\mathbf{Q}_p$. Then $\mathrm{Sp}_J(\mathscr{O}_L)^*$ is a compact $\mathscr{O}_L$-module, which is torsion-free.

If S is a profinite set and A an abelian group, let

$$D(S, A) = \mathrm{Hom}(\mathrm{LC}(S, \mathbf{Z}), A) = \mathrm{LC}(S, A)^*$$

be the space of A-valued locally constant distributions on S. We recall the interpretation of $\mathrm{Sp}_i(\mathbf{Z}_p)^*$ in terms of distributions. Recall that $\mathscr{H}$ denotes the compact space of K-rational hyperplanes in K^{d+1}. If $H \in \mathscr{H}$, let ℓ_H be a unimodular equation for H (thus ℓ_H is a linear form with integer coefficients, at least one of them being a unit). Let

$\mathrm{LC}^c(\mathscr{H}^{i+1}, \mathbf{Z})$ be the space of locally constant functions $f : \mathscr{H}^{i+1} \to \mathbf{Z}$ such that, for all $H_0, ..., H_{i+1} \in \mathscr{H}$,

$$f(H_1, ..., H_{i+1}) - f(H_0, H_2, ..., H_{i+1}) + \cdots + (-1)^{i+1} f(H_0, ..., H_i) = 0$$

and, if ℓ_{H_j}, $0 \leq j \leq i$, are linearly dependent, then $f(H_0, ..., H_i) = 0$. The work of Schneider-Stuhler [10, Sec. 5.4.1] gives a G-equivariant isomorphism

$$\mathrm{Sp}_i(\mathbf{Z}) \simeq \mathrm{LC}^c(\mathscr{H}^{i+1}, \mathbf{Z}).$$

It follows that the inclusion $\mathrm{LC}^c(\mathscr{H}^{i+1}, \mathbf{Z}) \subset \mathrm{LC}(\mathscr{H}^{i+1}, \mathbf{Z})$ gives rise to a strict exact sequence

$$0 \to D(\mathscr{H}^{i+1}, A)_{\mathrm{deg}} \to D(\mathscr{H}^{i+1}, A) \to \mathrm{Hom}(\mathrm{Sp}_i(\mathbf{Z}), A) \to 0, \tag{4.4}$$

where $D(\mathscr{H}^{i+1}, A)_{\mathrm{deg}}$ is the space of degenerate distributions (which is defined via the exact sequence above).

4.2. **Integral de Rham cohomology of Drinfeld symmetric spaces.** Recall the following acyclicity result of Grosse-Klönne, which played a crucial role in [10].

Theorem 4.5. (Grosse-Klönne, [14, Th. 4.5], [16, Prop. 4.5]) *For $i > 0$, $j \geq 0$, we have $H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}_{\mathscr{O}_K}, \Omega^j_{\mathfrak{X}_{\mathscr{O}_K}}) = 0$ and $d = 0$ on $H^0_{\mathrm{\acute{e}t}}(\mathfrak{X}_{\mathscr{O}_K}, \Omega^j_{\mathfrak{X}_{\mathscr{O}_K}})$. In particular, we have a natural quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathfrak{X}_{\mathscr{O}_K}) \simeq \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(\mathfrak{X}_{\mathscr{O}_K}, \Omega^\bullet_{\mathfrak{X}_{\mathscr{O}_K}}) \simeq \bigoplus_{i \geq 0} \Gamma_{\mathrm{\acute{e}t}}(\mathfrak{X}_{\mathscr{O}_K}, \Omega^i_{\mathfrak{X}_{\mathscr{O}_K}})[-i].$$

Using it and some extra work, we have obtained the following description of $H^i_{\mathrm{dR}}(\mathfrak{X}_{\mathscr{O}_K})$:

Theorem 4.6. (Colmez-Dospinescu-Nizioł, [10, Th. 6.26]) *Let $i \geq 0$. There are natural de Rham and Hodge–Tate regulator maps*

$$\begin{aligned} r_{\mathrm{dR}} &: D(\mathscr{H}^{i+1}, \mathscr{O}_K) \to H^i_{\mathrm{dR}}(\mathfrak{X}_{\mathscr{O}_K}), \\ r_{\mathrm{HT}} &: D(\mathscr{H}^{i+1}, \mathscr{O}_K) \to H^0(\mathfrak{X}_{\mathscr{O}_K}, \Omega^i_{\mathfrak{X}_{\mathscr{O}_K}}) \end{aligned}$$

that induce topological G-equivariant isomorphisms in the commutative diagram:

$$\begin{tikzcd} \mathrm{Sp}_i(\mathscr{O}_K)^* \arrow[r, "\sim", "r_{\mathrm{dR}}"'] \arrow[dr, "\sim", "r_{\mathrm{HT}}"'] & H^i_{\mathrm{dR}}(\mathfrak{X}_{\mathscr{O}_K}) \\ & H^0(\mathfrak{X}_{\mathscr{O}_K}, \Omega^i_{\mathfrak{X}_{\mathscr{O}_K}}) \arrow[u, "\wr"] \end{tikzcd}$$

Proof. (Sketch) Our starting point was the computation of Schneider-Stuhler [21, chap. 3,4]: a G-equivariant topological isomorphism

$$\mathrm{Sp}_i(K)^* \xrightarrow[\sim]{\alpha_S} H^i_{\mathrm{dR}}(X_K).$$

Iovita-Spiess [18] made this isomorphism explicit: they proved that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & D(\mathscr{H}^{i+1},K)_{\mathrm{deg}} \arrow[r] & D(\mathscr{H}^{i+1},K) \arrow[r] \arrow[dr, "r_{\mathrm{dR}}"] & \mathrm{Sp}_i(K)^* \arrow[r] \arrow[d, "\alpha_S", "\wr"'] & 0 \\ & & & H^i_{\mathrm{dR}}(X_K) & \end{tikzcd}$$

With a help from a detailed analysis of the integral Hyodo-Kato cohomology of the special fiber of $\mathfrak{X}_{\mathscr{O}_K}$ and some representation theory[11] this computation can be lifted to $\mathscr{O}_K$. $\square$

The following computation follows immediately:

Corollary 4.7. *Let $i \geq 0$.*

(1) *The de Rham regulator r_{dR} induces a topological G-equivariant isomorphism*

$$r_{\mathrm{dR}} : \mathrm{Sp}_i(\mathscr{O}_K)^* \widehat{\otimes}_{\mathscr{O}_K} \mathscr{O}_C \xrightarrow{\sim} H^i_{\mathrm{dR}}(\mathfrak{X}_{\mathscr{O}_K}) \widehat{\otimes}_{\mathscr{O}_K} \mathscr{O}_C \xrightarrow{\sim} H^i_{\mathrm{dR}}(\mathfrak{X}).$$

(2) *The Hodge–Tate regulator r_{HT} induces a topological G-equivariant isomorphism*

$$r_{\mathrm{HT}} : \mathrm{Sp}_i(\mathscr{O}_K)^* \widehat{\otimes}_{\mathscr{O}_K} \mathscr{O}_C \xrightarrow{\sim} H^0_{\mathrm{\acute{e}t}}(\mathfrak{X}_{\mathscr{O}_K}, \Omega^i_{\mathfrak{X}_{\mathscr{O}_K}}) \widehat{\otimes}_{\mathscr{O}_K} \mathscr{O}_C \xrightarrow{\sim} H^0_{\mathrm{\acute{e}t}}(\mathfrak{X}, \Omega^i_{\mathfrak{X}}).$$

4.3. **Integrating symbols.** Let $i \geq 1$. In this section, our goal is to construct natural compatible regulator maps

$$r_{\mathrm{\acute{e}t}} : \mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^i_{\mathrm{\acute{e}t}}(X, \mathbf{Z}_p(i)), \quad r_{\mathrm{inf}} : \mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$$

that are compatible with the classical étale and A_{inf}-regulators. We will show later that (the linearizations of) both regulators are $G \times \mathscr{G}_K$-equivariant isomorphisms. The maps $r_{\mathrm{\acute{e}t}}$, r_{inf} are constructed by interpreting elements of $\mathrm{Sp}_i(\mathbf{Z}_p)^*$ as suitable distributions (see the discussion in Section 4.1.2), and integrating étale and A_{inf}-symbols of invertible functions on $\mathbb{H}^d_K$ against them. This idea appears in Iovita-Spiess [18] and was also heavily used in [10].

[11]We used here two facts: (a) $\mathrm{Sp}_i(\mathscr{O}_K)$ is, up to a K^*-homothety, the unique G-stable lattice in $\mathrm{Sp}_i(K)$; (b) $\mathrm{Sp}_i(k)$, the reduction mod p of $\mathrm{Sp}_i(\mathscr{O}_K)$, is irreducible.

4.3.1. *Integrating étale symbols.* We start with the construction of the étale regulator map

$$r_{\text{ét}}: \mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^i_{\text{ét}}(X, \mathbf{Z}_p(i)).$$

Fix a cohomological degree i and set $M := H^i_{\text{ét}}(X, \mathbf{Z}_p(i))$. For $H_0, ..., H_i \in \mathscr{H}$, let

$$\psi_{\text{ét}}(H_0, ..., H_i) := r_{\text{ét}}\left(\frac{\ell_{H_1}}{\ell_{H_0}} \otimes ... \otimes \frac{\ell_{H_i}}{\ell_{H_0}}\right) \in M,$$

where $r_{\text{ét}}: \mathscr{O}(\mathbb{H}^d_C)^{*,\otimes i} \to M$ is the étale regulator map. It is clear that this definition is independent of the choice of the unimodular equations for $H_0, ..., H_i$.

Proposition 4.8. *Let $i \geq 1$.*

(1) *Let δ_x denote the Dirac distribution at x. There is a unique continuous $\mathbf{Z}_p$-linear map*
$$r_{\text{ét}}: D(\mathscr{H}^{i+1}, \mathbf{Z}_p) \to H^i_{\text{ét}}(X, \mathbf{Z}_p(i))$$
such that $r_{\text{ét}}(\delta_{(H_0,...,H_i)}) = \psi_{\text{ét}}(H_0, ..., H_i)$ for all $H_0, ..., H_i \in \mathscr{H}$.

(2) *The map $r_{\text{ét}}$ factors through the quotient $\mathrm{Sp}_i(\mathbf{Z}_p)^*$ of $D(\mathscr{H}^{i+1}, \mathbf{Z}_p)$ and induces a natural map of $\mathbf{Z}_p$-modules*
$$r_{\text{ét}}: \mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^i_{\text{ét}}(X, \mathbf{Z}_p(i)).$$

Proof. Uniqueness in (1) is clear since the $\mathbf{Z}_p$-submodule of $D(\mathscr{H}^{i+1}, \mathbf{Z}_p)$ spanned by the Dirac distributions is dense.

Existence in (1) requires more work. Let $\{U_n\}_{n\geq 1}$ be the standard admissible affinoid covering of X (see [10, proof of Th. 5.8]). Let $\Pi(n)$ be the profinite étale fundamental group of U_n. Denote by $\mathrm{R}\Gamma(\Pi(n), \mathbf{Z}_p(i))$ the complex of nonhomogenous continuous cochains representing the continuous group cohomology of $\Pi(n)$. By the $K(\pi, 1)$-Theorem of Scholze [22, Th. 1.2] this complex also represents $\mathrm{R}\Gamma_{\text{ét}}(U_n, \mathbf{Z}_p(i))$. Since the action of $\Pi(n)$ on $\mathbf{Z}_p(1)$ is trivial the local étale Chern class map factors as

$$c^{\text{ét}}_{1,n}: \mathscr{O}(U_n)^* \to \mathrm{Hom}(\Pi(n), \mathbf{Z}_p(1)) \to \mathrm{R}\Gamma(\Pi(n), \mathbf{Z}_p(1))[1].$$

The global étale Chern class is represented by the composition

$$\begin{tikzcd} c^{\text{ét}}_{1,n}: \mathscr{O}(X)^* \arrow[r] & \varprojlim_n \mathscr{O}(U_n)^* \arrow[r] & \mathrm{holim}_n \mathrm{Hom}(\Pi(n), \mathbf{Z}_p(1)) \arrow[d] \\ & \mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Z}_p(1))[1] \arrow[r, "\sim"] & \mathrm{holim}_n \mathrm{R}\Gamma(\Pi(n), \mathbf{Z}_p(1))[1] \end{tikzcd}$$

The étale regulator $r_{\text{ét}}: \mathscr{O}(X)^{*,\otimes i} \to \mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Z}_p(i))[i]$ is then represented by the cup product: $r_{\text{ét}} := c^{\text{ét}}_1 \cup \cdots \cup c^{\text{ét}}_1$.

The composition

$$\Psi_i : \mathscr{H}^{i+1} \to \mathscr{O}(X)^{*,\otimes i} \xrightarrow{r_{\text{ét}}} \mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Z}_p(i))[i] \tag{4.9}$$

represents the map $\psi_{\text{ét}}$. We claim that it is continuous. Indeed, it suffices to show that so are the induced maps $\Psi_{i,n} : \mathscr{H}^{i+1} \to \mathrm{R}\Gamma(\Pi(n), \mathbf{Z}_p(i))[i]$, for $n \geq 1$. Or, by continuty of the cup product that so are the maps $\Psi_{1,n}$. Or, simplifying further, that so are the maps

$$\Psi_{1,n} : \mathscr{H}^2 \to \mathscr{O}(X)^* \xrightarrow{r_{\text{ét}}} \mathrm{Hom}(\Pi(n), \mathbf{Z}_p(1)). \tag{4.10}$$

To show this, write $\mathscr{H} = \varprojlim_m \mathscr{H}_m$, where $\mathscr{H}_m$ is the set of m-equivalence classes of K-rational hyperplanes[12] and set

$$M_n := \mathrm{Hom}(\Pi(n), \mathbf{Z}_p(1)).$$

It suffices to show that, for each $k \geq 1$, there is an m such that the map

$$\Psi_{1,n,k} : \mathscr{H}^2 \xrightarrow{\Psi_1} M_n \to M_n/p^k M_n$$

factors through the projection $\mathscr{H}^2 \to \mathscr{H}_m^2$. Taking into account the construction of $\Psi_{1,n}$, it suffices to show that, for m large enough, if two hyperplanes H_0, H_1 are m-equivalent, then $r_{\text{ét}}(\ell_{H_1}/\ell_{H_0}) \in p^k M_n$. But this is clear, since in this case ℓ_{H_1}/ℓ_{H_0} has a $p^{r_n m}$'th root in $\mathscr{O}(U_n)^*$, for some constant $r_n > 0$ depending only on U_n, and since $r_{\text{ét}}$ is a homomorphism, we have $r_{\text{ét}}(\ell_{H_1}/\ell_{H_0}) \in p^{r_n m} M_n$.

Since $\mathrm{Hom}(\Pi(n), \mathbf{Z}_p(1))$ is a Banach space and the map $\Psi_{i,n}$, defined below (4.9), is continuous on $\mathscr{H}^{i+1}$, it defines, by integration against distibutions, a continuous map

$$r_{\text{ét},n} : D(\mathscr{H}^{i+1}, \mathbf{Z}_p) \to \mathrm{R}\Gamma_{\text{ét}}(U_n, \mathbf{Z}_p(i))[i]$$

such that $r_{\text{ét},n}(\delta_{(H_0,\ldots,H_i)}) = \psi_{\text{ét},n}(H_0, \ldots, H_i)$, for all $H_0, \ldots, H_i \in \mathscr{H}$, where $\psi_{\text{ét},n}$ is the analog of $\psi_{\text{ét}}$ for U_n. The construction being compatible with the change of n we get the existence of the map in (1) by setting $r_{\text{ét}} := \varprojlim_n r_{\text{ét},n}$ and passing to cohomology.

For (2) we need to check the factorization of the regulator $r_{\text{ét}}$ from (1) through the quotient by the degenerate distributions. That is, we need to show that, for any $\mu \in D(\mathscr{H}^{i+1}, \mathbf{Z}_p)_{\deg}$, we have $r_{\text{ét}}(\mu) = 0$. For that, by the construction of $r_{\text{ét}}(\mu)$, it suffices to check that:

(1) for all $H_0, ..., H_{i+1} \in \mathscr{H}$, we have

$$\psi_{\text{ét}}(H_1, ..., H_{i+1}) - \psi_{\text{ét}}(H_0, H_2, ..., H_{i+1}) + ... + (-1)^{i+1}\psi_{\text{ét}}(H_0, ..., H_i) = 0 \tag{4.11}$$

[12] Recall that two hyperplanes H_1, H_2 are called m-equivalent (i.e., $[H_1] = [H_2] \in \mathscr{H}_m$) if they have unimodular equations ℓ_1, ℓ_2 such that $\ell_1 = \ell_2$ modulo ϖ^m, where ϖ is a uniformizer of K.

(2) if the ℓ_{H_j}, $0 \le j \le i$, are linearly dependent, $\psi_{\text{ét}}(H_0, ..., H_i) = 0$.

To see (1) note that we can rewrite (4.11) as

$$\psi_{\text{ét}}(H_1, ..., H_{i+1}) = \psi_{\text{ét}}(H_0, H_2, ..., H_{i+1}) - ... + (-1)^i \psi_{\text{ét}}(H_0, ..., H_i).$$

Write $\psi_{n_1,...,n_{i+1}}$ for $\psi_{\text{ét}}(H_{n_1}, \dots, H_{n_{i+1}})$ and ℓ_j for ℓ_{H_j}. We compute, using the fact that $r_{\text{ét}}$ is alternate (this kills terms with two $\frac{\ell_0}{\ell_1}$ which allows us to go from line 1 to line 2, and introduces signs when we move 1 in front to go from line 3 to line 4),

$$\begin{aligned}
\psi_{1,...,i+1} &= r_{\text{ét}}\left(\tfrac{\ell_2}{\ell_1} \otimes \cdots \otimes \tfrac{\ell_{i+1}}{\ell_1}\right) = r_{\text{ét}}\left(\tfrac{\ell_2}{\ell_0}\tfrac{\ell_0}{\ell_1} \otimes \cdots \otimes \tfrac{\ell_{i+1}}{\ell_0}\tfrac{\ell_0}{\ell_1}\right) \\
&= r_{\text{ét}}\left(\tfrac{\ell_2}{\ell_0} \otimes \cdots \otimes \tfrac{\ell_{i+1}}{\ell_0}\right) \\
&\quad + \sum_{s=2}^{i+1} r_{\text{ét}}\left(\tfrac{\ell_2}{\ell_0} \otimes \ldots \otimes \tfrac{\ell_{s-1}}{\ell_0} \otimes \tfrac{\ell_0}{\ell_1} \otimes \tfrac{\ell_{s+1}}{\ell_0} \otimes \cdots \otimes \tfrac{\ell_{i+1}}{\ell_0}\right) \\
&= \psi_{0,2,...,i+1} + \sum_{s=2}^{i+1} (-1)\psi_{0,2,...,s-1,1,s+1,...,i+1} \\
&= \sum_{s=1}^{i+1} (-1)^{s-1} \psi_{0,1,...,\check{s},...,i+1}
\end{aligned}$$

as wanted.

(2) follows from the fact that the étale regulator satisfies the Steinberg relations. More precisely, if $x_j = \ell_j/\ell_0$, $0 \le j \le i$, where $\ell_0, \dots, \ell_i$ are linear equations of K-rational hyperplanes, it suffices to show that the symbol $\{x_1, \dots, x_i, 1+a_1x_1+\cdots+a_ix_i\}$ vanishes in the Milnor K-theory group $K^M_{i+1}(\mathscr{O}(X)^*)$ when $a_j \in K$. Note that the symbol $\{x_1, \dots, x_i, 1\}$ vanishes. We will reduce to this case by the following algorithm.

Step 1: up to reordering we may assume that $y_1 := (1 + a_1x_1) \neq 0$ (otherwise we are done). Then, using the Steinberg relations $\{z, 1-z\} = 0$ and the fact that $\{x, a\} = 0$, for $a \in K^*$, we compute

$$\{x_1, x_2, \dots, x_i, 1+a_1x_1+\cdots+a_ix_i\} = \{x_1, \tfrac{x_2}{y_1}, \dots, \tfrac{x_i}{y_1}, 1+\tfrac{a_2x_2}{y_1}+\cdots+\tfrac{a_ix_i}{y_1}\}.$$

Note that this makes sense since $\frac{x_j}{y_1} = \frac{\ell_j}{\ell_0+a_1\ell_1} \in \mathscr{O}(X)^*$ and, in fact, is again a quotient of two linear equations of K-hyperplanes.

Step 2: reorder the terms in the last symbol to make $\frac{x_2}{y_1}$ appear first and repeat.

□

4.3.2. *Integrating A_{inf}-symbols.* Let $i \ge 1$. We pass now to the $\mathbf{A}_{\text{inf}}$-regulator map

$$r_{\text{inf}} : A_{\text{inf}}\widehat{\otimes}_{\mathbf{Z}_p}\mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$$

that is compatible with the classical A_{inf}-regulator as well as with the étale regulator

$$r_{\text{ét}}: \mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^i_{\text{ét}}(X, \mathbf{Z}_p(i))$$

defined above. To start, we define the regulators

$$\begin{aligned} r_{\mathrm{inf}} &: D(\mathscr{H}^{i+1}, \mathbf{Z}_p) \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}),\\ r_{\mathrm{inf}} &: \mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})\end{aligned}$$

by setting $r_{\mathrm{inf}} := \gamma r_{\text{ét}}$, where $\gamma: H^i_{\text{ét}}(X, \mathbf{Z}_p(i)) \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$ is the canonical map from theorem 2.9 and the étale regulator

$$r_{\text{ét}}: D(\mathscr{H}^{i+1}, \mathbf{Z}_p) \to H^i_{\text{ét}}(X, \mathbf{Z}_p(i)) \tag{4.12}$$

is the map defined above.

Corollary 4.13. *Let $i \geq 1$. The above regulators extend uniquely to compatible continuous A_{inf}-linear maps*

$$\begin{aligned} r_{\mathrm{inf}} &: A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p} D(\mathscr{H}^{i+1}, \mathbf{Z}_p) \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}),\\ r_{\mathrm{inf}} &: A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p} \mathrm{Sp}_i(\mathbf{Z}_p)^* \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})\end{aligned}$$

that are compatible with the étale regulators.

Proof. Uniqueness is clear. To show the existence, let $\{U_n\}_{n\in\mathbf{N}}$ be the standard admissible affinoid covering of X. For $n \in \mathbf{N}$, set

$$r_{\mathrm{inf},n}: D(\mathscr{H}^{i+1}, \mathbf{Z}_p) \to \mathrm{R}\Gamma_{\text{ét}}(\mathscr{U}_n, A\Omega_{\mathscr{U}_n}\{i\})[i], \quad r_{\mathrm{inf,n}} := \gamma r_{\text{ét},n},$$

where $\mathscr{U}_n$ is the standard semistable formal model of U_n (see [10, Sec. 5.1]) and the map

$$r_{\text{ét},n}: D(\mathscr{H}^{i+1}, \mathbf{Z}_p) \to \mathrm{R}\Gamma_{\text{ét}}(U_n, \mathbf{Z}_p(i))[i]$$

was constructed above. The map $r_{\mathrm{inf},n}$ factors as

$$r_{\mathrm{inf},n}: D(\mathscr{H}^{i+1}, \mathbf{Z}_p) \to M_n \to \mathrm{R}\Gamma_{\text{ét}}(\mathscr{U}_n, A\Omega_{\mathscr{U}_n}\{i\})[i],$$

where $M_n := \mathrm{Hom}(\Pi(n), \mathbf{Z}_p(i))$ for the fundamental group $\Pi(n)$ of U_n.

Since $r_{\text{ét}} = \mathrm{holim}_n\, r_{\text{ét},n}$, we have the factorization

$$\begin{tikzcd} r_{\text{ét}}: D(\mathscr{H}^{i+1}, \mathbf{Z}_p) \arrow[r] & \varprojlim_n M_n \arrow[r, "\mathrm{holim}_n r_{\text{ét},n}"] & \mathrm{holim}_n \mathrm{R}\Gamma_{\text{ét}}(U_n, \mathbf{Z}_p(i))[i] \\ & & \mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Z}_p(i))[i] \arrow[u, "\wr"'] \end{tikzcd}$$

This induces the following composition of maps

$$r_{\mathrm{inf}} := \mathrm{holim}_n\, r_{\mathrm{inf},n}$$

$$\begin{tikzcd}
r_{\mathrm{inf}} : A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p} D(\mathscr{H}^{i+1}, \mathbf{Z}_p) \arrow[r] & A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p} \varprojlim_n M_n \arrow[r, "\sim"] & \varprojlim_n A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p} M_n \arrow[dll] \\
\mathrm{holim}_n \mathrm{R}\Gamma_{\text{ét}}(\mathscr{U}_n, A\Omega_{\mathfrak{X}}\{i\})[i] & \mathrm{R}\Gamma_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})[i] \arrow[l, "\sim"'] &
\end{tikzcd}$$

The existence of the first map and of the following isomorphism is clear. The third map exists because both $A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p} M_n$ and $\mathrm{R}\Gamma_{\text{ét}}(\mathscr{U}_n, A\Omega_{\mathfrak{X}}\{i\})$ are derived (p,μ)-adically complete. This proves the existence of the first regulator in the corollary. The existence of the second follows immediately from the fact that the map (4.12) factors through $\mathrm{Sp}_i(\mathbf{Z}_p)^*$ once we know that the sequence

$$0 \to A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p} D(\mathscr{H}^{i+1})_{\mathrm{deg}} \to A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p} D(\mathscr{H}^{i+1}) \to A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p} \mathrm{Sp}_i^* \to 0$$

(with $D(\mathscr{H}^{i+1})_{\mathrm{deg}} = D(\mathscr{H}^{i+1}, \mathbf{Z}_p)_{\mathrm{deg}}$, $D(\mathscr{H}^{i+1}) = D(\mathscr{H}^{i+1}, \mathbf{Z}_p)$, and $\mathrm{Sp}_i^* = \mathrm{Sp}_i(\mathbf{Z}_p)^*$) is strict exact. This sequence is obtained from the strict exact sequence (4.4) by tensoring with A_{inf}. Hence the only question is the strict surjection on the right, which follows from the fact that the sequence (4.4) is actually split (since all modules are duals of free modules). $\square$

4.4. **The A_{inf}-cohomology of Drinfeld symmetric spaces.** We are now ready to prove Theorem 4.1. If $i = 0$ both sides of (4.2) are naturally isomorphic to A_{inf}.

Let now $i \geq 1$. We will show that the map r_{inf} induces a φ^{-1}-equivariant topological isomorphism of A_{inf}-modules

$$r_{\mathrm{inf}} : A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p}\mathrm{Sp}_i(\mathbf{Z}_p)^* \xrightarrow{\sim} H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}).$$

Compatibility with the operator φ^{-1} follows from the fact that r_{inf} is A_{inf}-linear and it is induced from $r_{\text{ét}}$ hence maps $D(\mathscr{H}^{i+1}, \mathbf{Z}_p)$ to $H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1}$. For the rest of the claim, first, we show that the induced map

$$\overline{r}_{\mathrm{inf}} : (A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p}\mathrm{Sp}_i(\mathbf{Z}_p)^*)/\tilde{\xi} \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})/\tilde{\xi}$$

is a topological isomorphism. But this map fits into the following commutative diagram (of A_{inf}-linear continuous maps, where A_{inf} acts on $\mathscr{O}_C$

via $\tilde{\theta}$)

$$\begin{tikzcd}
(A_{\mathrm{inf}}\widehat{\otimes}_{\mathbf{Z}_p}\mathrm{Sp}_i(\mathbf{Z}_p)^*)/\tilde{\xi} \arrow[r, "\overline{r}_{\mathrm{inf}}"] \arrow[d, "\tilde{\theta}", "\wr"'] & H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})/\tilde{\xi} \arrow[d, hook, "\alpha"] \arrow[dd, bend left=60, "\overline{\iota}_{\mathrm{HK}}"] \\
\mathscr{O}_C\widehat{\otimes}_{\mathbf{Z}_p}\mathrm{Sp}_i(\mathbf{Z}_p)^* \arrow[dr, "r_{\mathrm{HT}}", "\sim"'] & H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}/\tilde{\xi}) \arrow[d, "\wr"'] \\
& H^0_{\mathrm{\acute{e}t}}(\mathfrak{X}, \Omega^i_{\mathfrak{X}})
\end{tikzcd}$$

The map α is the change-of-coefficients map; it is clearly injective. The lower right vertical map is an isomorphism because we have the local-global spectral sequence

$$E_2^{s,t} = H^s_{\mathrm{\acute{e}t}}(\mathfrak{X}, H^t(A\Omega_{\mathfrak{X}}\{i\}/\tilde{\xi})) \Rightarrow H^{s+t}_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\}/\tilde{\xi})$$

and, by Theorem 2.8 and Theorem 2.5, the isomorphisms $H^t(A\Omega_{\mathfrak{X}}\{i\}/\tilde{\xi}) \simeq H^t(\widetilde{\Omega}_{\mathfrak{X}}\{i\}) \simeq \Omega^t_{\mathfrak{X}}\{i-t\}$. Hence, by Theorem 4.5,

$$E_2^{s,t} = H^s_{\mathrm{\acute{e}t}}(\mathfrak{X}, \Omega^t\{i-t\}) = 0, \quad s \geq 1.$$

The above diagram commutes by Proposition 3.3. The slanted arrow is a topological isomorphism by Corollary 4.7. It follows that the map α is surjective, hence it is an isomorphism and so is, by the above diagram, the map $\overline{r}_{\mathrm{inf}}$. The latter is also a topological isomorphism because so is the map $\tilde{\theta}$ and the map $\overline{\iota}_{\mathrm{HK}}$ is a continuous isomorphism.

Next, we will show that $\overline{r}_{\mathrm{inf}}$ being a topological isomorphism implies that so is the original map r_{inf}. Let T be the homotopy fiber of r_{inf}. We claim that the complex

$$T \otimes^{\mathrm{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\tilde{\xi} \simeq 0. \tag{4.14}$$

Indeed, since $\overline{r}_{\mathrm{inf}}$ is an isomorphism, it suffices to show that the domain and the target of r_{inf} are $\tilde{\xi}$-torsion free. This is clear for the domain. For the target, note that the distinguished triangle

$$A\Omega_{\mathfrak{X}}\{i\} \xrightarrow{\tilde{\xi}} A\Omega_{\mathfrak{X}}\{i\} \to A\Omega_{\mathfrak{X}}\{i\}/\tilde{\xi}$$

yields an exact sequence

$$0 \to H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})/\tilde{\xi} \xrightarrow{\alpha} H^i_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}/\tilde{\xi}) \to H^{i+1}_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})[\tilde{\xi}] \to 0.$$

By the above, α is an isomorphism, hence $H^{i+1}_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})[\tilde{\xi}] = 0$. Since $i \geq 0$ was arbitrary, we deduce that, for all $j \geq 1$ and all i, $H^j_{\mathrm{\acute{e}t}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$ has no $\tilde{\xi}$-torsion, and this is clearly true for $j = 0$ as well.

Since T is derived $\tilde{\xi}$-complete (because so are the domain and the target of $r_{\rm inf}$, the latter using the derived $\tilde{\xi}$-completeness of $A\Omega_{\mathfrak{X}}$ and the preservation of this property by derived pushforward and passage to cohomology), by the derived Nakayama Lemma (see Section 2.1.1) we have $T \simeq 0$ as well. This finishes the proof that $r_{\rm inf}$ is an isomorphism.

Since the domain and the target of $r_{\rm inf}$ are $\tilde{\xi}$-torsion-free and the reduction $\overline{r}_{\rm inf}$ is a topological isomorphism so is $r_{\rm inf}$. This finishes the proof.

5. Integral p-adic étale cohomology of Drinfeld symmetric spaces

We are now ready to compute the integral p-adic étale cohomology of Drinfeld symmetric spaces. Let $X_K := \mathbb{H}^d_K$ be the Drinfeld symmetric space of dimension d over K and let $\mathfrak{X}_{\mathscr{O}_K}$ be its standard semistable formal model over $\mathscr{O}_K$. Let $X := X_K \times_K C$.

Theorem 5.1. *Let $i \geq 0$.*

(1) *There is a $G \times \mathscr{G}_K$-equivariant topological isomorphism*

$$r_{\text{ét}} : \mathrm{Sp}_i(\mathbf{Z}_p)^* \xrightarrow{\sim} H^i_{\text{ét}}(X, \mathbf{Z}_p(i)).$$

It is compatible with the rational isomorphism

$$r_{\text{ét}} : \mathrm{Sp}_i(\mathbf{Z}_p)^* \otimes \mathbf{Q}_p \xrightarrow{\sim} H^i_{\text{ét}}(X, \mathbf{Q}_p(i))$$

from [10].

(2) *There is a $G \times \mathscr{G}_K$-equivariant topological isomorphism*

$$\overline{r}_{\text{ét}} : \mathrm{Sp}_i(\mathbf{F}_p)^* \xrightarrow{\sim} H^i_{\text{ét}}(X, \mathbf{F}_p(i)).$$

Proof. Set $\mathfrak{X} := \mathfrak{X}_{\mathscr{O}_C}$. For $i = 0$ we set the regulators $r_{\text{ét}}$ and $\overline{r}_{\text{ét}}$ to be the identity on $\mathbf{Z}_p$ and $\mathbf{F}_p$ (after suitable identifications), respectively.

For $i > 0$, using the isomorphism $H^i_{\text{ét}}(X, \mathbf{Z}_p(i)) \xrightarrow{\sim} H^i_{\text{proét}}(X, \widehat{\mathbf{Z}}_p(i))$ [10, proof of Cor. 3.46], we pass to pro-étale cohomology. Now, by theorem 2.9, we have a natural short exact sequence

(5.2)
$$0 \to \frac{H^{i-1}_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})}{(1-\varphi^{-1})} \to H^i_{\text{proét}}(X, \widehat{\mathbf{Z}}_p(i)) \to H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1} \to 0.$$

By Theorem 4.1, we have a topological isomorphism

$$r_{\rm inf} : A_{\rm inf}\widehat{\otimes}_{\mathbf{Z}_p}\mathrm{Sp}_i(\mathbf{Z}_p)^* \xrightarrow{\sim} H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})$$

and this isomorphism is compatible with the action of φ^{-1}. We get topological isomorphisms

$$\begin{aligned}
H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1} &\simeq (A_{\inf}\widehat{\otimes}_{\mathbf{Z}_p}\mathrm{Sp}_i(\mathbf{Z}_p)^*)^{\varphi^{-1}=1}\\
&\simeq A_{\inf}^{\varphi^{-1}=1}\widehat{\otimes}_{\mathbf{Z}_p}\mathrm{Sp}_i(\mathbf{Z}_p)^* \simeq \mathrm{Sp}_i(\mathbf{Z}_p)^*,\\
H^{i-1}_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})/(1-\varphi^{-1}) &\simeq (A_{\inf}\{1\}\widehat{\otimes}_{\mathbf{Z}_p}\mathrm{Sp}_{i-1}(\mathbf{Z}_p)^*)/(1-\varphi^{-1})\\
&\simeq (A_{\inf}\{1\}/(1-\varphi^{-1}))\widehat{\otimes}_{\mathbf{Z}_p}\mathrm{Sp}_i(\mathbf{Z}_p)^* \simeq 0.
\end{aligned}$$

Hence, by the exact sequence (5.2), we get a natural continuous isomorphism $r_{\text{proét}} : \mathrm{Sp}_i(\mathbf{Z}_p)^* \stackrel{\sim}{\to} H^i_{\text{proét}}(X, \widehat{\mathbf{Z}}_p(i))$. Since its composition with the natural map $H^i_{\text{proét}}(X, \widehat{\mathbf{Z}}_p(i)) \stackrel{\sim}{\to} H^i_{\text{ét}}(\mathfrak{X}, A\Omega_{\mathfrak{X}}\{i\})^{\varphi^{-1}=1}$ is a topological isomorphism so is the map $r_{\text{proét}}$ itself, as wanted in claim (1).

The last sentence of claim (1) of the theorem is clear (since the integral and the rational étale regulators are compatible).

For claim (2), we define the regulator $\overline{r}_{\text{ét}}$ in an analogous way to its integral version $r_{\text{ét}}$ (with which it is compatible by construction). Since $\mathrm{Sp}_i(\mathbf{F}_p)^* \simeq \mathrm{Sp}_i(\mathbf{Z}_p)^* \otimes \mathbf{F}_p$ and $H^i_{\text{ét}}(X, \mathbf{F}_p(i)) \simeq H^i_{\text{ét}}(X, \mathbf{Z}_p(i)) \otimes \mathbf{F}_p$ (the latter isomorphism by claim (1), which shows that $H^i_{\text{ét}}(X, \mathbf{Z}_p(i))$ is p-torsion free), we have $\overline{r}_{\text{ét}} \simeq r_{\text{ét}} \otimes \mathrm{Id}_{\mathbf{F}_p}$. Hence, by claim (1), $\overline{r}_{\text{ét}}$ is an isomorphism, as wanted. □

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